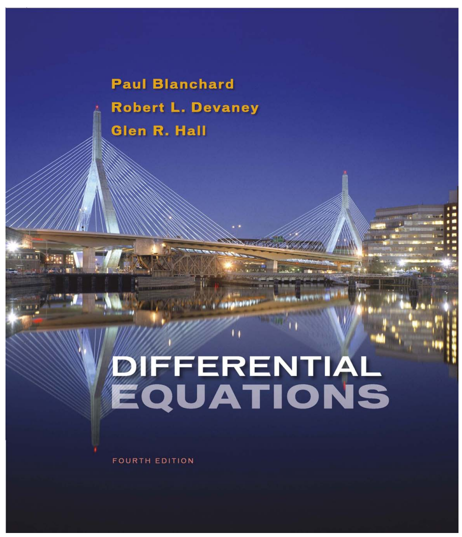
A Solution Manual For

DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012



Nasser M. Abbasi

May 15, 2024

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1.1 problem 1

Internal problem ID [12865]
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Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y+1}{1+t} = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{y+1}{1+t} \end{aligned}$$

Where $f(t) = \frac{1}{1+t}$ and $g(y) = y + 1$. Integrating both sides gives

$$\frac{1}{y+1}\, dy = \frac{1}{1+t}\, dt$$
$$\int \frac{1}{y+1}\, dy = \int \frac{1}{1+t}\, dt$$
$$\ln (y+1) = \ln (1+t) + c_1$$

Raising both side to exponential gives

$$y + 1 = e^{\ln(1+t)+c_1}$$

Which simplifies to

$$y + 1 = c_2(1 + t)$$

Summary
The solution(s) found are the following

$$y = c_2 e^{\ln(1+t)+c_1} - 1 \tag{1}$$

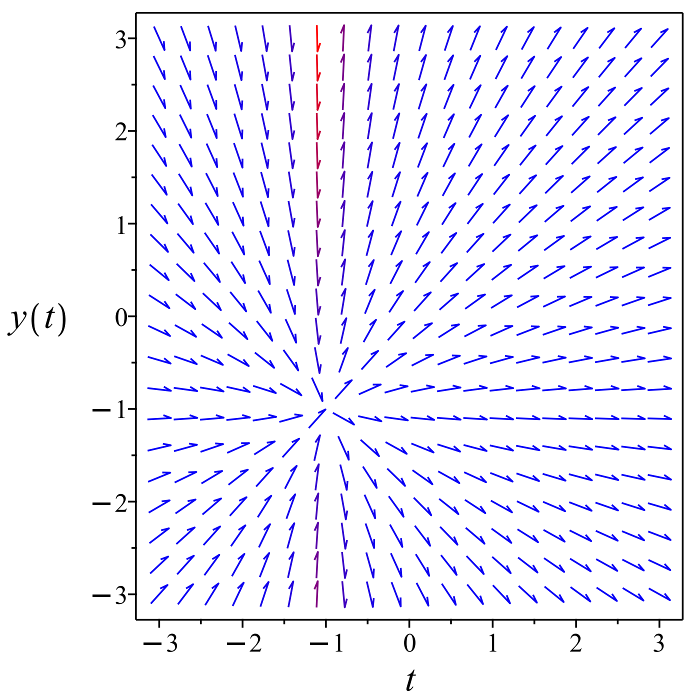


Figure 1: Slope field plot

Verification of solutions

$$y = c_2 e^{\ln(1+t)+c_1} - 1$$

Verified OK.

1.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned}p(t) &= -\frac{1}{1+t}\\ q(t) &= \frac{1}{1+t}\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{1+t} = \frac{1}{1+t}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{1+t}dt}\\ &= \frac{1}{1+t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{1}{1+t}\right)\\ \frac{d}{dt}\left(\frac{y}{1+t}\right) &= \left(\frac{1}{1+t}\right)\left(\frac{1}{1+t}\right)\\ d\left(\frac{y}{1+t}\right) &= \frac{1}{(1+t)^2}\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{1+t} &= \int \frac{1}{(1+t)^2}\, dt\\ \frac{y}{1+t} &= -\frac{1}{1+t} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{1+t}$ results in

$$y = -1 + c_1(1+t)$$

which simplifies to

$$y = c_1 t + c_1 - 1$$

Summary

The solution(s) found are the following

$$y = c_1 t + c_1 - 1 \tag{1}$$

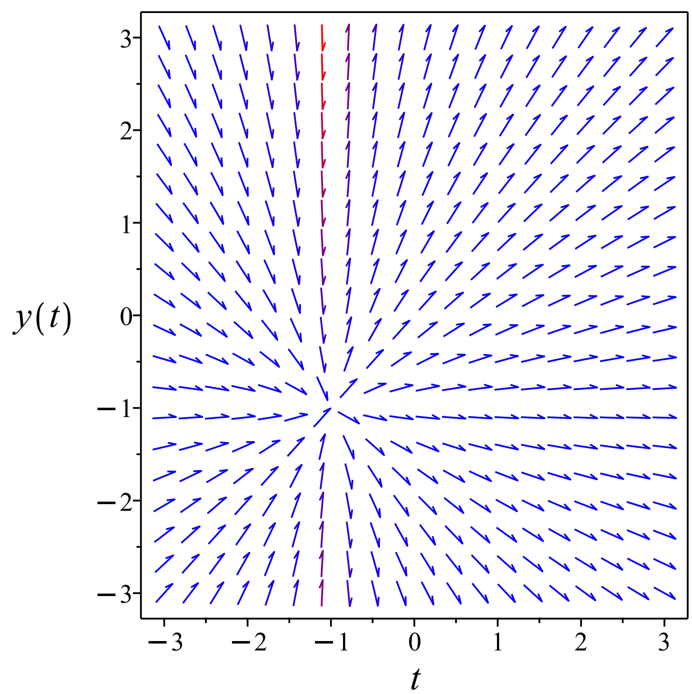


Figure 2: Slope field plot

Verification of solutions

$$y = c_1 t + c_1 - 1$$

Verified OK.

1.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) - \frac{u(t)\,t + 1}{1+t} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{-u+1}{t\,(1+t)} \end{aligned}$$

Where $f(t) = \frac{1}{t(1+t)}$ and $g(u) = -u+1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-u+1}\,du &= \frac{1}{t\,(1+t)}\,dt \\ \int \frac{1}{-u+1}\,du &= \int \frac{1}{t\,(1+t)}\,dt \\ -\ln(u-1) &= -\ln(1+t) + \ln(t) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{u-1} = \mathrm{e}^{-\ln(1+t)+\ln(t)+c_2}$$

Which simplifies to

$$\frac{1}{u-1} = c_3 \mathrm{e}^{-\ln(1+t)+\ln(t)}$$

Which simplifies to

$$u(t) = \frac{\left(\frac{c_3 \mathrm{e}^{c_2} t}{1+t} + 1\right)(1+t)\,\mathrm{e}^{-c_2}}{c_3 t}$$

Therefore the solution y is

$$\begin{aligned} y &= ut \\ &= \frac{\left(\frac{c_3 \mathrm{e}^{c_2} t}{1+t} + 1\right)(1+t)\,\mathrm{e}^{-c_2}}{c_3} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\frac{c_3 \mathrm{e}^{c_2 t}}{1+t} + 1\right)(1+t)\,\mathrm{e}^{-c_2}}{c_3} \tag{1}$$

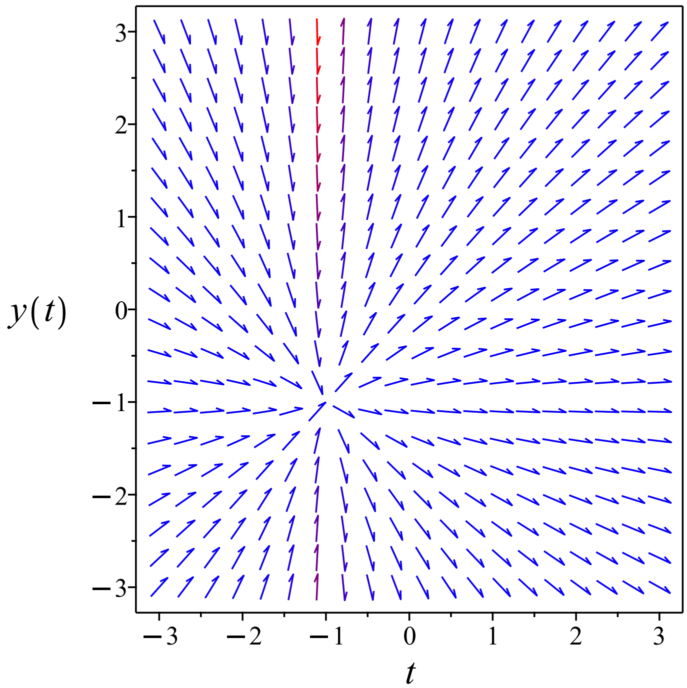


Figure 3: Slope field plot

Verification of solutions

$$y = \frac{\left(\frac{c_3 \mathrm{e}^{c_2 t}}{1+t} + 1\right)(1+t)\,\mathrm{e}^{-c_2}}{c_3}$$

Verified OK.

1.1.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = t + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0 + 1}{1 + X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= 0 \end{aligned}$$

Or

$$\frac{d}{dX}u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. Integrating both sides gives

$$\begin{aligned} u(X) &= \int 0 \,\mathrm{d}X \\ &= c_2 \end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X c_2$$

Using the solution for $Y(X)$

$$Y(X) = X c_2$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = t + x_0$$

Or

$$Y = y - 1$$
$$X = t - 1$$

Then the solution in y becomes

$$y + 1 = c_2(1 + t)$$

Summary

The solution(s) found are the following

$$y + 1 = c_2(1 + t) \tag{1}$$

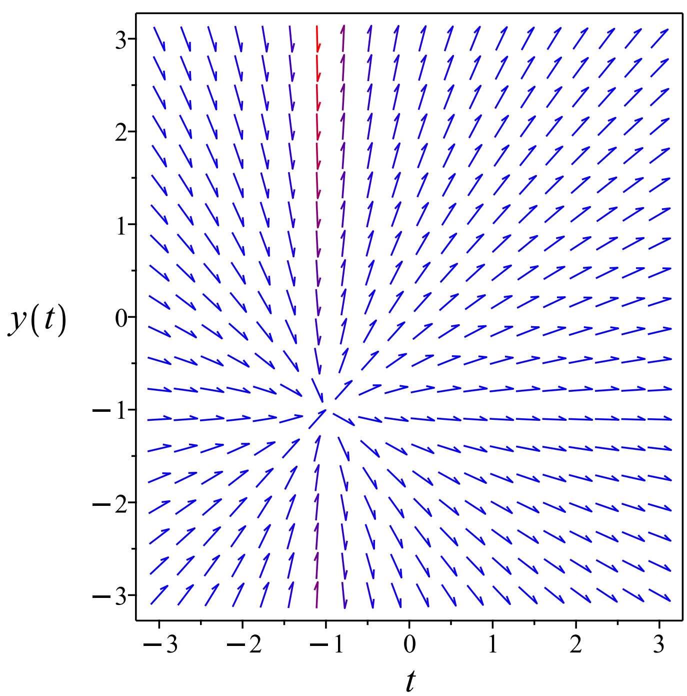


Figure 4: Slope field plot

Verification of solutions

$$y + 1 = c_2(1 + t)$$

Verified OK.

1.1.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y+1}{1+t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= 1 + t
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1+t} dy \end{aligned}$$

Which results in

$$S = \frac{y}{1+t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{y+1}{1+t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{y}{(1+t)^2} \\ S_y &= \frac{1}{1+t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(1+t)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(1+R)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{1+R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{1+t} = -\frac{1}{1+t} + c_1$$

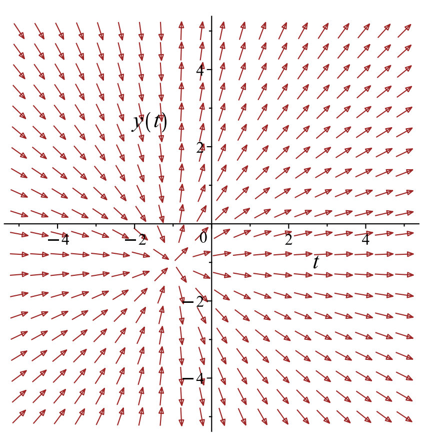
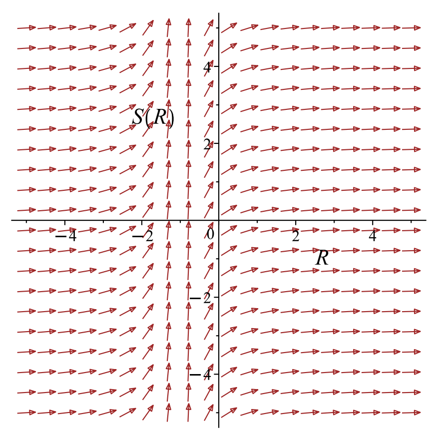
Which simplifies to

$$\frac{y}{1+t} = -\frac{1}{1+t} + c_1$$

Which gives

$$y = c_1 t + c_1 - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{y+1}{1+t}$ 	$R = t$ $S = \frac{y}{1+t}$	$\frac{dS}{dR} = \frac{1}{(1+R)^2}$ 

Summary

The solution(s) found are the following

$$y = c_1 t + c_1 - 1 \tag{1}$$

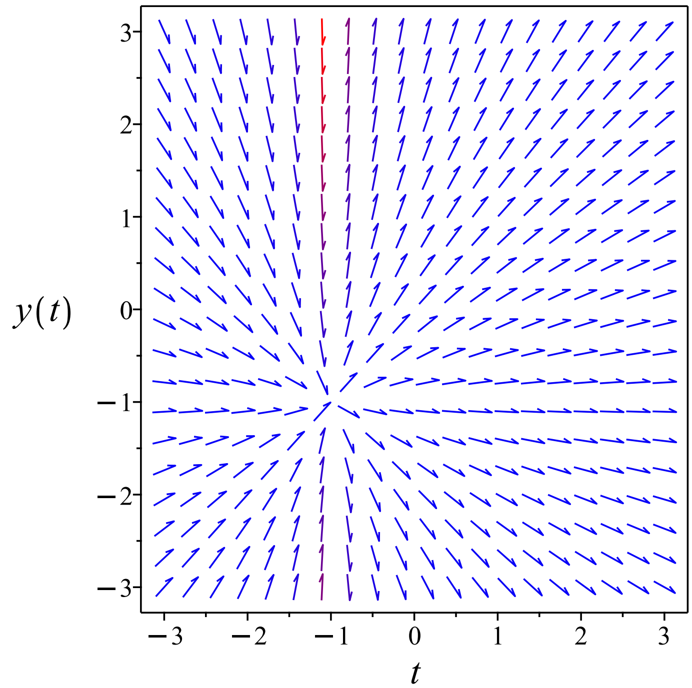


Figure 5: Slope field plot

Verification of solutions

$$y = c_1 t + c_1 - 1$$

Verified OK.

1.1.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y+1}\right)\mathrm{d}y &= \left(\frac{1}{1+t}\right)\mathrm{d}t \\ \left(-\frac{1}{1+t}\right)\mathrm{d}t + \left(\frac{1}{y+1}\right)\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -\frac{1}{1+t} \\ N(t, y) &= \frac{1}{y+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{1+t}\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{y+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int M\, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int -\frac{1}{1+t}\, \mathrm{d}t$$
$$\phi = -\ln(1+t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+1}$. Therefore equation (4) becomes

$$\frac{1}{y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{1}{y+1}\right) \mathrm{d}y$$
$$f(y) = \ln(y+1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(1+t) + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(1+t) + \ln(y+1)$$

The solution becomes

$$y = t\,\mathrm{e}^{c_1} + \mathrm{e}^{c_1} - 1$$

Summary

The solution(s) found are the following

$$y = t\,\mathrm{e}^{c_1} + \mathrm{e}^{c_1} - 1 \tag{1}$$

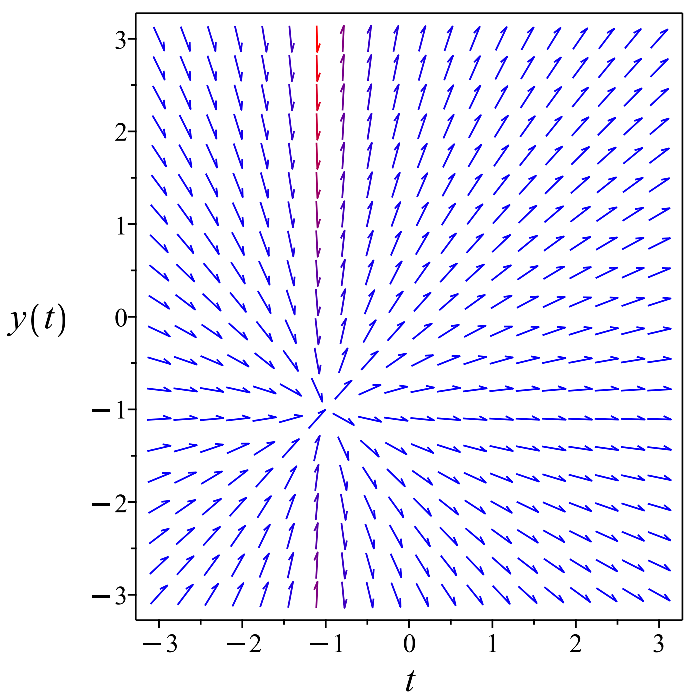


Figure 6: Slope field plot

Verification of solutions

$$y = t\,\mathrm{e}^{c_1} + \mathrm{e}^{c_1} - 1$$

Verified OK.

1.1.7 Maple step by step solution

Let's solve

$y' - \frac{y+1}{1+t} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1} = \frac{1}{1+t}$

- Integrate both sides with respect to t

  $\int \frac{y'}{y+1}dt = \int \frac{1}{1+t}dt + c_1$

- Evaluate integral

  $\ln(y+1) = \ln(1+t) + c_1$

- Solve for y

  $y = t\,\mathrm{e}^{c_1} + \mathrm{e}^{c_1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(t),t)=(y(t)+1)/(t+1),y(t), singsol=all)
```

$$y(t) = c_1 t + c_1 - 1$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 18

```
DSolve[y'[t]==(y[t]+1)/(t+1),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -1 + c_1(t + 1)$$
$$y(t) \to -1$$

1.2 problem 5

Internal problem ID [12866]
Internal file name [`OUTPUT/11518_Monday_November_06_2023_01_31_15_PM_71617828/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - t^2y^2 = 0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= y^2t^2 \end{aligned}$$

Where $f(t) = t^2$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\, dy = t^2\, dt$$

$$\int \frac{1}{y^2}\, dy = \int t^2\, dt$$

$$-\frac{1}{y} = \frac{t^3}{3} + c_1$$

Which results in

$$y = -\frac{3}{t^3 + 3c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{t^3 + 3c_1} \tag{1}$$

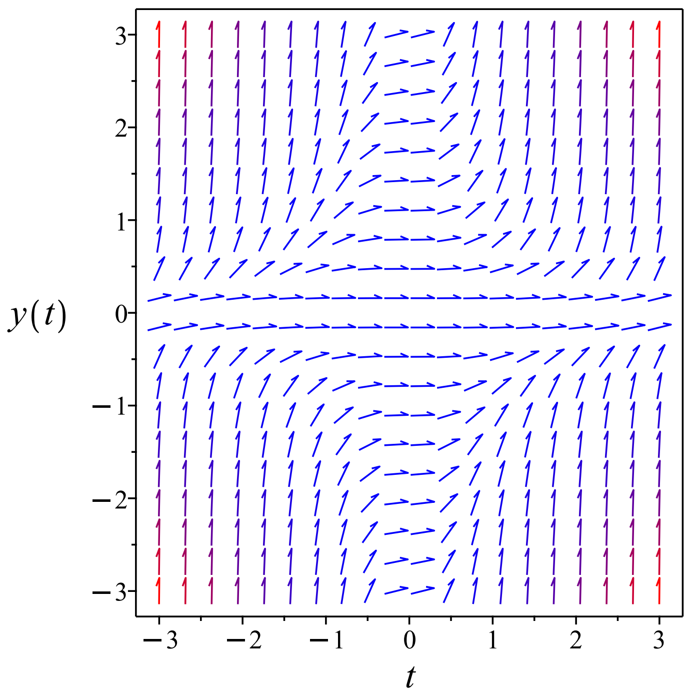


Figure 7: Slope field plot

Verification of solutions

$$y = -\frac{3}{t^3 + 3c_1}$$

Verified OK.

1.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y^2t^2 \\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= \frac{1}{t^2} \\ \eta(t,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{t^2}} dt\end{aligned}$$

Which results in

$$S = \frac{t^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = y^2 t^2$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= t^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{t^3}{3} = -\frac{1}{y} + c_1$$

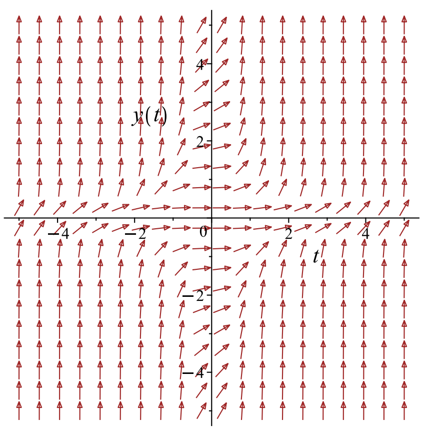
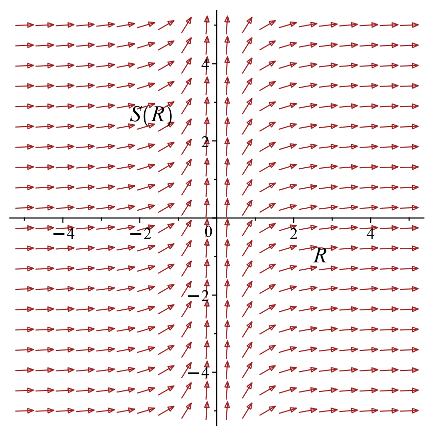
Which simplifies to

$$\frac{t^3}{3} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{3}{-t^3 + 3c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y^2 t^2$ 	$R = y$ $S = \frac{t^3}{3}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{3}{-t^3 + 3c_1} \tag{1}$$

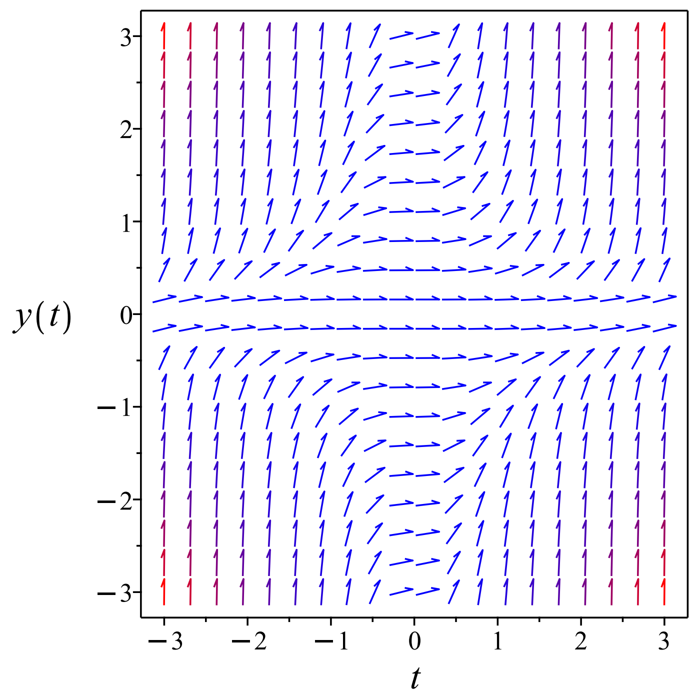


Figure 8: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{3}{-t^3 + 3c_1}$$

Verified OK.

1.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2}\right)\mathrm{d}y &= \left(t^2\right)\mathrm{d}t\\ \left(-t^2\right)\mathrm{d}t + \left(\frac{1}{y^2}\right)\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -t^2\\ N(t, y) &= \frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-t^2\right)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{y^2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t^2 \, \mathrm{d}t$$
$$\phi = -\frac{t^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2}\right)\mathrm{d}y$$
$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} - \frac{1}{y}$$

The solution becomes

$$y = -\frac{3}{t^3 + 3c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{t^3 + 3c_1} \tag{1}$$

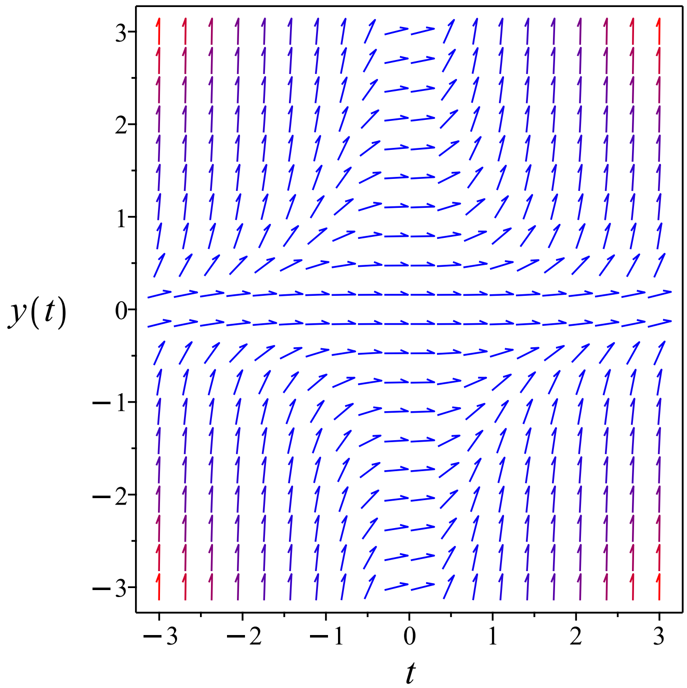


Figure 9: Slope field plot

Verification of solutions

$$y = -\frac{3}{t^3 + 3c_1}$$

Verified OK.

1.2.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= y^2 t^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 t^2$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = t^2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{t^2 u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2t \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$t^2 u''(t) - 2tu'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_2 t^3 + c_1$$

The above shows that

$$u'(t) = 3c_2 t^2$$

Using the above in (1) gives the solution

$$y = -\frac{3c_2}{c_2 t^3 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{3}{t^3 + c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{t^3 + c_3} \tag{1}$$

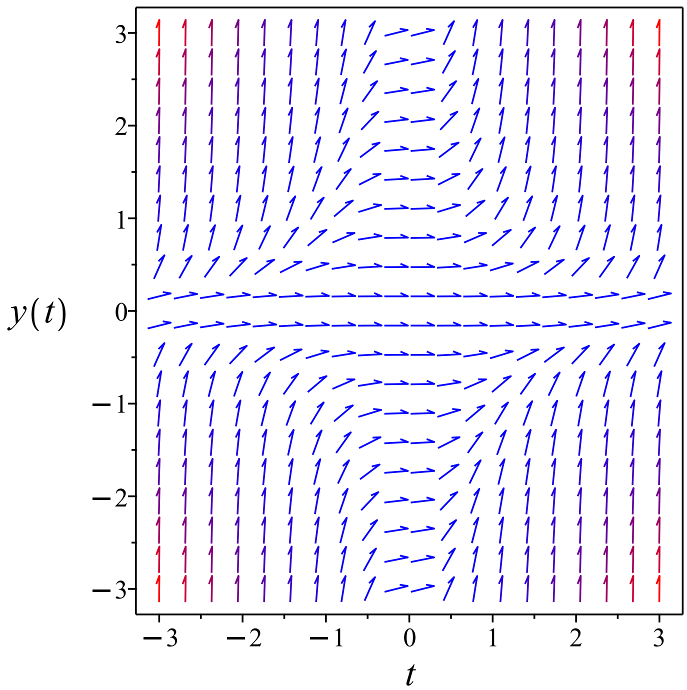


Figure 10: Slope field plot

Verification of solutions

$$y = -\frac{3}{t^3 + c_3}$$

Verified OK.

1.2.5 Maple step by step solution

Let's solve

$y' - t^2 y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = t^2$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int t^2 dt + c_1$

- Evaluate integral

  $-\frac{1}{y} = \frac{t^3}{3} + c_1$

- Solve for y

  $y = -\frac{3}{t^3+3c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(t),t)=(t*y(t))^2,y(t), singsol=all)
```

$$y(t) = -\frac{3}{t^3 - 3c_1}$$

✓ Solution by Mathematica

Time used: 0.214 (sec). Leaf size: 22

```
DSolve[y'[t]==(t*y[t])^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{3}{t^3 + 3c_1}$$
$$y(t) \to 0$$

1.3 problem 6

Internal problem ID [12867]
Internal file name [`OUTPUT/11519_Monday_November_06_2023_01_31_16_PM_12392343/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - t^4 y = 0}$$

1.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= t^4 y \end{aligned}$$

Where $f(t) = t^4$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= t^4\, dt \\
\int \frac{1}{y}\, dy &= \int t^4\, dt \\
\ln(y) &= \frac{t^5}{5} + c_1 \\
y &= e^{\frac{t^5}{5} + c_1} \\
&= c_1 e^{\frac{t^5}{5}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{t^5}{5}} \tag{1}$$

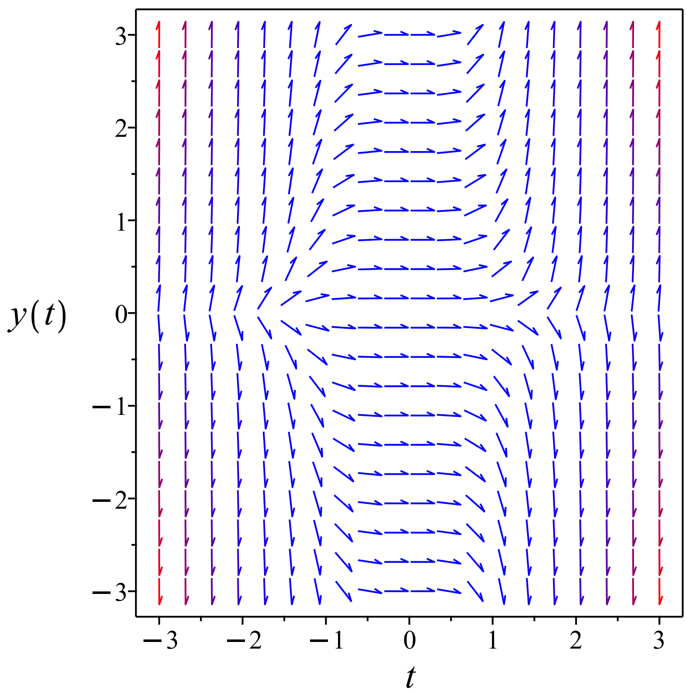


Figure 11: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{t^5}{5}}$$

Verified OK.

1.3.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -t^4 \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$y' - t^4 y = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -t^4 dt} \\ &= \mathrm{e}^{-\frac{t^5}{5}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\frac{t^5}{5}} y\right) &= 0 \end{aligned}$$

Integrating gives

$$\mathrm{e}^{-\frac{t^5}{5}} y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{t^5}{5}}$ results in

$$y = c_1 \mathrm{e}^{\frac{t^5}{5}}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{t^5}{5}} \tag{1}$$

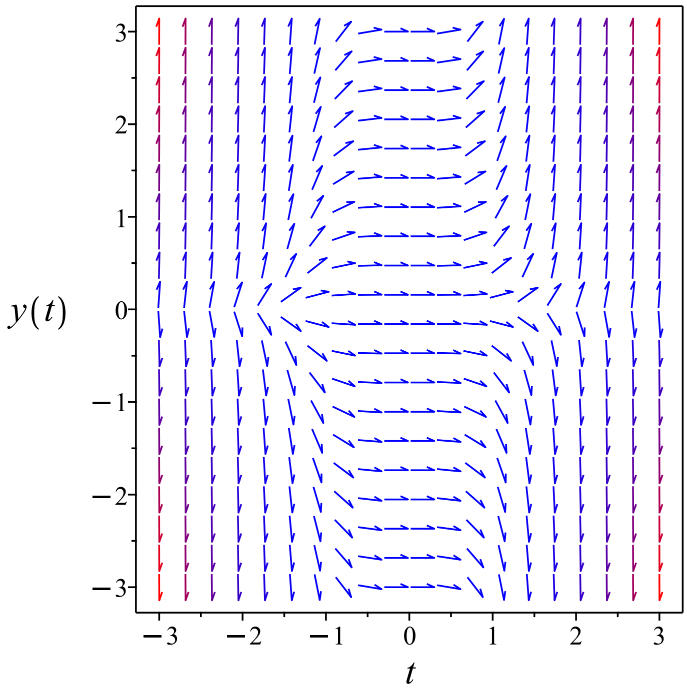


Figure 12: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{t^5}{5}}$$

Verified OK.

1.3.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) - t^5 u(t) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= \frac{u(t^5 - 1)}{t} \end{aligned}$$

Where $f(t) = \frac{t^5-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{t^5-1}{t}\, dt \\
\int \frac{1}{u}\, du &= \int \frac{t^5-1}{t}\, dt \\
\ln(u) &= \frac{t^5}{5} - \ln(t) + c_2 \\
u &= e^{\frac{t^5}{5} - \ln(t) + c_2} \\
&= c_2 e^{\frac{t^5}{5} - \ln(t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_2 e^{\frac{t^5}{5}}}{t}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= tu \\
&= c_2 e^{\frac{t^5}{5}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 e^{\frac{t^5}{5}} \tag{1}
$$

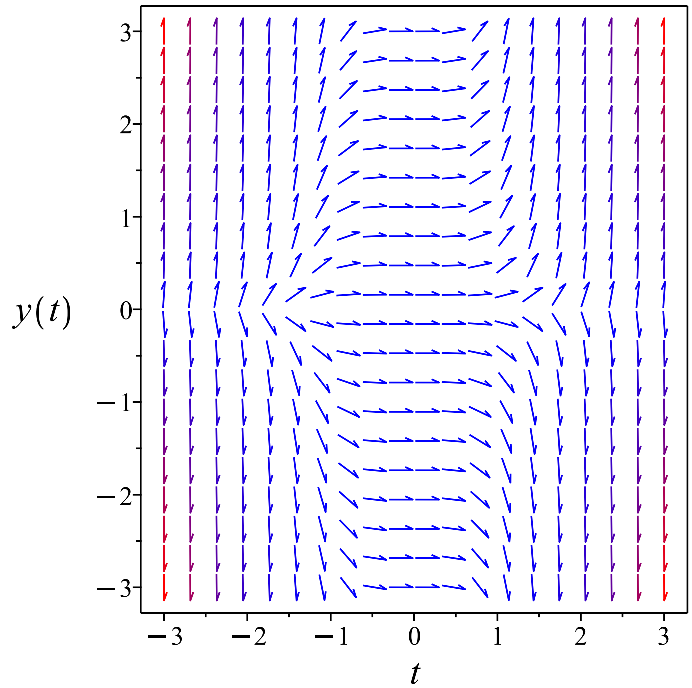


Figure 13: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{t^5}{5}}$$

Verified OK.

1.3.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = t^4 y$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t, y) = 0$$
$$\eta(t, y) = e^{\frac{t^5}{5}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{t^5}{5}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{t^5}{5}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = t^4 y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -t^4 e^{-\frac{t^5}{5}} y \\ S_y &= e^{-\frac{t^5}{5}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-\frac{t^5}{5}} y = c_1$$

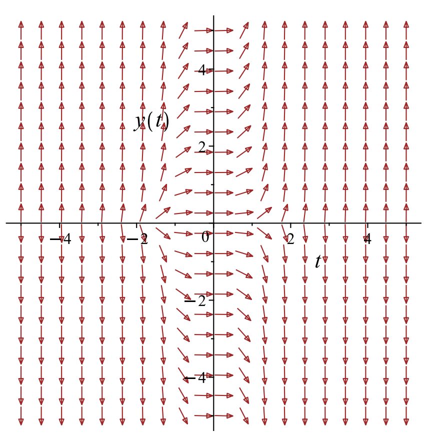
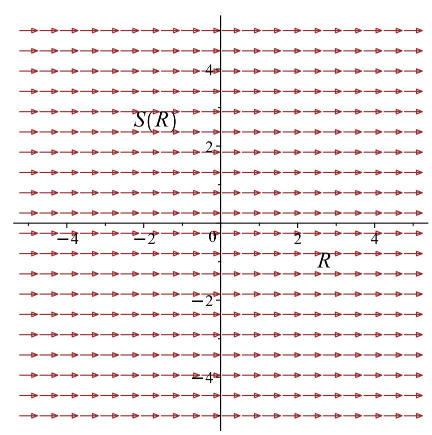
Which simplifies to

$$e^{-\frac{t^5}{5}} y = c_1$$

Which gives

$$y = c_1 e^{\frac{t^5}{5}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = t^4 y$ 	$R = t$ $S = e^{-\frac{t^5}{5}} y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{t^5}{5}} \tag{1}$$

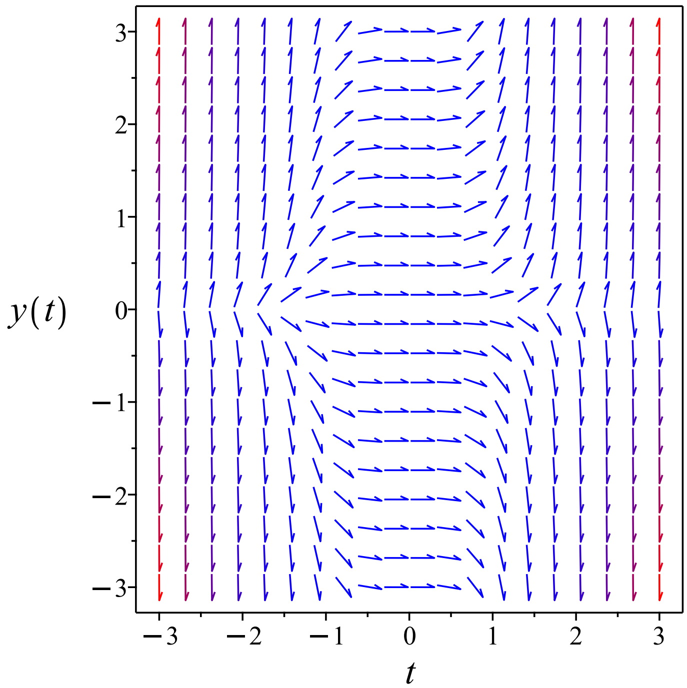


Figure 14: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{t^5}{5}}$$

Verified OK.

1.3.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y}\right) \mathrm{d}y &= \left(t^4\right) \mathrm{d}t \\
\left(-t^4\right) \mathrm{d}t + \left(\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -t^4 \\
N(t, y) &= \frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-t^4\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -t^4\,\mathrm{d}t\\ \phi &= -\frac{t^5}{5} + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y}\right)\mathrm{d}y\\ f(y) &= \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^5}{5} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^5}{5} + \ln(y)$$

The solution becomes

$$y = e^{\frac{t^5}{5} + c_1}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{t^5}{5} + c_1} \tag{1}$$

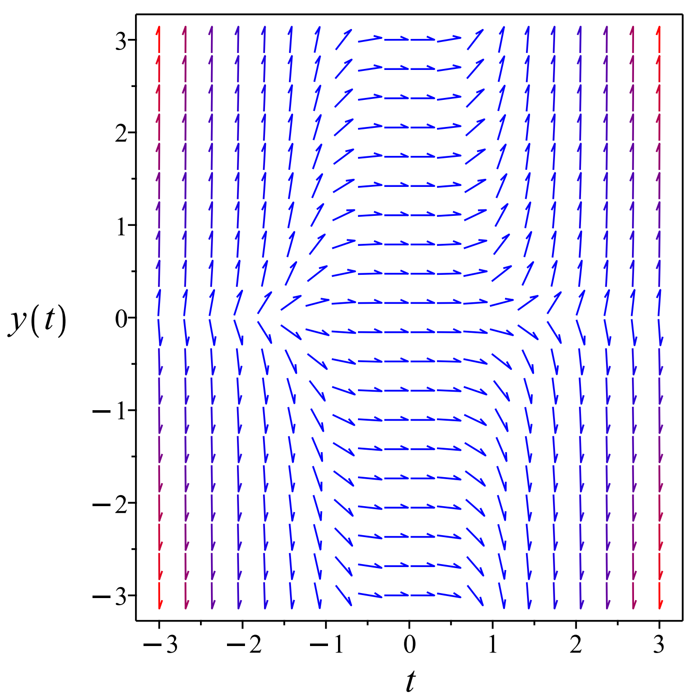


Figure 15: Slope field plot

Verification of solutions

$$y = e^{\frac{t^5}{5}+c_1}$$

Verified OK.

1.3.6 Maple step by step solution

Let's solve

$y' - t^4 y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = t^4$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int t^4 dt + c_1$

- Evaluate integral

  $\ln(y) = \frac{t^5}{5} + c_1$

- Solve for y

  $y = e^{\frac{t^5}{5}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)=t^4*y(t),y(t), singsol=all)
```

$$y(t) = c_1 e^{\frac{t^5}{5}}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 22

```
DSolve[y'[t]==t^4*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^{\frac{t^5}{5}}$$
$$y(t) \to 0$$

1.4 problem 7

Internal problem ID [12868]
Internal file name [`OUTPUT/11520_Monday_November_06_2023_01_31_17_PM_96256138/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - 2y = 1}$$

1.4.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{2y+1} dy = \int dt$$
$$\frac{\ln(2y+1)}{2} = t + c_1$$

Raising both side to exponential gives

$$\sqrt{2y+1} = e^{t+c_1}$$

Which simplifies to

$$\sqrt{2y+1} = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = \frac{c_2^2 e^{2t}}{2} - \frac{1}{2} \tag{1}$$

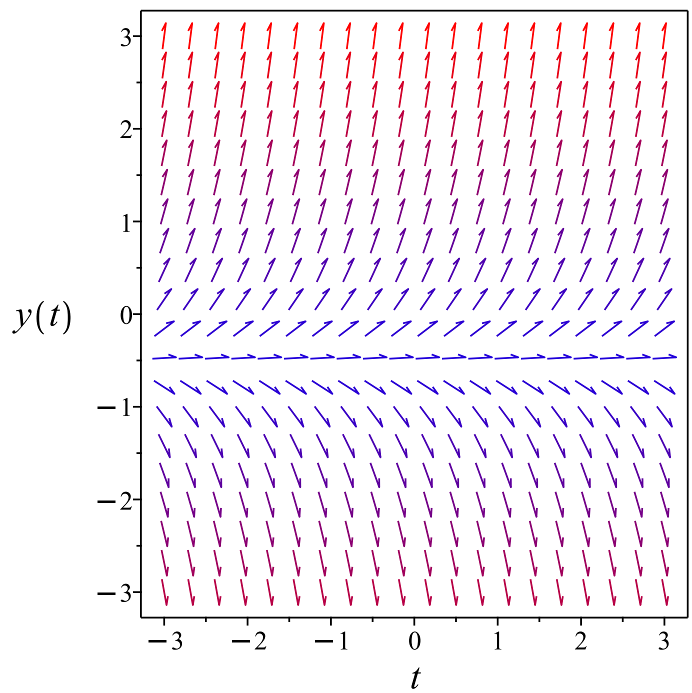


Figure 16: Slope field plot

Verification of solutions

$$y = \frac{c_2^2 e^{2t}}{2} - \frac{1}{2}$$

Verified OK.

1.4.2 Maple step by step solution

Let's solve

$y' - 2y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y+1} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{2y+1} dt = \int 1 dt + c_1$

- Evaluate integral

$\frac{\ln(2y+1)}{2} = t + c_1$

- Solve for y

  $y = -\frac{1}{2} + \frac{e^{2t+2c_1}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)=2*y(t)+1,y(t), singsol=all)
```

$$y(t) = -\frac{1}{2} + c_1 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 24

```
DSolve[y'[t]==2*y[t]+1,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{2} + c_1 e^{2t}$$
$$y(t) \to -\frac{1}{2}$$

1.5 problem 8

Internal problem ID [12869]
Internal file name [`OUTPUT/11521_Monday_November_06_2023_01_31_17_PM_12857361/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' + y = 2}$$

1.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y+2}dy = \int dt$$
$$-\ln(-y+2) = t + c_1$$

Raising both side to exponential gives

$$\frac{1}{-y+2} = e^{t+c_1}$$

Which simplifies to

$$\frac{1}{-y+2} = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-t}}{c_2} + 2 \tag{1}$$

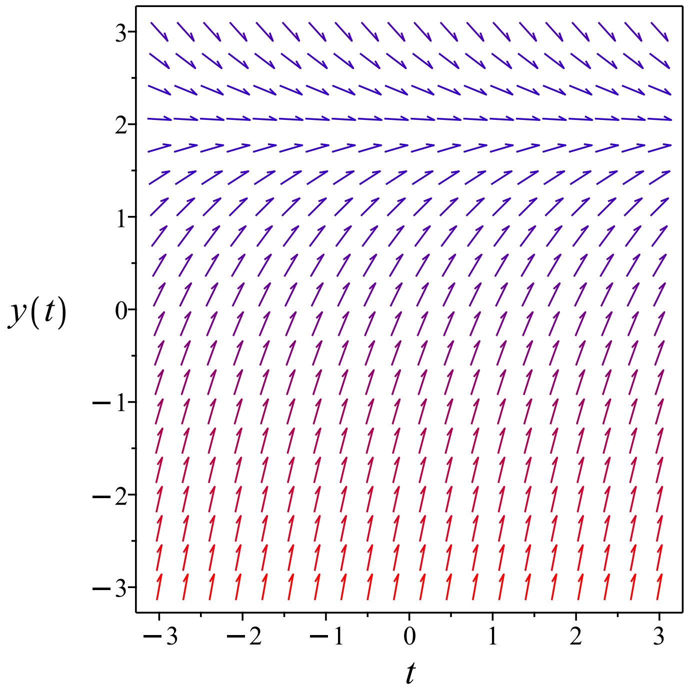


Figure 17: Slope field plot

Verification of solutions

$$y = -\frac{e^{-t}}{c_2} + 2$$

Verified OK.

1.5.2 Maple step by step solution

Let's solve

$y' + y = 2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2-y} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{2-y} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\ln(2-y) = t + c_1$$

- Solve for y

$$y = -e^{-t-c_1} + 2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)=2-y(t),y(t), singsol=all)
```

$$y(t) = 2 + e^{-t}c_1$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 20

```
DSolve[y'[t]==2-y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2 + c_1 e^{-t}$$
$$y(t) \to 2$$

1.6 problem 9

Internal problem ID [12870]
Internal file name [`OUTPUT/11522_Monday_November_06_2023_01_31_17_PM_23065635/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - e^{-y} = 0}$$

1.6.1 Solving as quadrature ode

Integrating both sides gives

$$\int e^{y} dy = t + c_1$$
$$e^{y} = t + c_1$$

Solving for y gives these solutions

$$y_1 = \ln(t + c_1)$$

Summary
The solution(s) found are the following

$$y = \ln(t + c_1) \tag{1}$$

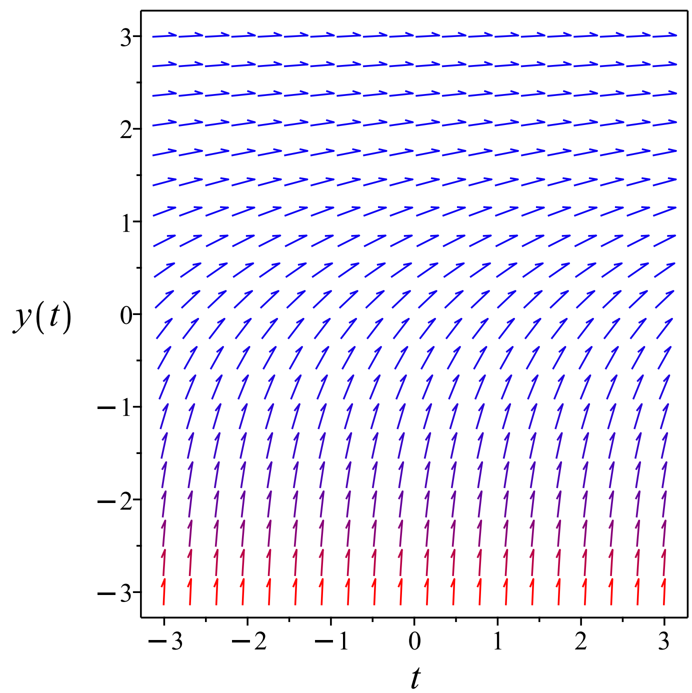


Figure 18: Slope field plot

Verification of solutions

$$y = \ln(t + c_1)$$

Verified OK.

1.6.2 Maple step by step solution

Let's solve

$y' - e^{-y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{e^{-y}} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{e^{-y}} dt = \int 1 dt + c_1$

- Evaluate integral

$$\frac{1}{e^{-y}} = t + c_1$$

- Solve for y

$$y = \ln(t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 8

```
dsolve(diff(y(t),t)=exp(-y(t)),y(t), singsol=all)
```

$$y(t) = \ln(t + c_1)$$

✓ Solution by Mathematica

Time used: 0.369 (sec). Leaf size: 10

```
DSolve[y'[t]==Exp[-y[t]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \log(t + c_1)$$

1.7 problem 10

Internal problem ID [12871]
Internal file name [`OUTPUT/11523_Monday_November_06_2023_01_31_18_PM_88496519/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x' - x^2 = 1}$$

1.7.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{x^2+1}dx = t + c_1$$
$$\arctan(x) = t + c_1$$

Solving for x gives these solutions

$$x_1 = \tan(t + c_1)$$

Summary
The solution(s) found are the following

$$x = \tan(t + c_1) \tag{1}$$

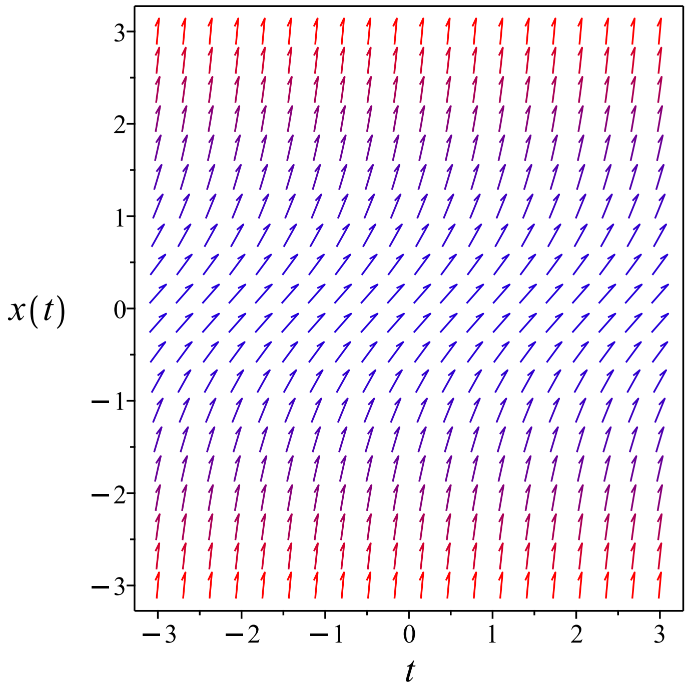


Figure 19: Slope field plot

Verification of solutions

$$x = \tan(t + c_1)$$

Verified OK.

1.7.2 Maple step by step solution

Let's solve

$x' - x^2 = 1$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{1+x^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{x'}{1+x^2} dt = \int 1 dt + c_1$

- Evaluate integral

$$\arctan(x) = t + c_1$$

- Solve for x

$$x = \tan(t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 8

```
dsolve(diff(x(t),t)=1+x(t)^2,x(t), singsol=all)
```

$$x(t) = \tan(t + c_1)$$

✓ Solution by Mathematica

Time used: 0.222 (sec). Leaf size: 24

```
DSolve[x'[t]==1+x[t]^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \tan(t + c_1)$$
$$x(t) \to -i$$
$$x(t) \to i$$

1.8 problem 11

Internal problem ID [12872]
Internal file name [`OUTPUT/11524_Monday_November_06_2023_01_31_18_PM_1173136/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2ty^2 - 3y^2 = 0}$$

1.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= y^2(2t+3) \end{aligned}$$

Where $f(t) = 2t + 3$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = 2t + 3\,dt$$

$$\int \frac{1}{y^2}\,dy = \int 2t + 3\,dt$$

$$-\frac{1}{y} = t^2 + c_1 + 3t$$

Which results in

$$y = -\frac{1}{t^2 + c_1 + 3t}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{t^2 + c_1 + 3t} \tag{1}$$

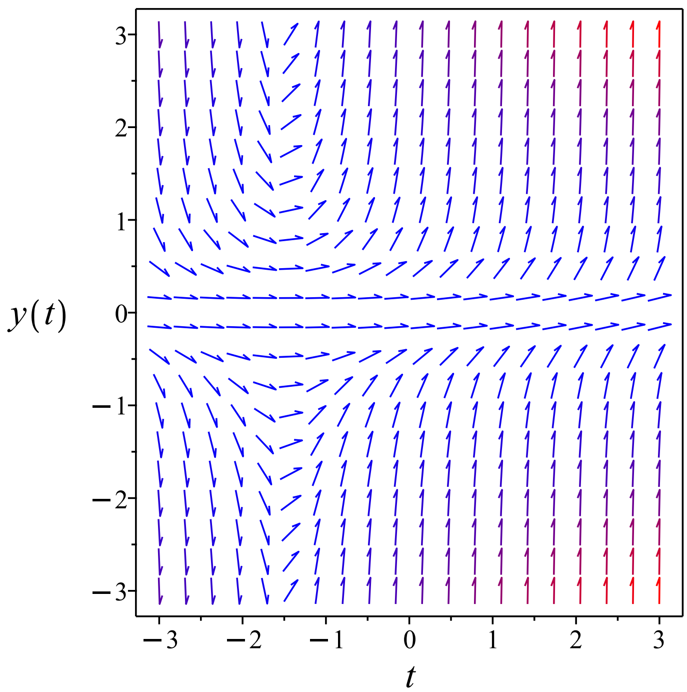


Figure 20: Slope field plot

Verification of solutions

$$y = -\frac{1}{t^2 + c_1 + 3t}$$

Verified OK.

1.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= 2t\,y^2 + 3y^2\\ y' &= \omega(t,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 14: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= \frac{1}{2t+3}\\ \eta(t,y) &= 0\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dt\\ &= \int \frac{1}{\frac{1}{2t+3}} dt\end{aligned}$$

Which results in

$$S = t^2 + 3t$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y)S_y}{R_t + \omega(t, y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = 2t\,y^2 + 3y^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 0\\ R_y &= 1\\ S_t &= 2t + 3\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$t^2 + 3t = -\frac{1}{y} + c_1$$

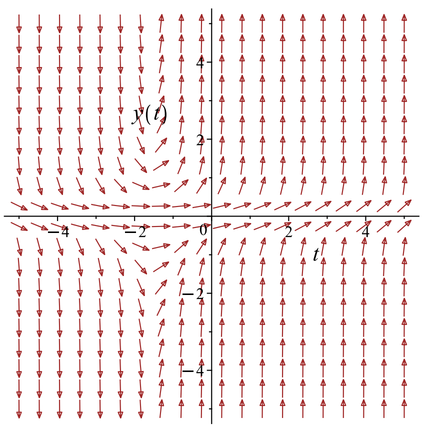
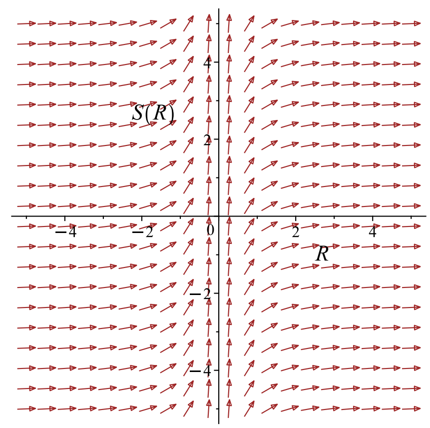
Which simplifies to

$$t^2 + 3t = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{1}{-t^2 + c_1 - 3t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2t\, y^2 + 3y^2$ 	$R = y$ $S = t^2 + 3t$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{1}{-t^2 + c_1 - 3t} \tag{1}$$

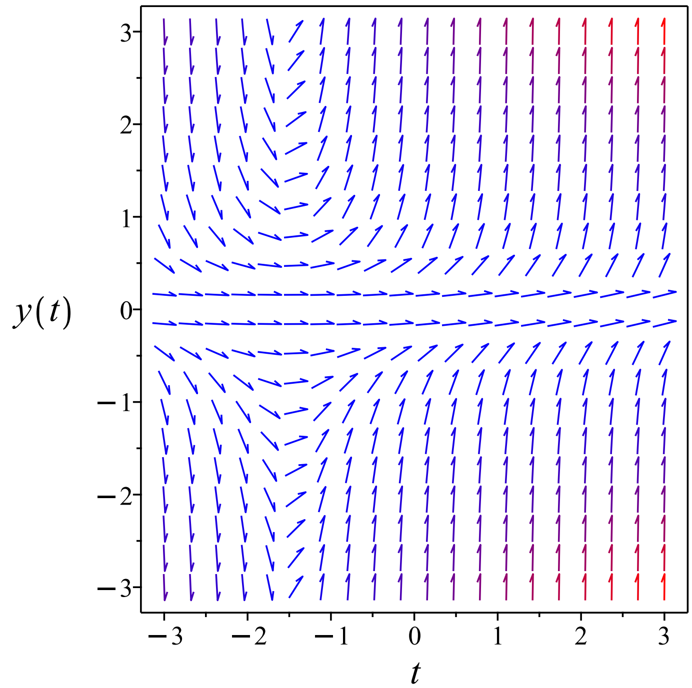


Figure 21: Slope field plot

Verification of solutions

$$y = \frac{1}{-t^2 + c_1 - 3t}$$

Verified OK.

1.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2}\right)\mathrm{d}y &= (2t+3)\,\mathrm{d}t\\ (-2t-3)\,\mathrm{d}t + \left(\frac{1}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -2t-3\\ N(t,y) &= \frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2t-3)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{y^2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -2t - 3 \, \mathrm{d}t$$
$$\phi = -t^2 - 3t + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y^2}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -t^2 - 3t - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t^2 - 3t - \frac{1}{y}$$

The solution becomes

$$y = -\frac{1}{t^2 + c_1 + 3t}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{t^2 + c_1 + 3t} \tag{1}$$

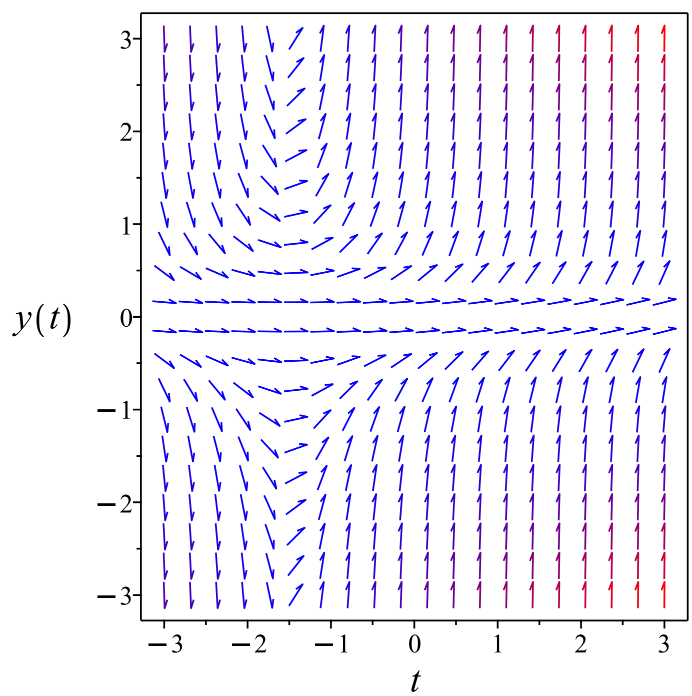


Figure 22: Slope field plot

Verification of solutions

$$y = -\frac{1}{t^2 + c_1 + 3t}$$

Verified OK.

1.8.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= 2t\, y^2 + 3y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2t\, y^2 + 3y^2$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = 2t + 3$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(2t + 3)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(2t + 3)\, u''(t) - 2u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + c_2\left(t + \frac{3}{2}\right)^2$$

The above shows that

$$u'(t) = c_2(2t + 3)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_1 + c_2\left(t + \frac{3}{2}\right)^2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{4}{4t^2 + 4c_3 + 12t + 9}$$

Summary

The solution(s) found are the following

$$y = -\frac{4}{4t^2 + 4c_3 + 12t + 9} \tag{1}$$

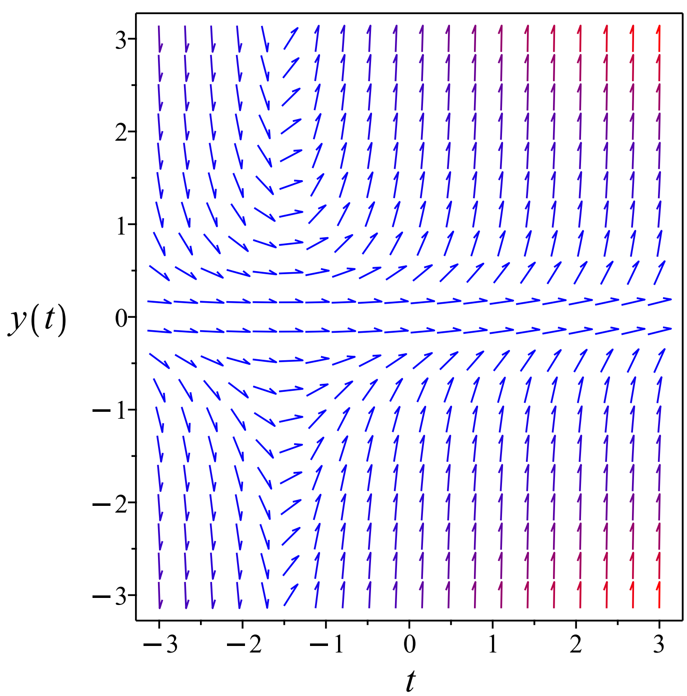


Figure 23: Slope field plot

Verification of solutions

$$y = -\frac{4}{4t^2 + 4c_3 + 12t + 9}$$

Verified OK.

1.8.5 Maple step by step solution

Let's solve

$y' - 2ty^2 - 3y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = 2t + 3$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int (2t + 3)\, dt + c_1$

- Evaluate integral

  $-\frac{1}{y} = t^2 + c_1 + 3t$

- Solve for y

  $y = -\frac{1}{t^2 + c_1 + 3t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)=2*t*y(t)^2+3*y(t)^2,y(t), singsol=all)
```

$$y(t) = \frac{1}{-t^2 + c_1 - 3t}$$

✓ Solution by Mathematica

Time used: 0.218 (sec). Leaf size: 23

```
DSolve[y'[t]==2*t*y[t]^2+3*y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{t^2 + 3t + c_1}$$
$$y(t) \to 0$$

1.9 problem 12

Internal problem ID [12873]
Internal file name [`OUTPUT/11525_Monday_November_06_2023_01_31_19_PM_50875666/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{t}{y} = 0}$$

1.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= \frac{t}{y} \end{aligned}$$

Where $f(t) = t$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = t\,dt$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int t\,dt$$

$$\frac{y^2}{2} = \frac{t^2}{2} + c_1$$

Which results in

$$y = \sqrt{t^2 + 2c_1}$$

$$y = -\sqrt{t^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{t^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{t^2 + 2c_1} \tag{2}$$

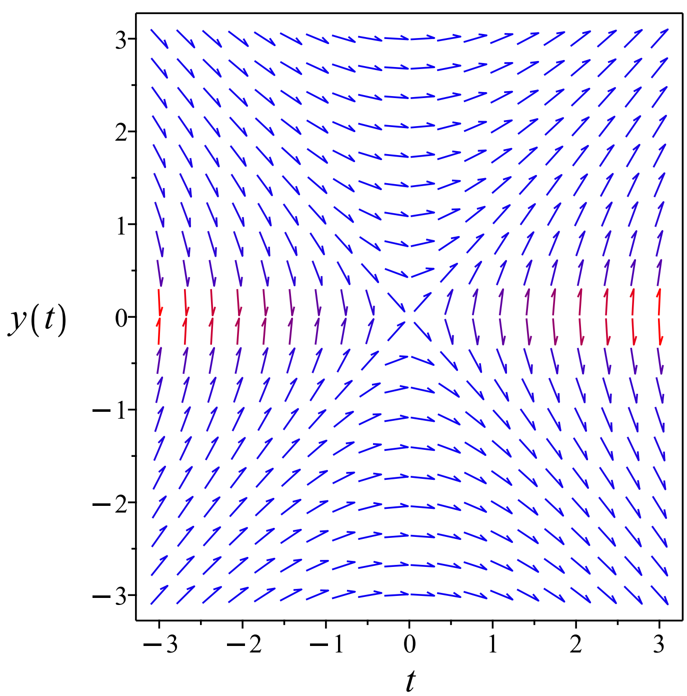


Figure 24: Slope field plot

Verification of solutions

$$y = \sqrt{t^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{t^2 + 2c_1}$$

Verified OK.

1.9.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t + u(t) - \frac{1}{u\,(t)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{u^2-1}{tu} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2-1}{u}}\, du &= -\frac{1}{t}\, dt \\ \int \frac{1}{\frac{u^2-1}{u}}\, du &= \int -\frac{1}{t}\, dt \\ \frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} &= -\ln(t) + c_2 \end{aligned}$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) &= -\ln(t) + 2c_2 \\ \ln(u-1) + \ln(u+1) &= (2)\,(-\ln(t) + 2c_2) \\ &= -2\ln(t) + 4c_2 \end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u-1)+\ln(u+1)} = \mathrm{e}^{-2\ln(t)+2c_2}$$

Which simplifies to

$$
\begin{aligned}
u^2 - 1 &= \frac{2c_2}{t^2} \\
&= \frac{c_3}{t^2}
\end{aligned}
$$

The solution is

$$
u(t)^2 - 1 = \frac{c_3}{t^2}
$$

Replacing $u(t)$ in the above solution by $\frac{y}{t}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y^2}{t^2} - 1 &= \frac{c_3}{t^2} \\
\frac{y^2}{t^2} - 1 &= \frac{c_3}{t^2}
\end{aligned}
$$

Which simplifies to

$$
-(t - y)(y + t) = c_3
$$

<u>Summary</u>
The solution(s) found are the following

$$
-(t - y)(y + t) = c_3 \tag{1}
$$

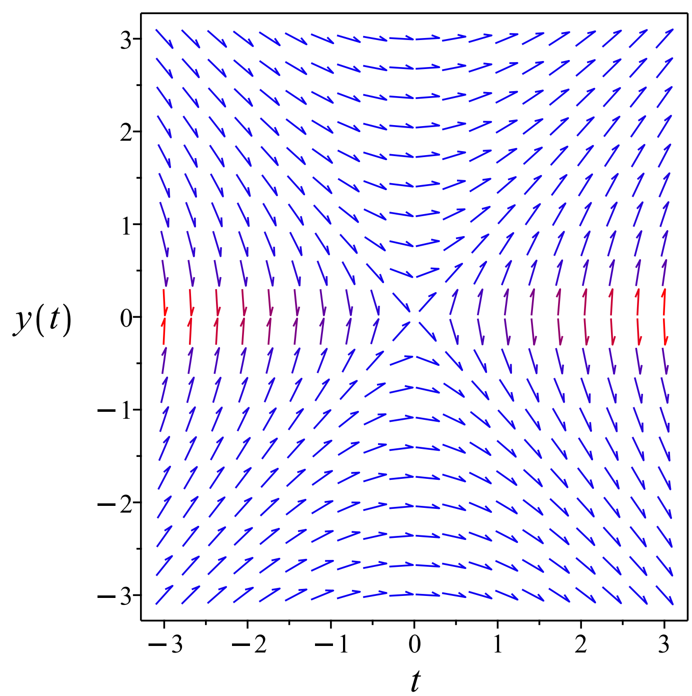


Figure 25: Slope field plot

Verification of solutions

$$-(t-y)(y+t)=c_3$$

Verified OK.

1.9.3 Solving as differentialType ode

Writing the ode as

$$y'=\frac{t}{y} \tag{1}$$

Which becomes

$$(y)\,dy=(t)\,dt \tag{2}$$

But the RHS is complete differential because

$$(t)\,dt=d\left(\frac{t^2}{2}\right)$$

Hence (2) becomes

$$(y)\,dy=d\left(\frac{t^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y=\sqrt{t^2+2c_1}+c_1$$
$$y=-\sqrt{t^2+2c_1}+c_1$$

Summary

The solution(s) found are the following

$$y=\sqrt{t^2+2c_1}+c_1 \tag{1}$$
$$y=-\sqrt{t^2+2c_1}+c_1 \tag{2}$$

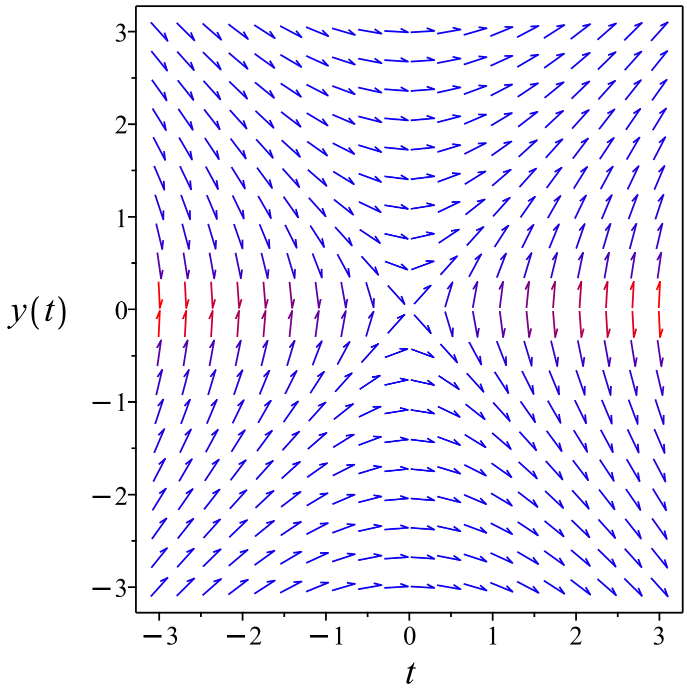


Figure 26: Slope field plot

Verification of solutions

$$y = \sqrt{t^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{t^2 + 2c_1} + c_1$$

Verified OK.

1.9.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{t}{y}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 17: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{1}{t} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{t}} dt \end{aligned}$$

Which results in

$$S = \frac{t^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{t}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= t \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

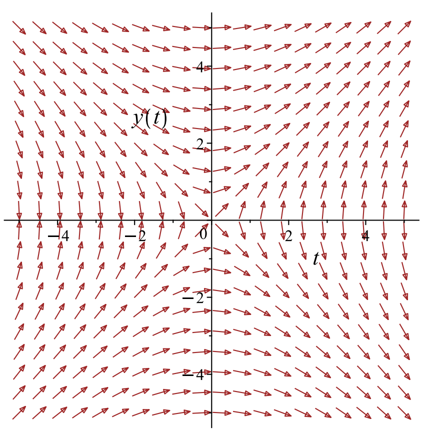
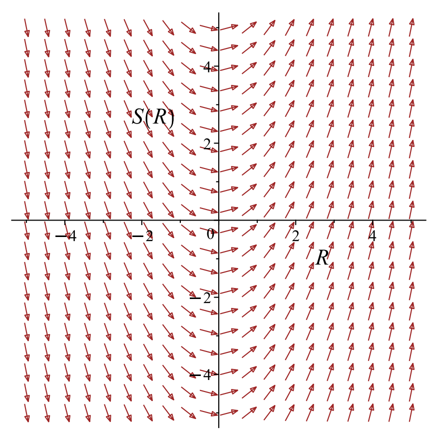
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{t^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{t^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{t}{y}$ 	$R = y$ $S = \frac{t^2}{2}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$\frac{t^2}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

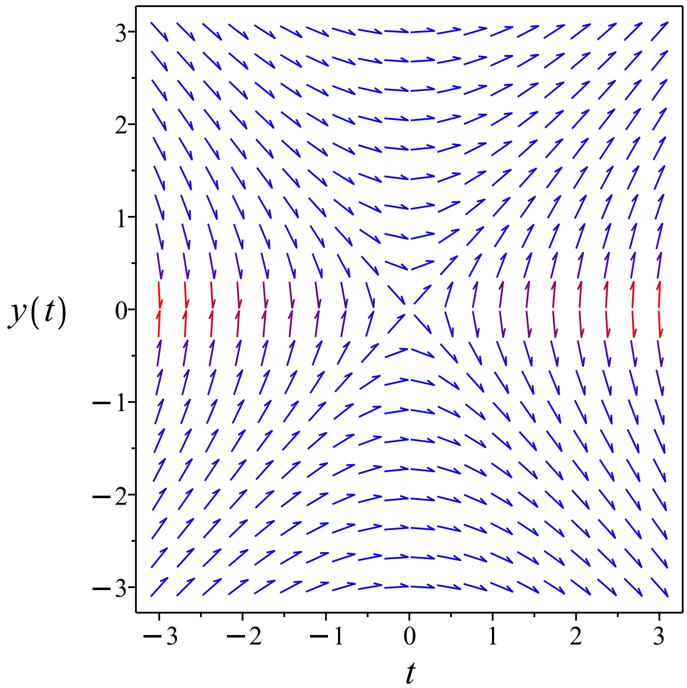


Figure 27: Slope field plot

Verification of solutions

$$\frac{t^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

1.9.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(y)\, \mathrm{d}y &= (t)\, \mathrm{d}t \\
(-t)\, \mathrm{d}t + (y)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -t \\
N(t, y) &= y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t \, \mathrm{d}t$$
$$\phi = -\frac{t^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y) \, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{t^2}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

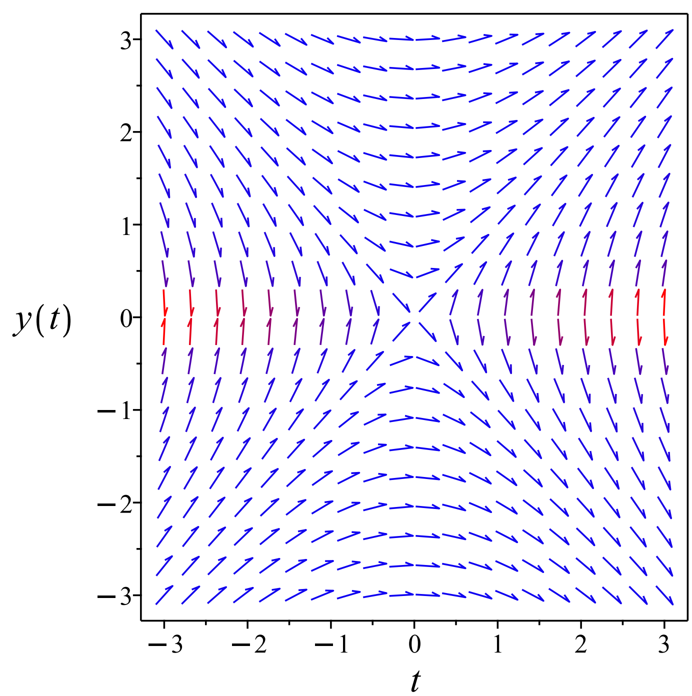


Figure 28: Slope field plot

Verification of solutions

$$-\frac{t^2}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

1.9.6 Maple step by step solution

Let's solve

$y' - \frac{t}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y = t$

- Integrate both sides with respect to t

  $\int y'y dt = \int t dt + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{t^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{t^2 + 2c_1}, y = -\sqrt{t^2 + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(t),t)=t/y(t),y(t), singsol=all)
```

$$y(t) = \sqrt{t^2 + c_1}$$
$$y(t) = -\sqrt{t^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 35

```
DSolve[y'[t]==t/y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\sqrt{t^2 + 2c_1}$$
$$y(t) \to \sqrt{t^2 + 2c_1}$$

1.10 problem 13

Internal problem ID [12874]
Internal file name [`OUTPUT/11526_Monday_November_06_2023_01_31_20_PM_81990854/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{t}{t^2 y + y} = 0}$$

1.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t) g(y) \\ &= \frac{t}{y\,(t^2+1)} \end{aligned}$$

Where $f(t) = \frac{t}{t^2+1}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = \frac{t}{t^2+1}\,dt$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int \frac{t}{t^2+1}\,dt$$

$$\frac{y^2}{2} = \frac{\ln(t^2+1)}{2} + c_1$$

Which results in

$$y = \sqrt{\ln(t^2+1) + 2c_1}$$

$$y = -\sqrt{\ln(t^2+1) + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{\ln(t^2+1) + 2c_1} \tag{1}$$

$$y = -\sqrt{\ln(t^2+1) + 2c_1} \tag{2}$$

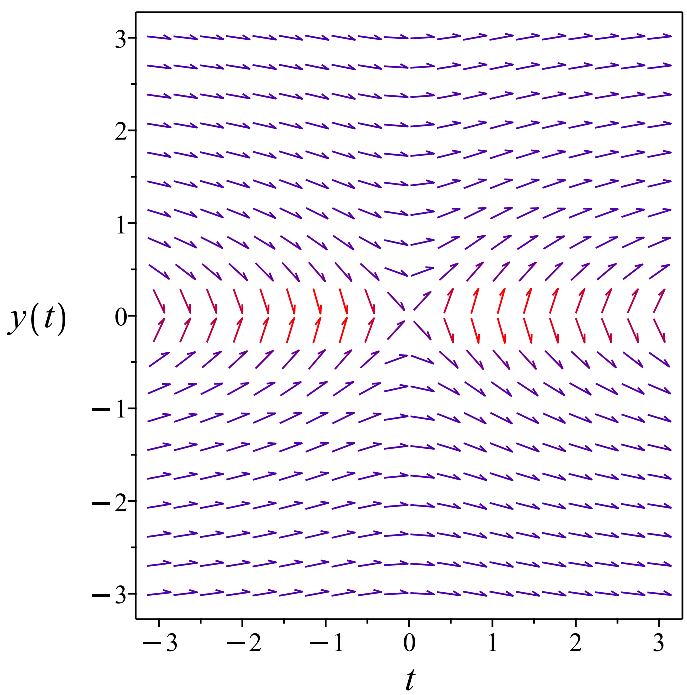


Figure 29: Slope field plot

Verification of solutions

$$y = \sqrt{\ln (t^2 + 1) + 2c_1}$$

Verified OK.

$$y = -\sqrt{\ln (t^2 + 1) + 2c_1}$$

Verified OK.

1.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{t}{y (t^2 + 1)}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 20: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{t^2 + 1}{t} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{t^2+1}{t}} dt \end{aligned}$$

Which results in

$$S = \frac{\ln(t^2+1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{t}{y(t^2+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= \frac{t}{t^2+1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

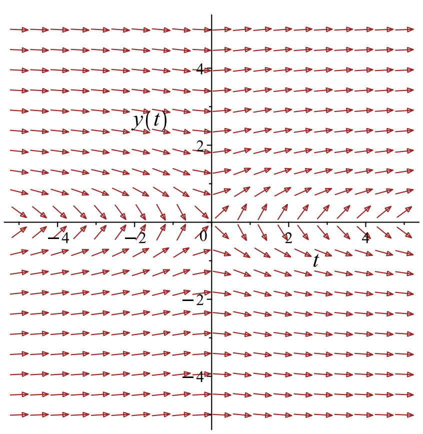
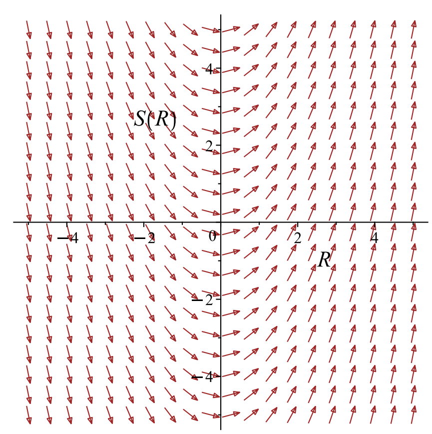
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{\ln (t^2 + 1)}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{\ln (t^2 + 1)}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{t}{y(t^2+1)}$ 	$R = y$ $S = \frac{\ln (t^2+1)}{2}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$\frac{\ln (t^2 + 1)}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

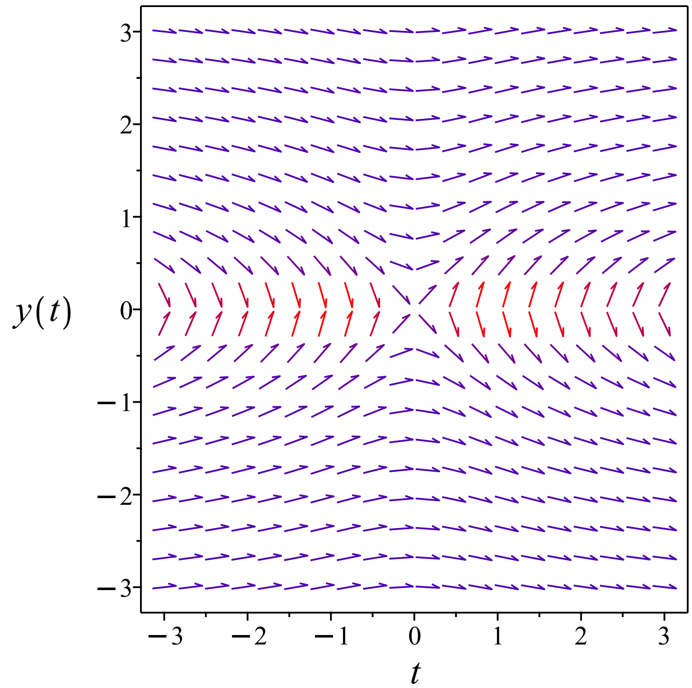


Figure 30: Slope field plot

Verification of solutions

$$\frac{\ln(t^2+1)}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

1.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y)\,\mathrm{d}y = \left(\frac{t}{t^2+1}\right)\mathrm{d}t$$
$$\left(-\frac{t}{t^2+1}\right)\mathrm{d}t + (y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -\frac{t}{t^2+1}$$
$$N(t, y) = y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{t}{t^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{t}{t^2+1}\,\mathrm{d}t\end{aligned}$$

$$\phi = -\frac{\ln(t^2+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int (y)\,\mathrm{d}y\\ f(y) &= \frac{y^2}{2} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t^2+1)}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t^2+1)}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(t^2+1)}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

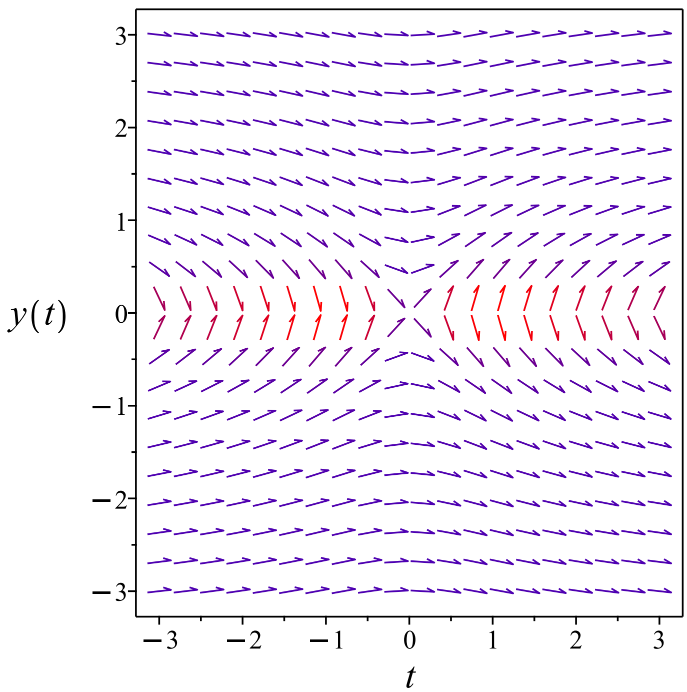


Figure 31: Slope field plot

Verification of solutions

$$-\frac{\ln(t^2+1)}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

1.10.4 Maple step by step solution

Let's solve

$y' - \frac{t}{t^2 y + y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' y = \frac{t}{t^2+1}$

- Integrate both sides with respect to t

  $\int y' y dt = \int \frac{t}{t^2+1} dt + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{\ln(t^2+1)}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{\ln(t^2+1) + 2c_1}, y = -\sqrt{\ln(t^2+1) + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(diff(y(t),t)=t/(t^2*y(t)+y(t)),y(t), singsol=all)
```

$$y(t) = \sqrt{\ln(t^2+1) + c_1}$$
$$y(t) = -\sqrt{\ln(t^2+1) + c_1}$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 41

```
DSolve[y'[t]==t/(t^2*y[t]+y[t]),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\sqrt{\log(t^2+1)+2c_1}$$
$$y(t) \to \sqrt{\log(t^2+1)+2c_1}$$

1.11 problem 14

Internal problem ID [12875]
Internal file name [`OUTPUT/11527_Monday_November_06_2023_01_31_20_PM_17276630/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - t y^{\frac{1}{3}} = 0}$$

1.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= t\, y^{\frac{1}{3}} \end{aligned}$$

Where $f(t) = t$ and $g(y) = y^{\frac{1}{3}}$. Integrating both sides gives

$$\frac{1}{y^{\frac{1}{3}}}\, dy = t\, dt$$

$$\int \frac{1}{y^{\frac{1}{3}}}\, dy = \int t\, dt$$

$$\frac{3y^{\frac{2}{3}}}{2} = \frac{t^2}{2} + c_1$$

The solution is

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{t^2}{2} - c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{t^2}{2} - c_1 = 0 \tag{1}$$

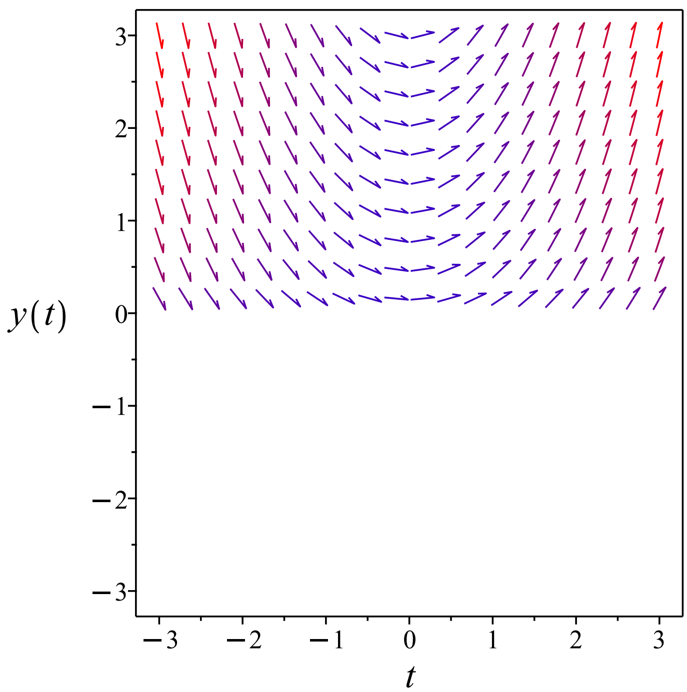


Figure 32: Slope field plot

Verification of solutions

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{t^2}{2} - c_1 = 0$$

Verified OK.

1.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= t\,y^{\frac{1}{3}} \\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 23: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,y) = \frac{1}{t}$$
$$\eta(t,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{t}} dt \end{aligned}$$

Which results in

$$S = \frac{t^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = t\, y^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= t \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3R^{\frac{2}{3}}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{t^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

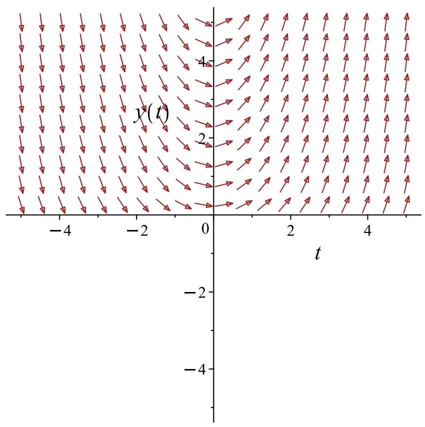
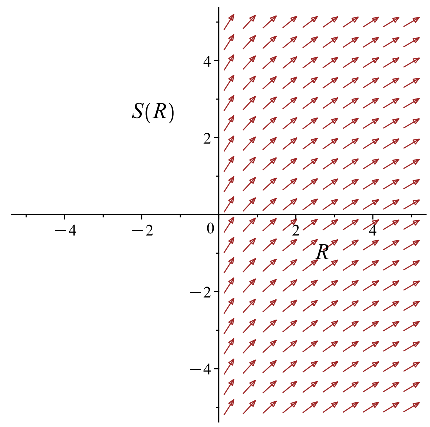
Which simplifies to

$$\frac{t^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

Which gives

$$y = \frac{(3t^2 - 6c_1)^{\frac{3}{2}}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = t\,y^{\frac{1}{3}}$ 	$R = y$ $S = \frac{t^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$ 

Summary

The solution(s) found are the following

$$y = \frac{(3t^2 - 6c_1)^{\frac{3}{2}}}{27} \tag{1}$$

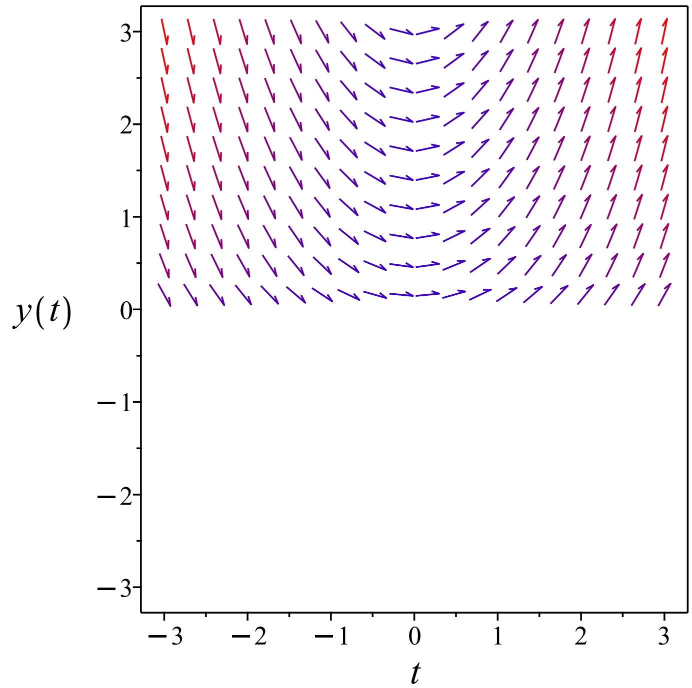


Figure 33: Slope field plot

Verification of solutions

$$y = \frac{(3t^2 - 6c_1)^{\frac{3}{2}}}{27}$$

Verified OK.

1.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^{\frac{1}{3}}}\right) \mathrm{d}y = (t) \, \mathrm{d}t$$
$$(-t) \, \mathrm{d}t + \left(\frac{1}{y^{\frac{1}{3}}}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -t$$
$$N(t, y) = \frac{1}{y^{\frac{1}{3}}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y^{\frac{1}{3}}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -t \,\mathrm{d}t \\
\phi &= -\frac{t^2}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$
\frac{1}{y^{\frac{1}{3}}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^{\frac{1}{3}}}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^{\frac{1}{3}}}\right)\mathrm{d}y$$

$$f(y) = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \frac{3y^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \frac{3y^{\frac{2}{3}}}{2}$$

The solution becomes

$$y = \frac{(3t^2 + 6c_1)^{\frac{3}{2}}}{27}$$

Summary

The solution(s) found are the following

$$y = \frac{(3t^2 + 6c_1)^{\frac{3}{2}}}{27} \tag{1}$$

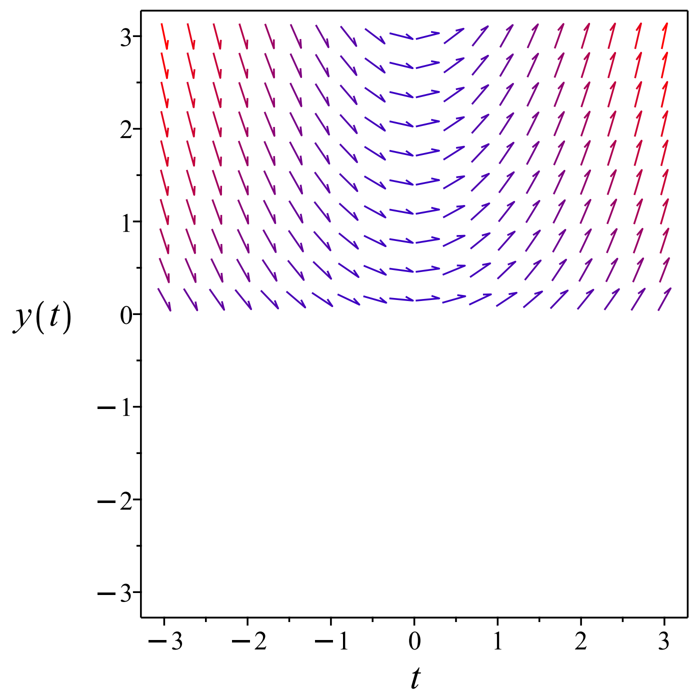


Figure 34: Slope field plot

Verification of solutions

$$y = \frac{(3t^2 + 6c_1)^{\frac{3}{2}}}{27}$$

Verified OK.

1.11.4 Maple step by step solution

Let's solve

$y' - ty^{\frac{1}{3}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = t$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^{\frac{1}{3}}} dt = \int t dt + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = \frac{t^2}{2} + c_1$

- Solve for y

  $y = \frac{(3t^2+6c_1)^{\frac{3}{2}}}{27}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)=t*y(t)^(1/3),y(t), singsol=all)
```

$$y(t)^{\frac{2}{3}} - \frac{t^2}{3} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.346 (sec). Leaf size: 31

```
DSolve[y'[t]==t*y[t]^(1/3),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{(t^2 + 2c_1)^{3/2}}{3\sqrt{3}}$$
$$y(t) \to 0$$

1.12 problem 15

Internal problem ID [12876]
Internal file name [`OUTPUT/11528_Monday_November_06_2023_01_31_21_PM_43900639/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \frac{1}{2y+1} = 0}$$

1.12.1 Solving as quadrature ode

Integrating both sides gives

$$\int (2y+1)\,dy = t + c_1$$
$$y^2 + y = t + c_1$$

Solving for y gives these solutions

$$y_1 = -\frac{1}{2} - \frac{\sqrt{1+4c_1+4t}}{2}$$
$$y_2 = -\frac{1}{2} + \frac{\sqrt{1+4c_1+4t}}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{2} - \frac{\sqrt{1+4c_1+4t}}{2} \tag{1}$$
$$y = -\frac{1}{2} + \frac{\sqrt{1+4c_1+4t}}{2} \tag{2}$$

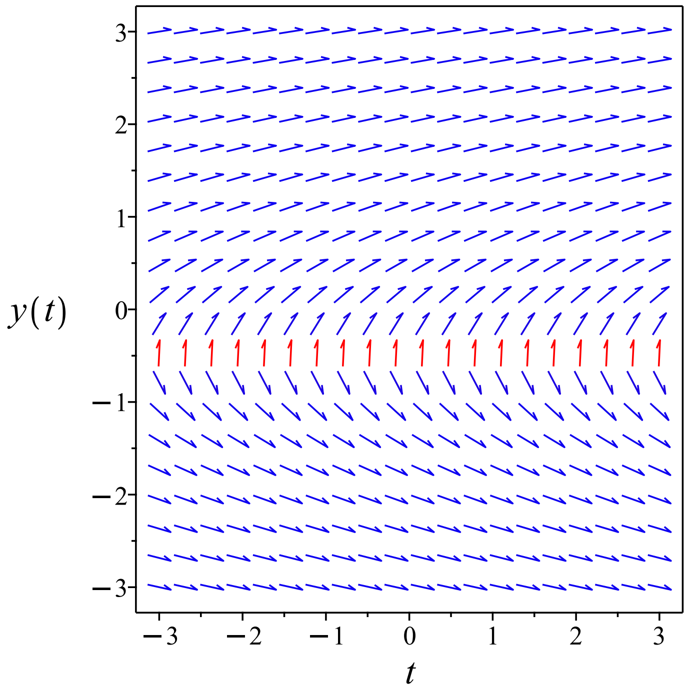


Figure 35: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{2} - \frac{\sqrt{1 + 4c_1 + 4t}}{2}$$

Verified OK.

$$y = -\frac{1}{2} + \frac{\sqrt{1 + 4c_1 + 4t}}{2}$$

Verified OK.

1.12.2 Maple step by step solution

Let's solve

$y' - \frac{1}{2y+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(2y + 1)\, y' = 1$

- Integrate both sides with respect to t

  $\int (2y+1)\, y' dt = \int 1 dt + c_1$

- Evaluate integral

  $y^2 + y = t + c_1$

- Solve for y

  $\left\{y = -\frac{1}{2} - \frac{\sqrt{1+4c_1+4t}}{2}, y = -\frac{1}{2} + \frac{\sqrt{1+4c_1+4t}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(t),t)=1/(2*y(t)+1),y(t), singsol=all)
```

$$y(t) = -\frac{1}{2} - \frac{\sqrt{1+4c_1+4t}}{2}$$
$$y(t) = -\frac{1}{2} + \frac{\sqrt{1+4c_1+4t}}{2}$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 49

```
DSolve[y'[t]==1/(2*y[t]+1),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}\left(-1 - \sqrt{4t+1+4c_1}\right)$$
$$y(t) \to \frac{1}{2}\left(-1 + \sqrt{4t+1+4c_1}\right)$$

1.13 problem 16

Internal problem ID [12877]
Internal file name [`OUTPUT/11529_Monday_November_06_2023_01_31_21_PM_97717993/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2y+1}{t} = 0}$$

1.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= \frac{2y+1}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(y) = 2y + 1$. Integrating both sides gives

$$\frac{1}{2y+1}\, dy = \frac{1}{t}\, dt$$

$$\int \frac{1}{2y+1} \, dy = \int \frac{1}{t} \, dt$$

$$\frac{\ln(2y+1)}{2} = \ln(t) + c_1$$

Raising both side to exponential gives

$$\sqrt{2y+1} = e^{\ln(t)+c_1}$$

Which simplifies to

$$\sqrt{2y+1} = c_2 t$$

Summary

The solution(s) found are the following

$$y = \frac{c_2^2 t^2 e^{2c_1}}{2} - \frac{1}{2} \tag{1}$$

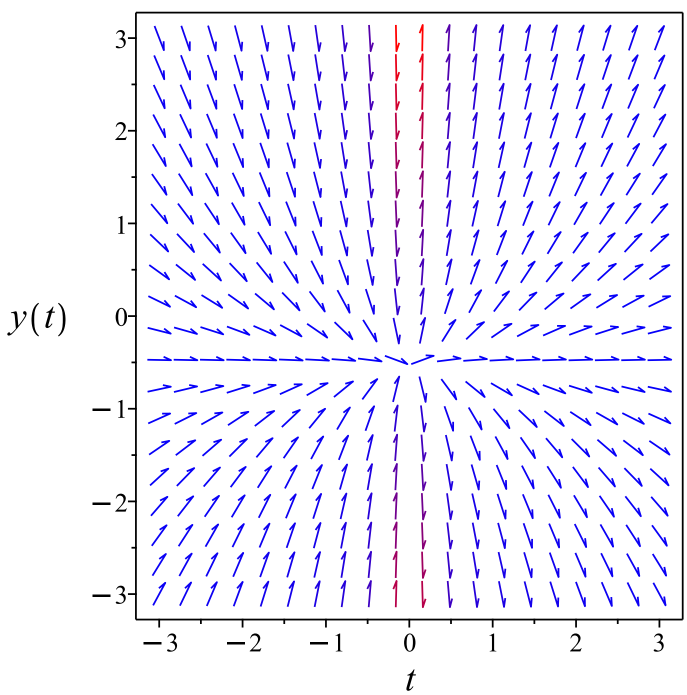


Figure 36: Slope field plot

Verification of solutions

$$y = \frac{c_2^2 t^2 e^{2c_1}}{2} - \frac{1}{2}$$

Verified OK.

1.13.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{2}{t} \\ q(t) &= \frac{1}{t} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{t} = \frac{1}{t}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{t}dt} \\ &= \frac{1}{t^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{1}{t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(\frac{1}{t}\right) \\ \mathrm{d}\left(\frac{y}{t^2}\right) &= \frac{1}{t^3}\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{t^2} &= \int \frac{1}{t^3}\,\mathrm{d}t \\ \frac{y}{t^2} &= -\frac{1}{2t^2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$y = -\frac{1}{2} + t^2 c_1$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{2} + t^2 c_1 \tag{1}$$

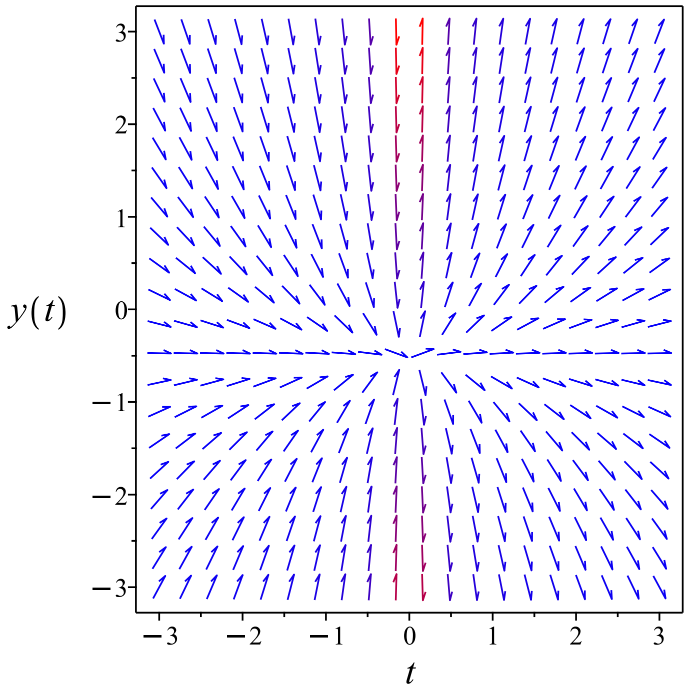


Figure 37: Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + t^2 c_1$$

Verified OK.

1.13.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = t + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 2y_0 + 1}{X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = -\frac{1}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X)}{X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{2Y}{X} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= 2u \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) X - u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= \frac{u}{X}
\end{aligned}
$$

Where $f(X) = \frac{1}{X}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{X}\,dX \\
\int \frac{1}{u}\,du &= \int \frac{1}{X}\,dX \\
\ln(u) &= \ln(X) + c_2 \\
u &= \mathrm{e}^{\ln(X)+c_2} \\
&= c_2 X
\end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X^2 c_2$$

Using the solution for $Y(X)$

$$Y(X) = X^2 c_2$$

And replacing back terms in the above solution using

$$\begin{aligned}
Y &= y + y_0 \\
X &= t + x_0
\end{aligned}$$

Or

$$\begin{aligned}
Y &= y - \frac{1}{2} \\
X &= t
\end{aligned}$$

Then the solution in y becomes

$$y + \frac{1}{2} = c_2 t^2$$

Summary

The solution(s) found are the following

$$y + \frac{1}{2} = c_2 t^2 \tag{1}$$

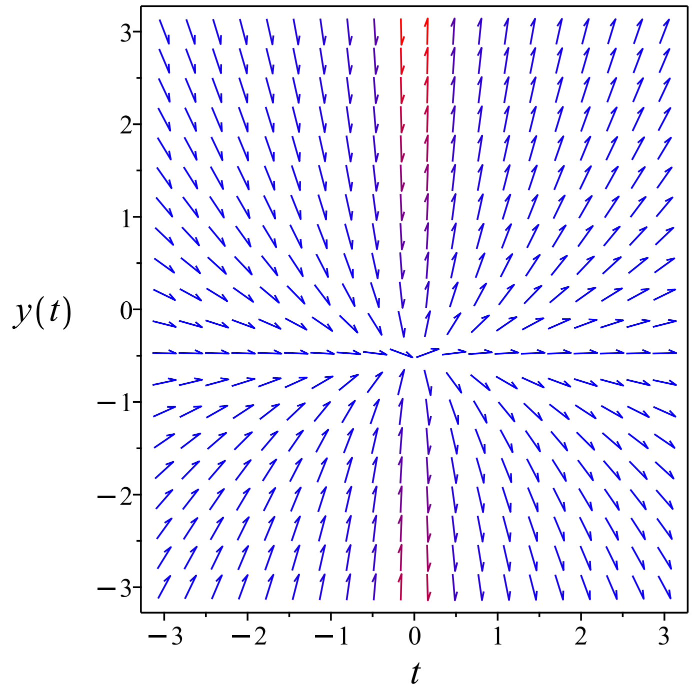


Figure 38: Slope field plot

Verification of solutions

$$y+\frac{1}{2}=c_2t^2$$

Verified OK.

1.13.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=\frac{2y+1}{t}$$
$$y'=\omega(t,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t+\omega(\eta_y-\xi_t)-\omega^2\xi_y-\omega_t\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 27: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= t^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{t^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{2y + 1}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{2y}{t^3} \\ S_y &= \frac{1}{t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{t^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{t^2} = -\frac{1}{2t^2} + c_1$$

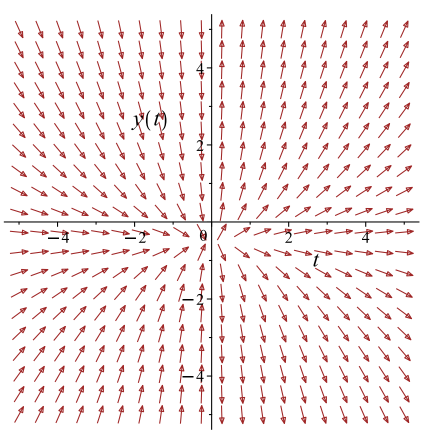
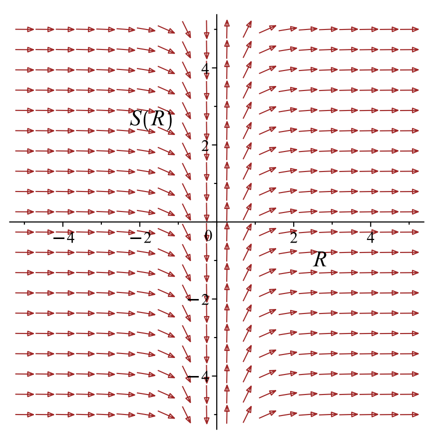
Which simplifies to

$$\frac{y}{t^2} = -\frac{1}{2t^2} + c_1$$

Which gives

$$y = -\frac{1}{2} + t^2 c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{2y+1}{t}$ 	$R = t$ $S = \frac{y}{t^2}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary
The solution(s) found are the following

$$y = -\frac{1}{2} + t^2 c_1 \tag{1}$$

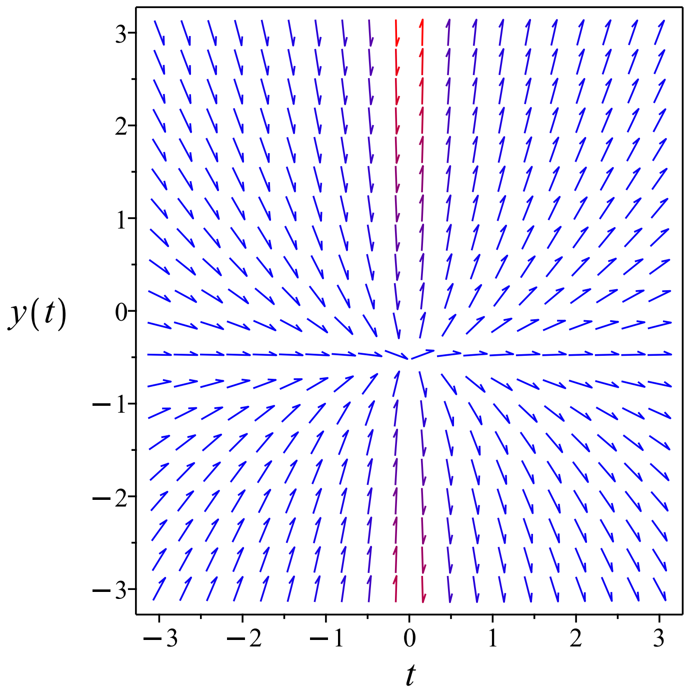


Figure 39: Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + t^2 c_1$$

Verified OK.

1.13.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2y+1}\right)\mathrm{d}y&=\left(\frac{1}{t}\right)\mathrm{d}t\\ \left(-\frac{1}{t}\right)\mathrm{d}t+\left(\frac{1}{2y+1}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{1}{t}\\ N(t,y)&=\frac{1}{2y+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{t}\right)\\ &=0\end{aligned}$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2y+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int M\, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int -\frac{1}{t}\, \mathrm{d}t$$
$$\phi = -\ln(t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y+1}$. Therefore equation (4) becomes

$$\frac{1}{2y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{1}{2y+1}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(2y+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(t) + \frac{\ln(2y+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) + \frac{\ln(2y+1)}{2}$$

The solution becomes

$$y = \frac{e^{2c_1} t^2}{2} - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{2c_1} t^2}{2} - \frac{1}{2} \tag{1}$$

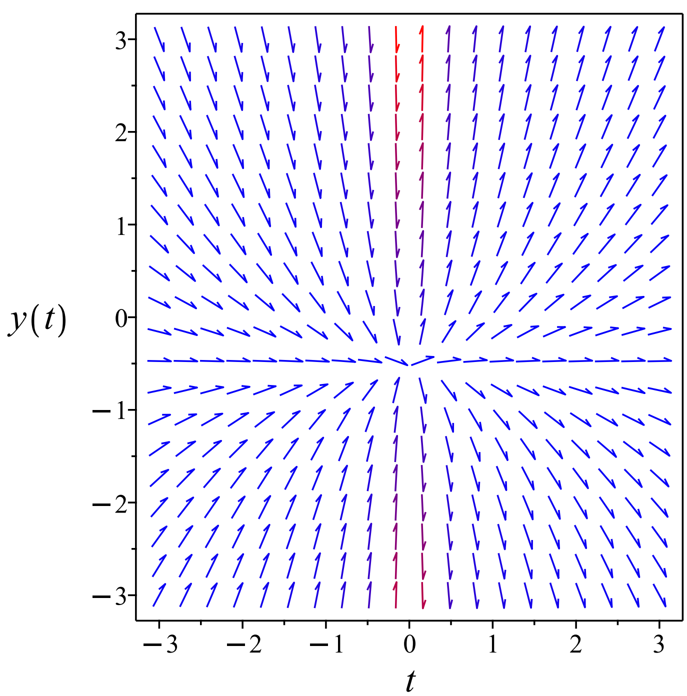


Figure 40: Slope field plot

Verification of solutions

$$y = \frac{e^{2c_1}t^2}{2} - \frac{1}{2}$$

Verified OK.

1.13.6 Maple step by step solution

Let's solve

$y' - \frac{2y+1}{t} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y+1} = \frac{1}{t}$

- Integrate both sides with respect to t

  $\int \frac{y'}{2y+1}dt = \int \frac{1}{t}dt + c_1$

- Evaluate integral

  $\frac{\ln(2y+1)}{2} = \ln(t) + c_1$

- Solve for y

  $y = \frac{e^{2c_1}t^2}{2} - \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(t),t)=(2*y(t)+1)/t,y(t), singsol=all)
```

$$y(t) = -\frac{1}{2} + t^2c_1$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 22

```
DSolve[y'[t]==(2*y[t]+1)/t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{2} + c_1 t^2$$
$$y(t) \to -\frac{1}{2}$$

1.14 problem 17

Internal problem ID [12878]
Internal file name [`OUTPUT/11530_Monday_November_06_2023_01_31_22_PM_26690807/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y(-y+1) = 0}$$

1.14.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y\,(y-1)}dy = \int dt$$
$$-\ln\,(y-1) + \ln\,(y) = t + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(y-1)+\ln(y)} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$\frac{y}{y-1} = c_2\mathrm{e}^t$$

Summary
The solution(s) found are the following

$$y = \frac{c_2\mathrm{e}^t}{-1 + c_2\mathrm{e}^t} \tag{1}$$

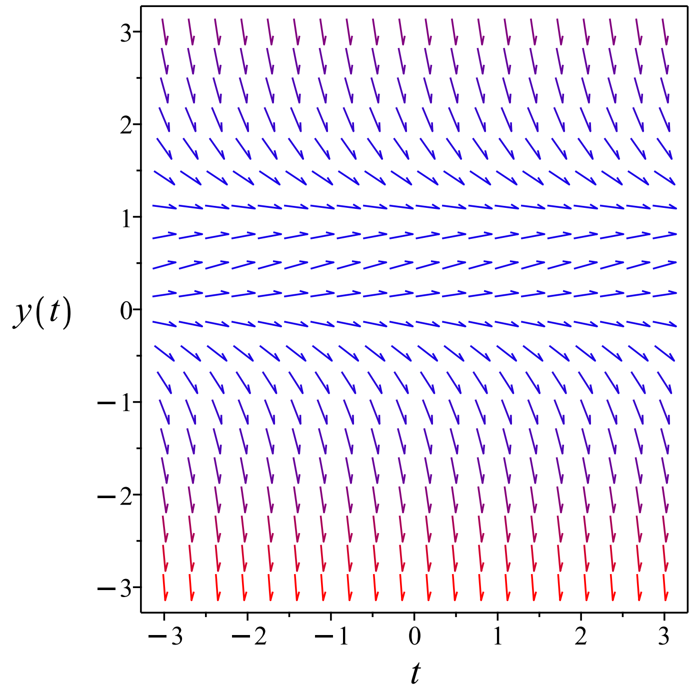


Figure 41: Slope field plot

Verification of solutions

$$y = \frac{c_2 e^t}{-1 + c_2 e^t}$$

Verified OK.

1.14.2 Maple step by step solution

Let's solve

$y' - y(-y+1) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(-y+1)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(-y+1)} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\ln(y-1)+\ln(y)=t+c_1$$

- Solve for y

$$y=\frac{e^{t+c_1}}{-1+e^{t+c_1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=y(t)*(1-y(t)),y(t), singsol=all)
```

$$y(t)=\frac{1}{1+e^{-t}c_1}$$

✓ Solution by Mathematica

Time used: 0.394 (sec). Leaf size: 29

```
DSolve[y'[t]==y[t]*(1-y[t]),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to\frac{e^t}{e^t+e^{c_1}}$$
$$y(t)\to 0$$
$$y(t)\to 1$$

1.15 problem 18

Internal problem ID [12879]
Internal file name [`OUTPUT/11531_Monday_November_06_2023_01_31_23_PM_37223931/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{4t}{1+3y^2} = 0}$$

1.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{4t}{3y^2+1} \end{aligned}$$

Where $f(t) = 4t$ and $g(y) = \frac{1}{3y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{3y^2+1}}\, dy = 4t\, dt$$

$$\int \frac{1}{\frac{1}{3y^2+1}}\, dy = \int 4t\, dt$$

$$y^3 + y = 2t^2 + c_1$$

Which results in

$$y = \frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{6} - \frac{2}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}$$

$$y = -\frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2}$$

$$y = -\frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{6} - \frac{2}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} \tag{1}$$

$$y = -\frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2} \tag{2}$$

$$y = -\frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2} \tag{3}$$

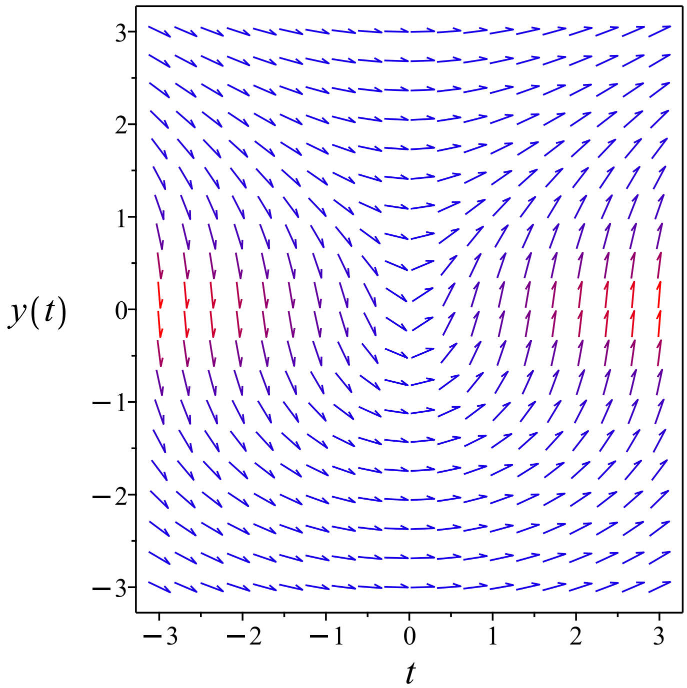


Figure 42: Slope field plot

Verification of solutions

$$y = \frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} - \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}$$

Verified OK.

$$y = -\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2}$$

Verified OK.

$$y = -\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2}$$

Verified OK.

1.15.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{4t}{1+3y^2} \tag{1}$$

Which becomes

$$\left(3y^2+1\right) dy = (4t)\, dt \tag{2}$$

But the RHS is complete differential because

$$(4t)\,dt = d\left(2t^2\right)$$

Hence (2) becomes

$$\left(3y^2+1\right)dy = d\left(2t^2\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} - \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c \cdots}\right.}$$

$$y = -\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+8 \cdots}\right.}$$

$$y = -\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+8 \cdots}\right.}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{6} - \frac{2}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} + c_1 \tag{1}$$

$$y = -\frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2} + c_1 \tag{3}$$

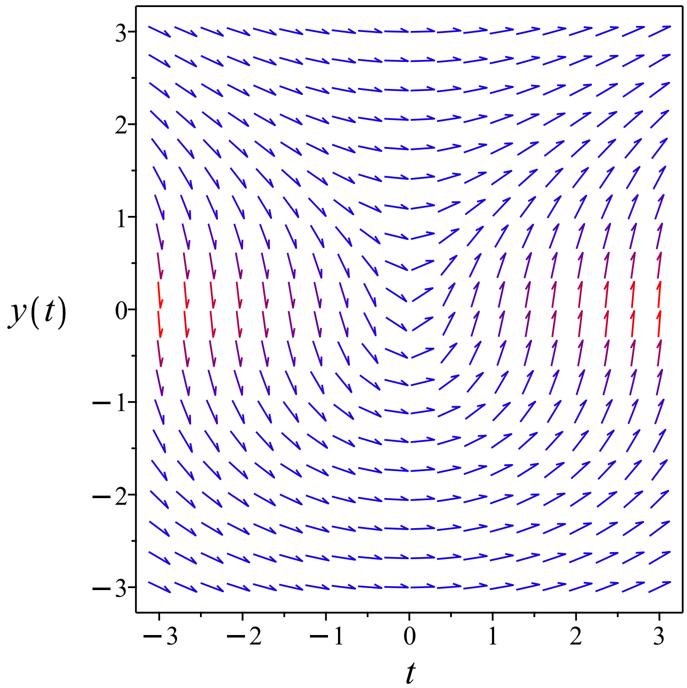


Figure 43: Slope field plot

Verification of solutions

$$y = \frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{6} - \frac{2}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} + c_1$$

Verified OK.

$$y = -\frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

$$y = -\frac{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}}{12} + \frac{1}{\left(216t^2 + 108c_1 + 12\sqrt{324t^4 + 324t^2c_1 + 81c_1^2 + 12}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}\right)}{2} + c_1$$

Verified OK.

1.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{4t}{3y^2+1} \\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 31: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,y) &= \frac{1}{4t} \\ \eta(t,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{4t}} dt \end{aligned}$$

Which results in

$$S = 2t^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y)S_y}{R_t + \omega(t, y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{4t}{3y^2 + 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= 4t \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3y^2 + 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2 + 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^3 + R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$2t^2 = y^3 + c_1 + y$$

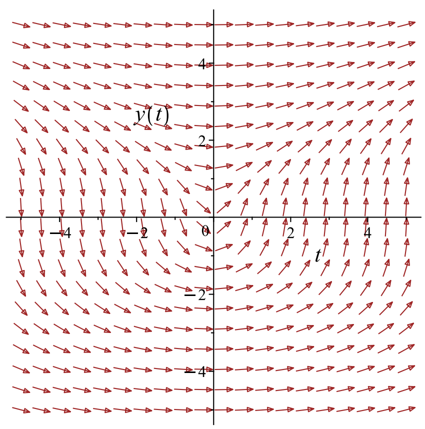
Which simplifies to

$$2t^2 = y^3 + c_1 + y$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dt} = \frac{4t}{3y^2+1}$



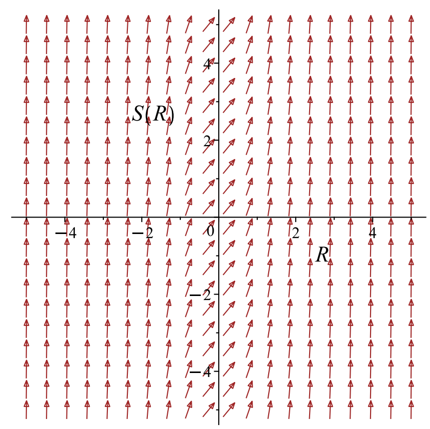
</td><td>

$R = y$

$S = 2t^2$

</td><td>

$\frac{dS}{dR} = 3R^2 + 1$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$2t^2 = y^3 + c_1 + y \tag{1}$$

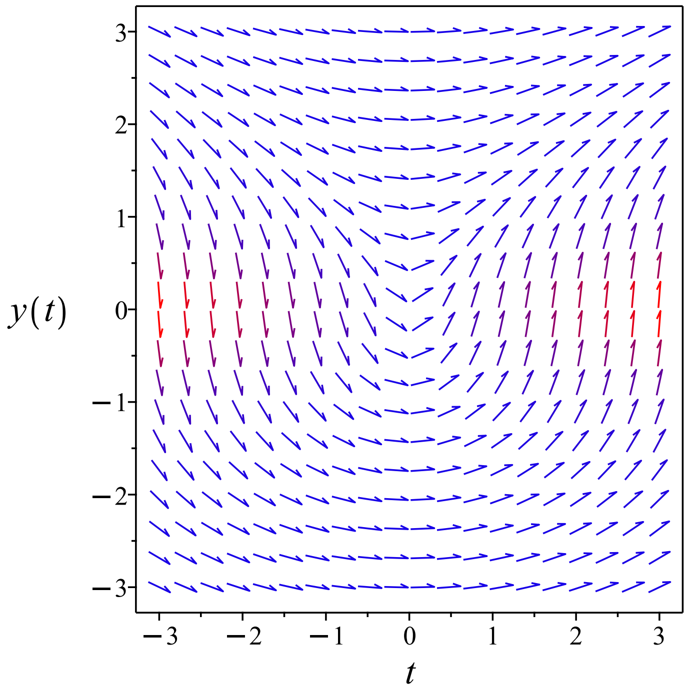


Figure 44: Slope field plot

Verification of solutions

$$2t^2 = y^3 + c_1 + y$$

Verified OK.

1.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{4} + \frac{3y^2}{4}\right)\mathrm{d}y &= (t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t + \left(\frac{1}{4} + \frac{3y^2}{4}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -t\\ N(t, y) &= \frac{1}{4} + \frac{3y^2}{4}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{4} + \frac{3y^2}{4}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int M\, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int -t\, \mathrm{d}t$$
$$\phi = -\frac{t^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{4} + \frac{3y^2}{4}$. Therefore equation (4) becomes

$$\frac{1}{4} + \frac{3y^2}{4} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{4} + \frac{3y^2}{4}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{4} + \frac{3y^2}{4}\right) \mathrm{d}y$$

$$f(y) = \frac{1}{4}y + \frac{1}{4}y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{2}t^2 + \frac{1}{4}y + \frac{1}{4}y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{2}t^2 + \frac{1}{4}y + \frac{1}{4}y^3$$

Summary

The solution(s) found are the following

$$-\frac{t^2}{2} + \frac{y}{4} + \frac{y^3}{4} = c_1 \tag{1}$$

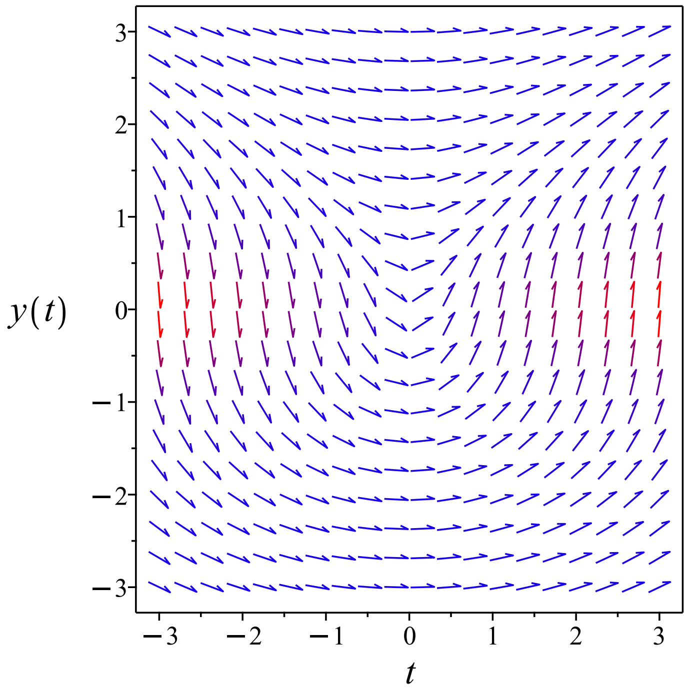


Figure 45: Slope field plot

Verification of solutions

$$-\frac{t^2}{2} + \frac{y}{4} + \frac{y^3}{4} = c_1$$

Verified OK.

1.15.5 Maple step by step solution

Let's solve

$y' - \frac{4t}{1+3y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(1 + 3y^2)\, y' = 4t$

- Integrate both sides with respect to t

  $\int (1 + 3y^2)\, y' dt = \int 4t dt + c_1$

- Evaluate integral

$$y^3 + y = 2t^2 + c_1$$

- Solve for y

$$y = \frac{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}{6} - \frac{2}{\left(216t^2+108c_1+12\sqrt{324t^4+324t^2c_1+81c_1^2+12}\right)^{\frac{1}{3}}}$$

<u>Maple trace</u>

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ <u>Solution by Maple</u>

Time used: 0.016 (sec). Leaf size: 282

```
dsolve(diff(y(t),t)=4*t/(1+3*y(t)^2),y(t), singsol=all)
```

$$y(t) = \frac{\left(27t^2+54c_1+3\sqrt{81t^4+324t^2c_1+324c_1^2+3}\right)^{\frac{2}{3}} - 3}{3\left(27t^2+54c_1+3\sqrt{81t^4+324t^2c_1+324c_1^2+3}\right)^{\frac{1}{3}}}$$

$$y(t) = -\frac{\left(1+i\sqrt{3}\right)\left(27t^2+54c_1+3\sqrt{81t^4+324t^2c_1+324c_1^2+3}\right)^{\frac{2}{3}} + 3i\sqrt{3} - 3}{6\left(27t^2+54c_1+3\sqrt{81t^4+324t^2c_1+324c_1^2+3}\right)^{\frac{1}{3}}}$$

$$y(t) = \frac{i\left(27t^2+54c_1+3\sqrt{81t^4+324t^2c_1+324c_1^2+3}\right)^{\frac{2}{3}}\sqrt{3} - \left(27t^2+54c_1+3\sqrt{81t^4+324t^2c_1+324c_1^2+3}\right)}{6\left(27t^2+54c_1+3\sqrt{81t^4+324t^2c_1+324c_1^2+3}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 3.132 (sec). Leaf size: 298

```
DSolve[y'[t]==4*t/(1+3*y[t]^2),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \frac{\sqrt[3]{54t^2 + \sqrt{108 + 729\left(2t^2 + c_1\right)^2} + 27c_1}}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}}{\sqrt[3]{54t^2 + \sqrt{108 + 729\left(2t^2 + c_1\right)^2} + 27c_1}}$$

$$y(t) \rightarrow \frac{\left(-1 + i\sqrt{3}\right)\sqrt[3]{54t^2 + \sqrt{108 + 729\left(2t^2 + c_1\right)^2} + 27c_1}}{6\sqrt[3]{2}} + \frac{1 + i\sqrt{3}}{2^{2/3}\sqrt[3]{54t^2 + \sqrt{108 + 729\left(2t^2 + c_1\right)^2} + 27c_1}}$$

$$y(t) \rightarrow \frac{1 - i\sqrt{3}}{2^{2/3}\sqrt[3]{54t^2 + \sqrt{108 + 729\left(2t^2 + c_1\right)^2} + 27c_1}} - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{54t^2 + \sqrt{108 + 729\left(2t^2 + c_1\right)^2} + 27c_1}}{6\sqrt[3]{2}}$$

1.16 problem 19

Internal problem ID [12880]
Internal file name [`OUTPUT/11532_Monday_November_06_2023_01_33_04_PM_52114133/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{v' - t^2 v + 2v = t^2 - 2}$$

1.16.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} v' &= F(t, v) \\ &= f(t)g(v) \\ &= \left(t^2 - 2\right)(v + 1) \end{aligned}$$

Where $f(t) = t^2 - 2$ and $g(v) = v + 1$. Integrating both sides gives

$$\frac{1}{v+1}\,dv = t^2 - 2\,dt$$

$$\int \frac{1}{v+1}\,dv = \int t^2 - 2\,dt$$

$$\ln(v + 1) = \frac{1}{3}t^3 - 2t + c_1$$

Raising both side to exponential gives

$$v + 1 = e^{\frac{1}{3}t^3 - 2t + c_1}$$

Which simplifies to

$$v + 1 = c_2 e^{\frac{1}{3}t^3 - 2t}$$

<u>Summary</u>
The solution(s) found are the following

$$v = c_2 e^{\frac{1}{3}t^3 - 2t + c_1} - 1 \tag{1}$$

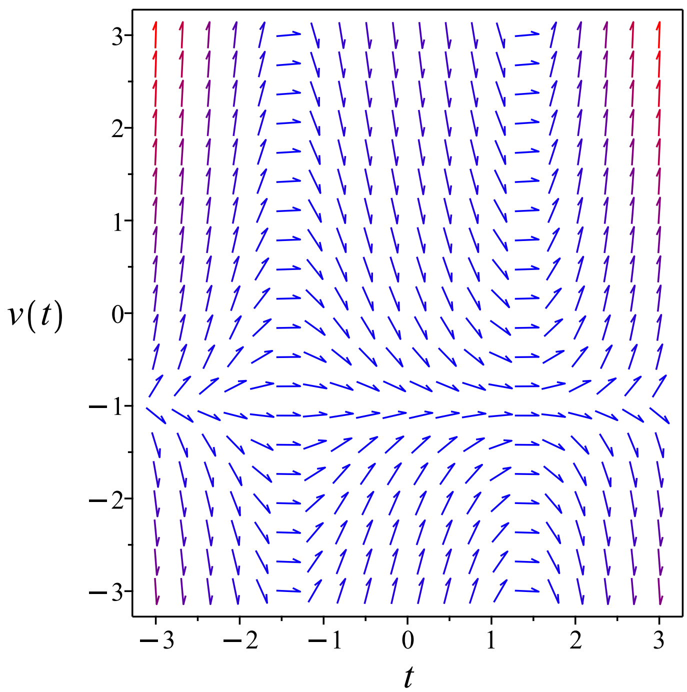


Figure 46: Slope field plot

<u>Verification of solutions</u>

$$v = c_2 e^{\frac{1}{3}t^3 - 2t + c_1} - 1$$

Verified OK.

1.16.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$v' + p(t)v = q(t)$$

Where here

$$\begin{aligned} p(t) &= -t^2 + 2 \\ q(t) &= t^2 - 2 \end{aligned}$$

Hence the ode is

$$v' + \left(-t^2 + 2\right) v = t^2 - 2$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int (-t^2+2) dt} \\ &= \mathrm{e}^{-\frac{1}{3}t^3+2t} \end{aligned}$$

Which simplifies to

$$\mu = \mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu v) &= (\mu)\left(t^2 - 2\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}} v\right) &= \left(\mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}}\right)\left(t^2 - 2\right) \\ \mathrm{d}\left(\mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}} v\right) &= \left(\left(t^2 - 2\right)\mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}}\right) \mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}} v &= \int \left(t^2 - 2\right)\mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}} \mathrm{d}t \\ \mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}} v &= -\mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}}$ results in

$$v = -\mathrm{e}^{\frac{t\left(t^2-6\right)}{3}} \mathrm{e}^{-\frac{t\left(t^2-6\right)}{3}} + c_1 \mathrm{e}^{\frac{t\left(t^2-6\right)}{3}}$$

which simplifies to

$$v = -1 + c_1 e^{\frac{t(t^2-6)}{3}}$$

Summary
The solution(s) found are the following

$$v = -1 + c_1 e^{\frac{t(t^2-6)}{3}} \tag{1}$$

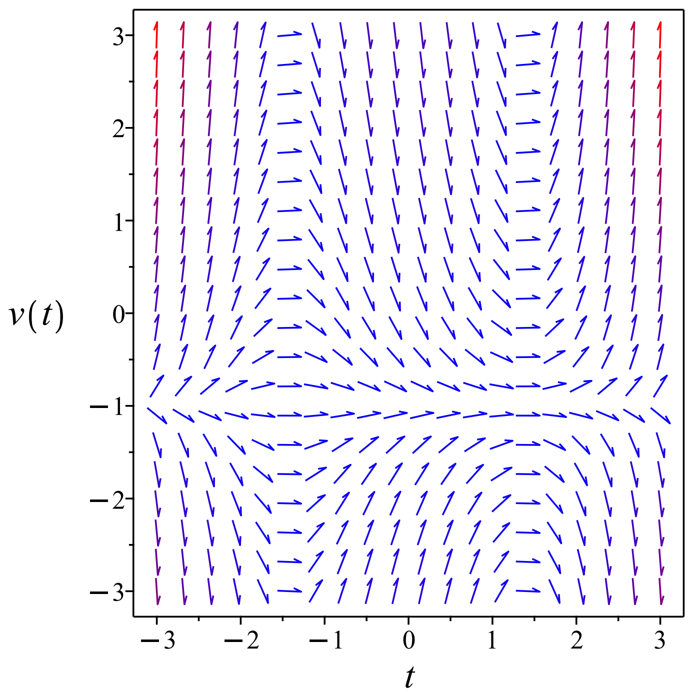


Figure 47: Slope field plot

Verification of solutions

$$v = -1 + c_1 e^{\frac{t(t^2-6)}{3}}$$

Verified OK.

1.16.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
v' &= t^2 v + t^2 - 2v - 2 \\
v' &= \omega(t, v)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_v - \xi_t) - \omega^2 \xi_v - \omega_t \xi - \omega_v \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 34: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, v) &= 0\\ \eta(t, v) &= e^{\frac{1}{3}t^3-2t}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, v) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dv}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial v}\right) S(t, v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{e^{\frac{1}{3}t^3-2t}} dy\end{aligned}$$

Which results in

$$S = e^{-\frac{1}{3}t^3+2t} v$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, v)S_v}{R_t + \omega(t, v)R_v} \tag{2}$$

Where in the above R_t, R_v, S_t, S_v are all partial derivatives and $\omega(t, v)$ is the right hand side of the original ode given by

$$\omega(t, v) = t^2 v + t^2 - 2v - 2$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1\\ R_v &= 0\\ S_t &= -\left(t^2 - 2\right) e^{-\frac{t\left(t^2-6\right)}{3}} v\\ S_v &= e^{-\frac{t\left(t^2-6\right)}{3}}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (t^2 - 2)\,\mathrm{e}^{-\frac{t(t^2-6)}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R^2 - 2)\,\mathrm{e}^{-\frac{R(R^2-6)}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-\frac{R(R^2-6)}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, v coordinates. This results in

$$\mathrm{e}^{-\frac{t(t^2-6)}{3}} v = -\mathrm{e}^{-\frac{t(t^2-6)}{3}} + c_1$$

Which simplifies to

$$\mathrm{e}^{-\frac{t(t^2-6)}{3}} v = -\mathrm{e}^{-\frac{t(t^2-6)}{3}} + c_1$$

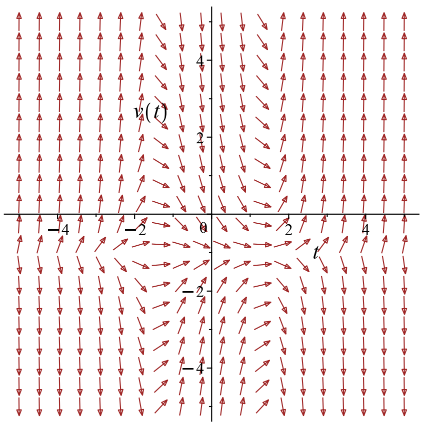
Which gives

$$v = -\left(\mathrm{e}^{-\frac{t(t^2-6)}{3}} - c_1\right)\mathrm{e}^{\frac{t(t^2-6)}{3}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, v coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dv}{dt} = t^2 v + t^2 - 2v - 2$



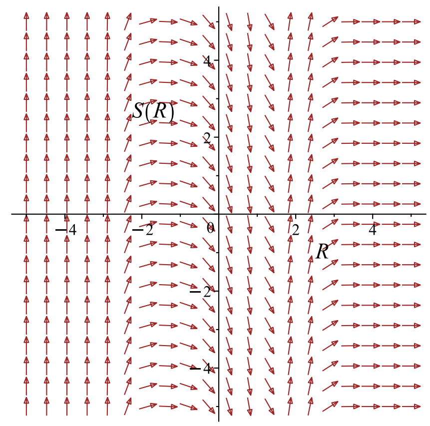
</td><td>

$R = t$

$S = \mathrm{e}^{-\frac{t(t^2-6)}{3}} v$

</td><td>

$\frac{dS}{dR} = (R^2 - 2)\,\mathrm{e}^{-\frac{R(R^2-6)}{3}}$



</td></tr>
</table>

<u>Summary</u>

The solution(s) found are the following

$$v = -\left(\mathrm{e}^{-\frac{t(t^2-6)}{3}} - c_1\right) \mathrm{e}^{\frac{t(t^2-6)}{3}} \tag{1}$$

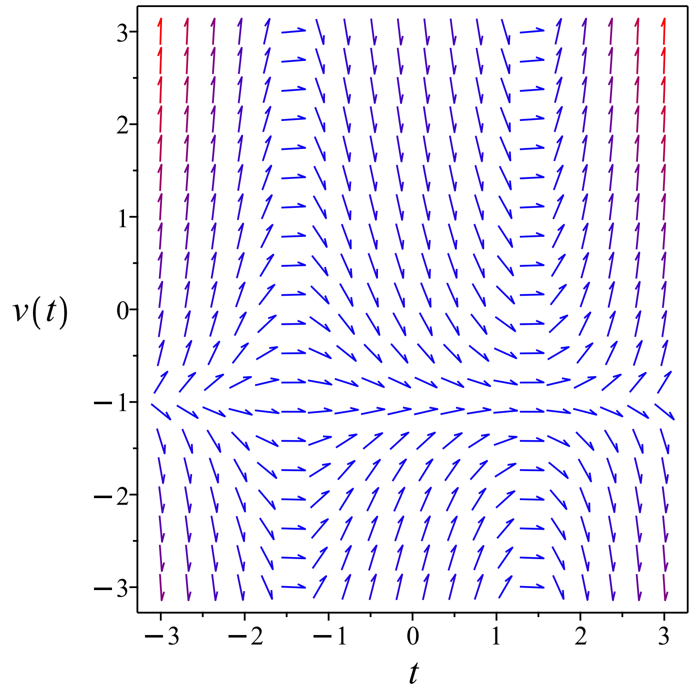


Figure 48: Slope field plot

Verification of solutions

$$v = -\left(e^{-\frac{t(t^2-6)}{3}} - c_1\right) e^{\frac{t(t^2-6)}{3}}$$

Verified OK.

1.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, v)\,\mathrm{d}t + N(t, v)\,\mathrm{d}v = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{v+1}\right) \mathrm{d}v = \left(t^2 - 2\right) \mathrm{d}t$$
$$\left(-t^2 + 2\right) \mathrm{d}t + \left(\frac{1}{v+1}\right) \mathrm{d}v = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, v) = -t^2 + 2$$
$$N(t, v) = \frac{1}{v+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}\left(-t^2 + 2\right) \\ &= 0 \end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{v+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, v)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial v} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t^2 + 2 \, \mathrm{d}t \\
\phi &= -\frac{1}{3}t^3 + 2t + f(v)
\end{aligned} \tag{3}
$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both t and v. Taking derivative of equation (3) w.r.t v gives

$$
\frac{\partial \phi}{\partial v} = 0 + f'(v) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = \frac{1}{v+1}$. Therefore equation (4) becomes

$$
\frac{1}{v+1} = 0 + f'(v) \tag{5}
$$

Solving equation (5) for $f'(v)$ gives

$$
f'(v) = \frac{1}{v+1}
$$

Integrating the above w.r.t v gives

$$
\begin{aligned}
\int f'(v) \, \mathrm{d}v &= \int \left(\frac{1}{v+1}\right) \mathrm{d}v \\
f(v) &= \ln(v+1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(v)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} + 2t + \ln(v+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} + 2t + \ln(v+1)$$

The solution becomes

$$v = e^{\frac{1}{3}t^3 - 2t + c_1} - 1$$

Summary

The solution(s) found are the following

$$v = e^{\frac{1}{3}t^3 - 2t + c_1} - 1 \tag{1}$$

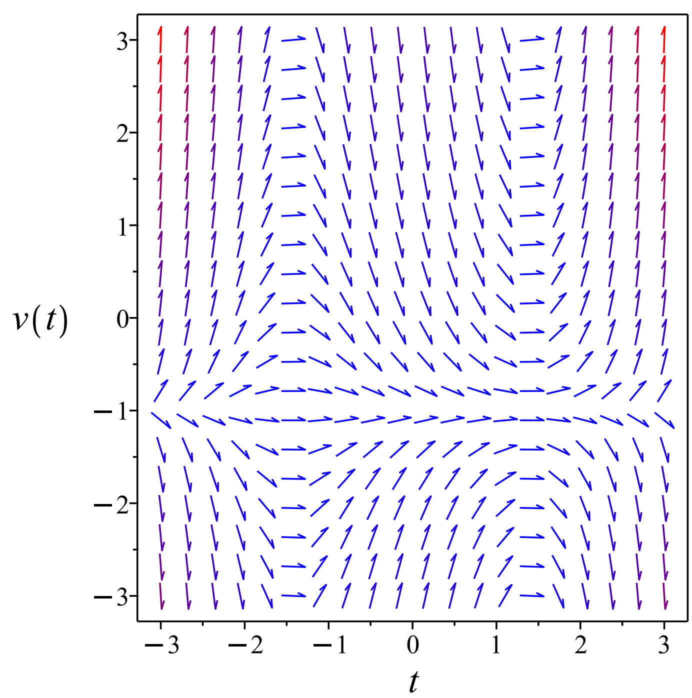


Figure 49: Slope field plot

Verification of solutions

$$v = e^{\frac{1}{3}t^3-2t+c_1} - 1$$

Verified OK.

1.16.5 Maple step by step solution

Let's solve

$v' - t^2v + 2v = t^2 - 2$

- Highest derivative means the order of the ODE is 1

  v'

- Separate variables

  $\frac{v'}{v+1} = t^2 - 2$

- Integrate both sides with respect to t

  $\int \frac{v'}{v+1}dt = \int (t^2 - 2)\, dt + c_1$

- Evaluate integral

  $\ln(v+1) = \frac{1}{3}t^3 - 2t + c_1$

- Solve for v

  $v = e^{\frac{1}{3}t^3-2t+c_1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(v(t),t)=t^2*v(t)-2-2*v(t)+t^2,v(t), singsol=all)
```

$$v(t) = -1 + e^{\frac{t(t^2-6)}{3}} c_1$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 27

```
DSolve[v'[t]==t^2*v[t]-2-2*v[t]+t^2,v[t],t,IncludeSingularSolutions -> True]
```

$$v(t) \to -1 + c_1 e^{\frac{1}{3}t(t^2-6)}$$
$$v(t) \to -1$$

1.17 problem 20

Internal problem ID [12881]
Internal file name [`OUTPUT/11533_Monday_November_06_2023_01_33_05_PM_95096190/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{1}{1 + ty + y + t} = 0}$$

1.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= \frac{1}{(y+1)(1+t)} \end{aligned}$$

Where $f(t) = \frac{1}{1+t}$ and $g(y) = \frac{1}{y+1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y+1}}\,dy = \frac{1}{1+t}\,dt$$

$$\int \frac{1}{\frac{1}{y+1}}\,dy = \int \frac{1}{1+t}\,dt$$

$$\frac{1}{2}y^2 + y = \ln(1+t) + c_1$$

Which results in

$$y = -1 + \sqrt{1 + 2\ln(1+t) + 2c_1}$$

$$y = -1 - \sqrt{1 + 2\ln(1+t) + 2c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -1 + \sqrt{1 + 2\ln(1+t) + 2c_1} \tag{1}$$

$$y = -1 - \sqrt{1 + 2\ln(1+t) + 2c_1} \tag{2}$$

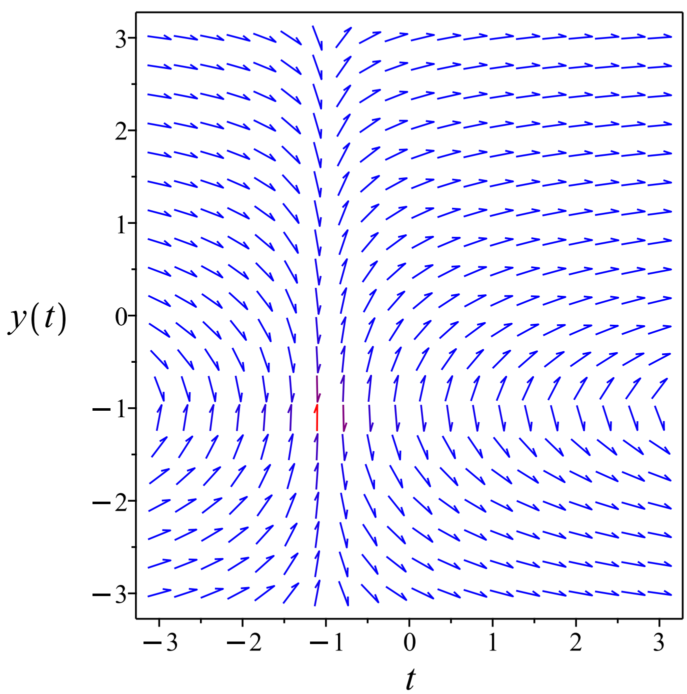


Figure 50: Slope field plot

Verification of solutions

$$y = -1 + \sqrt{1 + 2\ln(1+t) + 2c_1}$$

Verified OK.

$$y = -1 - \sqrt{1 + 2\ln(1+t) + 2c_1}$$

Verified OK.

1.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{1}{ty + t + y + 1} \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 37: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= 1 + t \\ \eta(t, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{1+t} dt \end{aligned}$$

Which results in

$$S = \ln(1+t)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{1}{ty + t + y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= \frac{1}{1+t} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y + 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R + 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 + R + c_1 \tag{4}$$

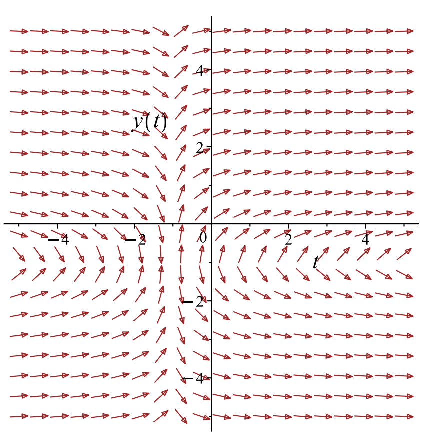
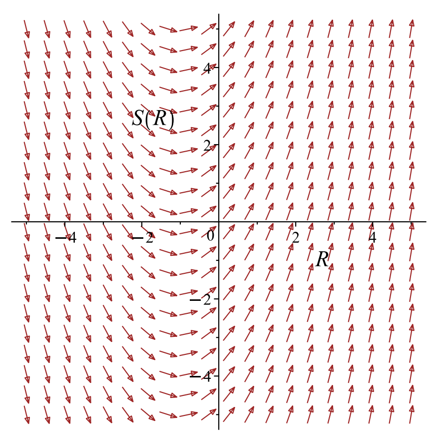
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\ln(1+t) = \frac{y^2}{2} + y + c_1$$

Which simplifies to

$$\ln(1+t) = \frac{y^2}{2} + y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{1}{ty+t+y+1}$ 	$R = y$ $S = \ln(1+t)$	$\frac{dS}{dR} = R + 1$ 

Summary

The solution(s) found are the following

$$\ln(1+t) = \frac{y^2}{2} + y + c_1 \tag{1}$$

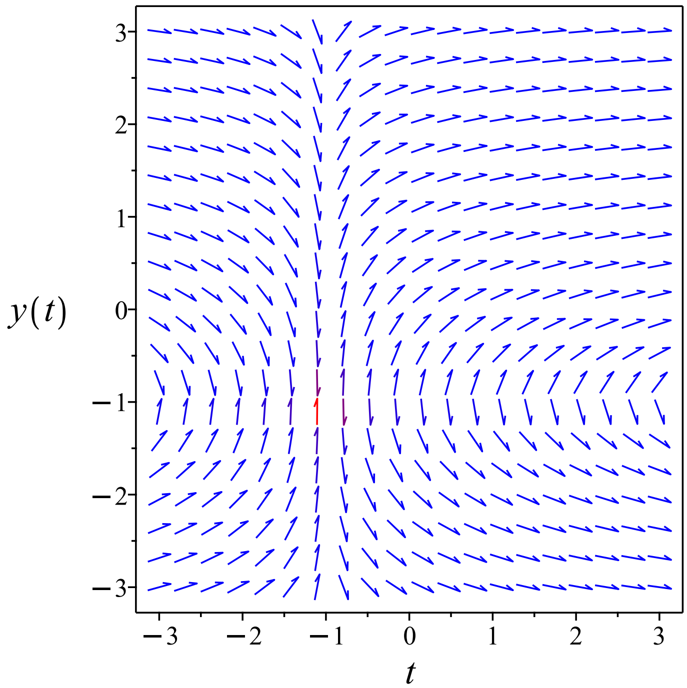


Figure 51: Slope field plot

Verification of solutions

$$\ln(1+t) = \frac{y^2}{2} + y + c_1$$

Verified OK.

1.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(y+1)\, \mathrm{d}y &= \left(\frac{1}{1+t}\right) \mathrm{d}t \\
\left(-\frac{1}{1+t}\right) \mathrm{d}t + (y+1)\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -\frac{1}{1+t} \\
N(t, y) &= y + 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{1+t}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(y+1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int M\, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int -\frac{1}{1+t}\, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\ln(1+t) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y + 1$. Therefore equation (4) becomes

$$
y + 1 = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y + 1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int (y+1)\, \mathrm{d}y \\
f(y) &= \frac{1}{2}y^2 + y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(1+t) + \frac{y^2}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(1+t) + \frac{y^2}{2} + y$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{y^2}{2} + y - \ln(1+t) = c_1 \tag{1}$$

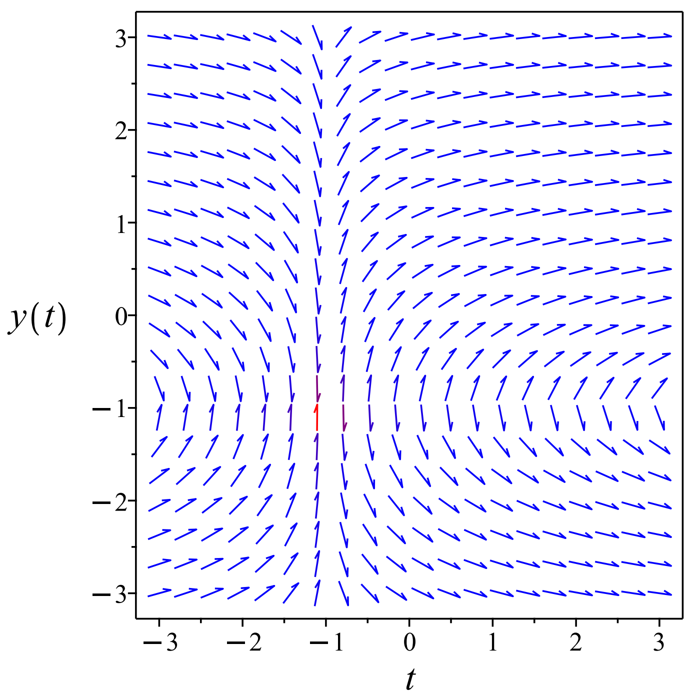


Figure 52: Slope field plot

<u>Verification of solutions</u>

$$\frac{y^2}{2} + y - \ln(1+t) = c_1$$

Verified OK.

1.17.4 Maple step by step solution

Let's solve

$y' - \frac{1}{1+ty+y+t} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(y+1) = \frac{1}{1+t}$

- Integrate both sides with respect to t

  $\int y'(y+1)\, dt = \int \frac{1}{1+t} dt + c_1$

- Evaluate integral

  $\frac{y^2}{2} + y = \ln(1+t) + c_1$

- Solve for y

  $\left\{y = -1 - \sqrt{1+2\ln(1+t)+2c_1}, y = -1 + \sqrt{1+2\ln(1+t)+2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve(diff(y(t),t)=1/(t*y(t)+t+y(t)+1),y(t), singsol=all)
```

$$y(t) = -1 - \sqrt{1+2\ln(t+1)+2c_1}$$
$$y(t) = -1 + \sqrt{1+2\ln(t+1)+2c_1}$$

✓ Solution by Mathematica

Time used: 0.217 (sec). Leaf size: 47

```
DSolve[y'[t]==1/(t*y[t]+t+y[t]+1),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -1 - \sqrt{2\log(t+1) + 1 + 2c_1}$$
$$y(t) \to -1 + \sqrt{2\log(t+1) + 1 + 2c_1}$$

1.18 problem 21

Internal problem ID [12882]
Internal file name [`OUTPUT/11534_Monday_November_06_2023_01_33_05_PM_35639577/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{e^t y}{1 + y^2} = 0}$$

1.18.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= \frac{e^t y}{y^2 + 1} \end{aligned}$$

Where $f(t) = e^t$ and $g(y) = \frac{y}{y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y^2+1}}\, dy = e^t\, dt$$

$$\int \frac{1}{\frac{y}{y^2+1}}\,dy = \int e^t\,dt$$

$$\frac{y^2}{2} + \ln(y) = e^t + c_1$$

Which results in

$$y = e^{-\frac{\mathrm{LambertW}\left(e^{2c_1+2\,e^t}\right)}{2}+c_1+e^t}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{\mathrm{LambertW}\left(e^{2c_1+2\,e^t}\right)}{2}+c_1+e^t} \tag{1}$$

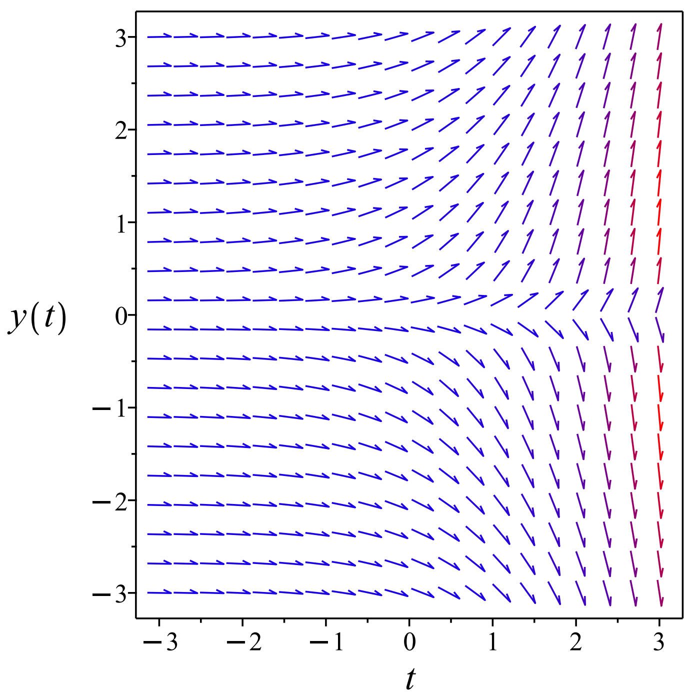


Figure 53: Slope field plot

Verification of solutions

$$y = e^{-\frac{\mathrm{LambertW}\left(e^{2c_1+2\,e^t}\right)}{2}+c_1+e^t}$$

Verified OK.

1.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{e^t y}{y^2+1} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 40: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,y) = e^{-t}$$
$$\eta(t,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{e^{-t}} dt \end{aligned}$$

Which results in

$$S = e^{t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{e^{t} y}{y^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= e^{t} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^2 + 1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2 + 1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^t = \frac{y^2}{2} + \ln(y) + c_1$$

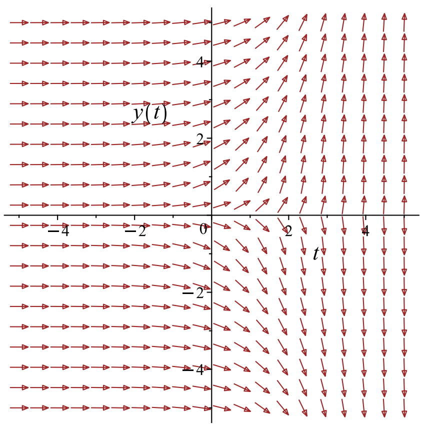
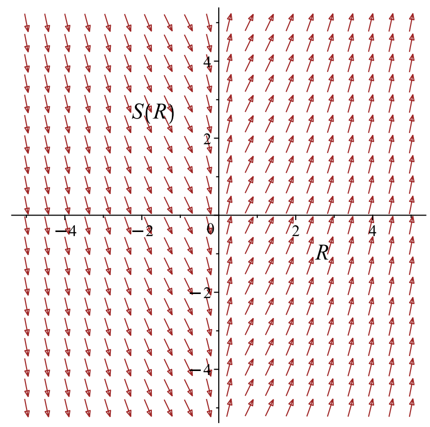
Which simplifies to

$$\mathrm{e}^t = \frac{y^2}{2} + \ln(y) + c_1$$

Which gives

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1 + 2\,\mathrm{e}^t}\right)}{2} - c_1 + \mathrm{e}^t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{e^t y}{y^2+1}$ 	$R = y$ $S = e^t$	$\frac{dS}{dR} = \frac{R^2+1}{R}$ 

Summary

The solution(s) found are the following

$$y = e^{-\frac{\mathrm{LambertW}\left(e^{-2c_1+2\,e^t}\right)}{2} - c_1 + e^t} \tag{1}$$

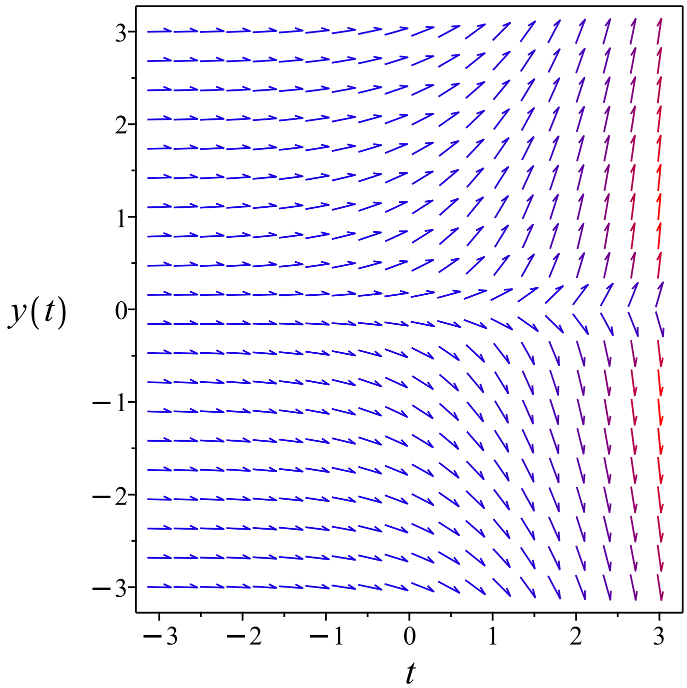


Figure 54: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1+2\,\mathrm{e}^t}\right)}{2}-c_1+\mathrm{e}^t}$$

Verified OK.

1.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y^2+1}{y}\right)\mathrm{d}y &= \left(\mathrm{e}^t\right)\mathrm{d}t \\ \left(-\mathrm{e}^t\right)\mathrm{d}t + \left(\frac{y^2+1}{y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -\mathrm{e}^t \\ N(t, y) &= \frac{y^2+1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^t\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{y^2+1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -\mathrm{e}^t \, \mathrm{d}t$$
$$\phi = -\mathrm{e}^t + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2+1}{y}$. Therefore equation (4) becomes

$$\frac{y^2+1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y^2+1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{y^2+1}{y}\right) \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{t} + \frac{y^2}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\mathrm{e}^{t} + \frac{y^2}{2} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(\mathrm{e}^{2c_1+2\,\mathrm{e}^{t}}\right)}{2}+c_1+\mathrm{e}^{t}}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(\mathrm{e}^{2c_1+2\,\mathrm{e}^{t}}\right)}{2}+c_1+\mathrm{e}^{t}} \tag{1}$$

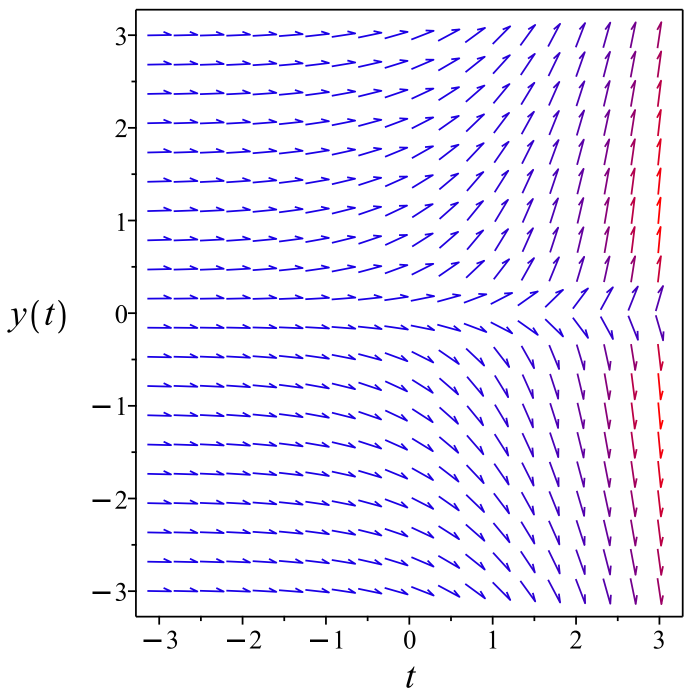


Figure 55: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(\mathrm{e}^{2c_1+2\,\mathrm{e}^t}\right)}{2}+c_1+\mathrm{e}^t}
$$

Verified OK.

1.18.4 Maple step by step solution

Let's solve

$y' - \frac{\mathrm{e}^t y}{1+y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1+y^2)}{y} = \mathrm{e}^t$

- Integrate both sides with respect to t

  $\int \frac{y'(1+y^2)}{y} dt = \int \mathrm{e}^t dt + c_1$

- Evaluate integral

  $\ln(y) + \frac{y^2}{2} = \mathrm{e}^t + c_1$

- Solve for y

  $y = \mathrm{e}^{-\frac{\mathit{LambertW}\left(\mathrm{e}^{2c_1+2\,\mathrm{e}^t}\right)}{2}+c_1+\mathrm{e}^t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 34

```
dsolve(diff(y(t),t)=exp(t)*y(t)/(1+y(t)^2),y(t), singsol=all)
```

$$y(t) = \frac{e^{e^t+c_1}}{\sqrt{\frac{e^{2c_1+2e^t}}{\mathrm{LambertW}\left(e^{2c_1+2e^t}\right)}}}$$

✓ Solution by Mathematica

Time used: 33.022 (sec). Leaf size: 46

```
DSolve[y'[t]==Exp[t]*y[t]/(1+y[t]^2),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\sqrt{W\left(e^{2(e^t+c_1)}\right)}$$
$$y(t) \to \sqrt{W\left(e^{2(e^t+c_1)}\right)}$$
$$y(t) \to 0$$

1.19 problem 22

Internal problem ID [12883]
Internal file name [`OUTPUT/11535_Monday_November_06_2023_01_33_06_PM_70014499/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = -4}$$

1.19.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4} dy = \int dt$$
$$\frac{\ln (y-2)}{4} - \frac{\ln (y+2)}{4} = t + c_1$$

The above can be written as

$$\left(\frac{1}{4}\right) (\ln (y-2) - \ln (y+2)) = t + c_1$$
$$\ln (y-2) - \ln (y+2) = (4)(t + c_1)$$
$$= 4t + 4c_1$$

Raising both side to exponential gives

$$e^{\ln(y-2)-\ln(y+2)} = 4c_1 e^{4t}$$

Which simplifies to

$$\frac{y-2}{y+2} = c_2 e^{4t}$$

Summary

The solution(s) found are the following

$$y = -\frac{2(c_2 e^{4t}+1)}{-1+c_2 e^{4t}} \tag{1}$$

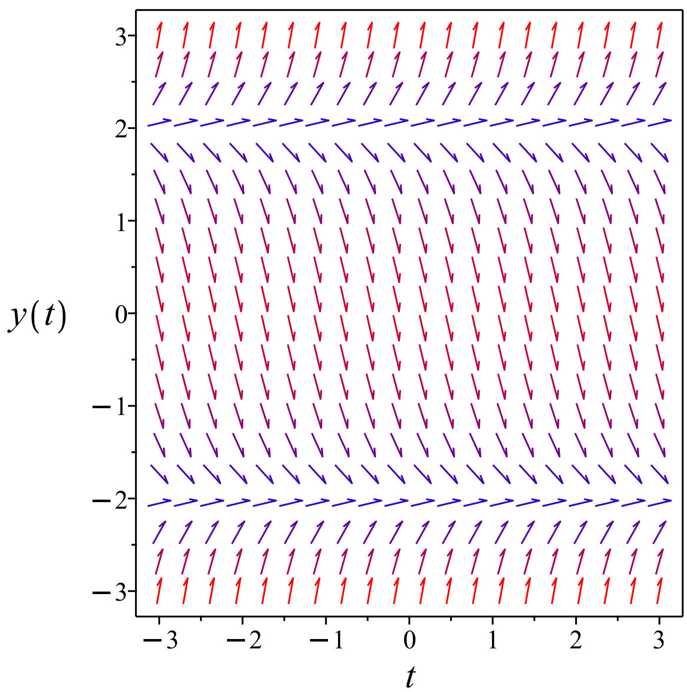


Figure 56: Slope field plot

Verification of solutions

$$y = -\frac{2(c_2 e^{4t}+1)}{-1+c_2 e^{4t}}$$

Verified OK.

1.19.2 Maple step by step solution

Let's solve

$y' - y^2 = -4$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4}dt = \int 1dt + c_1$

- Evaluate integral

  $\frac{\ln(y-2)}{4} - \frac{\ln(y+2)}{4} = t + c_1$

- Solve for y

  $y = -\frac{2(e^{4t+4c_1}+1)}{e^{4t+4c_1}-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve(diff(y(t),t)=y(t)^2-4,y(t), singsol=all)
```

$$y(t) = \frac{-2c_1 e^{4t} - 2}{-1 + c_1 e^{4t}}$$

✓ Solution by Mathematica

Time used: 1.053 (sec). Leaf size: 40

```
DSolve[y'[t]==y[t]^2-4,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{2 - 2e^{4(t+c_1)}}{1 + e^{4(t+c_1)}}$$
$$y(t) \to -2$$
$$y(t) \to 2$$

1.20 problem 23

Internal problem ID [12884]
Internal file name [`OUTPUT/11536_Monday_November_06_2023_01_33_06_PM_6700595/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{w' - \frac{w}{t} = 0}$$

1.20.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} w' &= F(t, w) \\ &= f(t)g(w) \\ &= \frac{w}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(w) = w$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{w}\,dw &= \frac{1}{t}\,dt \\
\int \frac{1}{w}\,dw &= \int \frac{1}{t}\,dt \\
\ln(w) &= \ln(t) + c_1 \\
w &= e^{\ln(t)+c_1} \\
&= c_1 t
\end{aligned}
$$

Summary

The solution(s) found are the following

$$w = c_1 t \tag{1}$$

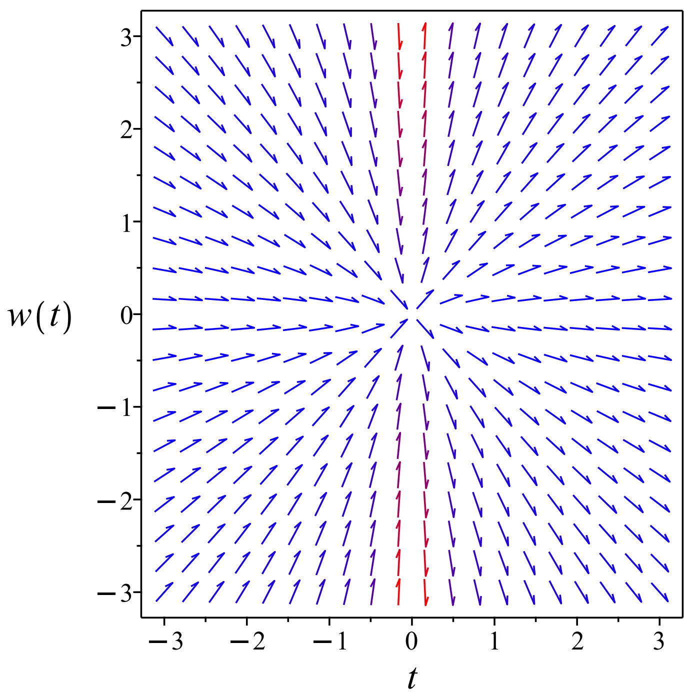


Figure 57: Slope field plot

Verification of solutions

$$w = c_1 t$$

Verified OK.

1.20.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w' + p(t)w = q(t)$$

Where here

$$\begin{aligned}p(t) &= -\frac{1}{t}\\ q(t) &= 0\end{aligned}$$

Hence the ode is

$$w' - \frac{w}{t} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{t}dt}\\ &= \frac{1}{t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}\mu w &= 0\\ \frac{d}{dt}\left(\frac{w}{t}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{w}{t} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t}$ results in

$$w = c_1 t$$

Summary
The solution(s) found are the following

$$w = c_1 t \tag{1}$$

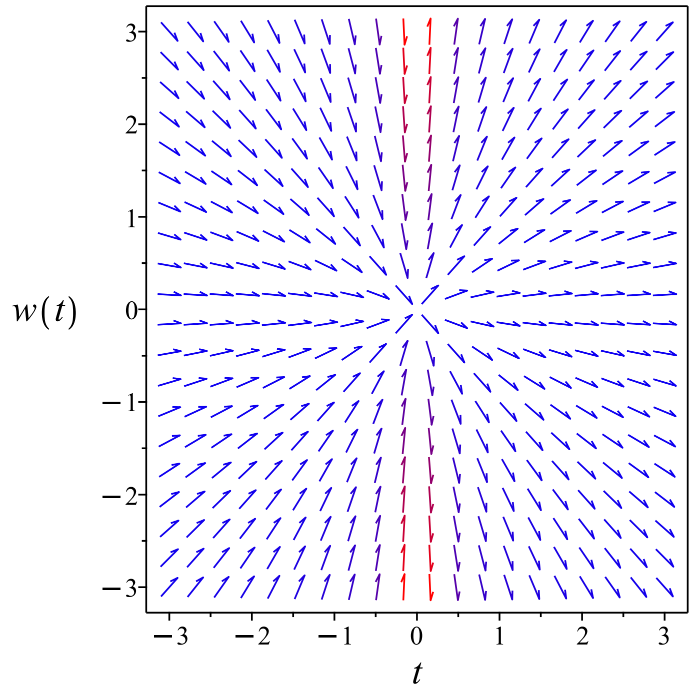


Figure 58: Slope field plot

Verification of solutions

$$w = c_1 t$$

Verified OK.

1.20.3 Solving as homogeneousTypeD2 ode

Using the change of variables $w = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t = 0$$

Integrating both sides gives

$$\begin{aligned} u(t) &= \int 0\, \mathrm{d}t \\ &= c_2 \end{aligned}$$

Therefore the solution w is

$$\begin{aligned} w &= tu \\ &= c_2 t \end{aligned}$$

Summary

The solution(s) found are the following

$$w = c_2 t \tag{1}$$

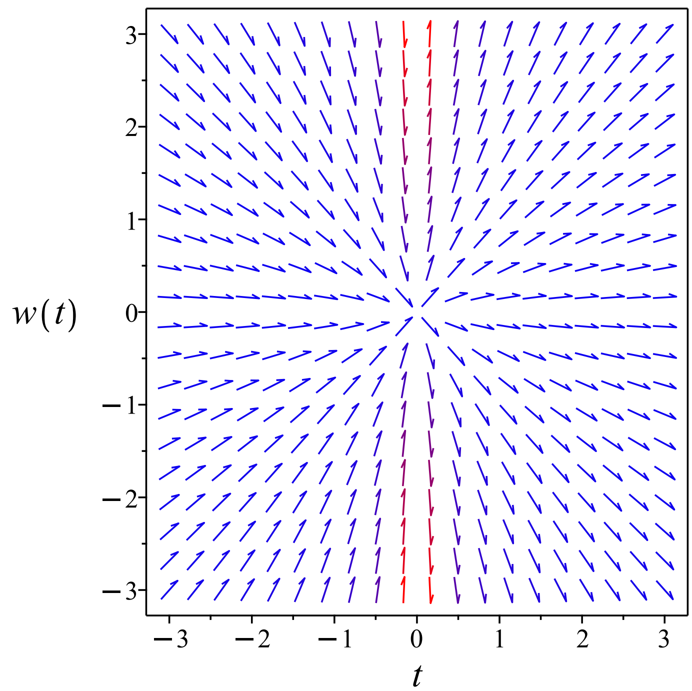


Figure 59: Slope field plot

Verification of solutions

$$w = c_2 t$$

Verified OK.

1.20.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$w' = \frac{w}{t}$$
$$w' = \omega(t, w)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_w - \xi_t) - \omega^2 \xi_w - \omega_t \xi - \omega_w \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 44: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t, w) = 0$$
$$\eta(t, w) = t \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, w) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dw}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial w}\right) S(t, w) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t} dy \end{aligned}$$

Which results in

$$S = \frac{w}{t}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, w) S_w}{R_t + \omega(t, w) R_w} \tag{2}$$

Where in the above R_t, R_w, S_t, S_w are all partial derivatives and $\omega(t, w)$ is the right hand side of the original ode given by

$$\omega(t, w) = \frac{w}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_w &= 0 \\ S_t &= -\frac{w}{t^2} \\ S_w &= \frac{1}{t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, w in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, w coordinates. This results in

$$\frac{w}{t} = c_1$$

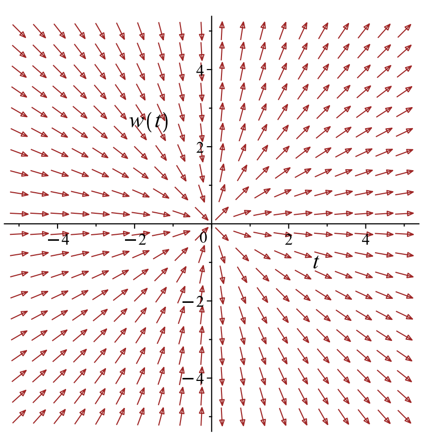
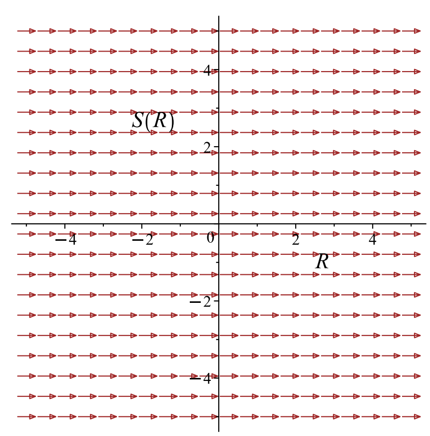
Which simplifies to

$$\frac{w}{t} = c_1$$

Which gives

$$w = c_1 t$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, w coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dw}{dt} = \frac{w}{t}$ 	$R = t$ $S = \frac{w}{t}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$w = c_1 t \tag{1}$$

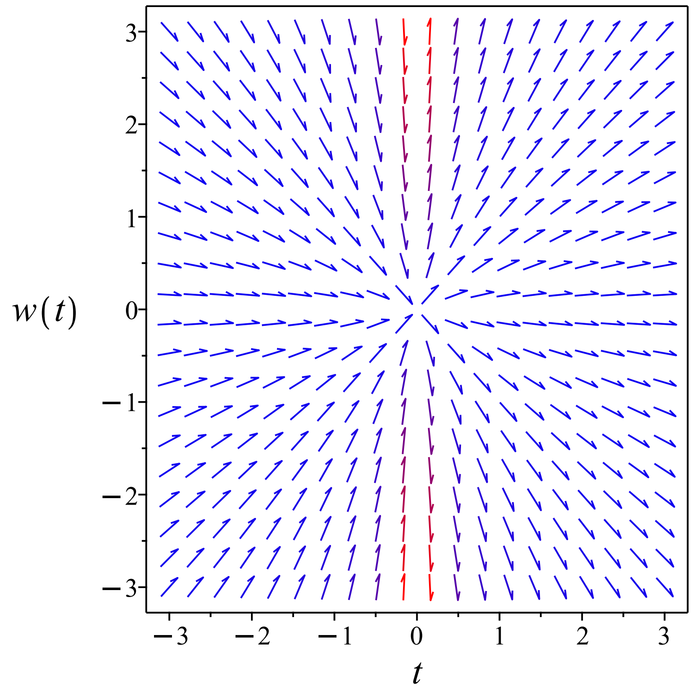


Figure 60: Slope field plot

Verification of solutions

$$w = c_1 t$$

Verified OK.

1.20.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, w) \, \mathrm{d}t + N(t, w) \, \mathrm{d}w = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{w}\right) \mathrm{d}w &= \left(\frac{1}{t}\right) \mathrm{d}t \\
\left(-\frac{1}{t}\right) \mathrm{d}t + \left(\frac{1}{w}\right) \mathrm{d}w &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, w) &= -\frac{1}{t} \\
N(t, w) &= \frac{1}{w}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial w} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial w} &= \frac{\partial}{\partial w}\left(-\frac{1}{t}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{w}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial w} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, w)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial w} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -\frac{1}{t}\,\mathrm{d}t$$
$$\phi = -\ln(t) + f(w) \tag{3}$$

Where $f(w)$ is used for the constant of integration since ϕ is a function of both t and w. Taking derivative of equation (3) w.r.t w gives

$$\frac{\partial \phi}{\partial w} = 0 + f'(w) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial w} = \frac{1}{w}$. Therefore equation (4) becomes

$$\frac{1}{w} = 0 + f'(w) \tag{5}$$

Solving equation (5) for $f'(w)$ gives

$$f'(w) = \frac{1}{w}$$

Integrating the above w.r.t w gives

$$\int f'(w)\,\mathrm{d}w = \int \left(\frac{1}{w}\right)\mathrm{d}w$$
$$f(w) = \ln(w) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(w)$ into equation (3) gives ϕ

$$\phi = -\ln(t) + \ln(w) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) + \ln(w)$$

The solution becomes

$$w = t\,e^{c_1}$$

Summary

The solution(s) found are the following

$$w = t\,e^{c_1} \tag{1}$$

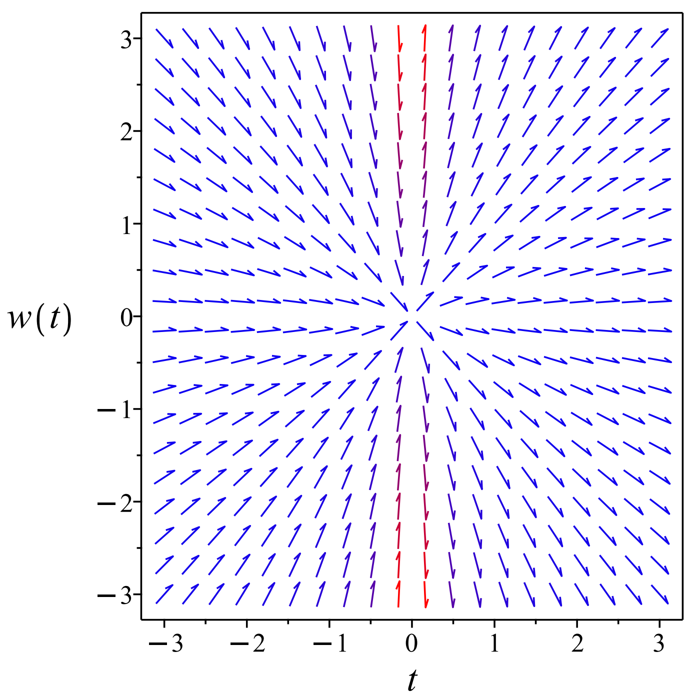


Figure 61: Slope field plot

Verification of solutions

$$w = t\,e^{c_1}$$

Verified OK.

1.20.6 Maple step by step solution

Let's solve

$w' - \frac{w}{t} = 0$

- Highest derivative means the order of the ODE is 1

  w'

- Separate variables

  $\frac{w'}{w} = \frac{1}{t}$

- Integrate both sides with respect to t

  $\int \frac{w'}{w} dt = \int \frac{1}{t} dt + c_1$

- Evaluate integral

  $\ln(w) = \ln(t) + c_1$

- Solve for w

  $w = t\,\mathrm{e}^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(diff(w(t),t)=w(t)/t,w(t), singsol=all)
```

$$w(t) = c_1 t$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 14

```
DSolve[w'[t]==w[t]/t,w[t],t,IncludeSingularSolutions -> True]
```

$$w(t) \to c_1 t$$
$$w(t) \to 0$$

1.21 problem 24

Internal problem ID [12885]
Internal file name [`OUTPUT/11537_Monday_November_06_2023_01_33_07_PM_3482729/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sec(y) = 0}$$

1.21.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\sec(y)} dy = x + c_1$$
$$\sin(y) = x + c_1$$

Solving for y gives these solutions

$$y_1 = \arcsin(x + c_1)$$

Summary
The solution(s) found are the following

$$y = \arcsin(x + c_1) \tag{1}$$

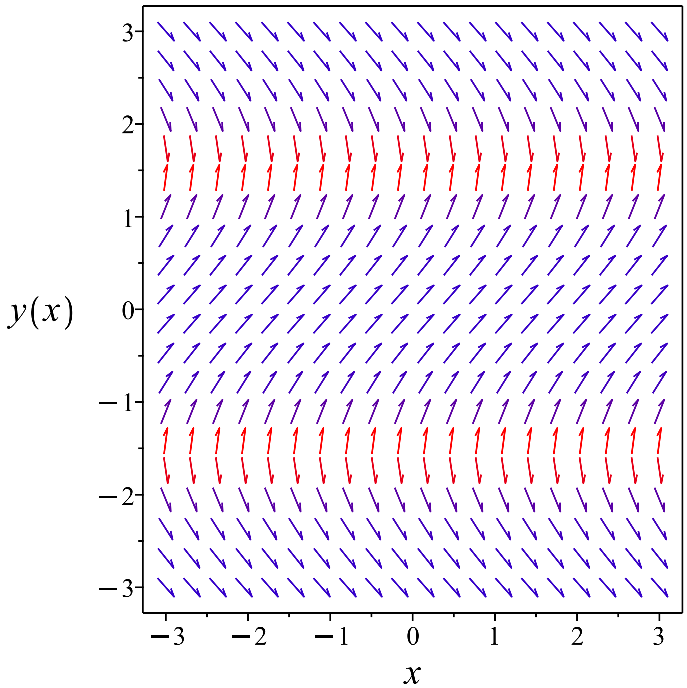


Figure 62: Slope field plot

Verification of solutions

$$y = \arcsin(x + c_1)$$

Verified OK.

1.21.2 Maple step by step solution

Let's solve

$y' - \sec(y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sec(y)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sec(y)} dx = \int 1 dx + c_1$

- Evaluate integral

$$\sin(y) = x + c_1$$

- Solve for y

$$y = \arcsin(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=sec(y(x)),y(x), singsol=all)
```

$$y(x) = \arcsin(c_1 + x)$$

✓ Solution by Mathematica

Time used: 0.35 (sec). Leaf size: 10

```
DSolve[y'[x]==Sec[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin(x + c_1)$$

1.22 problem 25

Internal problem ID [12886]
Internal file name [`OUTPUT/11538_Monday_November_06_2023_01_33_07_PM_97603790/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' + xt = 0}$$

With initial conditions

$$\left[x(0) = \frac{1}{\sqrt{\pi}}\right]$$

1.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= t \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$x' + xt = 0$$

The domain of $p(t) = t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

1.22.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= -tx \end{aligned}$$

Where $f(t) = -t$ and $g(x) = x$. Integrating both sides gives

$$\begin{aligned} \frac{1}{x} \, dx &= -t \, dt \\ \int \frac{1}{x} \, dx &= \int -t \, dt \\ \ln(x) &= -\frac{t^2}{2} + c_1 \\ x &= e^{-\frac{t^2}{2} + c_1} \\ &= e^{-\frac{t^2}{2}} c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \frac{1}{\sqrt{\pi}}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{\sqrt{\pi}} = c_1$$

$$c_1 = \frac{1}{\sqrt{\pi}}$$

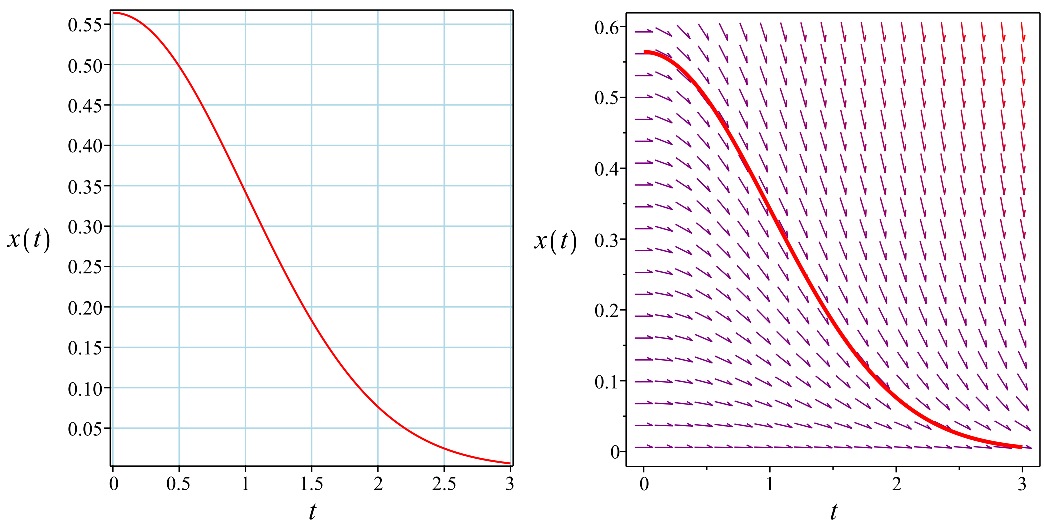
Substituting c_1 found above in the general solution gives

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

Verified OK.

1.22.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int t dt}$$
$$= e^{\frac{t^2}{2}}$$

The ode becomes

$$\frac{d}{dt}\mu x = 0$$
$$\frac{d}{dt}\left(e^{\frac{t^2}{2}} x\right) = 0$$

Integrating gives

$$e^{\frac{t^2}{2}} x = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{t^2}{2}}$ results in

$$x = e^{-\frac{t^2}{2}} c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \frac{1}{\sqrt{\pi}}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{\sqrt{\pi}} = c_1$$

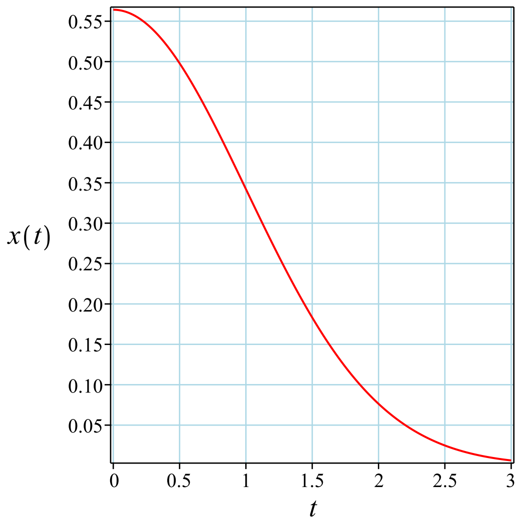
$$c_1 = \frac{1}{\sqrt{\pi}}$$

Substituting c_1 found above in the general solution gives

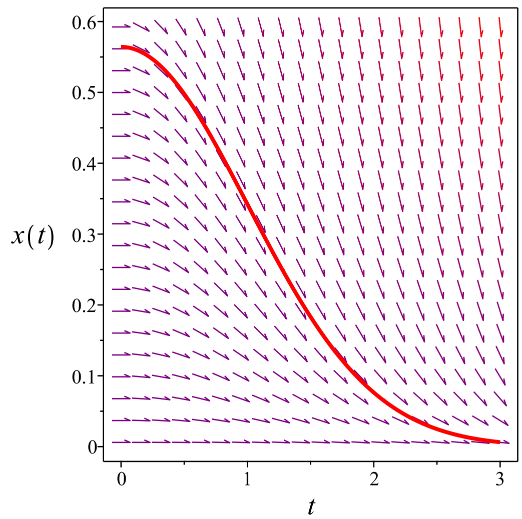
$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

Summary
The solution(s) found are the following

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

Verified OK.

1.22.4 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) + u(t)\,t^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= -\frac{u(t^2 + 1)}{t} \end{aligned}$$

Where $f(t) = -\frac{t^2+1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{t^2+1}{t}\, dt \\
\int \frac{1}{u}\, du &= \int -\frac{t^2+1}{t}\, dt \\
\ln(u) &= -\frac{t^2}{2} - \ln(t) + c_2 \\
u &= \mathrm{e}^{-\frac{t^2}{2} - \ln(t) + c_2} \\
&= c_2 \mathrm{e}^{-\frac{t^2}{2} - \ln(t)}
\end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_2 \mathrm{e}^{-\frac{t^2}{2}}}{t}$$

Therefore the solution x is

$$\begin{aligned}
x &= tu \\
&= c_2 \mathrm{e}^{-\frac{t^2}{2}}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $x = \frac{1}{\sqrt{\pi}}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{\sqrt{\pi}} = c_2$$

$$c_2 = \frac{1}{\sqrt{\pi}}$$

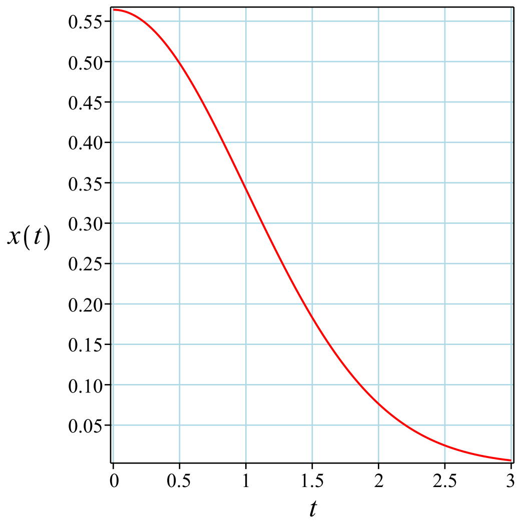
Substituting c_2 found above in the general solution gives

$$x = \frac{\mathrm{e}^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

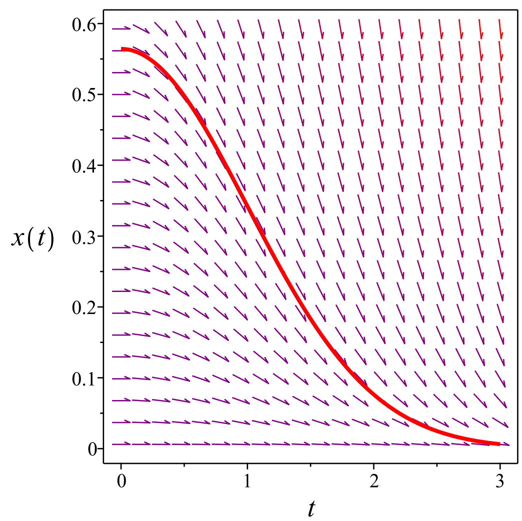
Summary

The solution(s) found are the following

$$x = \frac{\mathrm{e}^{-\frac{t^2}{2}}}{\sqrt{\pi}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

Verified OK.

1.22.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -tx$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 48: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0 \\
\eta(t,x) &= \mathrm{e}^{-\frac{t^2}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{t^2}{2}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{t^2}{2}} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = -tx$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= t\,\mathrm{e}^{\frac{t^2}{2}} x \\ S_x &= \mathrm{e}^{\frac{t^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^{\frac{t^2}{2}} x = c_1$$

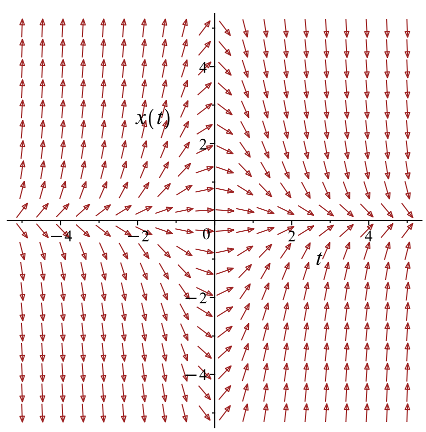
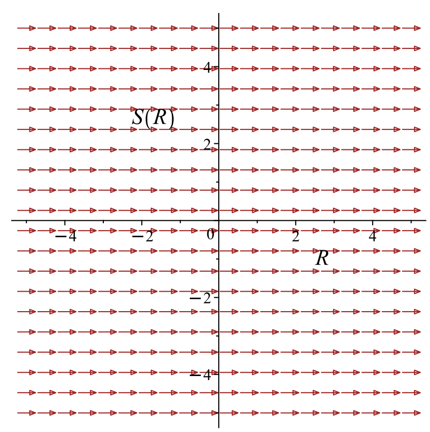
Which simplifies to

$$e^{\frac{t^2}{2}} x = c_1$$

Which gives

$$x = e^{-\frac{t^2}{2}} c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -tx$ 	$R = t$ $S = e^{\frac{t^2}{2}} x$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \frac{1}{\sqrt{\pi}}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{\sqrt{\pi}} = c_1$$

$$c_1 = \frac{1}{\sqrt{\pi}}$$

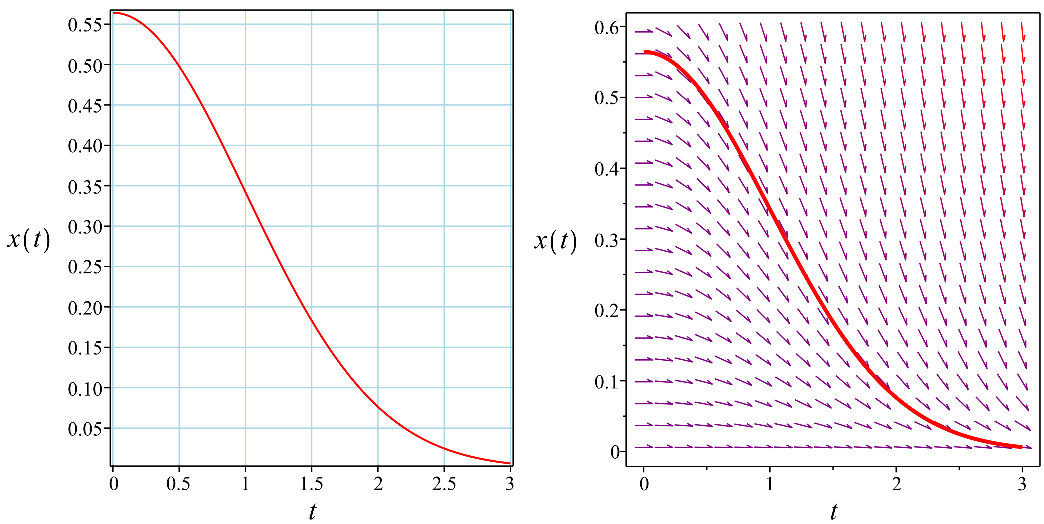
Substituting c_1 found above in the general solution gives

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

Verified OK.

1.22.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{x}\right)\mathrm{d}x = (t)\,\mathrm{d}t$$
$$(-t)\,\mathrm{d}t + \left(-\frac{1}{x}\right)\mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,x) &= -t \\
N(t,x) &= -\frac{1}{x}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-t) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -t \,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{t^2}{2} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -\frac{1}{x}$. Therefore equation (4) becomes

$$-\frac{1}{x} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{1}{x}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{1}{x}\right) \mathrm{d}x$$
$$f(x) = -\ln(x) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - \ln(x)$$

The solution becomes

$$x = \mathrm{e}^{-\frac{t^2}{2} - c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \frac{1}{\sqrt{\pi}}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{\sqrt{\pi}} = \mathrm{e}^{-c_1}$$

$$c_1 = \frac{\ln(\pi)}{2}$$

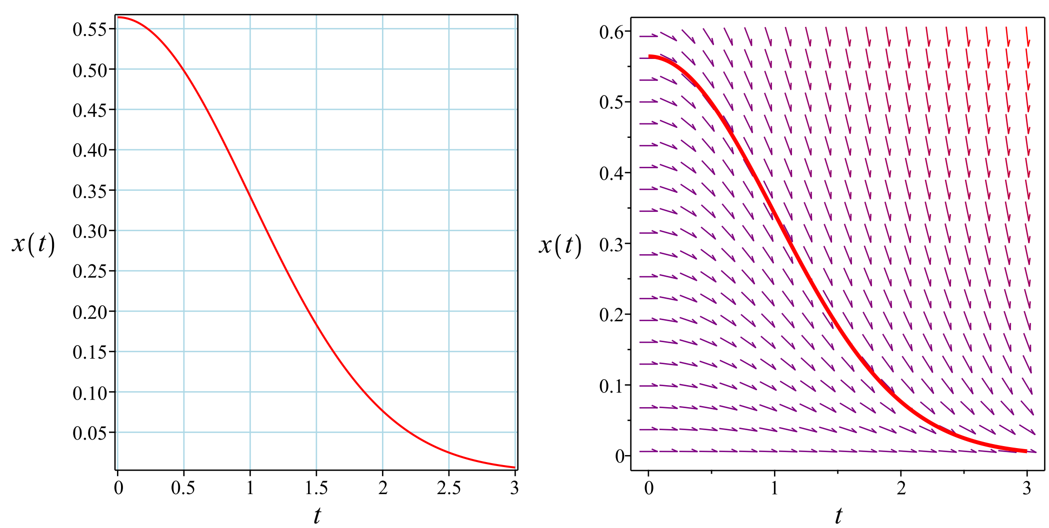
Substituting c_1 found above in the general solution gives

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

Verified OK.

1.22.7 Maple step by step solution

Let's solve

$\left[x' + xt = 0, x(0) = \frac{1}{\sqrt{\pi}}\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = -t$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int -t dt + c_1$

- Evaluate integral

  $\ln(x) = -\frac{t^2}{2} + c_1$

- Solve for x

  $x = \mathrm{e}^{-\frac{t^2}{2} + c_1}$

- Use initial condition $x(0) = \frac{1}{\sqrt{\pi}}$

  $\frac{1}{\sqrt{\pi}} = \mathrm{e}^{c_1}$

- Solve for c_1

  $c_1 = -\frac{\ln(\pi)}{2}$

- Substitute $c_1 = -\frac{\ln(\pi)}{2}$ into general solution and simplify

  $x = \frac{\mathrm{e}^{-\frac{t^2}{2}}}{\sqrt{\pi}}$

- Solution to the IVP

  $x = \frac{\mathrm{e}^{-\frac{t^2}{2}}}{\sqrt{\pi}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(x(t),t)=-x(t)*t,x(0) = 1/sqrt(Pi)],x(t), singsol=all)
```

$$x(t) = \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 20

```
DSolve[{x'[t]==-x[t]*t,{x[0]==1/Sqrt[Pi]}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{e^{-\frac{t^2}{2}}}{\sqrt{\pi}}$$

1.23 problem 26

Internal problem ID [12887]
Internal file name [`OUTPUT/11539_Monday_November_06_2023_01_33_08_PM_2720085/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - ty = 0}$$

With initial conditions

$$[y(0) = 3]$$

1.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -t \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$y' - ty = 0$$

The domain of $p(t) = -t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

1.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= ty \end{aligned}$$

Where $f(t) = t$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= t\, dt \\ \int \frac{1}{y}\, dy &= \int t\, dt \\ \ln(y) &= \frac{t^2}{2} + c_1 \\ y &= e^{\frac{t^2}{2} + c_1} \\ &= c_1 e^{\frac{t^2}{2}} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

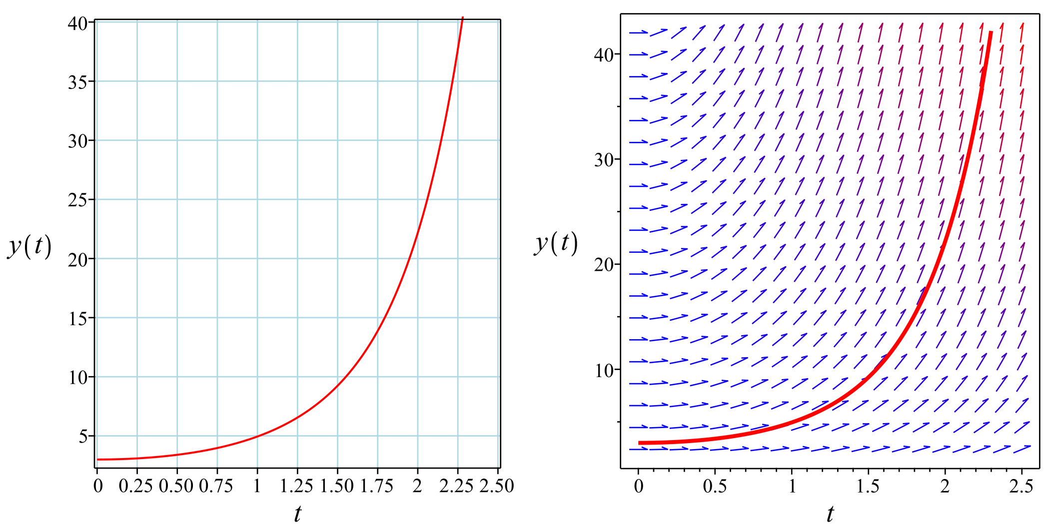
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}}$$

Summary
The solution(s) found are the following

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}}$$

Verified OK.

1.23.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = \mathrm{e}^{\int -t dt}$$
$$= \mathrm{e}^{-\frac{t^2}{2}}$$

The ode becomes

$$\frac{d}{dt}\mu y = 0$$

$$\frac{d}{dt}\left(e^{-\frac{t^2}{2}} y\right) = 0$$

Integrating gives

$$e^{-\frac{t^2}{2}} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{t^2}{2}}$ results in

$$y = c_1 e^{\frac{t^2}{2}}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

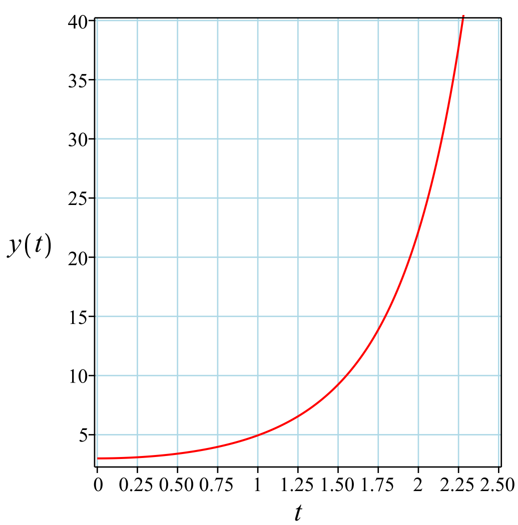
Substituting c_1 found above in the general solution gives

$$y = 3\, e^{\frac{t^2}{2}}$$

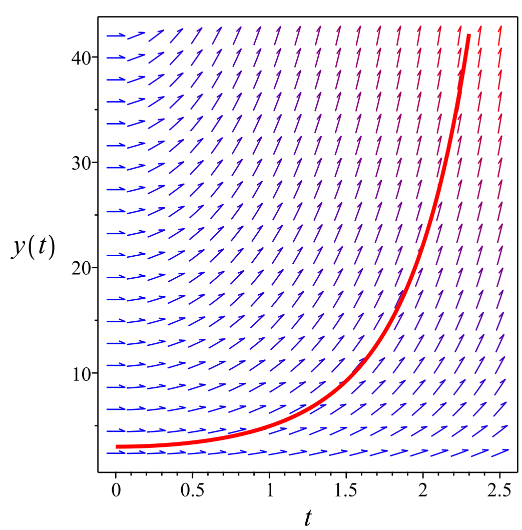
Summary

The solution(s) found are the following

$$y = 3\, e^{\frac{t^2}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}}$$

Verified OK.

1.23.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) - t^2 u(t) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= \frac{u(t^2 - 1)}{t} \end{aligned}$$

Where $f(t) = \frac{t^2-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{t^2 - 1}{t}\,dt \\ \int \frac{1}{u}\,du &= \int \frac{t^2 - 1}{t}\,dt \\ \ln(u) &= \frac{t^2}{2} - \ln(t) + c_2 \\ u &= \mathrm{e}^{\frac{t^2}{2} - \ln(t) + c_2} \\ &= c_2 \mathrm{e}^{\frac{t^2}{2} - \ln(t)} \end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_2 \mathrm{e}^{\frac{t^2}{2}}}{t}$$

Therefore the solution y is

$$\begin{aligned} y &= tu \\ &= \mathrm{e}^{\frac{t^2}{2}} c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_2$$

$$c_2 = 3$$

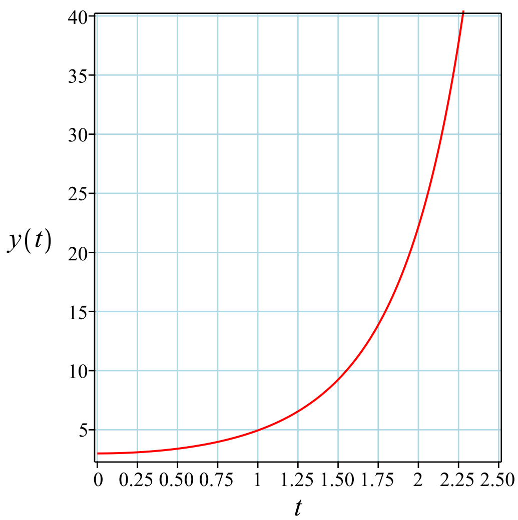
Substituting c_2 found above in the general solution gives

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}}$$

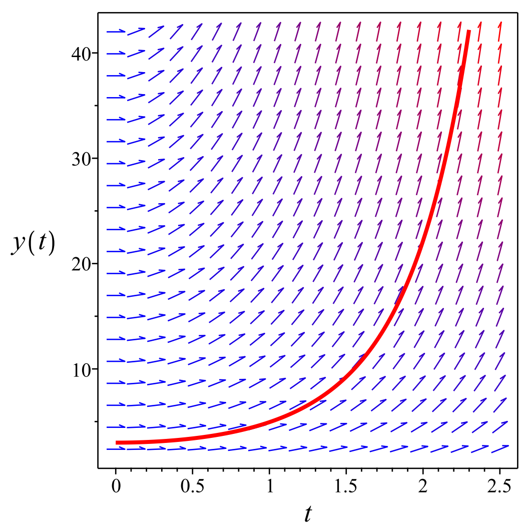
Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}}$$

Verified OK.

1.23.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = ty$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 51: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{\frac{t^2}{2}} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{t^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{t^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = ty$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -t\, e^{-\frac{t^2}{2}} y \\ S_y &= e^{-\frac{t^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-\frac{t^2}{2}} y = c_1$$

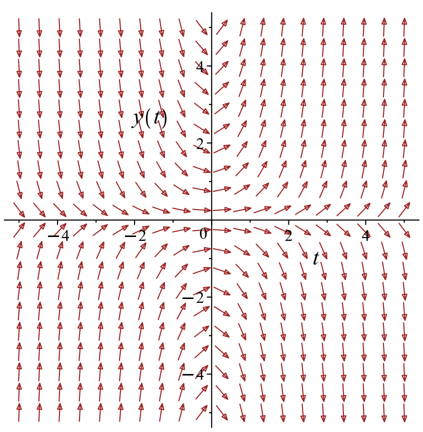
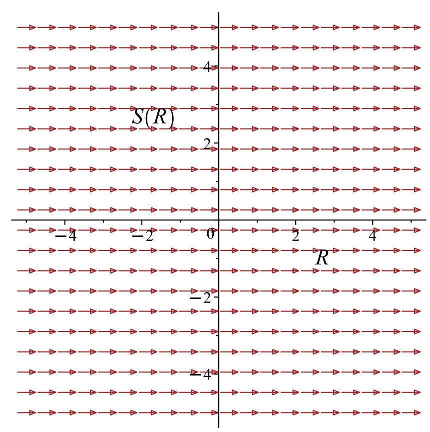
Which simplifies to

$$e^{-\frac{t^2}{2}} y = c_1$$

Which gives

$$y = c_1 e^{\frac{t^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = ty$ 	$R = t$ $S = e^{-\frac{t^2}{2}} y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

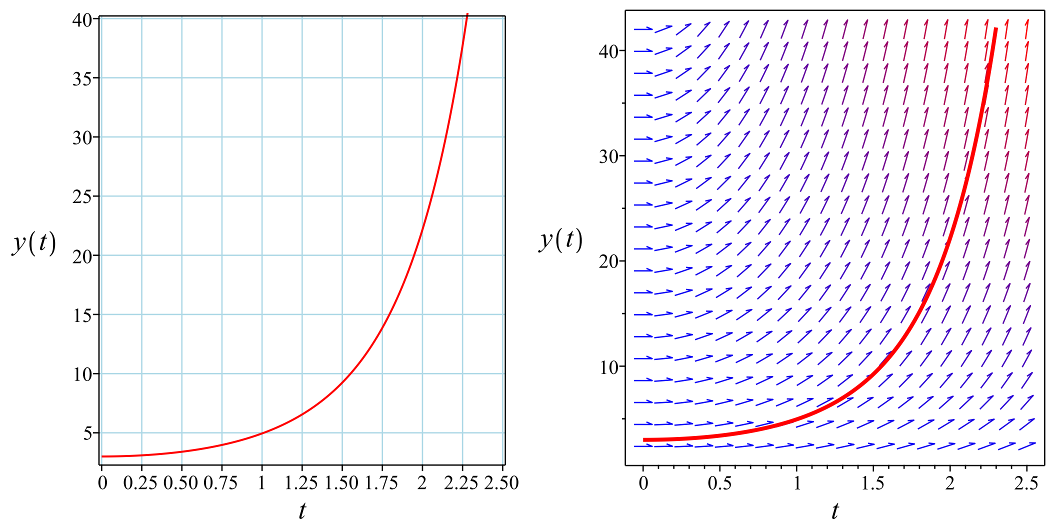
Substituting c_1 found above in the general solution gives

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}}$$

Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{\frac{t^2}{2}}$$

Verified OK.

1.23.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=(t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-t\\ N(t,y)&=\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-t)\\ &=0\end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t \, \mathrm{d}t \end{aligned}$$

$$\phi = -\frac{t^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\ f(y) &= \ln(y) + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \ln(y)$$

The solution becomes

$$y = e^{\frac{t^2}{2}+c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = e^{c_1}$$

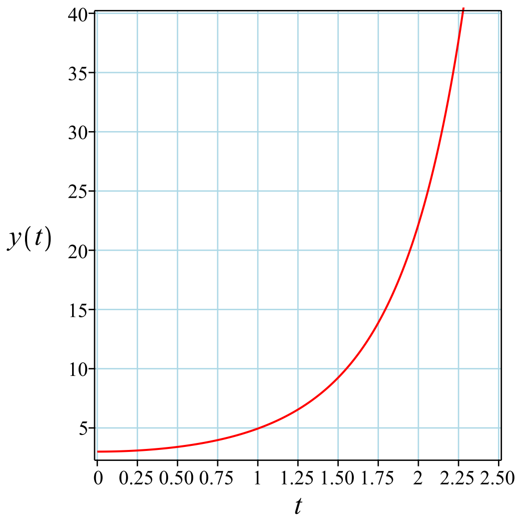
$$c_1 = \ln(3)$$

Substituting c_1 found above in the general solution gives

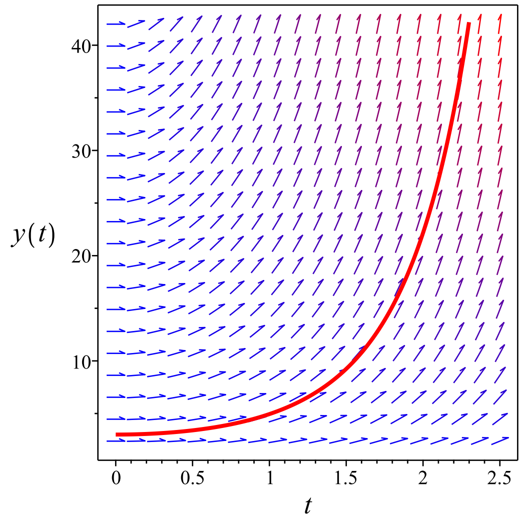
$$y = 3\,e^{\frac{t^2}{2}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 3\,e^{\frac{t^2}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3\,e^{\frac{t^2}{2}}$$

Verified OK.

1.23.7 Maple step by step solution

Let's solve

$[y' - ty = 0, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = t$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int t dt + c_1$

- Evaluate integral

  $\ln(y) = \frac{t^2}{2} + c_1$

- Solve for y

  $y = e^{\frac{t^2}{2} + c_1}$

- Use initial condition $y(0) = 3$

  $3 = e^{c_1}$

- Solve for c_1

  $c_1 = \ln(3)$

- Substitute $c_1 = \ln(3)$ into general solution and simplify

  $y = 3\,e^{\frac{t^2}{2}}$

- Solution to the IVP

  $y = 3\,e^{\frac{t^2}{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 12

```
dsolve([diff(y(t),t)=t*y(t),y(0) = 3],y(t), singsol=all)
```

$$y(t) = 3\,e^{\frac{t^2}{2}}$$

✓ Solution by Mathematica
Time used: 0.04 (sec). Leaf size: 16

```
DSolve[{y'[t]==t*y[t],{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 3e^{\frac{t^2}{2}}$$

1.24 problem 27

Internal problem ID [12888]
Internal file name [`OUTPUT/11540_Monday_November_06_2023_01_33_09_PM_56463063/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 = 0}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

1.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= -y^2 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-y^2) \\ &= -2y \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

1.24.2 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y^2} dy = t + c_1$$
$$\frac{1}{y} = t + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{1}{t + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{c_1}$$

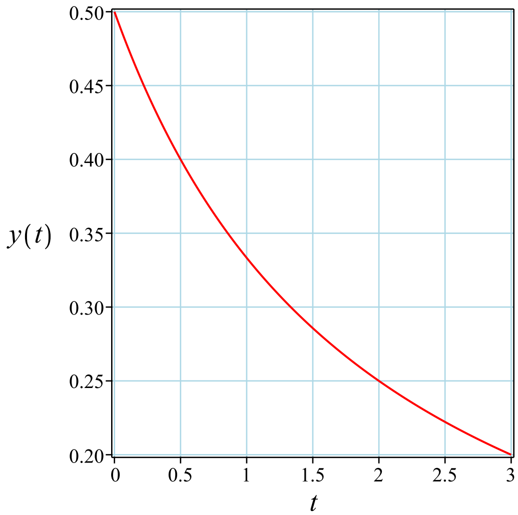
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

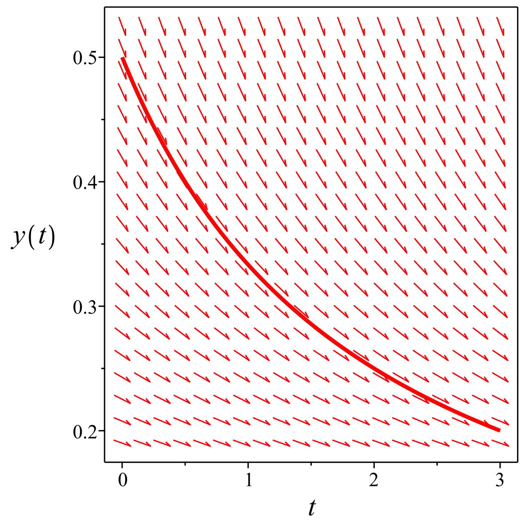
$$y = \frac{1}{t + 2}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{t + 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{t+2}$$

Verified OK.

1.24.3 Maple step by step solution

Let's solve

$\left[y' + y^2 = 0, y(0) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = -1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int (-1)\, dt + c_1$

- Evaluate integral

  $-\frac{1}{y} = -t + c_1$

- Solve for y

$y = -\frac{1}{-t+c_1}$

- Use initial condition $y(0) = \frac{1}{2}$

  $\frac{1}{2} = -\frac{1}{c_1}$

- Solve for c_1

  $c_1 = -2$

- Substitute $c_1 = -2$ into general solution and simplify

  $y = \frac{1}{t+2}$

- Solution to the IVP

  $y = \frac{1}{t+2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve([diff(y(t),t)=-y(t)^2,y(0) = 1/2],y(t), singsol=all)
```

$$y(t) = \frac{1}{t+2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 10

```
DSolve[{y'[t]==-y[t]^2,{y[0]==1/2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{t+2}$$

1.25 problem 28

Internal problem ID [12889]
Internal file name [`OUTPUT/11541_Monday_November_06_2023_01_33_09_PM_42758481/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - t^2 y^3 = 0}$$

With initial conditions

$$[y(0) = -1]$$

1.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^3 t^2 \end{aligned}$$

The t domain of $f(t, y)$ when $y = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t,y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^3t^2)\\ &= 3y^2t^2\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

1.25.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t,y)\\ &= f(t)g(y)\\ &= y^3t^2\end{aligned}$$

Where $f(t) = t^2$ and $g(y) = y^3$. Integrating both sides gives

$$\frac{1}{y^3}\,dy = t^2\,dt$$

$$\int \frac{1}{y^3}\,dy = \int t^2\,dt$$

$$-\frac{1}{2y^2} = \frac{t^3}{3} + c_1$$

Which results in

$$y = -\frac{3}{\sqrt{-6t^3 - 18c_1}}$$

$$y = \frac{3}{\sqrt{-6t^3 - 18c_1}}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{1}{\sqrt{-2c_1}}$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{\sqrt{-2c_1}}$$

$$c_1 = -\frac{1}{2}$$

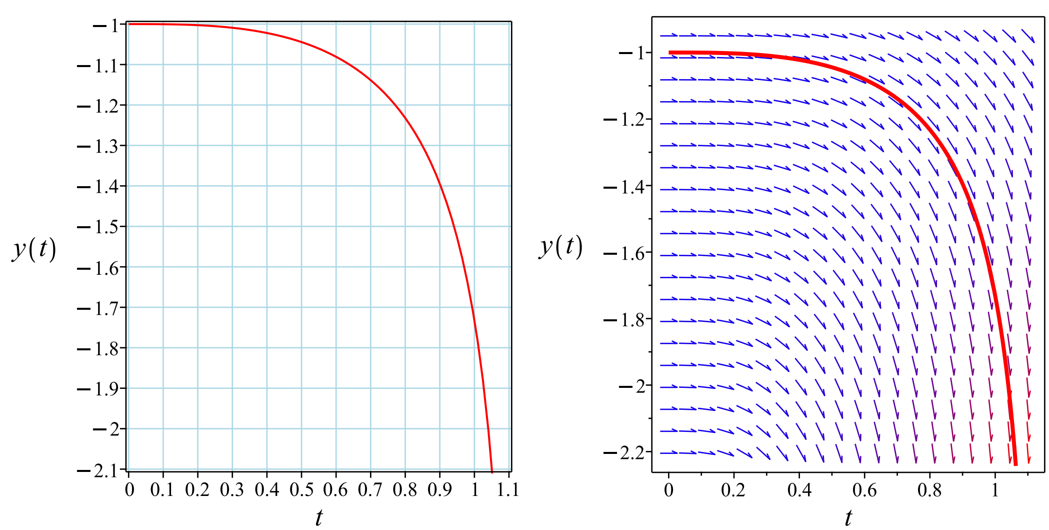
Substituting c_1 found above in the general solution gives

$$y = -\frac{3}{\sqrt{-6t^3 + 9}}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{\sqrt{-6t^3 + 9}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{\sqrt{-6t^3 + 9}}$$

Verified OK.

1.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y^3 t^2 \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 55: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= \frac{1}{t^2} \\ \eta(t, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{t^2}} dt \end{aligned}$$

Which results in

$$S = \frac{t^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = y^3 t^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= t^2 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

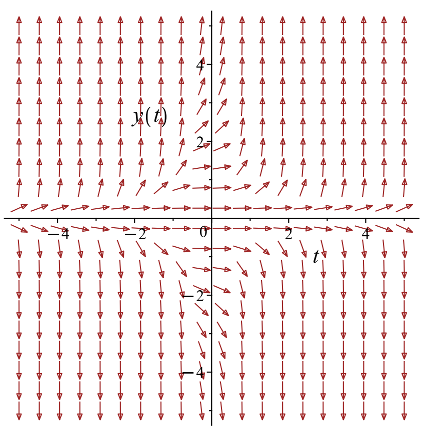
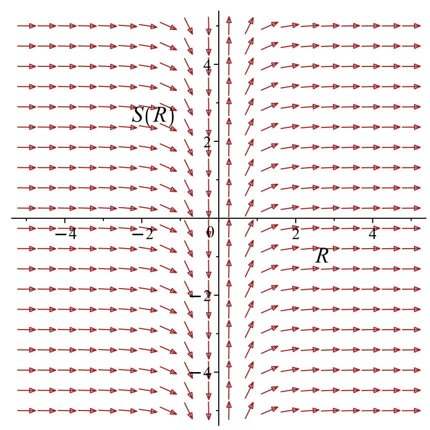
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{t^3}{3} = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$\frac{t^3}{3} = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y^3 t^2$ 	$R = y$ $S = \frac{t^3}{3}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 - \frac{1}{2}$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{t^3}{3} = \frac{y^2 - 1}{2y^2}$$

The above simplifies to

$$2y^2t^3 - 3y^2 + 3 = 0$$

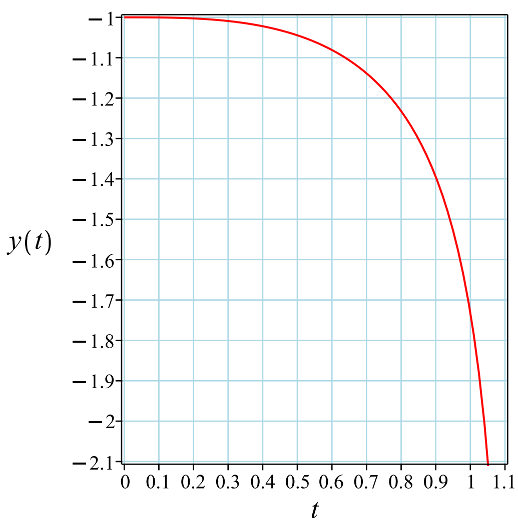
Solving for y from the above gives

$$y = -\frac{3}{\sqrt{-6t^3 + 9}}$$

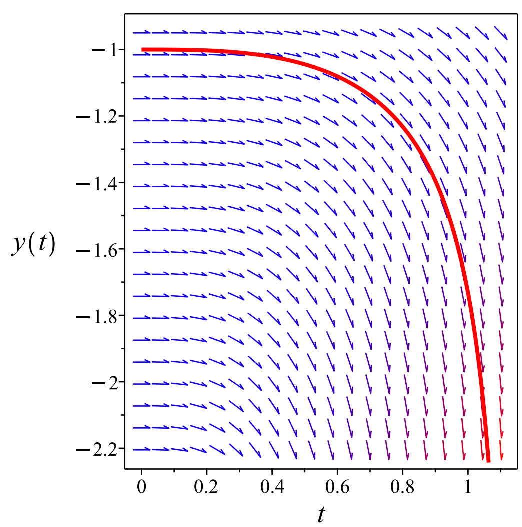
Summary

The solution(s) found are the following

$$y = -\frac{3}{\sqrt{-6t^3 + 9}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{\sqrt{-6t^3 + 9}}$$

Verified OK.

1.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^3}\right)\mathrm{d}y &= \left(t^2\right)\mathrm{d}t\\ \left(-t^2\right)\mathrm{d}t + \left(\frac{1}{y^3}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,y) &= -t^2 \\
N(t,y) &= \frac{1}{y^3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t^2) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y^3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -t^2 \,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{t^3}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{y^3}$. Therefore equation (4) becomes

$$\frac{1}{y^3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^3}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} - \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} - \frac{1}{2y^2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{t^3}{3} - \frac{1}{2y^2} = -\frac{1}{2}$$

The above simplifies to

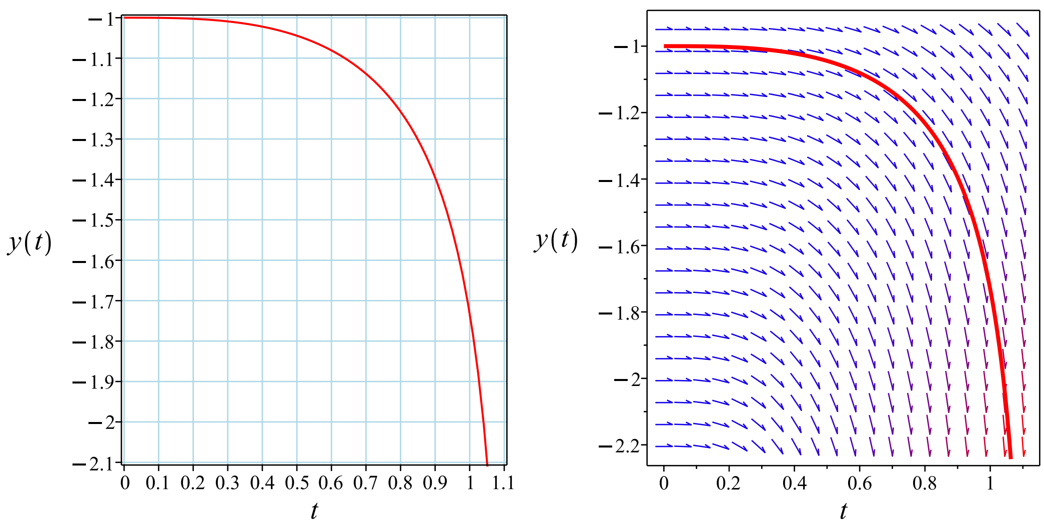
$$-2y^2t^3 + 3y^2 - 3 = 0$$

Solving for y from the above gives

$$y = -\frac{3}{\sqrt{-6t^3 + 9}}$$

Summary
The solution(s) found are the following

$$y = -\frac{3}{\sqrt{-6t^3 + 9}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{3}{\sqrt{-6t^3 + 9}}$$

Verified OK.

1.25.5 Maple step by step solution

Let's solve

$[y' - t^2 y^3 = 0, y(0) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = t^2$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^3} dt = \int t^2 dt + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = \frac{t^3}{3} + c_1$

- Solve for y

  $\left\{y = -\frac{3}{\sqrt{-6t^3 - 18c_1}}, y = \frac{3}{\sqrt{-6t^3 - 18c_1}}\right\}$

- Use initial condition $y(0) = -1$

  $-1 = -\frac{3}{\sqrt{-18c_1}}$

- Solve for c_1

  $c_1 = -\frac{1}{2}$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $y = -\frac{3}{\sqrt{-6t^3 + 9}}$

- Use initial condition $y(0) = -1$

  $-1 = \frac{3}{\sqrt{-18c_1}}$

- Solution does not satisfy initial condition
- Solution to the IVP

  $y = -\frac{3}{\sqrt{-6t^3 + 9}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)=t^2*y(t)^3,y(0) = -1],y(t), singsol=all)
```

$$y(t) = -\frac{3}{\sqrt{-6t^3+9}}$$

✓ Solution by Mathematica

Time used: 0.285 (sec). Leaf size: 20

```
DSolve[{y'[t]==t^2*y[t]^3,{y[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{\sqrt{1-\frac{2t^3}{3}}}$$

1.26 problem 29

Internal problem ID [12890]
Internal file name [`OUTPUT/11542_Monday_November_06_2023_01_33_10_PM_85978894/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 = 0}$$

With initial conditions

$$[y(0) = 0]$$

1.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= -y^2 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-y^2) \\ &= -2y \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.26.2 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y^2} dy = t + c_1$$
$$\frac{1}{y} = t + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{1}{t + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

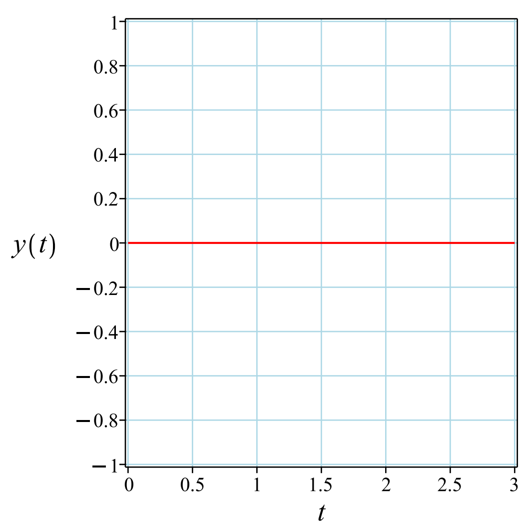
$$0 = \frac{1}{c_1}$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = \frac{1}{t+c_1} = y = 0$ and this result satisfies the given initial condition.

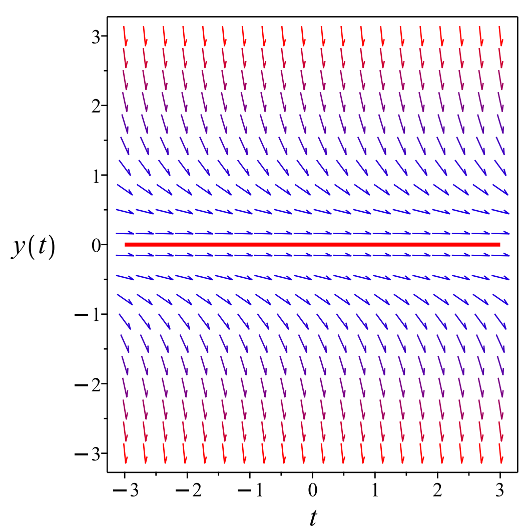
Summary

The solution(s) found are the following

$$y = 0$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

1.26.3 Maple step by step solution

Let's solve

$[y' + y^2 = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = -1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int (-1) \, dt + c_1$

- Evaluate integral

  $-\frac{1}{y} = -t + c_1$

- Solve for y

  $y = -\frac{1}{-t + c_1}$

- Use initial condition $y(0) = 0$

  $0 = -\frac{1}{c_1}$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(t),t)=-y(t)^2,y(0) = 0],y(t), singsol=all)
```

$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[t]==-y[t]^2,{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 0$$

1.27 problem 30

Internal problem ID [12891]
Internal file name [`OUTPUT/11543_Monday_November_06_2023_01_33_11_PM_4857332/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{t}{y - t^2 y} = 0}$$

With initial conditions

$$[y(0) = 4]$$

1.27.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= -\frac{t}{y\,(t^2 - 1)} \end{aligned}$$

The t domain of $f(t, y)$ when $y = 4$ is

$$\{-\infty \leq t < -1, -1 < t < 1, 1 < t \leq \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{t}{y\,(t^2-1)}\right)\\ &= \frac{t}{y^2\,(t^2-1)}\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = 4$ is

$$\{-\infty \le t < -1, -1 < t < 1, 1 < t \le \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Therefore solution exists and is unique.

1.27.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t, y)\\ &= f(t)g(y)\\ &= -\frac{t}{y\,(t^2-1)}\end{aligned}$$

Where $f(t) = -\frac{t}{t^2-1}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y}}\,dy &= -\frac{t}{t^2-1}\,dt\\ \int \frac{1}{\frac{1}{y}}\,dy &= \int -\frac{t}{t^2-1}\,dt\\ \frac{y^2}{2} &= -\frac{\ln(t-1)}{2} - \frac{\ln(1+t)}{2} + c_1\end{aligned}$$

Which results in

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + 2c_1}$$
$$y = -\sqrt{-\ln(t-1) - \ln(1+t) + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = -\sqrt{-i\pi + 2c_1}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \sqrt{-i\pi + 2c_1}$$

$$c_1 = \frac{i\pi}{2} + 8$$

Substituting c_1 found above in the general solution gives

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + i\pi + 16}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + i\pi + 16} \tag{1}$$

Verification of solutions

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + i\pi + 16}$$

Verified OK.

1.27.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{t}{y(t^2-1)}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 59: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= -\frac{t^2 - 1}{t} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial y}\right)S(t,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta=0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{-\frac{t^2-1}{t}} dt \end{aligned}$$

Which results in

$$S = -\frac{\ln(t-1)}{2} - \frac{\ln(1+t)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\frac{t}{y(t^2-1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= -\frac{t}{t^2-1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

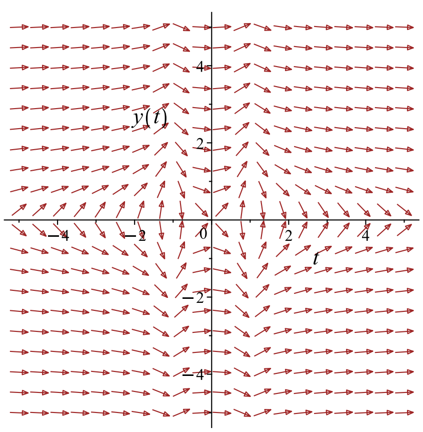
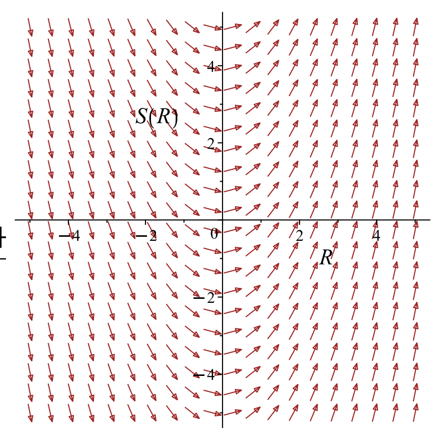
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{\ln(t-1)}{2} - \frac{\ln(1+t)}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{\ln(t-1)}{2} - \frac{\ln(1+t)}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{t}{y(t^2-1)}$ 	$R = y$ $S = -\frac{\ln(t-1)}{2} - \frac{\ln(1+}{2}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{i\pi}{2} = 8 + c_1$$

$$c_1 = -\frac{i\pi}{2} - 8$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(t-1)}{2} - \frac{\ln(1+t)}{2} = \frac{y^2}{2} - \frac{i\pi}{2} - 8$$

Solving for y from the above gives

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + i\pi + 16}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + i\pi + 16} \tag{1}$$

Verification of solutions

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + i\pi + 16}$$

Verified OK. `{positive}`

1.27.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y)\,\mathrm{d}y &= \left(\frac{t}{t^2-1}\right)\mathrm{d}t\\ \left(-\frac{t}{t^2-1}\right)\mathrm{d}t + (-y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -\frac{t}{t^2-1}\\ N(t,y) &= -y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{t}{t^2-1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(-y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -\frac{t}{t^2 - 1} \, \mathrm{d}t$$
$$\phi = -\frac{\ln(t-1)}{2} - \frac{\ln(1+t)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y$. Therefore equation (4) becomes

$$-y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-y) \, \mathrm{d}y$$
$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t-1)}{2} - \frac{\ln(1+t)}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t-1)}{2} - \frac{\ln(1+t)}{2} - \frac{y^2}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{i\pi}{2} - 8 = c_1$$

$$c_1 = -\frac{i\pi}{2} - 8$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(t-1)}{2} - \frac{\ln(1+t)}{2} - \frac{y^2}{2} = -\frac{i\pi}{2} - 8$$

Solving for y from the above gives

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + i\pi + 16}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + i\pi + 16} \tag{1}$$

Verification of solutions

$$y = \sqrt{-\ln(t-1) - \ln(1+t) + i\pi + 16}$$

Verified OK. `{positive}`

1.27.5 Maple step by step solution

Let's solve

$\left[y' - \frac{t}{y - t^2 y} = 0, y(0) = 4\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y = -\frac{t}{(t-1)(1+t)}$

- Integrate both sides with respect to t

  $\int y'ydt = \int -\frac{t}{(t-1)(1+t)}dt + c_1$

- Evaluate integral

  $\frac{y^2}{2} = -\frac{\ln((t-1)(1+t))}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{-\ln\left((t-1)(1+t)\right) + 2c_1}, y = -\sqrt{-\ln\left((t-1)(1+t)\right) + 2c_1}\right\}$

- Use initial condition $y(0) = 4$

  $4 = \sqrt{-\mathrm{I}\pi + 2c_1}$

- Solve for c_1

  $c_1 = \frac{\mathrm{I}\pi}{2} + 8$

- Substitute $c_1 = \frac{\mathrm{I}\pi}{2} + 8$ into general solution and simplify

  $y = \sqrt{-\ln(t^2 - 1) + \mathrm{I}\pi + 16}$

- Use initial condition $y(0) = 4$

  $4 = -\sqrt{-\mathrm{I}\pi + 2c_1}$

- Solution does not satisfy initial condition

- Solution to the IVP

  $y = \sqrt{-\ln(t^2 - 1) + \mathrm{I}\pi + 16}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 24

```
dsolve([diff(y(t),t)=t/(y(t)-t^2*y(t)),y(0) = 4],y(t), singsol=all)
```

$$y(t) = \sqrt{i\pi - \ln(t-1) - \ln(t+1) + 16}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 24

```
DSolve[{y'[t]==t/(y[t]-t^2*y[t]),{y[0]==4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \sqrt{-\log(t^2-1) + i\pi + 16}$$

1.28 problem 31

Internal problem ID [12892]
Internal file name [`OUTPUT/11544_Monday_November_06_2023_01_33_12_PM_92578670/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 2y = 1}$$

With initial conditions

$$[y(0) = 3]$$

1.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = 1$$

Hence the ode is

$$y' - 2y = 1$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.28.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{2y+1} dy = \int dt$$
$$\frac{\ln(2y+1)}{2} = t + c_1$$

Raising both side to exponential gives

$$\sqrt{2y+1} = e^{t+c_1}$$

Which simplifies to

$$\sqrt{2y+1} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{c_2^2}{2} - \frac{1}{2}$$

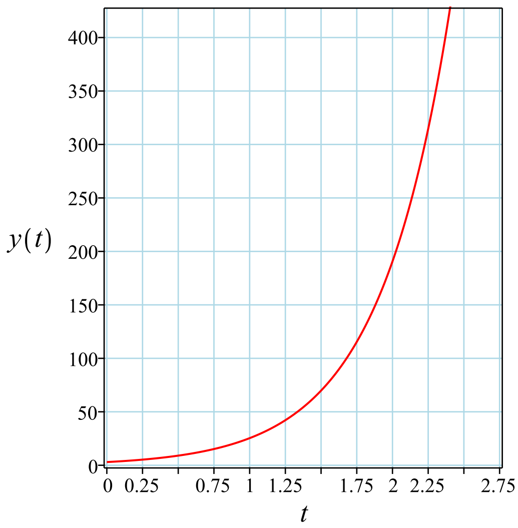
$$c_2 = -\sqrt{7}$$

Substituting c_2 found above in the general solution gives

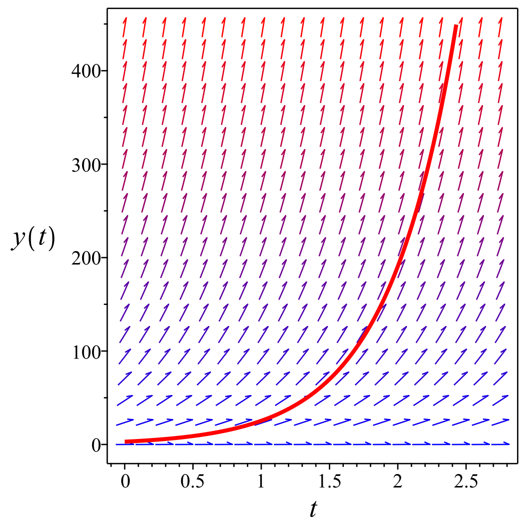
$$y = \frac{7\,e^{2t}}{2} - \frac{1}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{7\,e^{2t}}{2} - \frac{1}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{7\,e^{2t}}{2} - \frac{1}{2}$$

Verified OK.

1.28.3 Maple step by step solution

Let's solve

$[y' - 2y = 1, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y+1} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{2y+1} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{\ln(2y+1)}{2} = t + c_1$

- Solve for y

$y = -\frac{1}{2} + \frac{e^{2t+2c_1}}{2}$

- Use initial condition $y(0) = 3$

  $3 = -\frac{1}{2} + \frac{e^{2c_1}}{2}$

- Solve for c_1

  $c_1 = \frac{\ln(7)}{2}$

- Substitute $c_1 = \frac{\ln(7)}{2}$ into general solution and simplify

  $y = \frac{7\,e^{2t}}{2} - \frac{1}{2}$

- Solution to the IVP

  $y = \frac{7\,e^{2t}}{2} - \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(t),t)=2*y(t)+1,y(0) = 3],y(t), singsol=all)
```

$$y(t) = -\frac{1}{2} + \frac{7\,e^{2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 18

```
DSolve[{y'[t]==2*y[t]+1,{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}\left(7e^{2t} - 1\right)$$

1.29 problem 32

Internal problem ID [12893]
Internal file name [`OUTPUT/11545_Monday_November_06_2023_01_33_12_PM_41829624/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - ty^2 - 2y^2 = 0}$$

With initial conditions

$$[y(0) = 1]$$

1.29.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= t\,y^2 + 2y^2 \end{aligned}$$

The t domain of $f(t, y)$ when $y = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(t\,y^2 + 2y^2\right) \\
&= 2ty + 4y
\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.29.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(t, y) \\
&= f(t)g(y) \\
&= y^2(t + 2)
\end{aligned}$$

Where $f(t) = t + 2$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y^2}\,dy &= t + 2\,dt \\
\int \frac{1}{y^2}\,dy &= \int t + 2\,dt \\
-\frac{1}{y} &= \frac{1}{2}t^2 + 2t + c_1
\end{aligned}$$

Which results in

$$y = -\frac{2}{t^2 + 2c_1 + 4t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1}$$

$$c_1 = -1$$

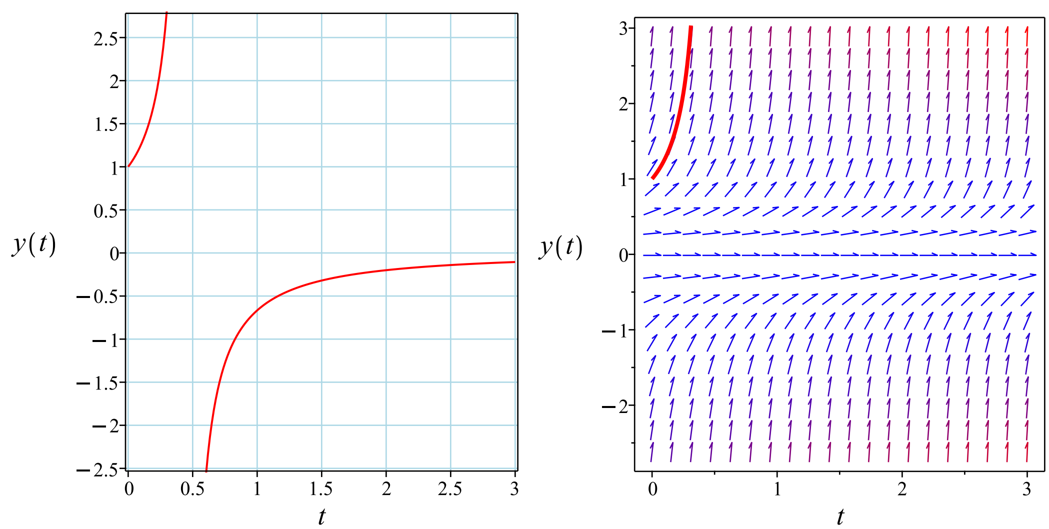
Substituting c_1 found above in the general solution gives

$$y = -\frac{2}{t^2 + 4t - 2}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{t^2 + 4t - 2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{2}{t^2 + 4t - 2}$$

Verified OK.

1.29.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= t\,y^2 + 2y^2 \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 63: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= \frac{1}{t+2} \\
\eta(t,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{\frac{1}{t+2}} dt
\end{aligned}
$$

Which results in

$$
S = \frac{1}{2}t^2 + 2t
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = t\,y^2 + 2y^2
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= t+2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{1}{2}t^2 + 2t = -\frac{1}{y} + c_1$$

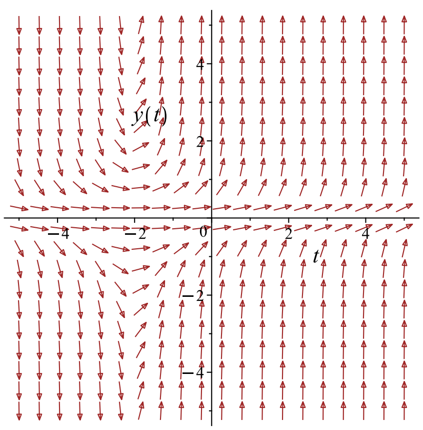
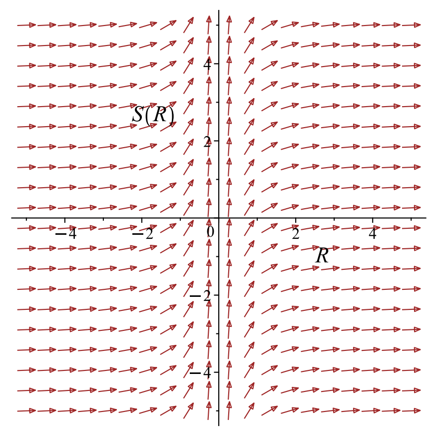
Which simplifies to

$$\frac{1}{2}t^2 + 2t = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{2}{-t^2 + 2c_1 - 4t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = t\,y^2 + 2y^2$ 	$R = y$ $S = \frac{1}{2}t^2 + 2t$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_1}$$

$$c_1 = 1$$

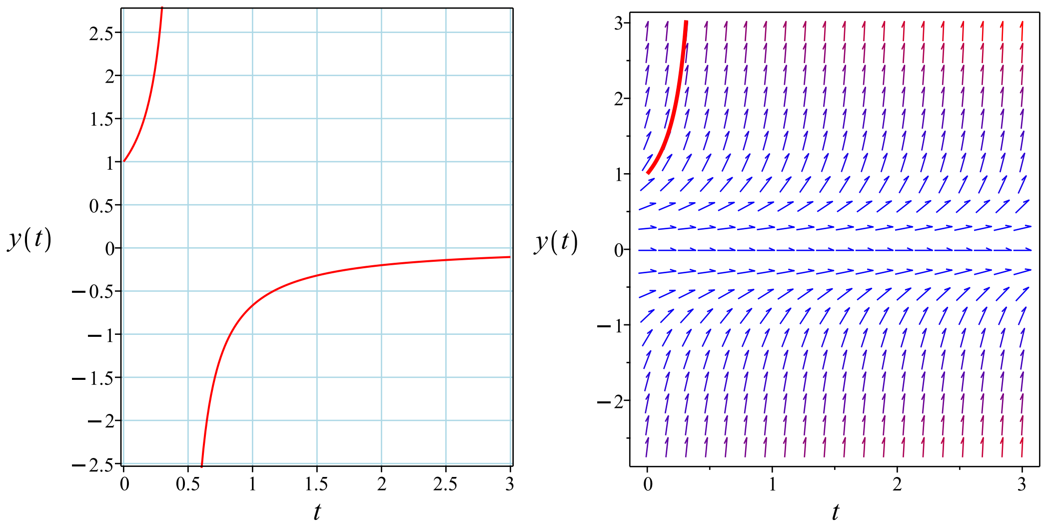
Substituting c_1 found above in the general solution gives

$$y = -\frac{2}{t^2 + 4t - 2}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{2}{t^2 + 4t - 2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{2}{t^2 + 4t - 2}$$

Verified OK.

1.29.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2}\right)\mathrm{d}y&=(t+2)\,\mathrm{d}t\\(-t-2)\,\mathrm{d}t+\left(\frac{1}{y^2}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-t-2\\N(t,y)&=\frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-t-2)\\&=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t}&=\frac{\partial}{\partial t}\left(\frac{1}{y^2}\right)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, dt = \int M \, dt$$

$$\int \frac{\partial \phi}{\partial t} \, dt = \int -t - 2 \, dt$$

$$\phi = -\frac{1}{2}t^2 - 2t + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(\frac{1}{y^2}\right) dy$$

$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - 2t - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - 2t - \frac{1}{y}$$

The solution becomes

$$y = -\frac{2}{t^2 + 2c_1 + 4t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1}$$

$$c_1 = -1$$

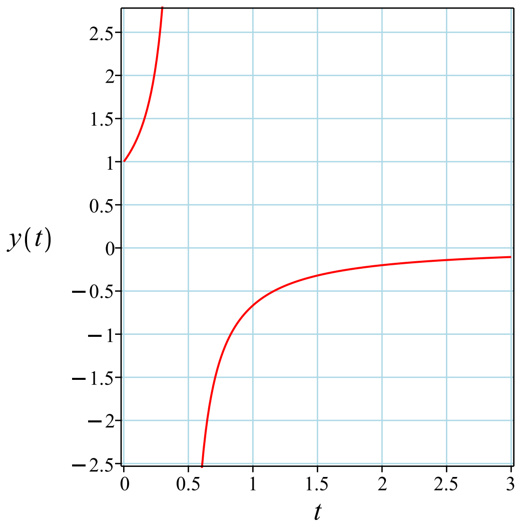
Substituting c_1 found above in the general solution gives

$$y = -\frac{2}{t^2 + 4t - 2}$$

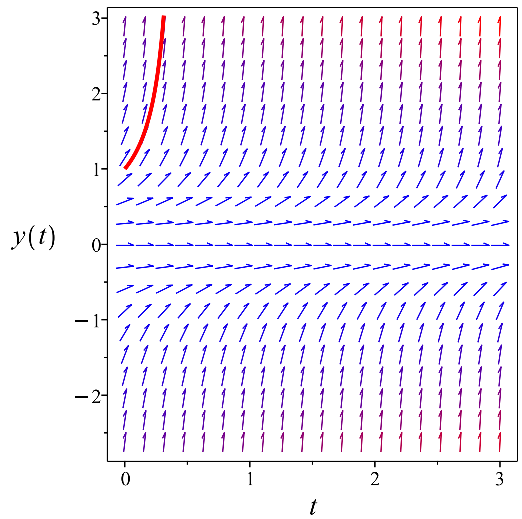
Summary

The solution(s) found are the following

$$y = -\frac{2}{t^2 + 4t - 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2}{t^2 + 4t - 2}$$

Verified OK.

1.29.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= t\,y^2 + 2y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = t\,y^2 + 2y^2$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = t + 2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(t+2)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(t+2)\, u''(t) - u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + c_2 (t+2)^2$$

The above shows that

$$u'(t) = 2c_2(t+2)$$

Using the above in (1) gives the solution

$$y = -\frac{2c_2}{c_1 + c_2\,(t+2)^2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2}{t^2 + c_3 + 4t + 4}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{2}{c_3 + 4}$$

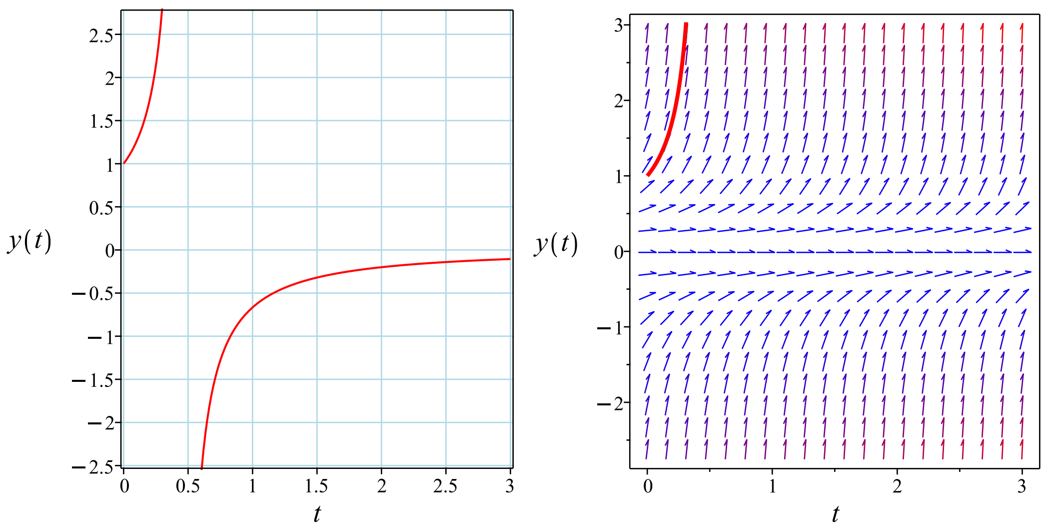
$$c_3 = -6$$

Substituting c_3 found above in the general solution gives

$$y = -\frac{2}{t^2 + 4t - 2}$$

Summary
The solution(s) found are the following

$$y = -\frac{2}{t^2 + 4t - 2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{2}{t^2 + 4t - 2}$$

Verified OK.

1.29.6 Maple step by step solution

Let's solve

$[y' - ty^2 - 2y^2 = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = t + 2$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int (t + 2)\, dt + c_1$

- Evaluate integral

  $-\frac{1}{y} = \frac{1}{2}t^2 + 2t + c_1$

- Solve for y

  $y = -\frac{2}{t^2 + 2c_1 + 4t}$

- Use initial condition $y(0) = 1$

  $1 = -\frac{1}{c_1}$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = -\frac{2}{t^2 + 4t - 2}$

- Solution to the IVP

  $y = -\frac{2}{t^2 + 4t - 2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 16

```
dsolve([diff(y(t),t)=t*y(t)^2+2*y(t)^2,y(0) = 1],y(t), singsol=all)
```

$$y(t) = -\frac{2}{t^2 + 4t - 2}$$

✓ Solution by Mathematica

Time used: 0.219 (sec). Leaf size: 17

```
DSolve[{y'[t]==t*y[t]^2+2*y[t]^2,{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{2}{t^2 + 4t - 2}$$

1.30 problem 33

Internal problem ID [12894]
Internal file name [`OUTPUT/11546_Monday_November_06_2023_01_33_13_PM_3073111/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' - \frac{t^2}{x + t^3 x} = 0}$$

With initial conditions

$$[x(0) = -2]$$

1.30.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(t,x) \\ &= \frac{t^2}{x\,(t^3+1)} \end{aligned}$$

The t domain of $f(t,x)$ when $x=-2$ is

$$\{t < -1 \lor -1 < t\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $f(t,x)$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{t^2}{x\,(t^3+1)}\right)\\ &= -\frac{t^2}{x^2\,(t^3+1)}\end{aligned}$$

The t domain of $\frac{\partial f}{\partial x}$ when $x = -2$ is

$$\{t < -1 \lor -1 < t\}$$

And the point $t_0 = 0$ is inside this domain. The x domain of $\frac{\partial f}{\partial x}$ when $t = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is inside this domain. Therefore solution exists and is unique.

1.30.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}x' &= F(t,x)\\ &= f(t)g(x)\\ &= \frac{t^2}{x\,(t^3+1)}\end{aligned}$$

Where $f(t) = \frac{t^2}{t^3+1}$ and $g(x) = \frac{1}{x}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{x}}\,dx &= \frac{t^2}{t^3+1}\,dt\\ \int \frac{1}{\frac{1}{x}}\,dx &= \int \frac{t^2}{t^3+1}\,dt\\ \frac{x^2}{2} &= \frac{\ln\left(t^3+1\right)}{3} + c_1\end{aligned}$$

Which results in

$$x = \frac{\sqrt{6\ln(t^3+1)+18c_1}}{3}$$

$$x = -\frac{\sqrt{6\ln(t^3+1)+18c_1}}{3}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -\sqrt{c_1}\,\sqrt{2}$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$x = -\frac{\sqrt{6\ln(t^3+1)+36}}{3}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = -2$ in the above solution gives an equation to solve for the constant of integration.

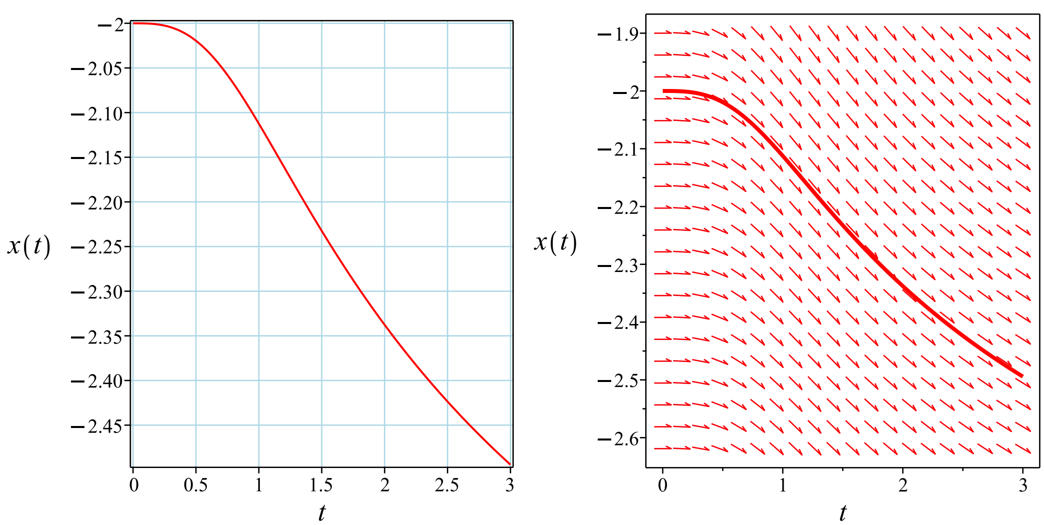
$$-2 = \sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

Warning: Unable to solve for constant of integration.

$$x = -\frac{\sqrt{6\ln(t^3+1) -}}{3}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -\frac{\sqrt{6\ln\left(t^3+1\right)+36}}{3}$$

Verified OK.

1.30.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{t^2}{x\left(t^3+1\right)}$$
$$x' = \omega(t,x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 66: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= \frac{t^3+1}{t^2} \\
\eta(t,x) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{t^3+1}{t^2}} dt \end{aligned}$$

Which results in

$$S = \frac{\ln(t^3+1)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{t^2}{x(t^3+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= \frac{t^2}{(t^2-t+1)(1+t)} \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

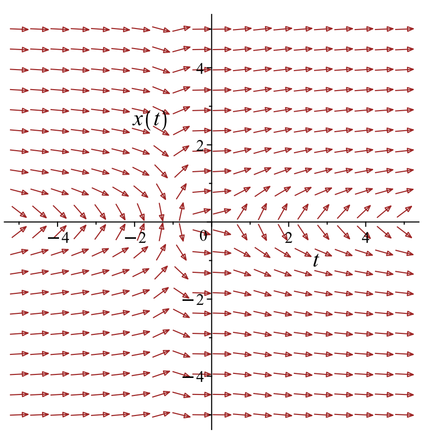
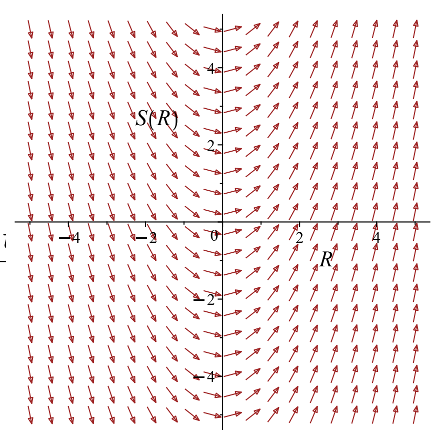
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{\ln(1+t)}{3} + \frac{\ln(t^2 - t + 1)}{3} = \frac{x^2}{2} + c_1$$

Which simplifies to

$$\frac{\ln(1+t)}{3} + \frac{\ln(t^2 - t + 1)}{3} = \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{t^2}{x(t^3+1)}$ 	$R = x$ $S = \frac{\ln(1+t)}{3} + \frac{\ln(t^2 -}{3}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = -2$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 2$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(1+t)}{3} + \frac{\ln(t^2-t+1)}{3} = \frac{x^2}{2} - 2$$

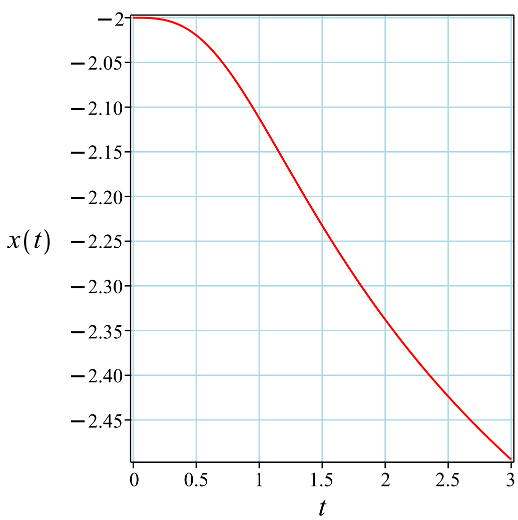
Solving for x from the above gives

$$x = -\frac{\sqrt{36 + 6\ln(1+t) + 6\ln(t^2-t+1)}}{3}$$

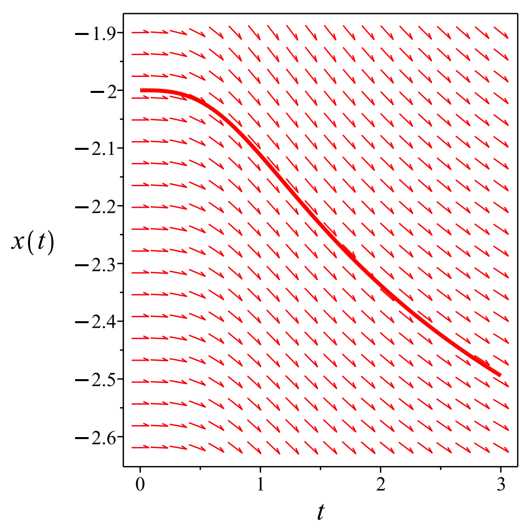
<u>Summary</u>

The solution(s) found are the following

$$x = -\frac{\sqrt{36 + 6\ln(1+t) + 6\ln(t^2-t+1)}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = -\frac{\sqrt{36 + 6\ln(1+t) + 6\ln(t^2-t+1)}}{3}$$

Verified OK.

1.30.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}x=\left(\frac{t^2}{t^3+1}\right)\mathrm{d}t$$

$$\left(-\frac{t^2}{t^3+1}\right)\mathrm{d}t+(x)\,\mathrm{d}x=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x) &= -\frac{t^2}{t^3+1}\\N(t,x) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{t^2}{t^3+1}\right)\\&= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(x)\\&= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -\frac{t^2}{t^3+1}\,\mathrm{d}t$$

$$\phi = -\frac{\ln(t^3+1)}{3} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = x$. Therefore equation (4) becomes

$$x = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = x$$

Integrating the above w.r.t x gives

$$\int f'(x) \, \mathrm{d}x = \int (x) \, \mathrm{d}x$$

$$f(x) = \frac{x^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t^3+1)}{3} + \frac{x^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t^3+1)}{3} + \frac{x^2}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = -2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(t^3+1)}{3} + \frac{x^2}{2} = 2$$

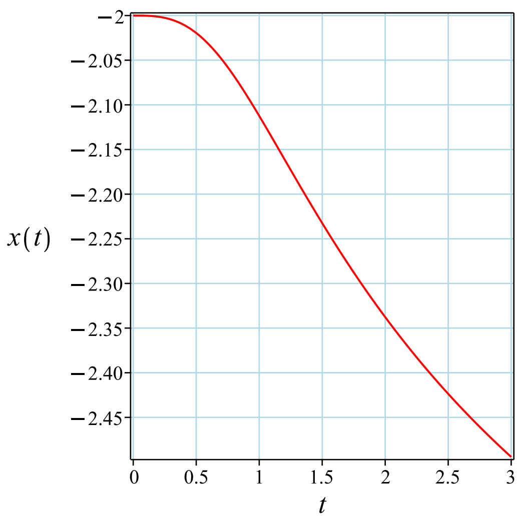
Solving for x from the above gives

$$x = -\frac{\sqrt{6\ln(t^3+1) + 36}}{3}$$

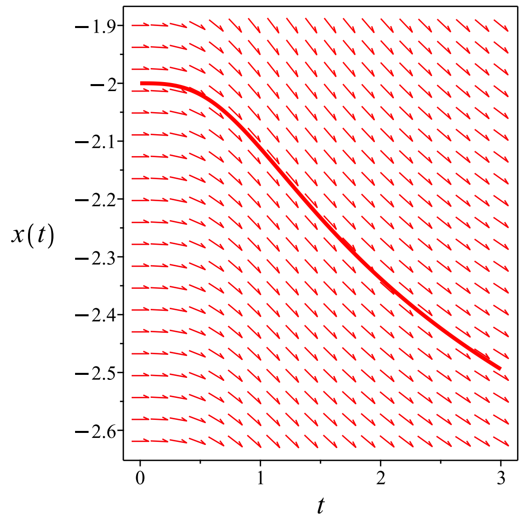
Summary

The solution(s) found are the following

$$x = -\frac{\sqrt{6 \ln (t^3 + 1) + 36}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{\sqrt{6 \ln (t^3 + 1) + 36}}{3}$$

Verified OK.

1.30.5 Maple step by step solution

Let's solve

$\left[x' - \frac{t^2}{x + t^3 x} = 0, x(0) = -2\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $x' x = \frac{t^2}{(t^2 - t + 1)(1 + t)}$

- Integrate both sides with respect to t

$$\int x' x dt = \int \frac{t^2}{(t^2-t+1)(1+t)} dt + c_1$$

- Evaluate integral

$$\frac{x^2}{2} = \frac{\ln((1+t)(t^2-t+1))}{3} + c_1$$

- Solve for x

$$\left\{x = -\frac{\sqrt{6\ln((1+t)(t^2-t+1))+18c_1}}{3}, x = \frac{\sqrt{6\ln((1+t)(t^2-t+1))+18c_1}}{3}\right\}$$

- Use initial condition $x(0) = -2$

$$-2 = -\frac{\sqrt{18}\sqrt{c_1}}{3}$$

- Solve for c_1

$$c_1 = 2$$

- Substitute $c_1 = 2$ into general solution and simplify

$$x = -\frac{\sqrt{6\ln((1+t)(t^2-t+1))+36}}{3}$$

- Use initial condition $x(0) = -2$

$$-2 = \frac{\sqrt{18}\sqrt{c_1}}{3}$$

- Solution does not satisfy initial condition
- Solution to the IVP

$$x = -\frac{\sqrt{6\ln((1+t)(t^2-t+1))+36}}{3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 18

```
dsolve([diff(x(t),t)=t^2/(x(t)+t^3*x(t)),x(0) = -2],x(t), singsol=all)
```

$$x(t) = -\frac{\sqrt{36 + 6\ln\left(t^3 + 1\right)}}{3}$$

✓ Solution by Mathematica

Time used: 0.202 (sec). Leaf size: 26

```
DSolve[{x'[t]==t^2/(x[t]+t^3*x[t]),{x[0]==-2}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\sqrt{\frac{2}{3}}\sqrt{\log\left(t^3 + 1\right) + 6}$$

1.31 problem 34

Internal problem ID [12895]
Internal file name [`OUTPUT/11547_Monday_November_06_2023_01_33_14_PM_20960949/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - \frac{1-y^2}{y} = 0}$$

With initial conditions

$$[y(0) = -2]$$

1.31.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= -\frac{y^2-1}{y} \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y^2-1}{y}\right)\\ &= -2 + \frac{y^2-1}{y^2}\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

1.31.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int -\frac{y}{y^2-1}dy &= \int dt\\ -\frac{\ln(y-1)}{2} - \frac{\ln(y+1)}{2} &= t + c_1\end{aligned}$$

The above can be written as

$$\begin{aligned}\left(-\frac{1}{2}\right)(\ln(y-1) + \ln(y+1)) &= t + c_1\\ \ln(y-1) + \ln(y+1) &= (-2)(t + c_1)\\ &= -2t - 2c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(y-1)+\ln(y+1)} = -2c_1 e^{-2t}$$

Which simplifies to

$$y^2 - 1 = c_2 e^{-2t}$$

Unable to solve for constant of integration due to RootOf in solution.

Summary

The solution(s) found are the following

$$y = \operatorname{RootOf}\left(\_Z^2 - c_2 e^{-2t} - 1\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{RootOf}\left(\_Z^2 - c_2 \mathrm{e}^{-2t} - 1\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 16

```
dsolve([diff(y(t),t)=(1-y(t)^2)/y(t),y(0) = -2],y(t), singsol=all)
```

$$y(t) = -\sqrt{3\,\mathrm{e}^{-2t} + 1}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[{y'[t]==(1-y[t]^2)/y[t],{y[0]==-2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\sqrt{3e^{-2t} + 1}$$

1.32 problem 35

Internal problem ID [12896]
Internal file name [`OUTPUT/11548_Monday_November_06_2023_01_33_16_PM_76225116/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (1 + y^2)\, t = 0}$$

With initial conditions

$$[y(0) = 1]$$

1.32.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= (y^2 + 1)\, t \end{aligned}$$

The t domain of $f(t, y)$ when $y = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\left(y^2+1\right)t\right)\\ &= 2ty\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.32.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t, y)\\ &= f(t)g(y)\\ &= \left(y^2+1\right)t\end{aligned}$$

Where $f(t) = t$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2+1}\, dy &= t\, dt\\ \int \frac{1}{y^2+1}\, dy &= \int t\, dt\\ \arctan\left(y\right) &= \frac{t^2}{2} + c_1\end{aligned}$$

Which results in

$$y = \tan\left(\frac{t^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \tan(c_1)$$

$$c_1 = \frac{\pi}{4}$$

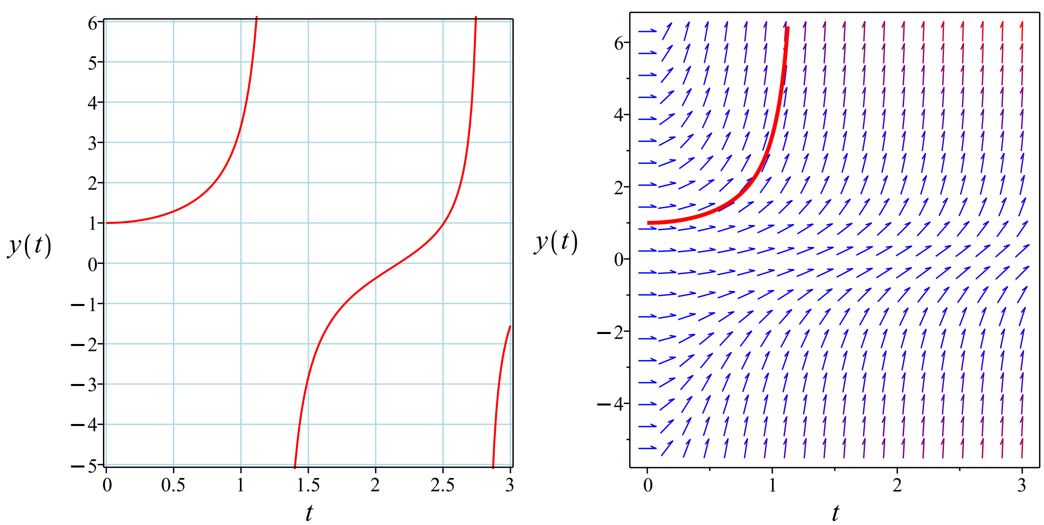
Substituting c_1 found above in the general solution gives

$$y = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right)$$

Verified OK.

1.32.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \left(y^2+1\right) t \\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 69: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t, y) = \frac{1}{t}$$
$$\eta(t, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$S = \int \frac{1}{\xi} dt$$
$$= \int \frac{1}{\frac{1}{t}} dt$$

Which results in

$$S = \frac{t^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = (y^2 + 1)\, t$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= t \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{t^2}{2} = \arctan(y) + c_1$$

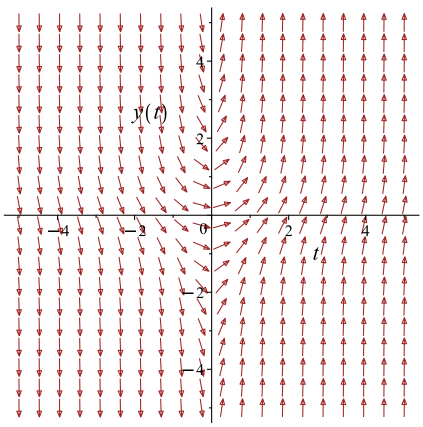
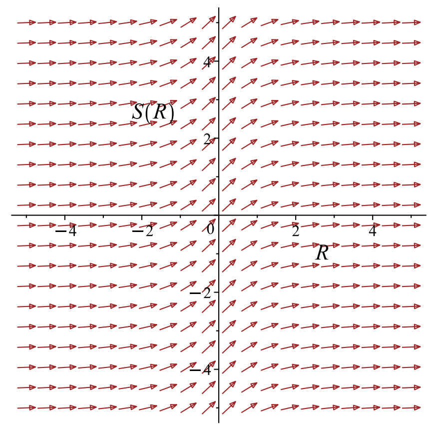
Which simplifies to

$$\frac{t^2}{2} = \arctan(y) + c_1$$

Which gives

$$y = -\tan\left(-\frac{t^2}{2} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = (y^2+1)\,t$ 	$R = y$ $S = \frac{t^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\tan(c_1)$$

$$c_1 = -\frac{\pi}{4}$$

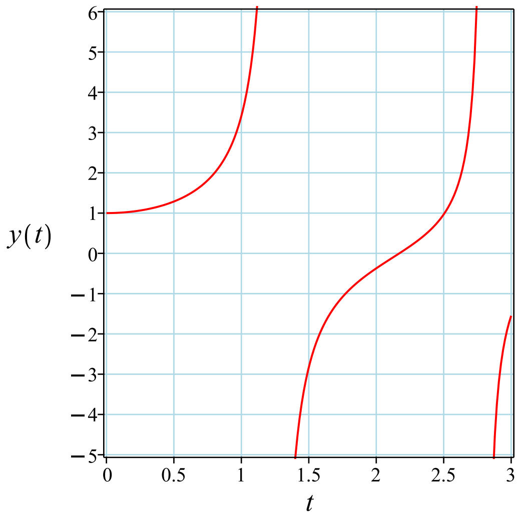
Substituting c_1 found above in the general solution gives

$$y = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right)$$

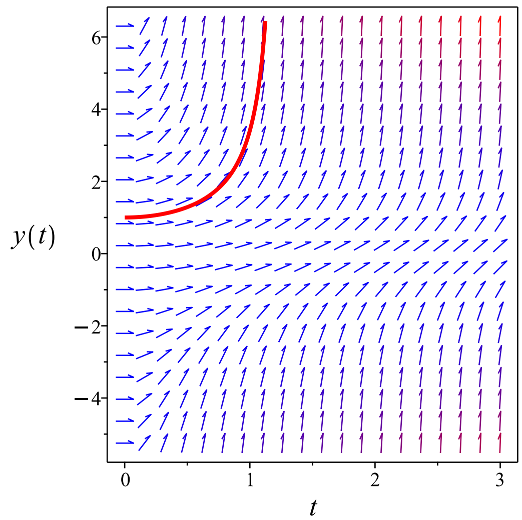
Summary

The solution(s) found are the following

$$y = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right)$$

Verified OK.

1.32.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2+1}\right)\mathrm{d}y &= (t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t + \left(\frac{1}{y^2+1}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -t\\ N(t, y) &= \frac{1}{y^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y^2+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t \, \mathrm{d}t$$

$$\phi = -\frac{t^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2 + 1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2 + 1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y^2 + 1}\right) \mathrm{d}y$$

$$f(y) = \arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \arctan(y)$$

The solution becomes

$$y = \tan\left(\frac{t^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \tan(c_1)$$

$$c_1 = \frac{\pi}{4}$$

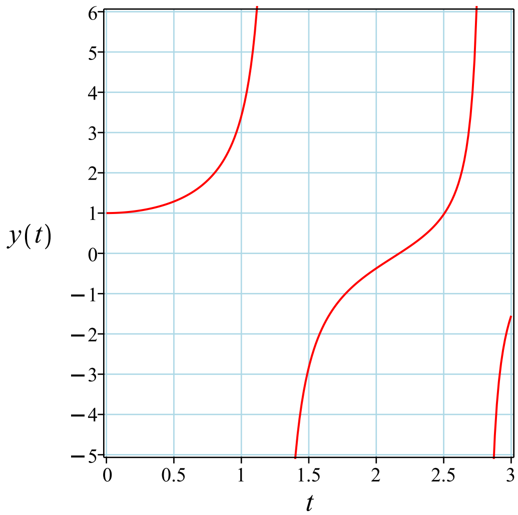
Substituting c_1 found above in the general solution gives

$$y = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right)$$

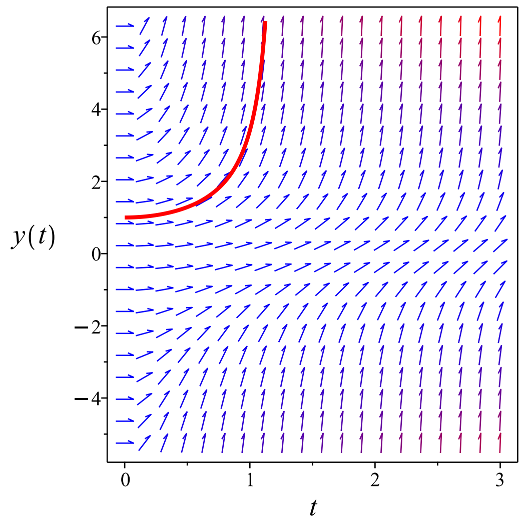
Summary

The solution(s) found are the following

$$y = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right)$$

Verified OK.

1.32.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= (y^2 + 1)\, t \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = t\, y^2 + t$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = t$, $f_1(t) = 0$ and $f_2(t) = t$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{t u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(t) - (f_2' + f_1f_2)\,u'(t) + f_2^2f_0u(t) = 0 \tag{2}$$

But

$$\begin{aligned}f_2' &= 1\\ f_1f_2 &= 0\\ f_2^2f_0 &= t^3\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$tu''(t) - u'(t) + t^3u(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 \sin\left(\frac{t^2}{2}\right) + c_2 \cos\left(\frac{t^2}{2}\right)$$

The above shows that

$$u'(t) = t\left(c_1 \cos\left(\frac{t^2}{2}\right) - c_2 \sin\left(\frac{t^2}{2}\right)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 \cos\left(\frac{t^2}{2}\right) - c_2 \sin\left(\frac{t^2}{2}\right)}{c_1 \sin\left(\frac{t^2}{2}\right) + c_2 \cos\left(\frac{t^2}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \cos\left(\frac{t^2}{2}\right) + \sin\left(\frac{t^2}{2}\right)}{c_3 \sin\left(\frac{t^2}{2}\right) + \cos\left(\frac{t^2}{2}\right)}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -c_3$$

$$c_3 = -1$$

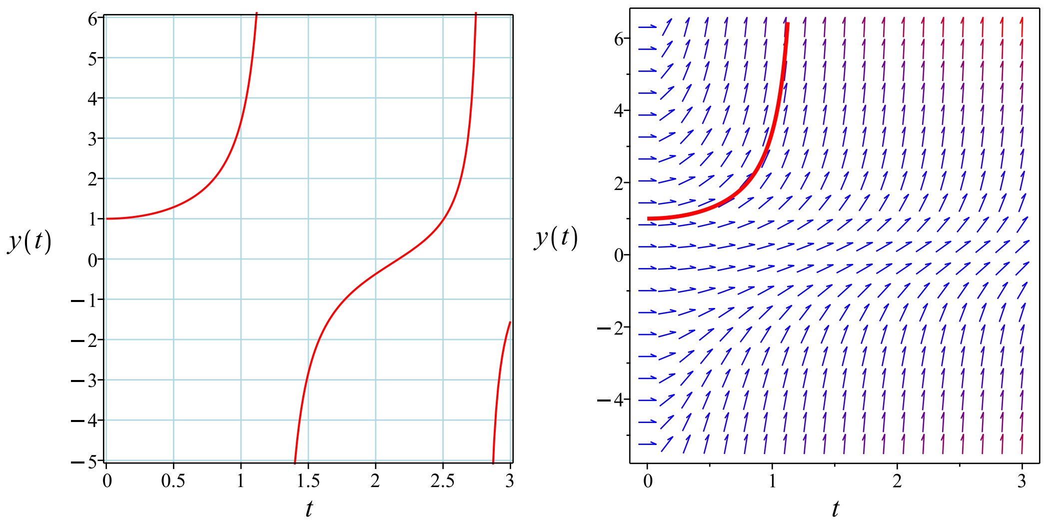
Substituting c_3 found above in the general solution gives

$$y = \frac{-\sin\left(\frac{t^2}{2}\right) - \cos\left(\frac{t^2}{2}\right)}{\sin\left(\frac{t^2}{2}\right) - \cos\left(\frac{t^2}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\sin\left(\frac{t^2}{2}\right) - \cos\left(\frac{t^2}{2}\right)}{\sin\left(\frac{t^2}{2}\right) - \cos\left(\frac{t^2}{2}\right)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-\sin\left(\frac{t^2}{2}\right) - \cos\left(\frac{t^2}{2}\right)}{\sin\left(\frac{t^2}{2}\right) - \cos\left(\frac{t^2}{2}\right)}$$

Verified OK.

1.32.6 Maple step by step solution

Let's solve

$\left[y' - (1 + y^2)\, t = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = t$

- Integrate both sides with respect to t

  $\int \frac{y'}{1+y^2} dt = \int t dt + c_1$

- Evaluate integral

  $\arctan (y) = \frac{t^2}{2} + c_1$

- Solve for y

  $y = \tan \left(\frac{t^2}{2} + c_1\right)$

- Use initial condition $y(0) = 1$

  $1 = \tan (c_1)$

- Solve for c_1

  $c_1 = \frac{\pi}{4}$

- Substitute $c_1 = \frac{\pi}{4}$ into general solution and simplify

  $y = \tan \left(\frac{t^2}{2} + \frac{\pi}{4}\right)$

- Solution to the IVP

  $y = \tan \left(\frac{t^2}{2} + \frac{\pi}{4}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 14

```
dsolve([diff(y(t),t)=(y(t)^2+1)*t,y(0) = 1],y(t), singsol=all)
```

$$y(t) = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.29 (sec). Leaf size: 17

```
DSolve[{y'[t]==(y[t]^2+1)*t,{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \tan\left(\frac{1}{4}(2t^2 + \pi)\right)$$

1.33 problem 36

Internal problem ID [12897]
Internal file name [`OUTPUT/11549_Monday_November_06_2023_01_33_17_PM_75412810/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \frac{1}{2y+3} = 0}$$

With initial conditions

$$[y(0) = 1]$$

1.33.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \frac{1}{2y+3} \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\left\{y < -\frac{3}{2} \vee -\frac{3}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{2y+3}\right) \\
&= -\frac{2}{(2y+3)^2}
\end{aligned}
$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$
\left\{y < -\frac{3}{2} \lor -\frac{3}{2} < y\right\}
$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.33.2 Solving as quadrature ode

Integrating both sides gives

$$
\begin{aligned}
\int (2y+3)\, dy &= t + c_1 \\
y^2 + 3y &= t + c_1
\end{aligned}
$$

Solving for y gives these solutions

$$
\begin{aligned}
y_1 &= -\frac{3}{2} - \frac{\sqrt{9 + 4t + 4c_1}}{2} \\
y_2 &= -\frac{3}{2} + \frac{\sqrt{9 + 4t + 4c_1}}{2}
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = -\frac{3}{2} + \frac{\sqrt{9 + 4c_1}}{2}
$$

$$
c_1 = 4
$$

Substituting c_1 found above in the general solution gives

$$
y = -\frac{3}{2} + \frac{\sqrt{25 + 4t}}{2}
$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

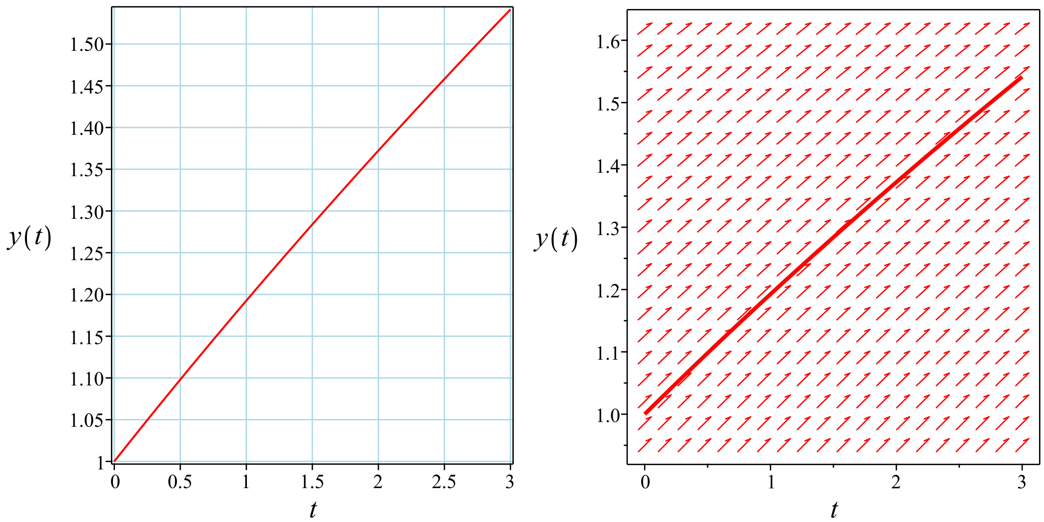
$$
1 = -\frac{3}{2} - \frac{\sqrt{9 + 4c_1}}{2}
$$

Warning: Unable to solve for constant of integration.

Summary

The solution(s) found are the following

$$y = -\frac{3}{2} + \frac{\sqrt{25+4}}{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{2} + \frac{\sqrt{25+4t}}{2}$$

Verified OK.

1.33.3 Maple step by step solution

Let's solve

$\left[y' - \frac{1}{2y+3} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(2y+3) = 1$

- Integrate both sides with respect to t

$\int y'(2y+3)\,dt = \int 1dt + c_1$

- Evaluate integral

  $y^2 + 3y = t + c_1$

- Solve for y

  $\left\{y = -\frac{3}{2} - \frac{\sqrt{9+4t+4c_1}}{2}, y = -\frac{3}{2} + \frac{\sqrt{9+4t+4c_1}}{2}\right\}$

- Use initial condition $y(0) = 1$

  $1 = -\frac{3}{2} - \frac{\sqrt{9+4c_1}}{2}$

- Solution does not satisfy initial condition
- Use initial condition $y(0) = 1$

  $1 = -\frac{3}{2} + \frac{\sqrt{9+4c_1}}{2}$

- Solve for c_1

  $c_1 = 4$

- Substitute $c_1 = 4$ into general solution and simplify

  $y = -\frac{3}{2} + \frac{\sqrt{25+4t}}{2}$

- Solution to the IVP

  $y = -\frac{3}{2} + \frac{\sqrt{25+4t}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)=1/(2*y(t)+3),y(0) = 1],y(t), singsol=all)
```

$$y(t) = -\frac{3}{2} + \frac{\sqrt{25 + 4t}}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 20

```
DSolve[{y'[t]==1/(2*y[t]+3),{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}\left(\sqrt{4t + 25} - 3\right)$$

1.34 problem 37

Internal problem ID [12898]
Internal file name [`OUTPUT/11550_Monday_November_06_2023_01_33_17_PM_86722850/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2ty^2 - 3t^2y^2 = 0}$$

With initial conditions

$$[y(1) = -1]$$

1.34.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= 3y^2t^2 + 2t\,y^2 \end{aligned}$$

The t domain of $f(t, y)$ when $y = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The y domain of $f(t, y)$ when $t = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3y^2t^2 + 2t\,y^2\right)\\ &= 6y\,t^2 + 4ty\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

1.34.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t, y)\\ &= f(t)g(y)\\ &= \left(3t^2 + 2t\right)y^2\end{aligned}$$

Where $f(t) = 3t^2 + 2t$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2}\,dy &= 3t^2 + 2t\,dt\\ \int \frac{1}{y^2}\,dy &= \int 3t^2 + 2t\,dt\\ -\frac{1}{y} &= t^3 + t^2 + c_1\end{aligned}$$

Which results in

$$y = -\frac{1}{t^3 + t^2 + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{c_1 + 2}$$

$$c_1 = -1$$

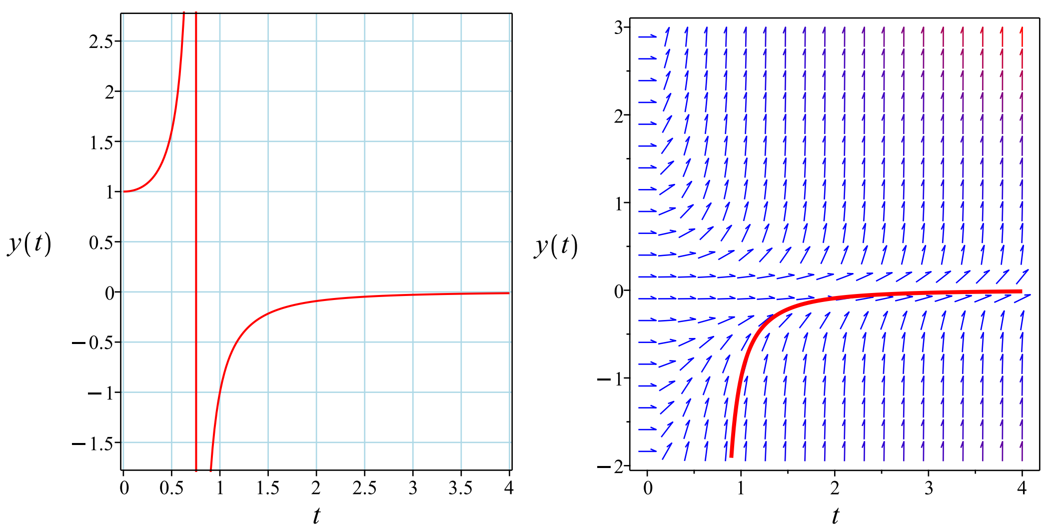
Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{t^3 + t^2 - 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Verified OK.

1.34.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 3y^2t^2 + 2t\,y^2 \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 73: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= \frac{1}{3t^2 + 2t} \\ \eta(t, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{3t^2+2t}} dt \end{aligned}$$

Which results in

$$S = t^3 + t^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = 3y^2 t^2 + 2t\, y^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= 3t^2 + 2t \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$t^2(1+t) = -\frac{1}{y} + c_1$$

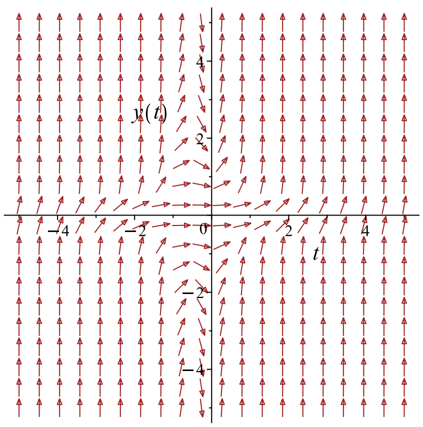
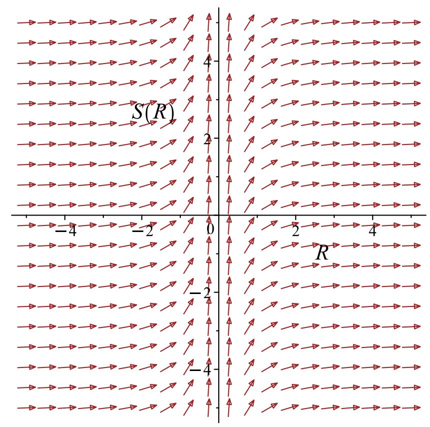
Which simplifies to

$$t^2(1+t) = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{1}{-t^3 - t^2 + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 3y^2t^2 + 2t\,y^2$ 	$R = y$ $S = t^2(1 + t)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{1}{-2 + c_1}$$

$$c_1 = 1$$

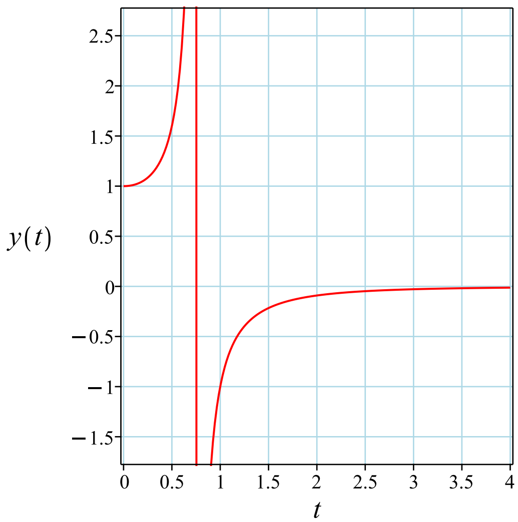
Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{t^3 + t^2 - 1}$$

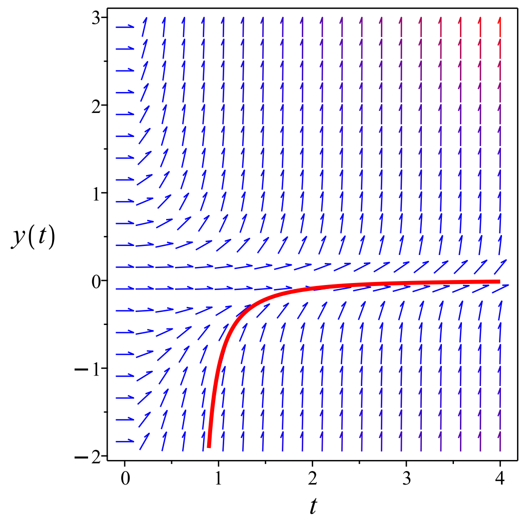
Summary

The solution(s) found are the following

$$y = -\frac{1}{t^3 + t^2 - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Verified OK.

1.34.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{1}{y^2}\right)\mathrm{d}y &= \left(3t^2+2t\right)\mathrm{d}t\\
\left(-3t^2-2t\right)\mathrm{d}t + \left(\frac{1}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(t,y) &= -3t^2-2t\\
N(t,y) &= \frac{1}{y^2}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3t^2-2t\right)\\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y^2}\right)\\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -3t^2 - 2t \, \mathrm{d}t$$

$$\phi = -t^3 - t^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y^2}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -t^3 - t^2 - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t^3 - t^2 - \frac{1}{y}$$

The solution becomes

$$y = -\frac{1}{t^3 + t^2 + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{c_1 + 2}$$

$$c_1 = -1$$

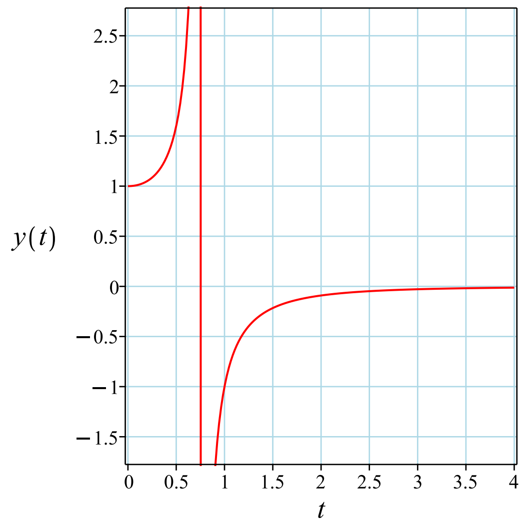
Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{t^3 + t^2 - 1}$$

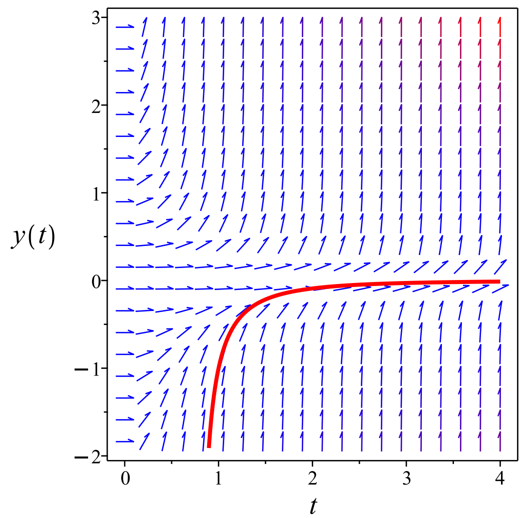
Summary

The solution(s) found are the following

$$y = -\frac{1}{t^3 + t^2 - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Verified OK.

1.34.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= 3y^2t^2 + 2t\,y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 3y^2t^2 + 2t\,y^2$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = 3t^2 + 2t$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(3t^2 + 2t)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 6t + 2 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(3t^2 + 2t) u''(t) - (6t + 2) u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + t^2(1 + t) c_2$$

The above shows that

$$u'(t) = c_2 t(3t + 2)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 t(3t + 2)}{(3t^2 + 2t)(c_1 + t^2(1 + t) c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{t^3 + t^2 + c_3}$$

Initial conditions are used to solve for c_3. Substituting $t = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{c_3 + 2}$$

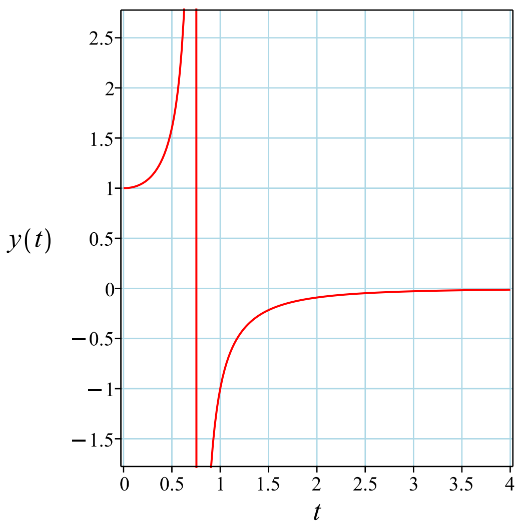
$$c_3 = -1$$

Substituting c_3 found above in the general solution gives

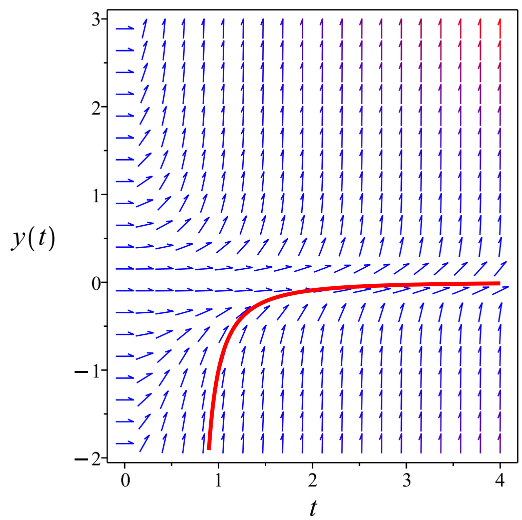
$$y = -\frac{1}{t^3 + t^2 - 1}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{t^3 + t^2 - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Verified OK.

1.34.6 Maple step by step solution

Let's solve

$[y' - 2ty^2 - 3t^2y^2 = 0, y(1) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = t(3t + 2)$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int t(3t + 2)\, dt + c_1$
- Evaluate integral

  $-\frac{1}{y} = t^3 + t^2 + c_1$
- Solve for y

  $y = -\frac{1}{t^3+t^2+c_1}$
- Use initial condition $y(1) = -1$

  $-1 = -\frac{1}{c_1+2}$
- Solve for c_1

  $c_1 = -1$
- Substitute $c_1 = -1$ into general solution and simplify

  $y = -\frac{1}{t^3+t^2-1}$
- Solution to the IVP

  $y = -\frac{1}{t^3+t^2-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 16

```
dsolve([diff(y(t),t)=2*t*y(t)^2+3*t^2*y(t)^2,y(1) = -1],y(t), singsol=all)
```

$$y(t) = -\frac{1}{t^3 + t^2 - 1}$$

✓ Solution by Mathematica

Time used: 0.222 (sec). Leaf size: 17

```
DSolve[{y'[t]==2*t*y[t]^2+3*t^2*y[t]^2,{y[1]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{t^3 + t^2 - 1}$$

1.35 problem 38

Internal problem ID [12899]
Internal file name [`OUTPUT/11551_Monday_November_06_2023_01_33_18_PM_36492103/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.2. page 33
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \frac{y^2 + 5}{y} = 0}$$

With initial conditions

$$[y(0) = -2]$$

1.35.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \frac{y^2 + 5}{y} \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2+5}{y}\right) \\ &= 2 - \frac{y^2+5}{y^2}\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

1.35.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{y}{y^2+5} dy &= \int dt \\ \frac{\ln(y^2+5)}{2} &= t + c_1\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y^2+5} = e^{t+c_1}$$

Which simplifies to

$$\sqrt{y^2+5} = c_2 e^t$$

Unable to solve for constant of integration due to RootOf in solution.

Summary

The solution(s) found are the following

$$y = \text{RootOf}\left(\_Z^2 - c_2^2 e^{2t} + 5\right) \tag{1}$$

Verification of solutions

$$y = \text{RootOf}\left(\_Z^2 - c_2^2 e^{2t} + 5\right)$$

Verified OK.

1.35.3 Maple step by step solution

Let's solve

$\left[y' - \frac{y^2+5}{y} = 0, y(0) = -2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{y^2+5} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'y}{y^2+5}dt = \int 1dt + c_1$

- Evaluate integral

  $\frac{\ln(y^2+5)}{2} = t + c_1$

- Solve for y

  $\left\{y = \sqrt{-5 + \mathrm{e}^{2t+2c_1}}, y = -\sqrt{-5 + \mathrm{e}^{2t+2c_1}}\right\}$

- Use initial condition $y(0) = -2$

  $-2 = \sqrt{-5 + \mathrm{e}^{2c_1}}$

- Solution does not satisfy initial condition

- Use initial condition $y(0) = -2$

  $-2 = -\sqrt{-5 + \mathrm{e}^{2c_1}}$

- Solve for c_1

  $c_1 = \ln(3)$

- Substitute $c_1 = \ln(3)$ into general solution and simplify

  $y = -\sqrt{-5 + 9\,\mathrm{e}^{2t}}$

- Solution to the IVP

  $y = -\sqrt{-5 + 9\,\mathrm{e}^{2t}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 16

```
dsolve([diff(y(t),t)=(y(t)^2+5)/y(t),y(0) = -2],y(t), singsol=all)
```

$$y(t) = -\sqrt{9\,\mathrm{e}^{2t} - 5}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 20

```
DSolve[{y'[t]==(y[t]^2+5)/y[t],{y[0]==-2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\sqrt{9e^{2t} - 5}$$

2 Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47

2.1 problem 1

Internal problem ID [12900]
Internal file name [`OUTPUT/11552_Monday_November_06_2023_01_33_19_PM_34935870/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = t^2 + t}$$

2.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int t^2 + t \, \mathrm{d}t \\ &= \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1 \tag{1}$$

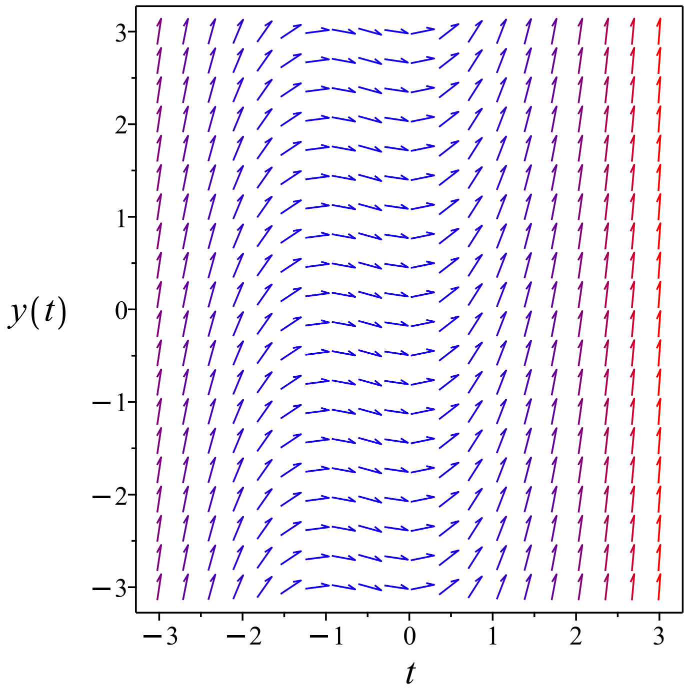


Figure 95: Slope field plot

Verification of solutions

$$y = \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1$$

Verified OK.

2.1.2 Maple step by step solution

Let's solve

$y' = t^2 + t$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to t

  $\int y'dt = \int (t^2 + t)\, dt + c_1$

- Evaluate integral

  $y = \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1$

- Solve for y

$$y = \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)=t^2+t,y(t), singsol=all)
```

$$y(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 22

```
DSolve[y'[t]==t^2+t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{t^3}{3} + \frac{t^2}{2} + c_1$$

2.2 problem 2

Internal problem ID [12901]
Internal file name [`OUTPUT/11553_Tuesday_November_07_2023_11_26_56_PM_64086885/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = t^2 + 1}$$

2.2.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int t^2 + 1 \, \mathrm{d}t \\ &= \frac{1}{3}t^3 + t + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{3}t^3 + t + c_1 \tag{1}$$

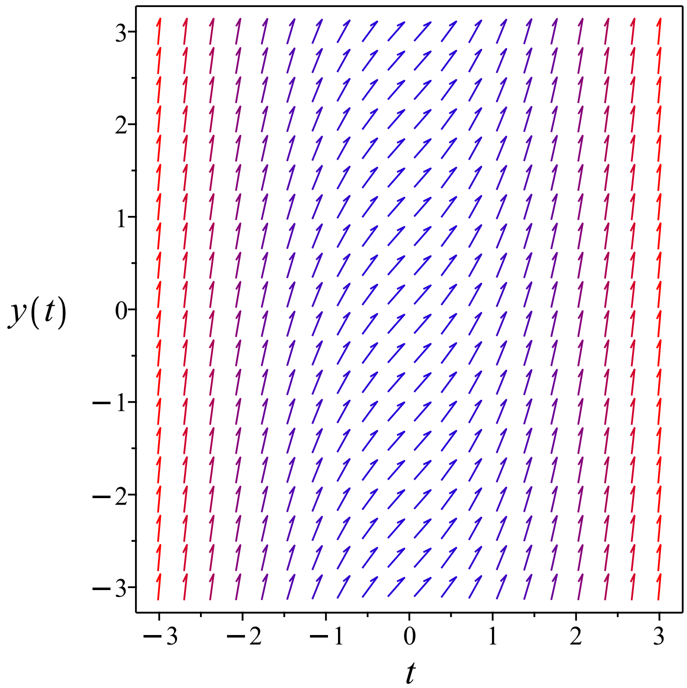


Figure 96: Slope field plot

Verification of solutions

$$y = \frac{1}{3}t^3 + t + c_1$$

Verified OK.

2.2.2 Maple step by step solution

Let's solve

$y' = t^2 + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to t

  $\int y' dt = \int (t^2 + 1)\, dt + c_1$

- Evaluate integral

  $y = \frac{1}{3}t^3 + t + c_1$

- Solve for y

$$y = \frac{1}{3}t^3 + t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)=t^2+1,y(t), singsol=all)
```

$$y(t) = \frac{1}{3}t^3 + t + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 16

```
DSolve[y'[t]==t^2+1,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{t^3}{3} + t + c_1$$

2.3 problem 3

Internal problem ID [12902]
Internal file name [`OUTPUT/11554_Tuesday_November_07_2023_11_26_59_PM_59055855/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + 2y = 1}$$

2.3.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{1-2y} dy = \int dt$$
$$-\frac{\ln(1-2y)}{2} = t + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{1-2y}} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{1-2y}} = c_2 \mathrm{e}^{t}$$

Summary
The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-2t}}{2c_2^2} + \frac{1}{2} \tag{1}$$

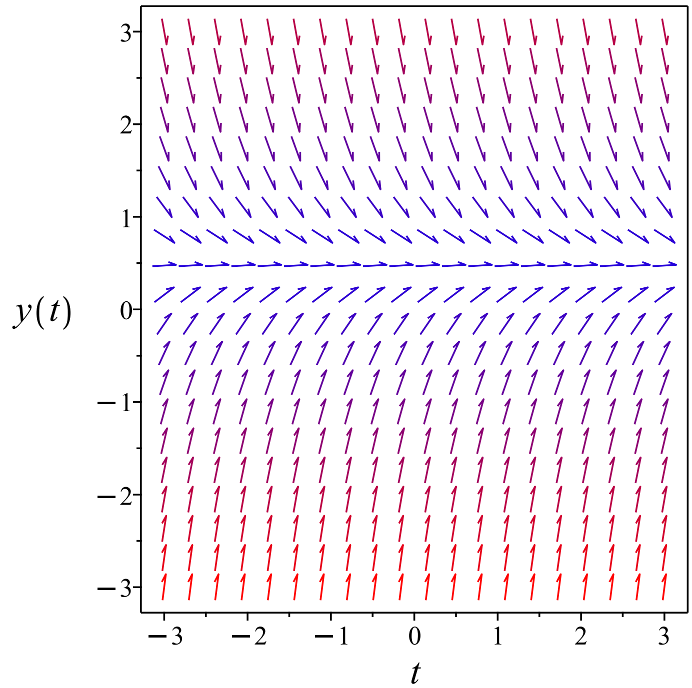


Figure 97: Slope field plot

Verification of solutions

$$y = -\frac{e^{-2t}}{2c_2^2} + \frac{1}{2}$$

Verified OK.

2.3.2 Maple step by step solution

Let's solve

$y' + 2y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1-2y} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{1-2y} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\frac{\ln(1-2y)}{2} = t + c_1$$

- Solve for y

$$y = -\frac{e^{-2t-2c_1}}{2} + \frac{1}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)=1-2*y(t),y(t), singsol=all)
```

$$y(t) = e^{-2t}c_1 + \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 24

```
DSolve[y'[t]==1-2*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2} + c_1 e^{-2t}$$
$$y(t) \to \frac{1}{2}$$

2.4 problem 4

Internal problem ID [12903]
Internal file name [`OUTPUT/11555_Tuesday_November_07_2023_11_27_00_PM_5441983/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - 4y^2 = 0}$$

2.4.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{4y^2} dy = t + c_1$$
$$-\frac{1}{4y} = t + c_1$$

Solving for y gives these solutions

$$y_1 = -\frac{1}{4(t + c_1)}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{4(t + c_1)} \tag{1}$$

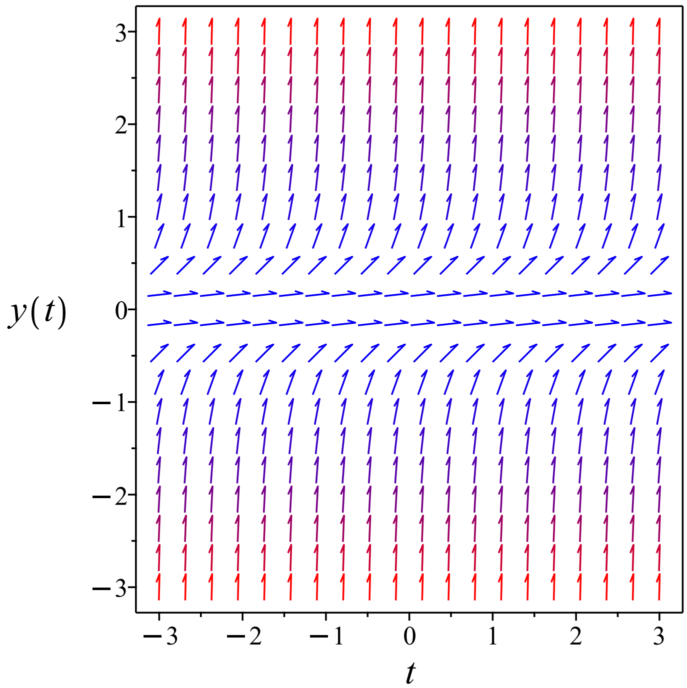


Figure 98: Slope field plot

Verification of solutions

$$y = -\frac{1}{4(t + c_1)}$$

Verified OK.

2.4.2 Maple step by step solution

Let's solve

$y' - 4y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = 4$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int 4dt + c_1$

- Evaluate integral

$-\frac{1}{y} = 4t + c_1$

- Solve for y

$y = -\frac{1}{4t+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(t),t)=4*y(t)^2,y(t), singsol=all)
```

$$y(t) = \frac{1}{-4t + c_1}$$

✓ Solution by Mathematica

Time used: 0.157 (sec). Leaf size: 20

```
DSolve[y'[t]==4*y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{4t + c_1}$$
$$y(t) \to 0$$

2.5 problem 5

Internal problem ID [12904]
Internal file name [`OUTPUT/11556_Tuesday_November_07_2023_11_27_01_PM_72649872/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 2y(-y + 1) = 0}$$

2.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{2y\left(y-1\right)}dy = \int dt$$
$$-\frac{\ln\left(y-1\right)}{2} + \frac{\ln\left(y\right)}{2} = t + c_1$$

The above can be written as

$$\left(-\frac{1}{2}\right)\left(\ln\left(y-1\right) - \ln\left(y\right)\right) = t + c_1$$
$$\ln\left(y-1\right) - \ln\left(y\right) = \left(-2\right)\left(t + c_1\right)$$
$$= -2t - 2c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(y)} = -2c_1 e^{-2t}$$

Which simplifies to

$$\frac{y-1}{y} = c_2 e^{-2t}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{-1 + c_2 e^{-2t}} \tag{1}$$

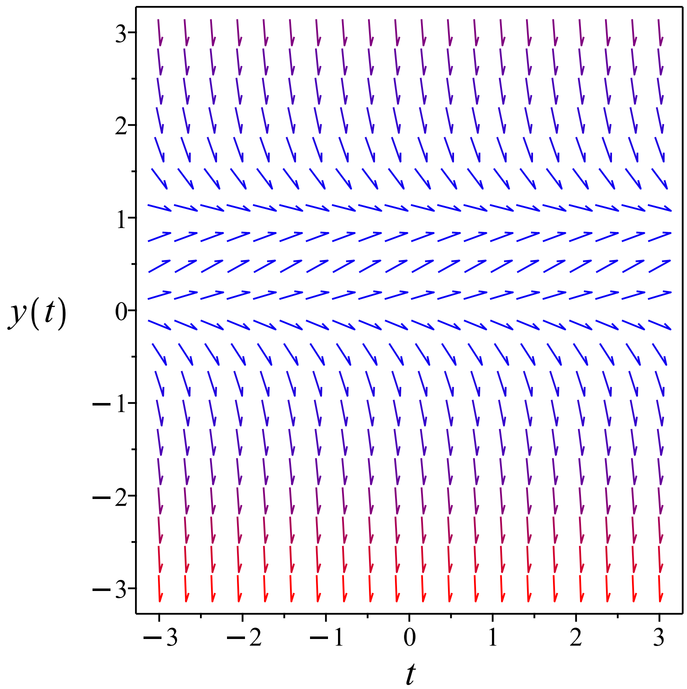


Figure 99: Slope field plot

Verification of solutions

$$y = -\frac{1}{-1 + c_2 e^{-2t}}$$

Verified OK.

2.5.2 Maple step by step solution

Let's solve

$y' - 2y(-y+1) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(-y+1)} = 2$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(-y+1)} dt = \int 2dt + c_1$

- Evaluate integral

  $-\ln(y-1) + \ln(y) = 2t + c_1$

- Solve for y

  $y = \frac{e^{2t+c_1}}{-1+e^{2t+c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=2*y(t)*(1-y(t)),y(t), singsol=all)
```

$$y(t) = \frac{1}{e^{-2t}c_1 + 1}$$

✓ Solution by Mathematica

Time used: 0.404 (sec). Leaf size: 33

```
DSolve[y'[t]==2*y[t]*(1-y[t]),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{2t}}{e^{2t} + e^{c_1}}$$
$$y(t) \to 0$$
$$y(t) \to 1$$

2.6 problem 6

Internal problem ID [12905]
Internal file name [`OUTPUT/11557_Tuesday_November_07_2023_11_27_01_PM_98842774/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 1 + t}$$

2.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = 1 + t$$

Hence the ode is

$$y' - y = 1 + t$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int (-1)dt}$$
$$= \mathrm{e}^{-t}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) = (\mu)(1+t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-t} y) = (\mathrm{e}^{-t})(1+t)$$
$$\mathrm{d}(\mathrm{e}^{-t} y) = ((1+t)\,\mathrm{e}^{-t})\ \mathrm{d}t$$

Integrating gives

$$\mathrm{e}^{-t} y = \int (1+t)\,\mathrm{e}^{-t}\,\mathrm{d}t$$
$$\mathrm{e}^{-t} y = -(t+2)\,\mathrm{e}^{-t} + c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t}$ results in

$$y = -\mathrm{e}^{t}(t+2)\,\mathrm{e}^{-t} + c_1 \mathrm{e}^{t}$$

which simplifies to

$$y = -t - 2 + c_1 \mathrm{e}^{t}$$

Summary

The solution(s) found are the following

$$y = -t - 2 + c_1 \mathrm{e}^{t} \tag{1}$$

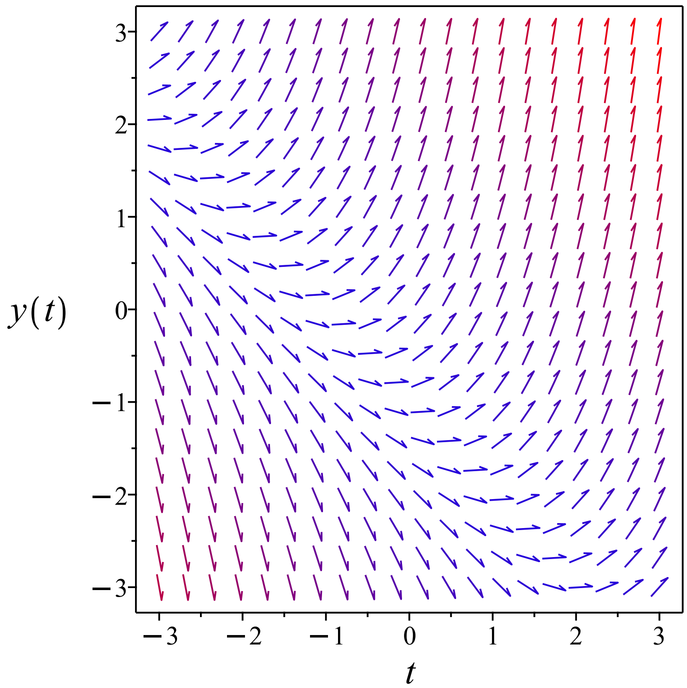


Figure 100: Slope field plot

Verification of solutions

$$y = -t - 2 + c_1 e^{t}$$

Verified OK.

2.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + t + 1$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 82: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^t
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^t} dy \end{aligned}$$

Which results in

$$S = e^{-t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = y + t + 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -e^{-t} y \\ S_y &= e^{-t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (1 + t) e^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (1 + R) e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R+2)\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-t} y = -(t+2)\,\mathrm{e}^{-t} + c_1$$

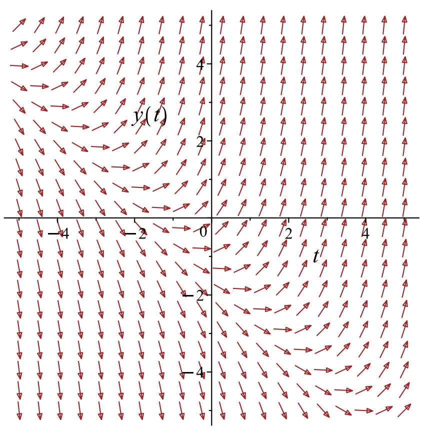
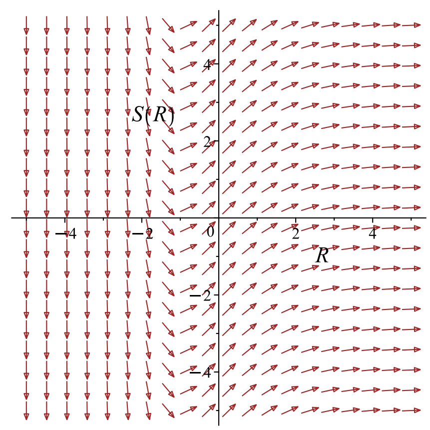
Which simplifies to

$$(t+y+2)\,\mathrm{e}^{-t} - c_1 = 0$$

Which gives

$$y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y + t + 1$ 	$R = t$ $S = \mathrm{e}^{-t} y$	$\frac{dS}{dR} = (1+R)\,\mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{t} \tag{1}$$

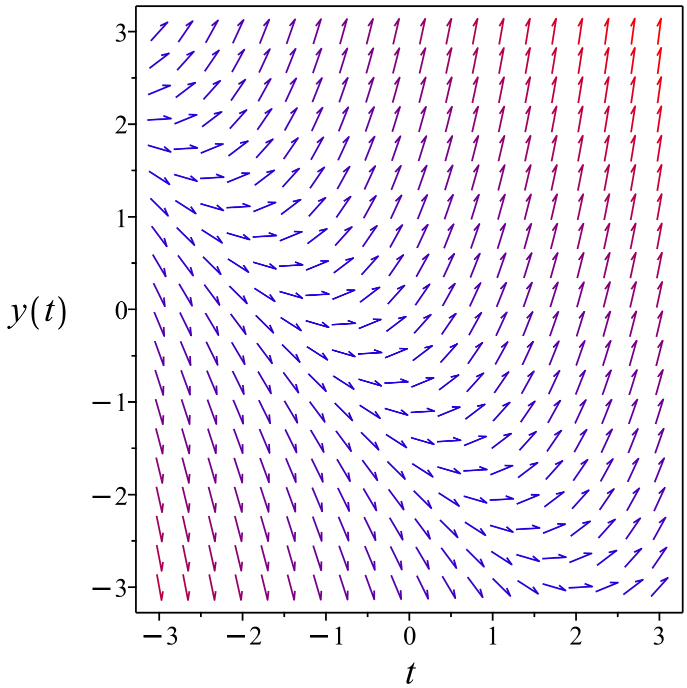


Figure 101: Slope field plot

Verification of solutions

$$y = -\left(t\,e^{-t} + 2\,e^{-t} - c_1\right) e^{t}$$

Verified OK.

2.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (y + t + 1)\,\mathrm{d}t$$
$$(-y - t - 1)\,\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -y - t - 1$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y - t - 1)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -1 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-t} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-t}(-y - t - 1) \\
&= -\mathrm{e}^{-t}(y + t + 1)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-t}(1) \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{-t}(y + t + 1)\right) + \left(\mathrm{e}^{-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\mathrm{e}^{-t}(y+t+1)\,\mathrm{d}t \\
\phi &= (t+y+2)\,\mathrm{e}^{-t} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-t} = \mathrm{e}^{-t} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = (t+y+2)\,\mathrm{e}^{-t} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = (t+y+2)\,\mathrm{e}^{-t}
$$

The solution becomes

$$
y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{t}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = -\left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t} - c_1\right)\mathrm{e}^{t} \tag{1}
$$

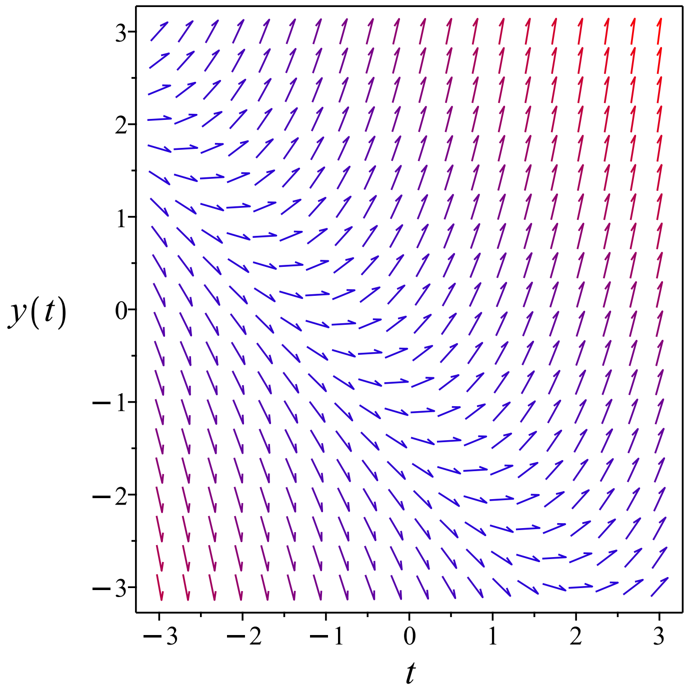


Figure 102: Slope field plot

Verification of solutions

$$y = -\left(t\,e^{-t} + 2\,e^{-t} - c_1\right)e^{t}$$

Verified OK.

2.6.4 Maple step by step solution

Let's solve

$y' - y = 1 + t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + t + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 1 + t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - y) = \mu(t)\,(1 + t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y'-y)=\mu'(t)\,y+\mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t)=-\mu(t)$
- Solve to find the integrating factor

  $\mu(t)=\mathrm{e}^{-t}$
- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\mu(t)\,(1+t)\,dt+c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y=\int\mu(t)\,(1+t)\,dt+c_1$
- Solve for y

  $y=\frac{\int\mu(t)(1+t)dt+c_1}{\mu(t)}$
- Substitute $\mu(t)=\mathrm{e}^{-t}$

  $y=\frac{\int(1+t)\mathrm{e}^{-t}dt+c_1}{\mathrm{e}^{-t}}$
- Evaluate the integrals on the rhs

  $y=\frac{-(t+2)\mathrm{e}^{-t}+c_1}{\mathrm{e}^{-t}}$
- Simplify

  $y=-t-2+c_1\mathrm{e}^{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(t),t)=y(t)+t+1,y(t), singsol=all)
```

$$y(t) = -t - 2 + c_1 e^t$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 16

```
DSolve[y'[t]==y[t]+t+1,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -t + c_1 e^t - 2$$

2.7 problem 7

Internal problem ID [12906]
Internal file name [`OUTPUT/11558_Tuesday_November_07_2023_11_27_02_PM_74390849/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 3y(-y+1) = 0}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

2.7.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= -3y(y-1) \end{aligned}$$

The y domain of $f(t,y)$ when $t=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-3y(y-1)) \\ &= 3-6y \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

2.7.2 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{3y(y-1)} dy = \int dt$$
$$-\frac{\ln(y-1)}{3} + \frac{\ln(y)}{3} = t + c_1$$

The above can be written as

$$\left(-\frac{1}{3}\right)(\ln(y-1) - \ln(y)) = t + c_1$$
$$\ln(y-1) - \ln(y) = (-3)(t + c_1)$$
$$= -3t - 3c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(y)} = -3\,e^{-3t}c_1$$

Which simplifies to

$$\frac{y-1}{y} = c_2 e^{-3t}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\frac{1}{-1 + c_2}$$

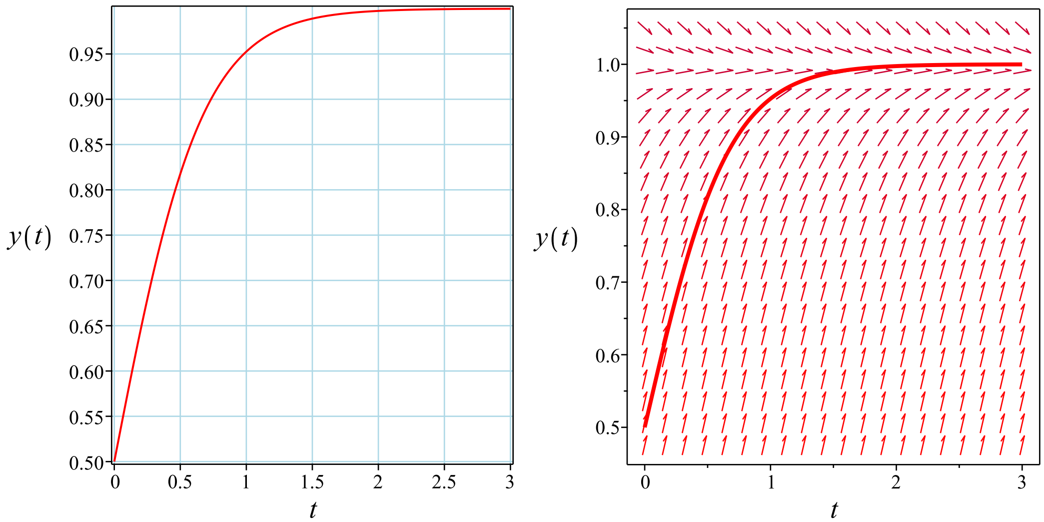
$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

$$y = \frac{1}{1 + e^{-3t}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{1 + e^{-3t}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{1 + e^{-3t}}$$

Verified OK.

2.7.3 Maple step by step solution

Let's solve

$\left[y' - 3y(-y+1) = 0, y(0) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(-y+1)} = 3$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(-y+1)} dt = \int 3dt + c_1$

- Evaluate integral

  $-\ln (y-1)+\ln (y)=3t+c_1$

- Solve for y

  $y=\frac{e^{3t+c_1}}{-1+e^{3t+c_1}}$

- Use initial condition $y(0)=\frac{1}{2}$

  $\frac{1}{2}=\frac{e^{c_1}}{-1+e^{c_1}}$

- Solve for c_1

  $c_1=\mathrm{I}\pi$

- Substitute $c_1=\mathrm{I}\pi$ into general solution and simplify

  $y=\frac{e^{3t}}{1+e^{3t}}$

- Solution to the IVP

  $y=\frac{e^{3t}}{1+e^{3t}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(t),t)=3*y(t)*(1-y(t)),y(0) = 1/2],y(t), singsol=all)
```

$$y(t) = \frac{1}{1 + e^{-3t}}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 20

```
DSolve[{y'[t]==3*y[t]*(1-y[t]),{y[0]==1/2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{3t}}{e^{3t} + 1}$$

2.8 problem 8

Internal problem ID [12907]
Internal file name [`OUTPUT/11559_Tuesday_November_07_2023_11_27_03_PM_30439987/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = -t}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

2.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= -t \end{aligned}$$

Hence the ode is

$$y' - 2y = -t$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.8.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-2)dt} \\ &= e^{-2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)(-t) \\ \frac{d}{dt}(e^{-2t}y) &= (e^{-2t})(-t) \\ d(e^{-2t}y) &= (-t\,e^{-2t})\,dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-2t}y &= \int -t\,e^{-2t}\,dt \\ e^{-2t}y &= \frac{(2t+1)\,e^{-2t}}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2t}$ results in

$$y = \frac{e^{2t}(2t+1)\,e^{-2t}}{4} + c_1 e^{2t}$$

which simplifies to

$$y = \frac{t}{2} + \frac{1}{4} + c_1 e^{2t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{4} + c_1$$

$$c_1 = \frac{1}{4}$$

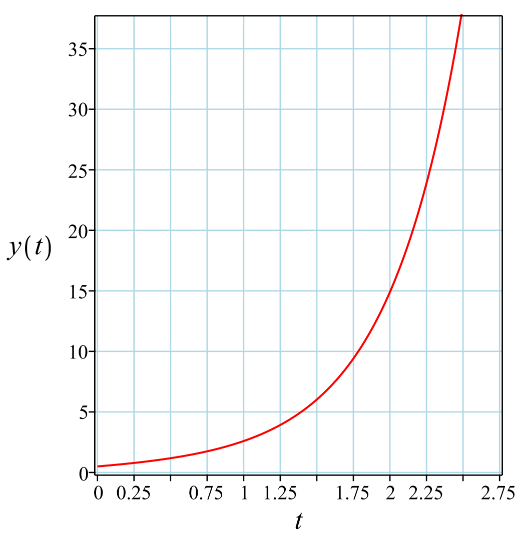
Substituting c_1 found above in the general solution gives

$$y = \frac{t}{2} + \frac{1}{4} + \frac{e^{2t}}{4}$$

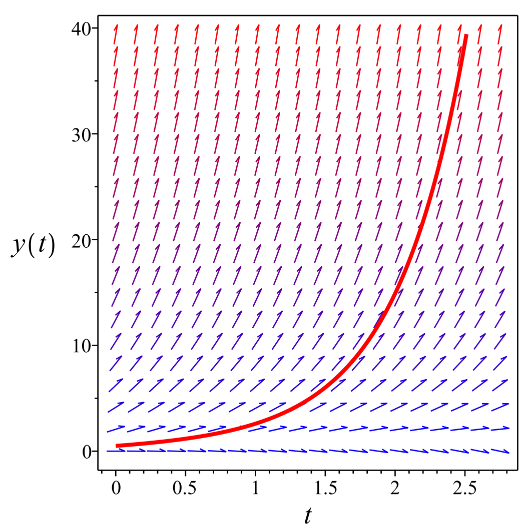
Summary

The solution(s) found are the following

$$y = \frac{t}{2} + \frac{1}{4} + \frac{e^{2t}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{t}{2} + \frac{1}{4} + \frac{e^{2t}}{4}$$

Verified OK.

2.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 2y - t \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 86: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= e^{2t}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{2t}} dy
\end{aligned}
$$

Which results in

$$
S = e^{-2t} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = 2y - t
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= -2\,e^{-2t} y \\
S_y &= e^{-2t}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -t\,\mathrm{e}^{-2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R\,\mathrm{e}^{-2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(2R+1)\,\mathrm{e}^{-2R}}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-2t}y = \frac{(2t+1)\,\mathrm{e}^{-2t}}{4} + c_1$$

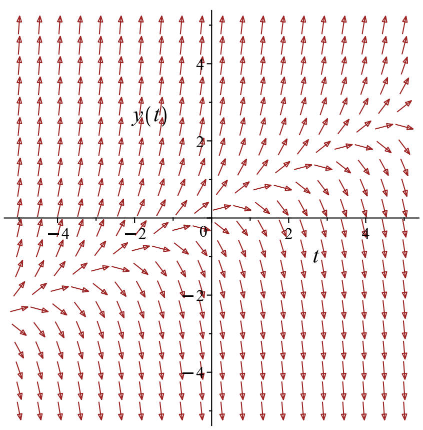
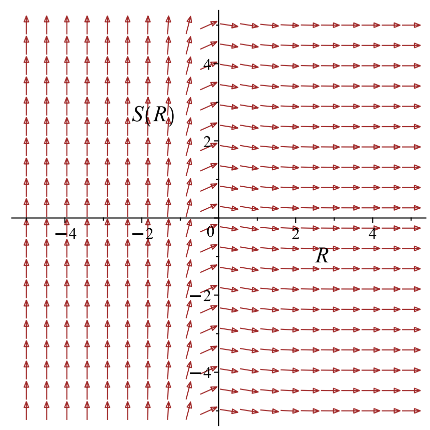
Which simplifies to

$$\mathrm{e}^{-2t}y = \frac{(2t+1)\,\mathrm{e}^{-2t}}{4} + c_1$$

Which gives

$$y = \frac{(2t\,\mathrm{e}^{-2t} + \mathrm{e}^{-2t} + 4c_1)\,\mathrm{e}^{2t}}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2y - t$ 	$R = t$ $S = \mathrm{e}^{-2t} y$	$\frac{dS}{dR} = -R\,\mathrm{e}^{-2R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{4} + c_1$$

$$c_1 = \frac{1}{4}$$

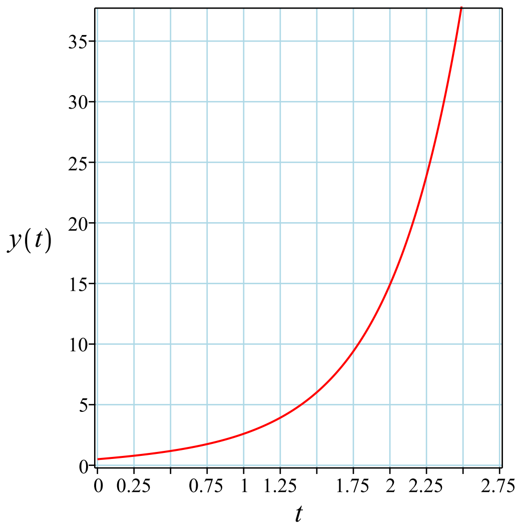
Substituting c_1 found above in the general solution gives

$$y = \frac{t}{2} + \frac{1}{4} + \frac{\mathrm{e}^{2t}}{4}$$

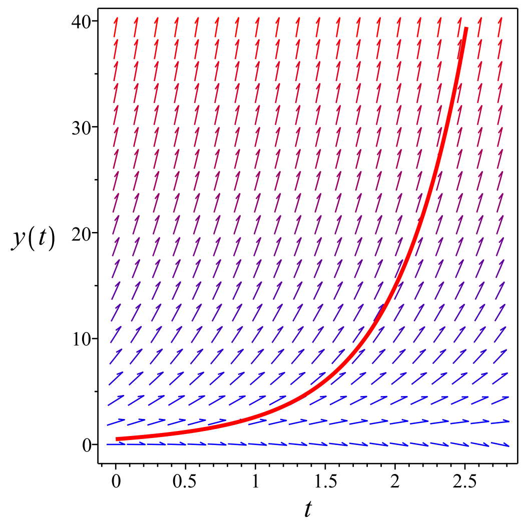
Summary

The solution(s) found are the following

$$y = \frac{t}{2} + \frac{1}{4} + \frac{\mathrm{e}^{2t}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{t}{2} + \frac{1}{4} + \frac{e^{2t}}{4}$$

Verified OK.

2.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= (2y - t)\,\mathrm{d}t \\ (-2y + t)\,\mathrm{d}t + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t,y) &= -2y + t \\ N(t,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2y + t) \\ &= -2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((-2) - (0)) \\ &= -2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -2\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2t}\\ &= \mathrm{e}^{-2t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-2t}(-2y+t)\\ &= (-2y+t)\,\mathrm{e}^{-2t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-2t}(1)\\ &= \mathrm{e}^{-2t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \left((-2y+t)\,\mathrm{e}^{-2t}\right)+\left(\mathrm{e}^{-2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int\overline{M}\,\mathrm{d}t\\ \int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int(-2y+t)\,\mathrm{e}^{-2t}\,\mathrm{d}t\end{aligned}$$

$$\phi = -\frac{(2t-4y+1)\,\mathrm{e}^{-2t}}{4}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{-2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{-2t}$. Therefore equation (4) becomes

$$e^{-2t} = e^{-2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(2t - 4y + 1)\, e^{-2t}}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(2t - 4y + 1)\, e^{-2t}}{4}$$

The solution becomes

$$y = \frac{(2t\, e^{-2t} + e^{-2t} + 4c_1)\, e^{2t}}{4}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{4} + c_1$$

$$c_1 = \frac{1}{4}$$

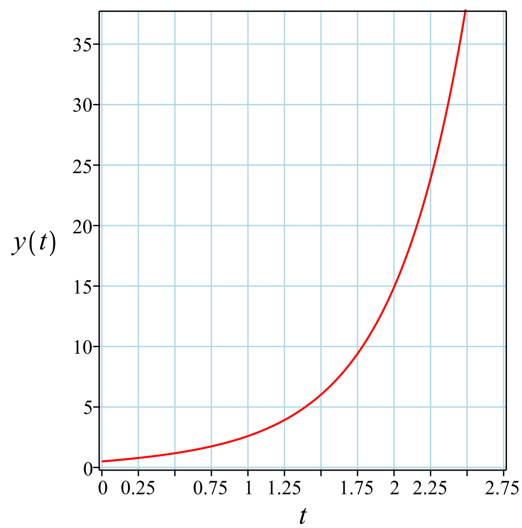
Substituting c_1 found above in the general solution gives

$$y = \frac{t}{2} + \frac{1}{4} + \frac{e^{2t}}{4}$$

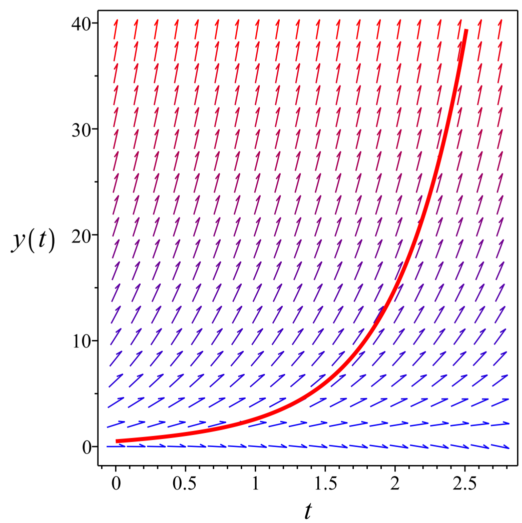
Summary

The solution(s) found are the following

$$y = \frac{t}{2} + \frac{1}{4} + \frac{e^{2t}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{t}{2} + \frac{1}{4} + \frac{e^{2t}}{4}$$

Verified OK.

2.8.5 Maple step by step solution

Let's solve

$\left[y' - 2y = -t, y(0) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = 2y - t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = -t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - 2y\right) = -\mu(t)\, t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - 2y\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-2t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int -\mu(t)\, t dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int -\mu(t)\, t dt + c_1$

- Solve for y

  $y = \frac{\int -\mu(t) t dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-2t}$

  $y = \frac{\int -t\, \mathrm{e}^{-2t} dt + c_1}{\mathrm{e}^{-2t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{(2t+1)\mathrm{e}^{-2t}}{4} + c_1}{\mathrm{e}^{-2t}}$

- Simplify

  $y = \frac{t}{2} + \frac{1}{4} + c_1 \mathrm{e}^{2t}$

- Use initial condition $y(0) = \frac{1}{2}$

  $\frac{1}{2} = \frac{1}{4} + c_1$

- Solve for c_1

  $c_1 = \frac{1}{4}$

- Substitute $c_1 = \frac{1}{4}$ into general solution and simplify

  $y = \frac{t}{2} + \frac{1}{4} + \frac{\mathrm{e}^{2t}}{4}$

- Solution to the IVP

  $y = \frac{t}{2} + \frac{1}{4} + \frac{e^{2t}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)=2*y(t)-t,y(0) = 1/2],y(t), singsol=all)
```

$$y(t) = \frac{t}{2} + \frac{1}{4} + \frac{e^{2t}}{4}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 19

```
DSolve[{y'[t]==2*y[t]-t,{y[0]==1/2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{4}\left(2t + e^{2t} + 1\right)$$

2.9 problem 9

Internal problem ID [12908]
Internal file name [`OUTPUT/11560_Tuesday_November_07_2023_11_27_03_PM_52077065/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \left(y+\frac{1}{2}\right)(y+t) = 0}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

2.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= \frac{(2y+1)(t+y)}{2} \end{aligned}$$

The t domain of $f(t,y)$ when $y=\frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t,y)$ when $t=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{(2y+1)(t+y)}{2}\right)\\ &= t+2y+\frac{1}{2}\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

2.9.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t,y)\\ &= \frac{(2y+1)(t+y)}{2}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = ty + y^2 + \frac{1}{2}t + \frac{1}{2}y$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = \frac{t}{2}$, $f_1(t) = t + \frac{1}{2}$ and $f_2(t) = 1$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{u}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$
\begin{aligned}
f_2' &= 0\\
f_1 f_2 &= t + \frac{1}{2}\\
f_2^2 f_0 &= \frac{t}{2}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
u''(t) - \left(t + \frac{1}{2}\right) u'(t) + \frac{t u(t)}{2} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(t) = \mathrm{e}^{\frac{t}{2}} \left(c_1 + \mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right) c_2\right)
$$

The above shows that

$$
u'(t) = \frac{\mathrm{e}^{\frac{t}{2}} \left(i\mathrm{e}^{\frac{(2t-1)^2}{8}} \sqrt{2}\, c_2 + \frac{\left(c_1+\mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right) c_2\right)\sqrt{\pi}}{2}\right)}{\sqrt{\pi}}
$$

Using the above in (1) gives the solution

$$
y = -\frac{i\mathrm{e}^{\frac{(2t-1)^2}{8}} \sqrt{2}\, c_2 + \frac{\left(c_1+\mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right) c_2\right)\sqrt{\pi}}{2}}{\sqrt{\pi}\, \left(c_1 + \mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right) c_2\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = -\frac{2\left(i\mathrm{e}^{\frac{(2t-1)^2}{8}} \sqrt{2} + \frac{\left(c_3+\mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right)\right)\sqrt{\pi}}{2}\right)}{\sqrt{\pi}\, \left(2c_3 + 2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right)\right)}
$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{1}{2} = \frac{2i\mathrm{e}^{\frac{1}{8}}\sqrt{2} - \sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right) + \sqrt{\pi}\, c_3}{-2\sqrt{\pi}\, c_3 + 2\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right)}
$$

$$c_3 = \frac{-i\mathrm{e}^{\frac{1}{8}}\sqrt{2}+\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right)}{\sqrt{\pi}}$$

Substituting c_3 found above in the general solution gives

$$y = \frac{i\mathrm{e}^{\frac{1}{8}}\sqrt{2}-2i\mathrm{e}^{\frac{(2t-1)^2}{8}}\sqrt{2}-\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right)-\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right)}{-2i\mathrm{e}^{\frac{1}{8}}\sqrt{2}+2\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right)+2\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{i\mathrm{e}^{\frac{1}{8}}\sqrt{2}-2i\mathrm{e}^{\frac{(2t-1)^2}{8}}\sqrt{2}-\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right)-\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right)}{-2i\mathrm{e}^{\frac{1}{8}}\sqrt{2}+2\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right)+2\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{i\mathrm{e}^{\frac{1}{8}}\sqrt{2}-2i\mathrm{e}^{\frac{(2t-1)^2}{8}}\sqrt{2}-\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right)-\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right)}{-2i\mathrm{e}^{\frac{1}{8}}\sqrt{2}+2\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right)+2\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful`
```

✓ Solution by Maple

Time used: 0.188 (sec). Leaf size: 65

```
dsolve([diff(y(t),t)=(y(t)+1/2)*(y(t)+t),y(0) = 1/2],y(t), singsol=all)
```

$$y(t) = \frac{\sqrt{\pi}\, \mathrm{e}^{-\frac{1}{8}} \sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right) + \sqrt{\pi}\, \mathrm{e}^{-\frac{1}{8}} \sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right) + 4i\mathrm{e}^{\frac{t(t-1)}{2}} - 2i}{-2\sqrt{\pi}\, \mathrm{e}^{-\frac{1}{8}} \sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}}{4}\right) - 2\sqrt{\pi}\, \mathrm{e}^{-\frac{1}{8}} \sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2t-1)}{4}\right) + 4i}$$

✓ Solution by Mathematica

Time used: 0.332 (sec). Leaf size: 124

```
DSolve[{y'[t]==(y[t]+1/2)*(y[t]+t),{y[0]==1/2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{-\sqrt{2\pi}\mathrm{erfi}\left(\frac{1-2t}{2\sqrt{2}}\right) + \sqrt{2\pi}\mathrm{erfi}\left(\frac{1}{2\sqrt{2}}\right) + 4e^{\frac{1}{8}(1-2t)^2} - 2\sqrt[8]{e}}{2\sqrt{2\pi}\mathrm{erfi}\left(\frac{1-2t}{2\sqrt{2}}\right) - 2\sqrt{2\pi}\mathrm{erfi}\left(\frac{1}{2\sqrt{2}}\right) + 4\sqrt[8]{e}}$$

2.10 problem 10

Internal problem ID [12909]
Internal file name [`OUTPUT/11561_Tuesday_November_07_2023_11_27_05_PM_54658325/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (1 + t)\, y = 0}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

2.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 - t \\ q(t) &= 0 \end{aligned}$$

Hence the ode is

$$y' + (-1 - t)\, y = 0$$

The domain of $p(t) = -1 - t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

2.10.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= (1 + t)\, y \end{aligned}$$

Where $f(t) = 1 + t$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= 1 + t\, dt \\ \int \frac{1}{y}\, dy &= \int 1 + t\, dt \\ \ln(y) &= \frac{1}{2}t^2 + t + c_1 \\ y &= \mathrm{e}^{\frac{1}{2}t^2 + t + c_1} \\ &= c_1 \mathrm{e}^{t + \frac{1}{2}t^2} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

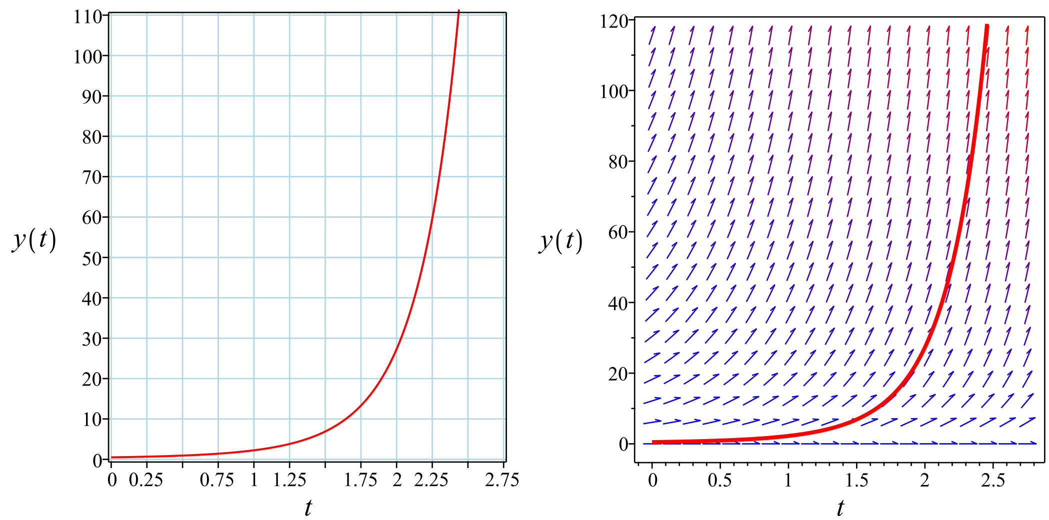
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{t+\frac{1}{2}t^2}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{t+\frac{1}{2}t^2}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{t+\frac{1}{2}t^2}}{2}$$

Verified OK.

2.10.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1-t)dt} \\
&= \mathrm{e}^{-t-\frac{1}{2}t^2}
\end{aligned}
$$

Which simplifies to

$$
\mu = \mathrm{e}^{-\frac{t(t+2)}{2}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\frac{t(t+2)}{2}} y\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\mathrm{e}^{-\frac{t(t+2)}{2}} y = c_1
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{t(t+2)}{2}}$ results in

$$
y = c_1 \mathrm{e}^{\frac{t(t+2)}{2}}
$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{1}{2} = c_1
$$

$$
c_1 = \frac{1}{2}
$$

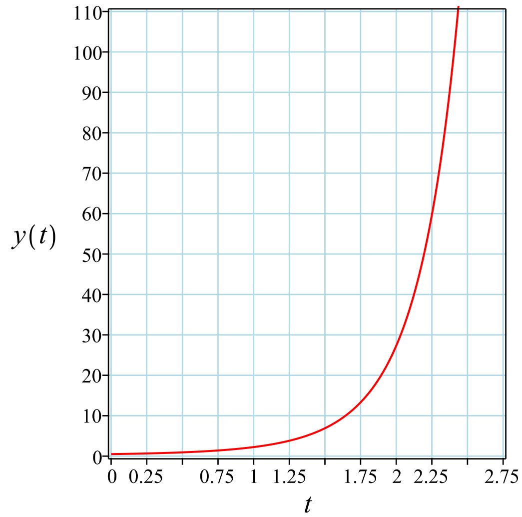
Substituting c_1 found above in the general solution gives

$$
y = \frac{\mathrm{e}^{\frac{t(t+2)}{2}}}{2}
$$

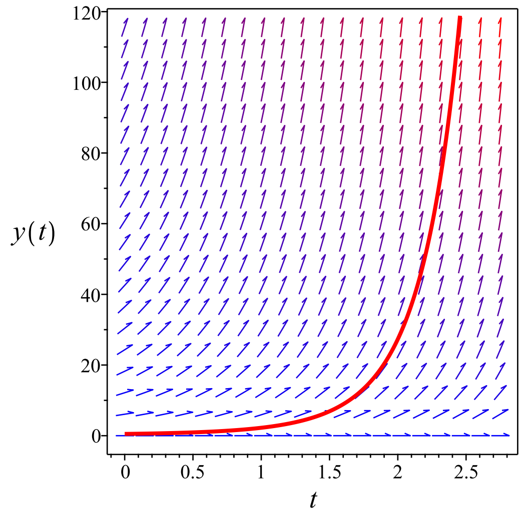
Summary

The solution(s) found are the following

$$
y = \frac{\mathrm{e}^{\frac{t(t+2)}{2}}}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{t(t+2)}{2}}}{2}$$

Verified OK.

2.10.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t + u(t) - (1+t)\, u(t)\, t = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= \frac{u(t^2 + t - 1)}{t} \end{aligned}$$

Where $f(t) = \frac{t^2+t-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{t^2+t-1}{t}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{t^2+t-1}{t}\,dt \\
\ln(u) &= \frac{t^2}{2} + t - \ln(t) + c_2 \\
u &= \mathrm{e}^{\frac{t^2}{2}+t-\ln(t)+c_2} \\
&= c_2 \mathrm{e}^{\frac{t^2}{2}+t-\ln(t)}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ut \\
&= t c_2 \mathrm{e}^{\frac{t^2}{2}+t-\ln(t)}
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{1}{2} = c_2
$$

$$
c_2 = \frac{1}{2}
$$

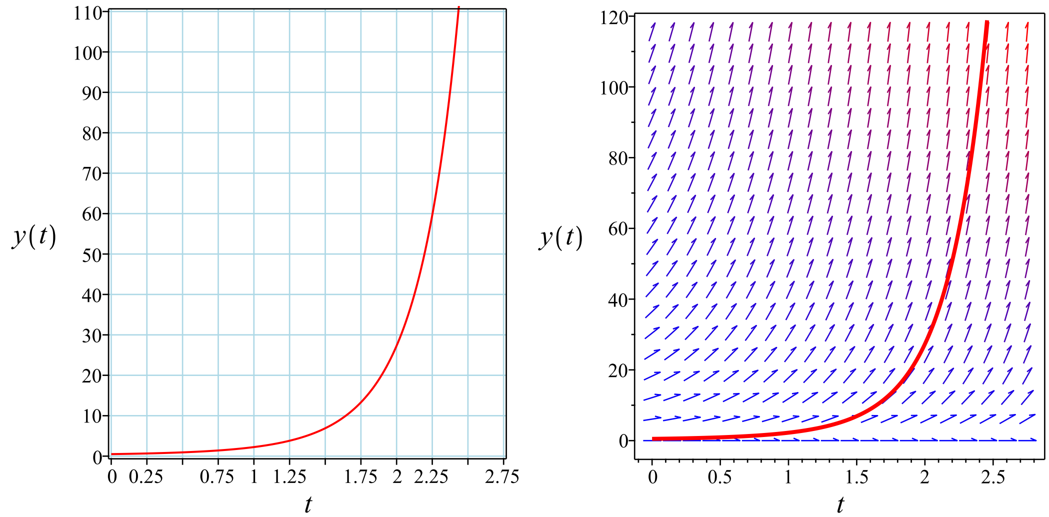
Substituting c_2 found above in the general solution gives

$$
y = \frac{\mathrm{e}^{\frac{t(t+2)}{2}}}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\mathrm{e}^{\frac{t(t+2)}{2}}}{2} \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{t(t+2)}{2}}}{2}$$

Verified OK.

2.10.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = (1 + t) y$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 89: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= e^{t+\frac{1}{2}t^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{t+\frac{1}{2}t^2}} dy \end{aligned}$$

Which results in

$$S = e^{-t-\frac{1}{2}t^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = (1+t)\,y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -(1+t)\,e^{-\frac{t(t+2)}{2}} y \\ S_y &= e^{-\frac{t(t+2)}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-\frac{t(t+2)}{2}} y = c_1$$

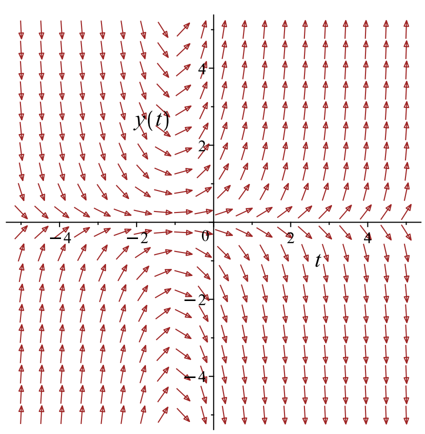
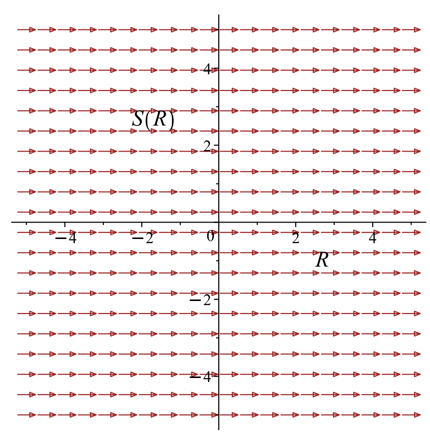
Which simplifies to

$$e^{-\frac{t(t+2)}{2}} y = c_1$$

Which gives

$$y = c_1 e^{\frac{t(t+2)}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = (1+t)\, y$ 	$R = t$ $S = e^{-\frac{t(t+2)}{2}} y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

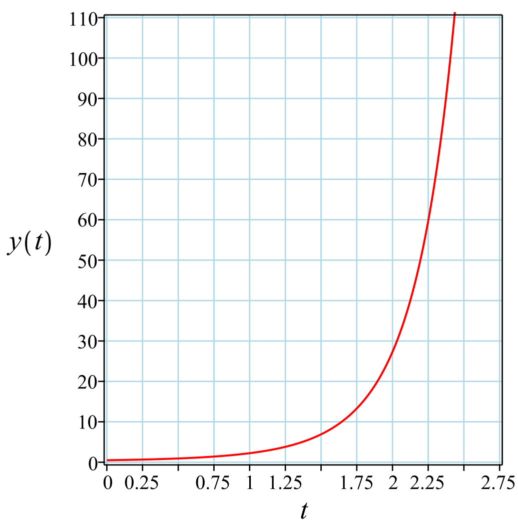
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{\frac{t(t+2)}{2}}}{2}$$

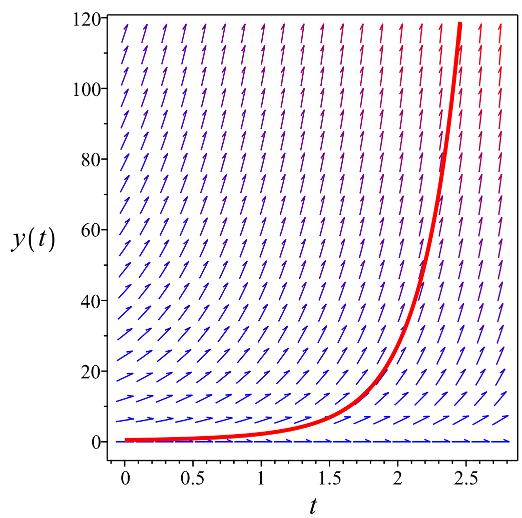
Summary

The solution(s) found are the following

$$y = \frac{e^{\frac{t(t+2)}{2}}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{t(t+2)}{2}}}{2}$$

Verified OK.

2.10.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=(1+t)\,\mathrm{d}t\\ (-1-t)\,\mathrm{d}t+\left(\frac{1}{y}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t,y) &= -1-t \\ N(t,y) &= \frac{1}{y} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1-t) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -1-t \, \mathrm{d}t \end{aligned}$$

$$\phi = -t - \frac{1}{2}t^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -t - \frac{t^2}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t - \frac{t^2}{2} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\frac{1}{2}t^2 + t + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \mathrm{e}^{c_1}$$

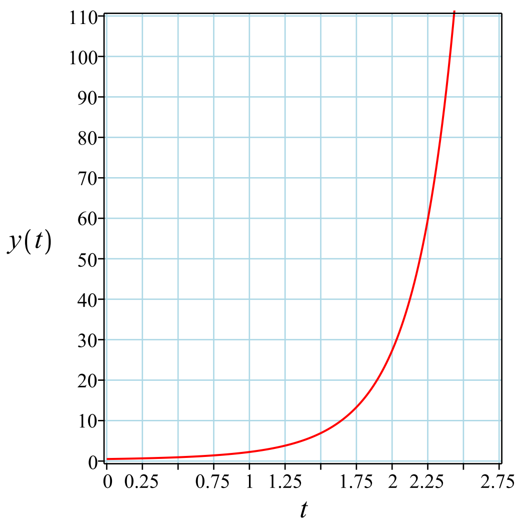
$$c_1 = -\ln(2)$$

Substituting c_1 found above in the general solution gives

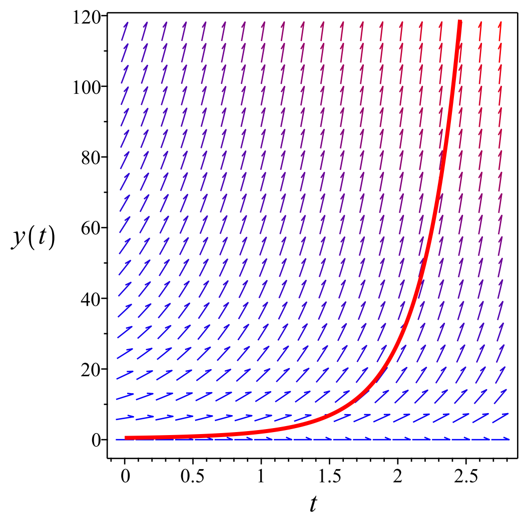
$$y = \frac{e^{\frac{t(t+2)}{2}}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{\frac{t(t+2)}{2}}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{t(t+2)}{2}}}{2}$$

Verified OK.

2.10.7 Maple step by step solution

Let's solve

$\left[y' - (1+t)\,y = 0, y(0) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1 + t$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int (1 + t)\, dt + c_1$

- Evaluate integral

  $\ln (y) = \frac{1}{2}t^2 + t + c_1$

- Solve for y

  $y = e^{\frac{1}{2}t^2 + t + c_1}$

- Use initial condition $y(0) = \frac{1}{2}$

  $\frac{1}{2} = e^{c_1}$

- Solve for c_1

  $c_1 = -\ln (2)$

- Substitute $c_1 = -\ln (2)$ into general solution and simplify

  $y = \frac{e^{\frac{t(t+2)}{2}}}{2}$

- Solution to the IVP

  $y = \frac{e^{\frac{t(t+2)}{2}}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(y(t),t)=(t+1)*y(t),y(0) = 1/2],y(t), singsol=all)
```

$$y(t) = \frac{e^{\frac{t(t+2)}{2}}}{2}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 19

```
DSolve[{y'[t]==(t+1)*y[t],{y[0]==1/2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{\frac{1}{2}t(t+2)}$$

2.11 problem 15 b(1)

Internal problem ID [12910]
Internal file name [`OUTPUT/11562_Tuesday_November_07_2023_11_27_05_PM_28153444/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 15 b(1).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{S' - S^3 + 2S^2 - S = 0}$$

With initial conditions

$$\left[S(0) = \frac{1}{2}\right]$$

2.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} S' &= f(t, S) \\ &= S^3 - 2S^2 + S \end{aligned}$$

The S domain of $f(t, S)$ when $t = 0$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial S} &= \frac{\partial}{\partial S}(S^3 - 2S^2 + S) \\ &= 3S^2 - 4S + 1 \end{aligned}$$

The S domain of $\frac{\partial f}{\partial S}$ when $t=0$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

2.11.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{S^3 - 2S^2 + S} dS = \int dt$$
$$\int^S \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $S = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{\frac{1}{2}} \frac{1}{\_a(\_a^2 - 2\_a + 1)} d\_a = c_1$$

$$c_1 = \int^{\frac{1}{2}} \frac{1}{\_a(\_a - 1)^2} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^S \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t + \int^{\frac{1}{2}} \frac{1}{\_a(\_a - 1)^2} d\_a$$

Solving for S from the above gives

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a(\_a - 1)^2} d\_a\right) + t + \int^{\frac{1}{2}} \frac{1}{\_a(\_a - 1)^2} d\_a\right)$$

Summary

The solution(s) found are the following

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a(\_a - 1)^2} d\_a\right) + t + \int^{\frac{1}{2}} \frac{1}{\_a(\_a - 1)^2} d\_a\right) \tag{1}$$

Verification of solutions

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a(\_a - 1)^2} d\_a\right) + t + \int^{\frac{1}{2}} \frac{1}{\_a(\_a - 1)^2} d\_a\right)$$

Verified OK.

2.11.3 Maple step by step solution

Let's solve

$\left[S' - S^3 + 2S^2 - S = 0, S(0) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  S'

- Separate variables

  $\frac{S'}{S^3 - 2S^2 + S} = 1$

- Integrate both sides with respect to t

  $\int \frac{S'}{S^3 - 2S^2 + S} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + c_1$

- Use initial condition $S(0) = \frac{1}{2}$

  $2 - \mathrm{I}\pi = c_1$

- Solve for c_1

  $c_1 = 2 - \mathrm{I}\pi$

- Substitute $c_1 = 2 - \mathrm{I}\pi$ into general solution and simplify

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + 2 - \mathrm{I}\pi$

- Solution to the IVP

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + 2 - \mathrm{I}\pi$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 1.39 (sec). Leaf size: 37

```
dsolve([diff(S(t),t)=S(t)^3-2*S(t)^2+S(t),S(0) = 1/2],S(t), singsol=all)
```

$$S(t) = \mathrm{e}^{\mathrm{RootOf}\left(-i\pi\,\mathrm{e}^{\_Z}-\ln\left(\mathrm{e}^{\_Z}+1\right)\mathrm{e}^{\_Z}+\_Z\,\mathrm{e}^{\_Z}+t\,\mathrm{e}^{\_Z}+2\,\mathrm{e}^{\_Z}+1\right)}+1$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{S'[t]==S[t]^3-2*S[t]^2+S[t],{S[0]==1/2}},S[t],t,IncludeSingularSolutions -> True]
```

$$\{\}$$

2.12 problem 15 b(2)

Internal problem ID [12911]
Internal file name [`OUTPUT/11563_Tuesday_November_07_2023_11_27_07_PM_54585174/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 15 b(2).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{S' - S^3 + 2S^2 - S = 0}$$

With initial conditions

$$\left[S(1) = \frac{1}{2}\right]$$

2.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} S' &= f(t, S) \\ &= S^3 - 2S^2 + S \end{aligned}$$

The S domain of $f(t, S)$ when $t = 1$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial S} &= \frac{\partial}{\partial S}\left(S^3 - 2S^2 + S\right) \\ &= 3S^2 - 4S + 1 \end{aligned}$$

The S domain of $\frac{\partial f}{\partial S}$ when $t=1$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

2.12.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{S^3 - 2S^2 + S} dS = \int dt$$
$$\int^S \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t=1$ and $S=\frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{\frac{1}{2}} \frac{1}{\_a\,(\_a^2 - 2\_a + 1)} d\_a = 1 + c_1$$

$$c_1 = -1 + \int^{\frac{1}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^S \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t - 1 + \int^{\frac{1}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a$$

Solving for S from the above gives

$$S = \mathrm{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\,(\_a-1)^2} d\_a\right) + t - 1 + \int^{\frac{1}{2}} \frac{1}{\_a\,(\_a-1)^2} d\_a\right)$$

Summary

The solution(s) found are the following

$$S = \mathrm{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\,(\_a-1)^2} d\_a\right) + t - 1 + \int^{\frac{1}{2}} \frac{1}{\_a\,(\_a-1)^2} d\_a\right) \tag{1}$$

Verification of solutions

$$S = \mathrm{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\,(\_a-1)^2} d\_a\right) + t - 1 + \int^{\frac{1}{2}} \frac{1}{\_a\,(\_a-1)^2} d\_a\right)$$

Verified OK.

2.12.3 Maple step by step solution

Let's solve

$\left[S' - S^3 + 2S^2 - S = 0, S(1) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  S'

- Separate variables

  $\frac{S'}{S^3 - 2S^2 + S} = 1$

- Integrate both sides with respect to t

  $\int \frac{S'}{S^3 - 2S^2 + S} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + c_1$

- Use initial condition $S(1) = \frac{1}{2}$

  $2 - \mathrm{I}\pi = 1 + c_1$

- Solve for c_1

  $c_1 = 1 - \mathrm{I}\pi$

- Substitute $c_1 = 1 - \mathrm{I}\pi$ into general solution and simplify

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + 1 - \mathrm{I}\pi$

- Solution to the IVP

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + 1 - \mathrm{I}\pi$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.735 (sec). Leaf size: 35

```
dsolve([diff(S(t),t)=S(t)^3-2*S(t)^2+S(t),S(1) = 1/2],S(t), singsol=all)
```

$$S(t) = e^{\mathrm{RootOf}\left(-i\pi\, e^{\_Z} - \ln\left(e^{\_Z}+1\right) e^{\_Z} + \_Z\, e^{\_Z} + t\, e^{\_Z} + e^{\_Z} + 1\right)} + 1$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{S'[t]==S[t]^3-2*S[t]^2+S[t],{S[1]==1/2}},S[t],t,IncludeSingularSolutions -> True]
```

$\{\}$

2.13 problem 15 b(3)

Internal problem ID [12912]
Internal file name [`OUTPUT/11564_Tuesday_November_07_2023_11_27_08_PM_32998932/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 15 b(3).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{S' - S^3 + 2S^2 - S = 0}$$

With initial conditions

$$[S(0) = 1]$$

2.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} S' &= f(t, S) \\ &= S^3 - 2S^2 + S \end{aligned}$$

The S domain of $f(t, S)$ when $t = 0$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial S} &= \frac{\partial}{\partial S}\left(S^3 - 2S^2 + S\right) \\ &= 3S^2 - 4S + 1 \end{aligned}$$

The S domain of $\frac{\partial f}{\partial S}$ when $t = 0$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.13.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{S^3 - 2S^2 + S} dS = \int dt$$
$$\int^{S} \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $S = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{1} \frac{1}{\_a\left(\_a^2 - 2\_a + 1\right)} d\_a = c_1$$

$$c_1 = \int^{1} \frac{1}{\_a\left(\_a - 1\right)^2} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^{S} \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t + \int^{1} \frac{1}{\_a\left(\_a - 1\right)^2} d\_a$$

Solving for S from the above gives

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\left(\_a - 1\right)^2} d\_a\right) + t + \int^{1} \frac{1}{\_a\left(\_a - 1\right)^2} d\_a\right)$$

Summary

The solution(s) found are the following

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\left(\_a - 1\right)^2} d\_a\right) + t + \int^{1} \frac{1}{\_a\left(\_a - 1\right)^2} d\_a\right) \tag{1}$$

Verification of solutions

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\left(\_a - 1\right)^2} d\_a\right) + t + \int^{1} \frac{1}{\_a\left(\_a - 1\right)^2} d\_a\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(S(t),t)=S(t)^3-2*S(t)^2+S(t),S(0) = 1],S(t), singsol=all)
```

$$S(t) = 1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{S'[t]==S[t]^3-2*S[t]^2+S[t],{S[0]==1}},S[t],t,IncludeSingularSolutions -> True]
```

$$S(t) \to 1$$

2.14 problem 15 b(4)

Internal problem ID [12913]
Internal file name [`OUTPUT/11565_Tuesday_November_07_2023_11_27_09_PM_69487583/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 15 b(4).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{S' - S^3 + 2S^2 - S = 0}$$

With initial conditions

$$\left[S(0) = \frac{3}{2}\right]$$

2.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} S' &= f(t, S) \\ &= S^3 - 2S^2 + S \end{aligned}$$

The S domain of $f(t, S)$ when $t = 0$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = \frac{3}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial S} &= \frac{\partial}{\partial S}(S^3 - 2S^2 + S) \\ &= 3S^2 - 4S + 1 \end{aligned}$$

The S domain of $\frac{\partial f}{\partial S}$ when $t=0$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = \frac{3}{2}$ is inside this domain. Therefore solution exists and is unique.

2.14.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{S^3 - 2S^2 + S} dS = \int dt$$
$$\int^S \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t=0$ and $S = \frac{3}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{\frac{3}{2}} \frac{1}{\_a\,(\_a^2 - 2\_a + 1)} d\_a = c_1$$

$$c_1 = \int^{\frac{3}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^S \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t + \int^{\frac{3}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a$$

Solving for S from the above gives

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right) + t + \int^{\frac{3}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right)$$

Summary

The solution(s) found are the following

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right) + t + \int^{\frac{3}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right) \tag{1}$$

Verification of solutions

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right) + t + \int^{\frac{3}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right)$$

Verified OK.

2.14.3 Maple step by step solution

Let's solve

$\left[S' - S^3 + 2S^2 - S = 0, S(0) = \frac{3}{2}\right]$

- Highest derivative means the order of the ODE is 1

  S'

- Separate variables

  $\frac{S'}{S^3 - 2S^2 + S} = 1$

- Integrate both sides with respect to t

  $\int \frac{S'}{S^3 - 2S^2 + S} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + c_1$

- Use initial condition $S(0) = \frac{3}{2}$

  $-2 + \ln(2) + \ln\left(\frac{3}{2}\right) = c_1$

- Solve for c_1

  $c_1 = -2 + \ln(2) + \ln\left(\frac{3}{2}\right)$

- Substitute $c_1 = -2 + \ln(2) + \ln\left(\frac{3}{2}\right)$ into general solution and simplify

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t - 2 + \ln(3)$

- Solution to the IVP

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t - 2 + \ln(3)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 11.64 (sec). Leaf size: 41

```
dsolve([diff(S(t),t)=S(t)^3-2*S(t)^2+S(t),S(0) = 3/2],S(t), singsol=all)
```

$$S(t) = e^{\mathrm{RootOf}\left(-\ln\left(e^{-Z}+1\right)e^{-Z}+e^{-Z}\ln(3)+\_Z e^{-Z}+t e^{-Z}-2e^{-Z}+1\right)}+1$$

✓ Solution by Mathematica

Time used: 0.885 (sec). Leaf size: 31

```
DSolve[{S'[t]==S[t]^3-2*S[t]^2+S[t],{S[0]==3/2}},S[t],t,IncludeSingularSolutions -> True]
```

$$S(t) \rightarrow \text{InverseFunction}\left[-\frac{1}{\#1-1}-\log(\#1-1)+\log(\#1)\&\right][t-2+\log(3)]$$

2.15 problem 15 b(5)

Internal problem ID [12914]
Internal file name [`OUTPUT/11566_Tuesday_November_07_2023_11_27_10_PM_73851375/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 15 b(5).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{S' - S^3 + 2S^2 - S = 0}$$

With initial conditions

$$\left[S(0) = -\frac{1}{2}\right]$$

2.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} S' &= f(t, S) \\ &= S^3 - 2S^2 + S \end{aligned}$$

The S domain of $f(t, S)$ when $t = 0$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = -\frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial S} &= \frac{\partial}{\partial S}(S^3 - 2S^2 + S) \\ &= 3S^2 - 4S + 1 \end{aligned}$$

The S domain of $\frac{\partial f}{\partial S}$ when $t=0$ is

$$\{-\infty < S < \infty\}$$

And the point $S_0 = -\frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

2.15.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{S^3 - 2S^2 + S} dS = \int dt$$
$$\int^S \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t=0$ and $S = -\frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{-\frac{1}{2}} \frac{1}{\_a\,(\_a^2 - 2\_a + 1)} d\_a = c_1$$

$$c_1 = \int^{-\frac{1}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^S \frac{1}{\_a^3 - 2\_a^2 + \_a} d\_a = t + \int^{-\frac{1}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a$$

Solving for S from the above gives

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right) + t + \int^{-\frac{1}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right)$$

Summary
The solution(s) found are the following

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right) + t + \int^{-\frac{1}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right) \tag{1}$$

Verification of solutions

$$S = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right) + t + \int^{-\frac{1}{2}} \frac{1}{\_a\,(\_a - 1)^2} d\_a\right)$$

Verified OK.

2.15.3 Maple step by step solution

Let's solve

$\left[S' - S^3 + 2S^2 - S = 0, S(0) = -\frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  S'

- Separate variables

  $\frac{S'}{S^3 - 2S^2 + S} = 1$

- Integrate both sides with respect to t

  $\int \frac{S'}{S^3 - 2S^2 + S} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + c_1$

- Use initial condition $S(0) = -\frac{1}{2}$

  $\frac{2}{3} - \ln\left(\frac{3}{2}\right) - \ln(2) = c_1$

- Solve for c_1

  $c_1 = \frac{2}{3} - \ln\left(\frac{3}{2}\right) - \ln(2)$

- Substitute $c_1 = \frac{2}{3} - \ln\left(\frac{3}{2}\right) - \ln(2)$ into general solution and simplify

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + \frac{2}{3} - \ln(3)$

- Solution to the IVP

  $-\frac{1}{S-1} - \ln(S-1) + \ln(S) = t + \frac{2}{3} - \ln(3)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.61 (sec). Leaf size: 45

```
dsolve([diff(S(t),t)=S(t)^3-2*S(t)^2+S(t),S(0) = -1/2],S(t), singsol=all)
```

$$S(t) = e^{\mathrm{RootOf}\left(-3\ln\left(e^{\_Z}+1\right)e^{\_Z}-3\,e^{\_Z}\ln(3)+3\_Z\,e^{\_Z}+3t\,e^{\_Z}+2\,e^{\_Z}+3\right)}+1$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{S'[t]==S[t]^3-2*S[t]^2+S[t],{S[0]==-1/2}},S[t],t,IncludeSingularSolutions -> True]
```

{}

2.16 problem 16 (i)

Internal problem ID [12915]
Internal file name [`OUTPUT/11567_Tuesday_November_07_2023_11_27_11_PM_31375570/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 16 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 - y = 0}$$

2.16.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 + y} dy = \int dt$$
$$-\ln(y+1) + \ln(y) = t + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(y+1)+\ln(y)} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$\frac{y}{y+1} = c_2 \mathrm{e}^t$$

Summary
The solution(s) found are the following

$$y = -\frac{c_2 \mathrm{e}^t}{-1 + c_2 \mathrm{e}^t} \tag{1}$$

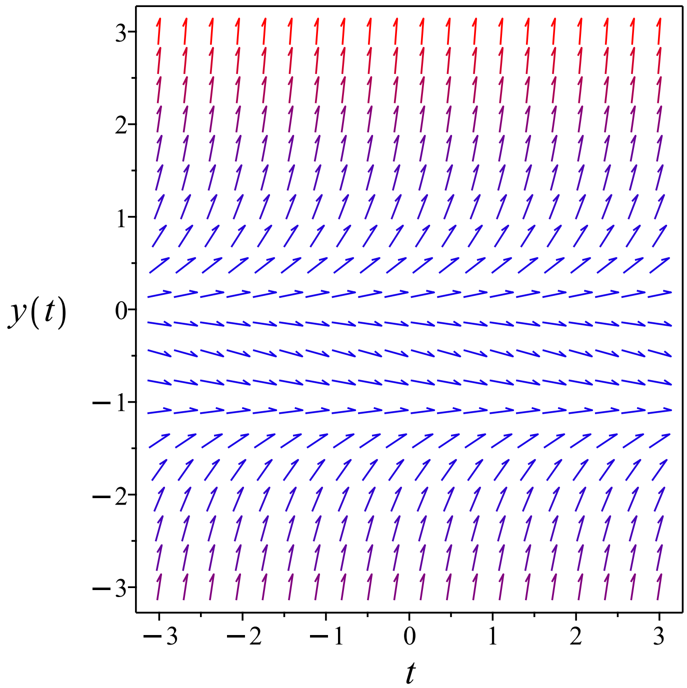


Figure 112: Slope field plot

Verification of solutions

$$y = -\frac{c_2 e^t}{-1 + c_2 e^t}$$

Verified OK.

2.16.2 Maple step by step solution

Let's solve

$y' - y^2 - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2+y} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2+y} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\ln(y+1)+\ln(y)=t+c_1$$

- Solve for y

$$y=-\frac{e^{t+c_1}}{-1+e^{t+c_1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=y(t)^2+y(t),y(t), singsol=all)
```

$$y(t)=\frac{1}{-1+e^{-t}c_1}$$

✓ Solution by Mathematica

Time used: 0.384 (sec). Leaf size: 33

```
DSolve[y'[t]==y[t]^2+y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to -\frac{e^{t+c_1}}{-1+e^{t+c_1}}$$
$$y(t)\to -1$$
$$y(t)\to 0$$

2.17 problem 16 (ii)

Internal problem ID [12916]
Internal file name [`OUTPUT/11568_Tuesday_November_07_2023_11_27_12_PM_13855943/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 16 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + y = 0}$$

2.17.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - y} dy = \int dt$$
$$\ln (y - 1) - \ln (y) = t + c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(y)} = e^{t+c_1}$$

Which simplifies to

$$\frac{y-1}{y} = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{-1 + c_2 e^t} \tag{1}$$

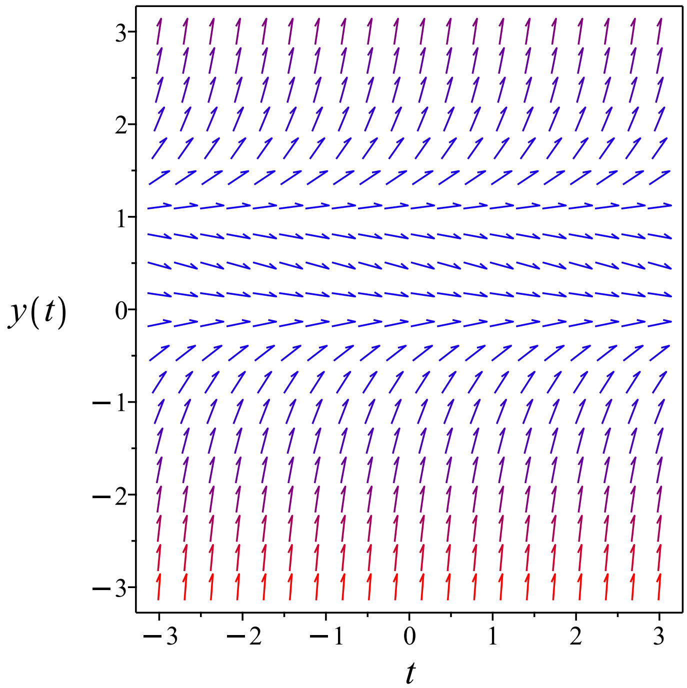


Figure 113: Slope field plot

Verification of solutions

$$y = -\frac{1}{-1 + c_2 e^t}$$

Verified OK.

2.17.2 Maple step by step solution

Let's solve

$y' - y^2 + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2 - y} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2 - y} dt = \int 1 dt + c_1$

- Evaluate integral

$$\ln(y-1) - \ln(y) = t + c_1$$

- Solve for y

$$y = -\frac{1}{-1+e^{t+c_1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)=y(t)^2-y(t),y(t), singsol=all)
```

$$y(t) = \frac{1}{1 + c_1 e^t}$$

✓ Solution by Mathematica

Time used: 0.294 (sec). Leaf size: 25

```
DSolve[y'[t]==y[t]^2-y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{1 + e^{t+c_1}}$$
$$y(t) \to 0$$
$$y(t) \to 1$$

2.18 problem 16 (iii)

Internal problem ID [12917]
Internal file name [`OUTPUT/11569_Tuesday_November_07_2023_11_27_12_PM_89430785/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 16 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^3 - y^2 = 0}$$

2.18.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^3 + y^2} dy = \int dt$$
$$\int^y \frac{1}{\_a^3 + \_a^2} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^y \frac{1}{\_a^3 + \_a^2} d\_a = t + c_1 \tag{1}$$

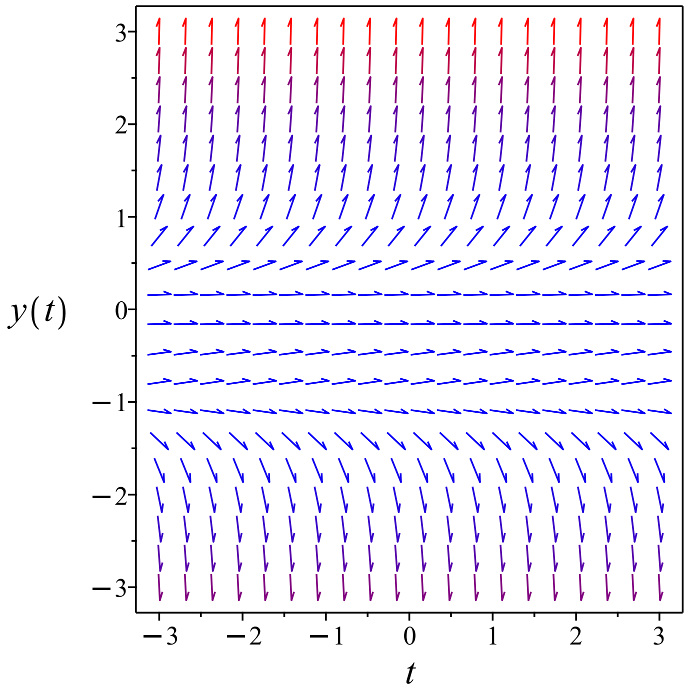


Figure 114: Slope field plot

Verification of solutions

$$\int^{y} \frac{1}{\_a^3 + \_a^2} d\_a = t + c_1$$

Verified OK.

2.18.2 Maple step by step solution

Let's solve

$y' - y^3 - y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3+y^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^3+y^2} dt = \int 1 dt + c_1$

- Evaluate integral

$$\ln(y+1) - \frac{1}{y} - \ln(y) = t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 18

```
dsolve(diff(y(t),t)=y(t)^3+y(t)^2,y(t), singsol=all)
```

$$y(t) = -\frac{1}{\mathrm{LambertW}\left(-c_1 \mathrm{e}^{t-1}\right) + 1}$$

✓ Solution by Mathematica

Time used: 0.318 (sec). Leaf size: 38

```
DSolve[y'[t]==y[t]^3+y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \text{InverseFunction}\left[-\frac{1}{\#1} - \log(\#1) + \log(\#1+1)\&\right][t + c_1]$$
$$y(t) \to -1$$
$$y(t) \to 0$$

2.19 problem 16 (iv)

Internal problem ID [12918]
Internal file name [`OUTPUT/11570_Tuesday_November_07_2023_11_27_13_PM_2375291/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 16 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = -t^2 + 2}$$

2.19.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int -t^2 + 2 \, \mathrm{d}t \\ &= -\frac{1}{3}t^3 + 2t + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{3}t^3 + 2t + c_1 \tag{1}$$

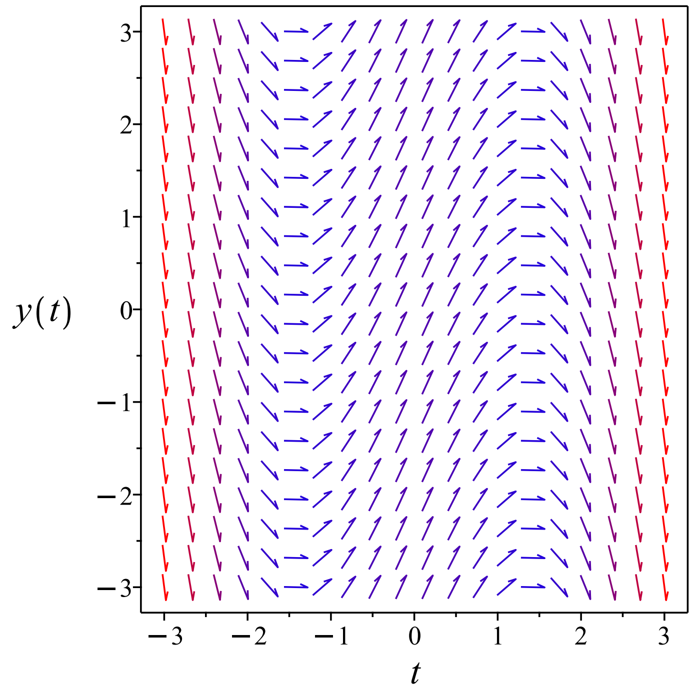


Figure 115: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{3}t^3 + 2t + c_1$$

Verified OK.

2.19.2 Maple step by step solution

Let's solve

$y' = -t^2 + 2$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to t

  $\int y' dt = \int (-t^2 + 2)\, dt + c_1$

- Evaluate integral

  $y = -\frac{1}{3}t^3 + 2t + c_1$

- Solve for y

$$y = -\frac{1}{3}t^3 + 2t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=2-t^2,y(t), singsol=all)
```

$$y(t) = -\frac{1}{3}t^3 + 2t + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 18

```
DSolve[y'[t]==2-t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{t^3}{3} + 2t + c_1$$

2.20 problem 16 (v)

Internal problem ID [12919]
Internal file name [`OUTPUT/11571_Tuesday_November_07_2023_11_27_14_PM_88612694/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 16 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - ty - ty^2 = 0}$$

2.20.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= ty(y+1) \end{aligned}$$

Where $f(t) = t$ and $g(y) = y(y+1)$. Integrating both sides gives

$$\frac{1}{y\,(y+1)}\,dy = t\,dt$$

$$\int \frac{1}{y(y+1)}\,dy = \int t\,dt$$

$$-\ln(y+1)+\ln(y) = \frac{t^2}{2}+c_1$$

Raising both side to exponential gives

$$e^{-\ln(y+1)+\ln(y)} = e^{\frac{t^2}{2}+c_1}$$

Which simplifies to

$$\frac{y}{y+1} = e^{\frac{t^2}{2}} c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{\frac{t^2}{2}} c_2}{-1+e^{\frac{t^2}{2}} c_2} \tag{1}$$

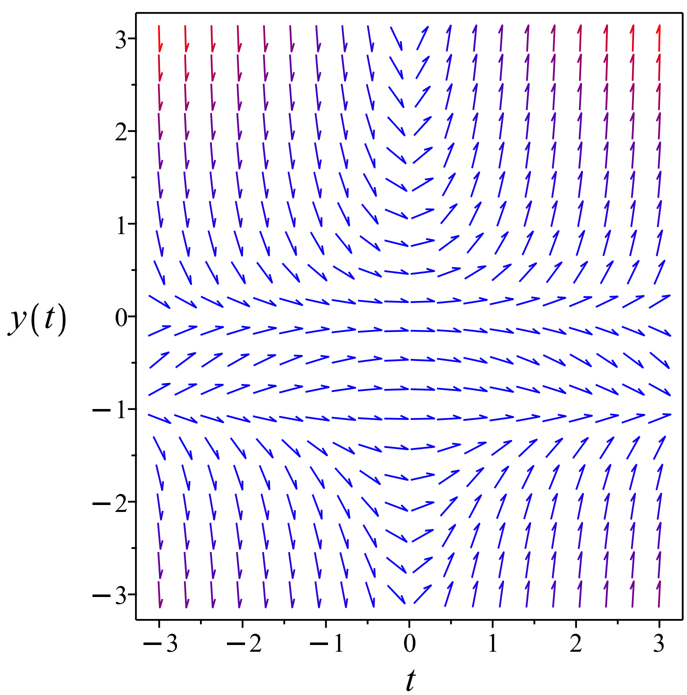


Figure 116: Slope field plot

Verification of solutions

$$y = -\frac{e^{\frac{t^2}{2}} c_2}{-1 + e^{\frac{t^2}{2}} c_2}$$

Verified OK.

2.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= t\,y^2 + ty \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 100: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{1}{t} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{t}} dt \end{aligned}$$

Which results in

$$S = \frac{t^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = t\, y^2 + ty$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= t \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y\,(y+1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R\,(R+1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R+1) + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{t^2}{2} = -\ln(y+1) + \ln(y) + c_1$$

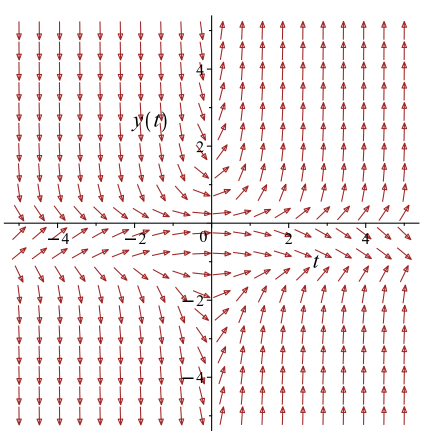
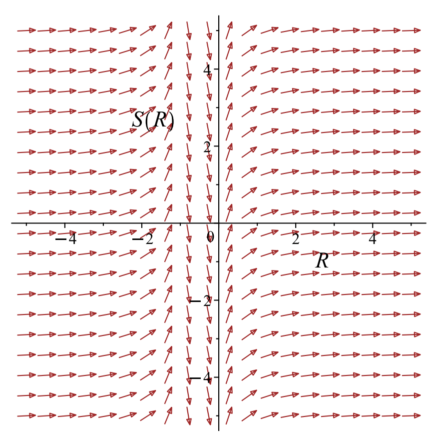
Which simplifies to

$$\frac{t^2}{2} = -\ln(y+1) + \ln(y) + c_1$$

Which gives

$$y = \frac{1}{e^{-\frac{t^2}{2}+c_1} - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = t\,y^2 + ty$ 	$R = y$ $S = \frac{t^2}{2}$	$\frac{dS}{dR} = \frac{1}{R(R+1)}$ 

Summary
The solution(s) found are the following

$$y = \frac{1}{e^{-\frac{t^2}{2}+c_1} - 1} \tag{1}$$

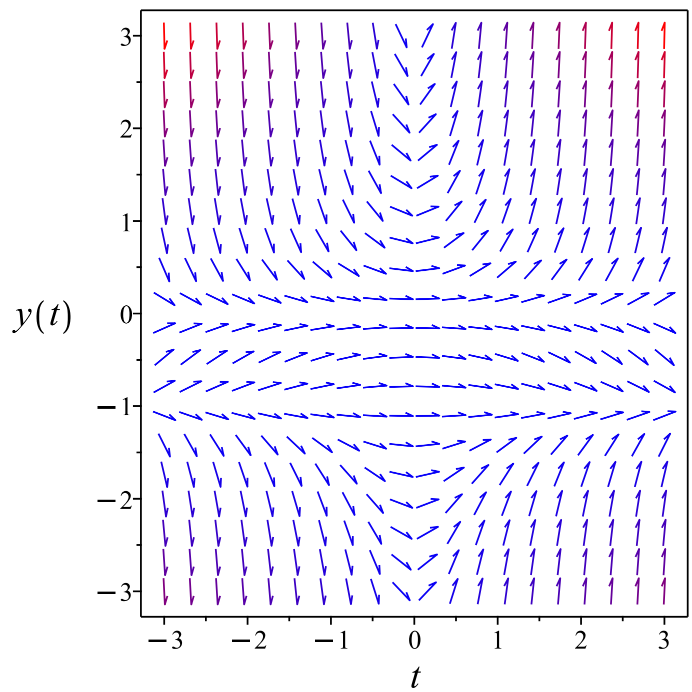


Figure 117: Slope field plot

Verification of solutions

$$y = \frac{1}{e^{-\frac{t^2}{2}+c_1} - 1}$$

Verified OK.

2.20.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= t\,y^2 + ty \end{aligned}$$

This is a Bernoulli ODE.

$$y' = ty + ty^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(t)y + f_1(t)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(t)y^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(t) &= t \\ f_1(t) &= t \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = \frac{t}{y} + t \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t t gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(t) &= w(t)\,t + t \\ w' &= -wt - t \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(t) + p(t)w(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= t \\ q(t) &= -t \end{aligned}$$

Hence the ode is

$$w'(t) + w(t)\, t = -t$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int t dt} \\ &= \mathrm{e}^{\frac{t^2}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu w) &= (\mu)\,(-t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{t^2}{2}} w\right) &= \left(\mathrm{e}^{\frac{t^2}{2}}\right)(-t) \\ \mathrm{d}\left(\mathrm{e}^{\frac{t^2}{2}} w\right) &= \left(-t\,\mathrm{e}^{\frac{t^2}{2}}\right)\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{\frac{t^2}{2}} w &= \int -t\,\mathrm{e}^{\frac{t^2}{2}}\,\mathrm{d}t \\ \mathrm{e}^{\frac{t^2}{2}} w &= -\mathrm{e}^{\frac{t^2}{2}} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{t^2}{2}}$ results in

$$w(t) = -\mathrm{e}^{-\frac{t^2}{2}}\mathrm{e}^{\frac{t^2}{2}} + \mathrm{e}^{-\frac{t^2}{2}} c_1$$

which simplifies to

$$w(t) = -1 + \mathrm{e}^{-\frac{t^2}{2}} c_1$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = -1 + \mathrm{e}^{-\frac{t^2}{2}} c_1$$

Or

$$y = \frac{1}{-1 + \mathrm{e}^{-\frac{t^2}{2}} c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{-1+\mathrm{e}^{-\frac{t^2}{2}} c_1} \tag{1}$$

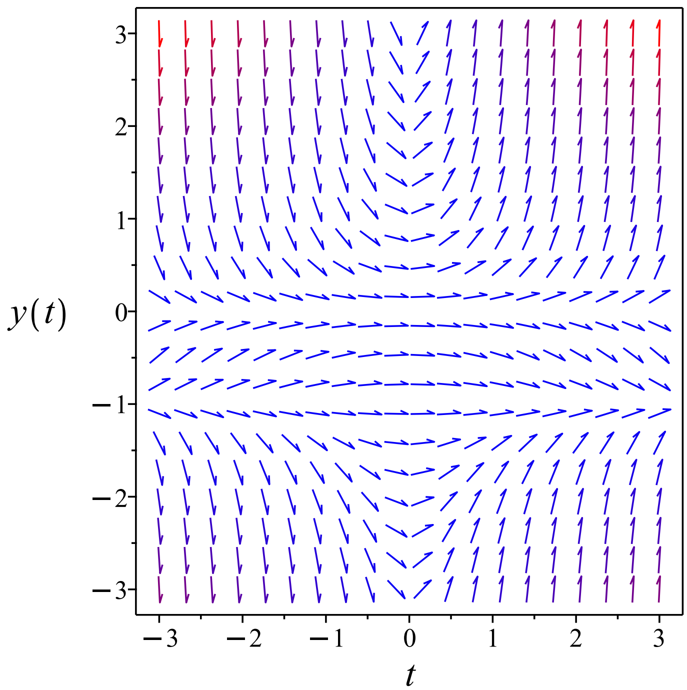


Figure 118: Slope field plot

Verification of solutions

$$y = \frac{1}{-1+\mathrm{e}^{-\frac{t^2}{2}} c_1}$$

Verified OK.

2.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y\,(y+1)}\right)\mathrm{d}y&=(t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t+\left(\frac{1}{y\,(y+1)}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-t\\ N(t,y)&=\frac{1}{y\,(y+1)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y\,(y+1)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -t\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{t^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y(y+1)}$. Therefore equation (4) becomes

$$
\frac{1}{y\,(y+1)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y\,(y+1)}
$$

Integrating the above w.r.t y gives

$$
\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y\,(y+1)}\right) \mathrm{d}y
$$
$$
f(y) = -\ln(y+1) + \ln(y) + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{t^2}{2} - \ln(y+1) + \ln(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{t^2}{2} - \ln(y+1) + \ln(y)
$$

The solution becomes

$$
y = -\frac{\mathrm{e}^{\frac{t^2}{2}+c_1}}{-1+\mathrm{e}^{\frac{t^2}{2}+c_1}}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\mathrm{e}^{\frac{t^2}{2}+c_1}}{-1+\mathrm{e}^{\frac{t^2}{2}+c_1}} \tag{1}
$$

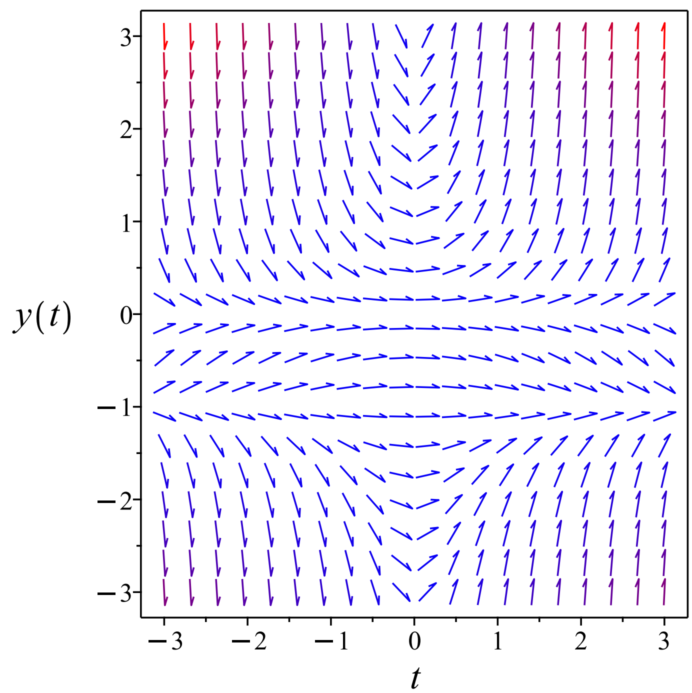


Figure 119: Slope field plot

Verification of solutions

$$y = -\frac{e^{\frac{t^2}{2}+c_1}}{-1+e^{\frac{t^2}{2}+c_1}}$$

Verified OK.

2.20.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= t\,y^2 + ty \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = t\,y^2 + ty$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = t$ and $f_2(t) = t$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{tu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1 f_2 &= t^2 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$t u''(t) - \left(t^2 + 1\right) u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + \mathrm{e}^{\frac{t^2}{2}} c_2$$

The above shows that

$$u'(t) = t\,\mathrm{e}^{\frac{t^2}{2}} c_2$$

Using the above in (1) gives the solution

$$y = -\frac{\mathrm{e}^{\frac{t^2}{2}} c_2}{c_1 + \mathrm{e}^{\frac{t^2}{2}} c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\mathrm{e}^{\frac{t^2}{2}}}{c_3 + \mathrm{e}^{\frac{t^2}{2}}}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{\frac{t^2}{2}}}{c_3 + e^{\frac{t^2}{2}}} \tag{1}$$

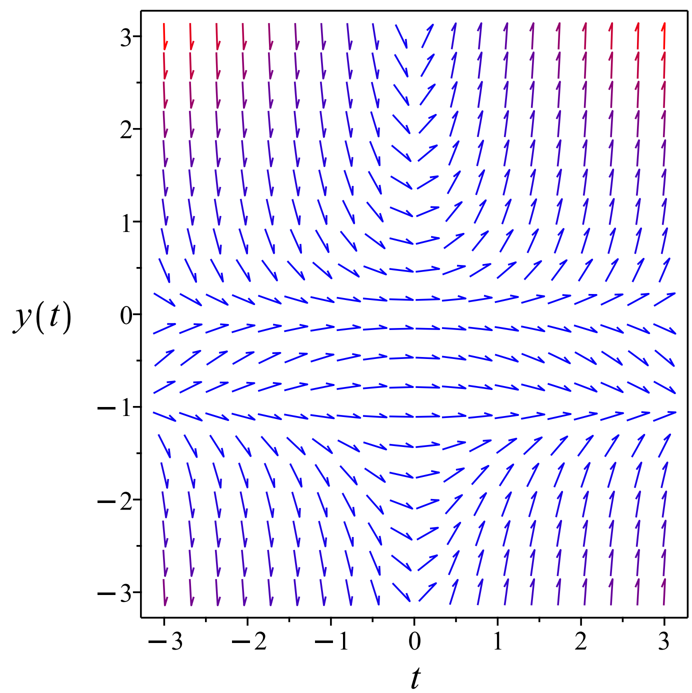


Figure 120: Slope field plot

Verification of solutions

$$y = -\frac{e^{\frac{t^2}{2}}}{c_3 + e^{\frac{t^2}{2}}}$$

Verified OK.

2.20.6 Maple step by step solution

Let's solve

$y' - ty - ty^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y+1)} = t$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(y+1)}dt = \int t dt + c_1$

- Evaluate integral

  $-\ln(y+1) + \ln(y) = \frac{t^2}{2} + c_1$

- Solve for y

  $y = -\frac{e^{\frac{t^2}{2}+c_1}}{-1+e^{\frac{t^2}{2}+c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)=t*y(t)+t*y(t)^2,y(t), singsol=all)
```

$$y(t) = \frac{1}{-1 + e^{-\frac{t^2}{2}} c_1}$$

✓ Solution by Mathematica

Time used: 0.396 (sec). Leaf size: 45

```
DSolve[y'[t]==t*y[t]+t*y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{e^{\frac{t^2}{2}+c_1}}{-1+e^{\frac{t^2}{2}+c_1}}$$
$$y(t) \to -1$$
$$y(t) \to 0$$

2.21 problem 16 (vi)

Internal problem ID [12920]
Internal file name [`OUTPUT/11572_Tuesday_November_07_2023_11_27_15_PM_80067051/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 16 (vi).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - t^2 y = t^2}$$

2.21.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= t^2(y+1) \end{aligned}$$

Where $f(t) = t^2$ and $g(y) = y + 1$. Integrating both sides gives

$$\frac{1}{y+1}\, dy = t^2\, dt$$
$$\int \frac{1}{y+1}\, dy = \int t^2\, dt$$

$$\ln(y+1) = \frac{t^3}{3} + c_1$$

Raising both side to exponential gives

$$y+1 = e^{\frac{t^3}{3}+c_1}$$

Which simplifies to

$$y+1 = c_2 e^{\frac{t^3}{3}}$$

Summary

The solution(s) found are the following

$$y = c_2 e^{\frac{t^3}{3}+c_1} - 1 \tag{1}$$

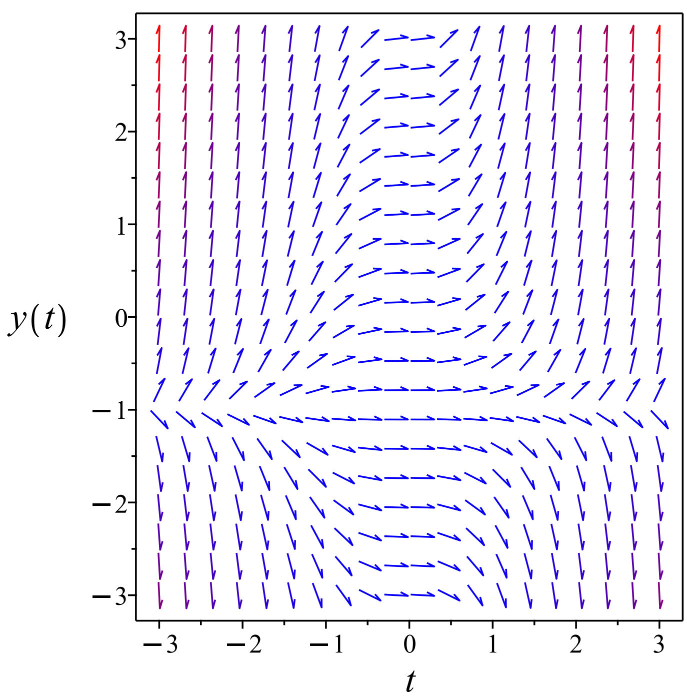


Figure 121: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{t^3}{3}+c_1} - 1$$

Verified OK.

2.21.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned}p(t) &= -t^2\\ q(t) &= t^2\end{aligned}$$

Hence the ode is

$$y' - t^2 y = t^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -t^2 dt}\\ &= e^{-\frac{t^3}{3}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(t^2\right)\\ \frac{d}{dt}\left(e^{-\frac{t^3}{3}} y\right) &= \left(e^{-\frac{t^3}{3}}\right)\left(t^2\right)\\ d\left(e^{-\frac{t^3}{3}} y\right) &= \left(t^2 e^{-\frac{t^3}{3}}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-\frac{t^3}{3}} y &= \int t^2 e^{-\frac{t^3}{3}}\, dt\\ e^{-\frac{t^3}{3}} y &= -e^{-\frac{t^3}{3}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{t^3}{3}}$ results in

$$y = -e^{\frac{t^3}{3}} e^{-\frac{t^3}{3}} + c_1 e^{\frac{t^3}{3}}$$

which simplifies to

$$y = -1 + c_1 e^{\frac{t^3}{3}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -1 + c_1 e^{\frac{t^3}{3}} \tag{1}$$

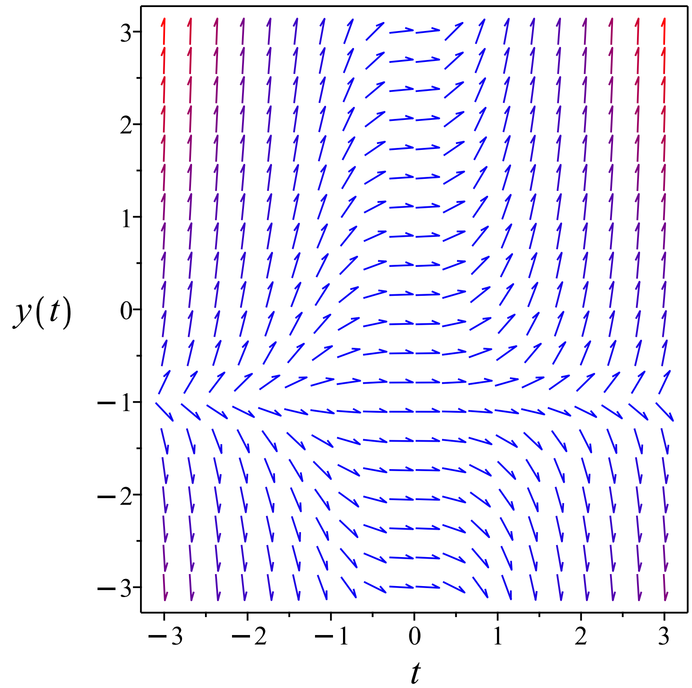


Figure 122: Slope field plot

Verification of solutions

$$y = -1 + c_1 e^{\frac{t^3}{3}}$$

Verified OK.

2.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y\,t^2 + t^2$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 103: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= e^{\frac{t^3}{3}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{t^3}{3}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\frac{t^3}{3}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = y\, t^2 + t^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -t^2 \mathrm{e}^{-\frac{t^3}{3}} y \\ S_y &= \mathrm{e}^{-\frac{t^3}{3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t^2 \mathrm{e}^{-\frac{t^3}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 \mathrm{e}^{-\frac{R^3}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -e^{-\frac{R^3}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-\frac{t^3}{3}} y = -e^{-\frac{t^3}{3}} + c_1$$

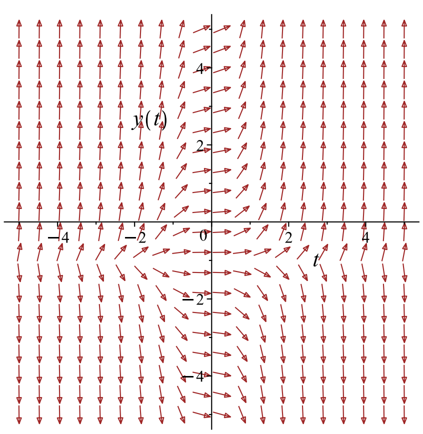
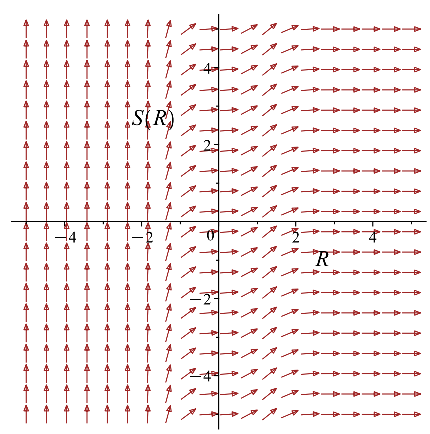
Which simplifies to

$$e^{-\frac{t^3}{3}} y = -e^{-\frac{t^3}{3}} + c_1$$

Which gives

$$y = -\left(e^{-\frac{t^3}{3}} - c_1\right) e^{\frac{t^3}{3}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y\,t^2 + t^2$ 	$R = t$ $S = e^{-\frac{t^3}{3}} y$	$\frac{dS}{dR} = R^2 e^{-\frac{R^3}{3}}$ 

Summary

The solution(s) found are the following

$$y = -\left(e^{-\frac{t^3}{3}} - c_1\right) e^{\frac{t^3}{3}} \tag{1}$$

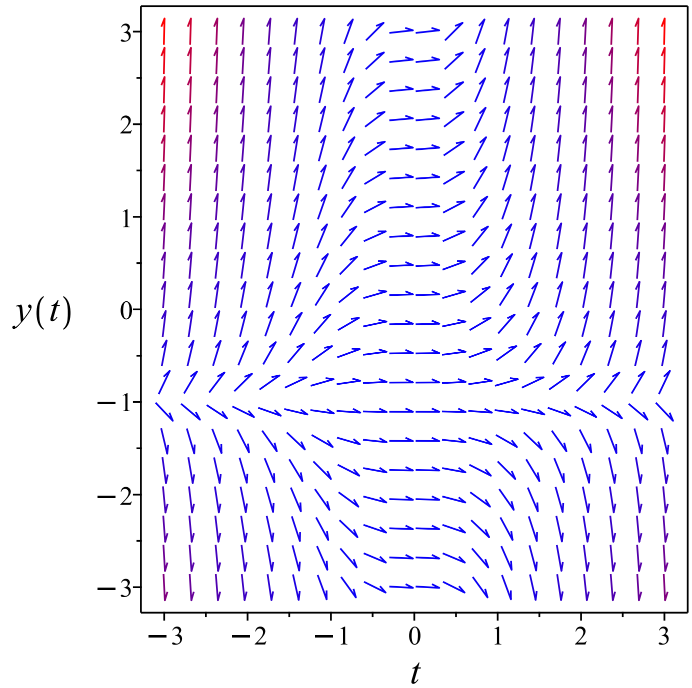


Figure 123: Slope field plot

<u>Verification of solutions</u>

$$y = -\left(e^{-\frac{t^3}{3}} - c_1\right) e^{\frac{t^3}{3}}$$

Verified OK.

2.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y+1}\right) \mathrm{d}y &= \left(t^2\right) \mathrm{d}t \\
\left(-t^2\right) \mathrm{d}t + \left(\frac{1}{y+1}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -t^2 \\
N(t, y) &= \frac{1}{y+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-t^2\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -t^2 \,\mathrm{d}t\end{aligned}$$

$$\phi = -\frac{t^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+1}$. Therefore equation (4) becomes

$$\frac{1}{y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y+1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y+1}\right) \mathrm{d}y\\ f(y) &= \ln(y+1) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} + \ln(y+1)$$

The solution becomes

$$y = e^{\frac{t^3}{3}+c_1} - 1$$

Summary
The solution(s) found are the following

$$y = e^{\frac{t^3}{3}+c_1} - 1 \tag{1}$$

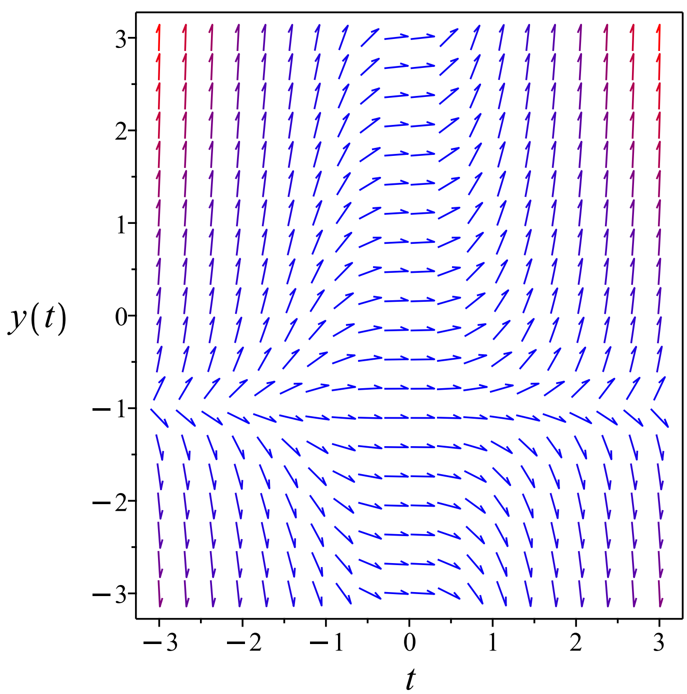


Figure 124: Slope field plot

Verification of solutions

$$y = e^{\frac{t^3}{3}+c_1} - 1$$

Verified OK.

2.21.5 Maple step by step solution

Let's solve

$y' - t^2 y = t^2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1} = t^2$

- Integrate both sides with respect to t

  $\int \frac{y'}{y+1} dt = \int t^2 dt + c_1$

- Evaluate integral

  $\ln(y+1) = \frac{t^3}{3} + c_1$

- Solve for y

  $y = e^{\frac{t^3}{3}+c_1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=t^2+t^2*y(t),y(t), singsol=all)
```

$$y(t) = -1 + c_1 e^{\frac{t^3}{3}}$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 24

```
DSolve[y'[t]==t^2+t^2*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -1 + c_1 e^{\frac{t^3}{3}}$$
$$y(t) \to -1$$

2.22 problem 16 (vii)

Internal problem ID [12921]
Internal file name [`OUTPUT/11573_Tuesday_November_07_2023_11_27_15_PM_89745885/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 16 (vii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - ty = t}$$

2.22.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(t, y) \\
&= f(t)g(y) \\
&= t(y+1)
\end{aligned}$$

Where $f(t) = t$ and $g(y) = y + 1$. Integrating both sides gives

$$\frac{1}{y+1}\, dy = t\, dt$$

$$\int \frac{1}{y+1}\, dy = \int t\, dt$$

$$\ln(y+1) = \frac{t^2}{2} + c_1$$

Raising both side to exponential gives

$$y+1 = e^{\frac{t^2}{2}+c_1}$$

Which simplifies to

$$y+1 = e^{\frac{t^2}{2}} c_2$$

Summary

The solution(s) found are the following

$$y = c_2 e^{\frac{t^2}{2}+c_1} - 1 \tag{1}$$

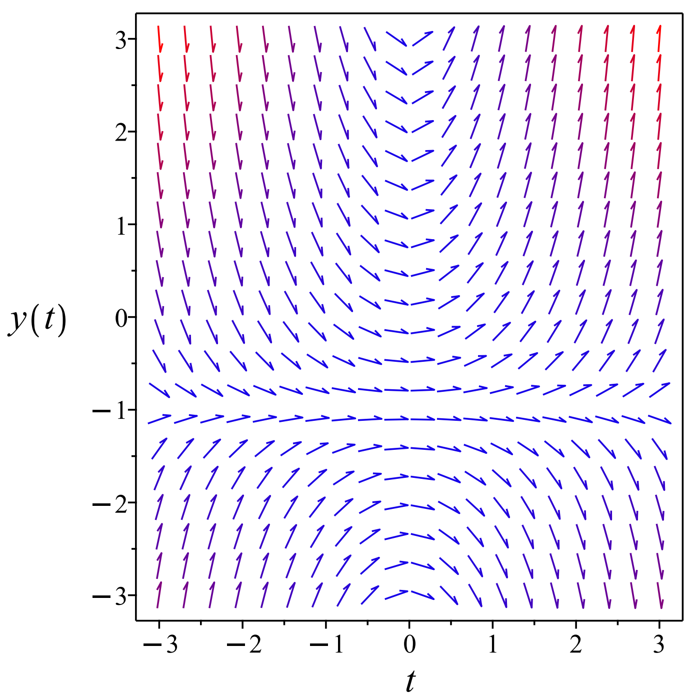


Figure 125: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{t^2}{2}+c_1} - 1$$

Verified OK.

2.22.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned}p(t) &= -t\\ q(t) &= t\end{aligned}$$

Hence the ode is

$$y' - ty = t$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -t dt}\\ &= e^{-\frac{t^2}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)(t)\\ \frac{d}{dt}\left(e^{-\frac{t^2}{2}} y\right) &= \left(e^{-\frac{t^2}{2}}\right)(t)\\ d\left(e^{-\frac{t^2}{2}} y\right) &= \left(t\, e^{-\frac{t^2}{2}}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-\frac{t^2}{2}} y &= \int t\, e^{-\frac{t^2}{2}}\, dt\\ e^{-\frac{t^2}{2}} y &= -e^{-\frac{t^2}{2}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{t^2}{2}}$ results in

$$y = -e^{-\frac{t^2}{2}} e^{\frac{t^2}{2}} + c_1 e^{\frac{t^2}{2}}$$

which simplifies to

$$y = -1 + c_1 e^{\frac{t^2}{2}}$$

Summary
The solution(s) found are the following

$$y = -1 + c_1 e^{\frac{t^2}{2}} \tag{1}$$

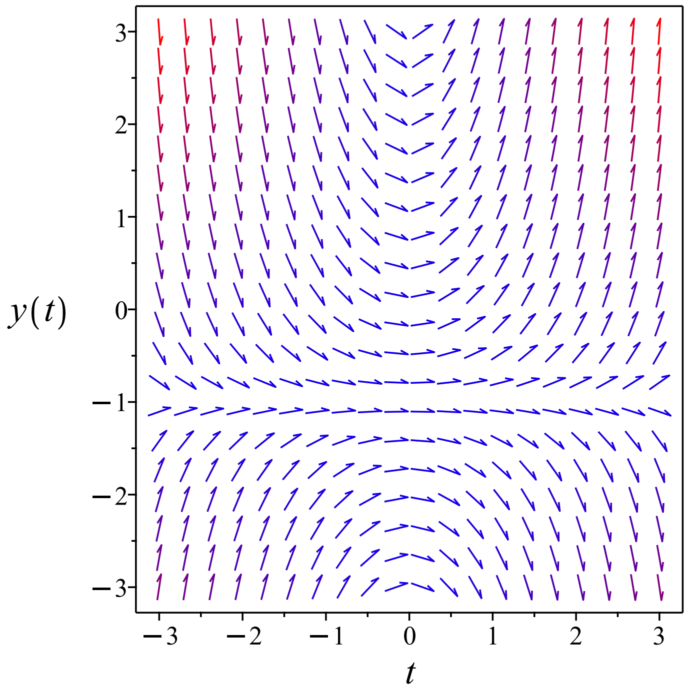


Figure 126: Slope field plot

Verification of solutions

$$y = -1 + c_1 e^{\frac{t^2}{2}}$$

Verified OK.

2.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = ty + t$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 106: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,y) &= 0 \\ \eta(t,y) &= \mathrm{e}^{\frac{t^2}{2}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{t^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{t^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = ty + t$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -t\, e^{-\frac{t^2}{2}} y \\ S_y &= e^{-\frac{t^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t\, e^{-\frac{t^2}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\, e^{-\frac{R^2}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -e^{-\frac{R^2}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-\frac{t^2}{2}} y = -e^{-\frac{t^2}{2}} + c_1$$

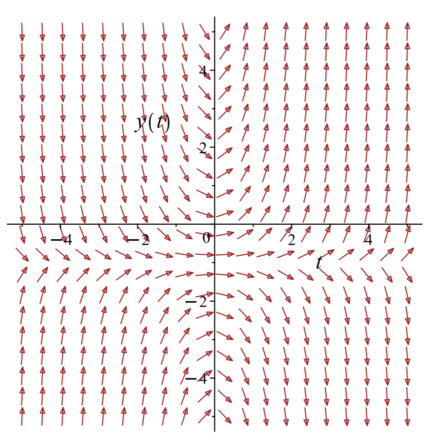
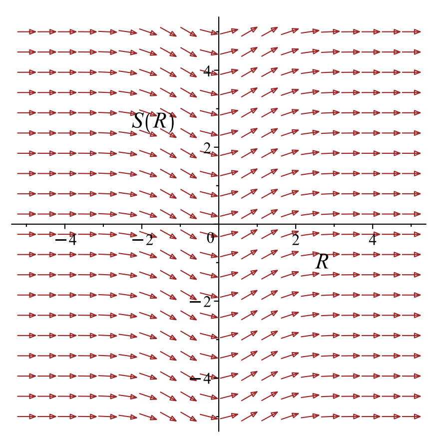
Which simplifies to

$$e^{-\frac{t^2}{2}} y = -e^{-\frac{t^2}{2}} + c_1$$

Which gives

$$y = -\left(e^{-\frac{t^2}{2}} - c_1\right) e^{\frac{t^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = ty + t$ 	$R = t$ $S = e^{-\frac{t^2}{2}} y$	$\frac{dS}{dR} = R e^{-\frac{R^2}{2}}$ 

Summary
The solution(s) found are the following

$$y = -\left(e^{-\frac{t^2}{2}} - c_1\right) e^{\frac{t^2}{2}} \tag{1}$$

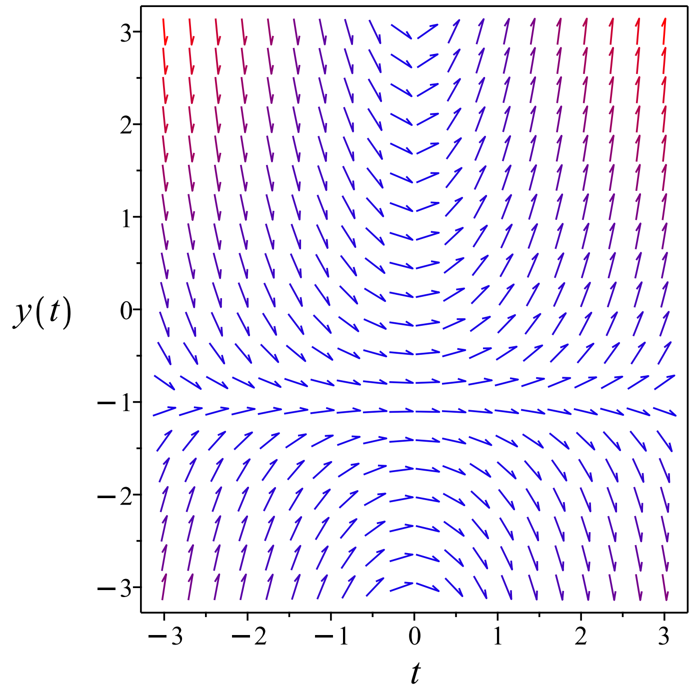


Figure 127: Slope field plot

<u>Verification of solutions</u>

$$y = -\left(e^{-\frac{t^2}{2}} - c_1\right) e^{\frac{t^2}{2}}$$

Verified OK.

2.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y+1}\right) \mathrm{d}y &= (t)\, \mathrm{d}t \\
(-t)\, \mathrm{d}t + \left(\frac{1}{y+1}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -t \\
N(t, y) &= \frac{1}{y+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -t\,\mathrm{d}t\end{aligned}$$

$$\phi = -\frac{t^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{y+1}$. Therefore equation (4) becomes

$$\frac{1}{y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y+1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y+1}\right)\mathrm{d}y\\ f(y) &= \ln(y+1) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \ln(y+1)$$

The solution becomes

$$y = -1 + e^{\frac{t^2}{2}+c_1}$$

Summary

The solution(s) found are the following

$$y = -1 + e^{\frac{t^2}{2}+c_1} \tag{1}$$

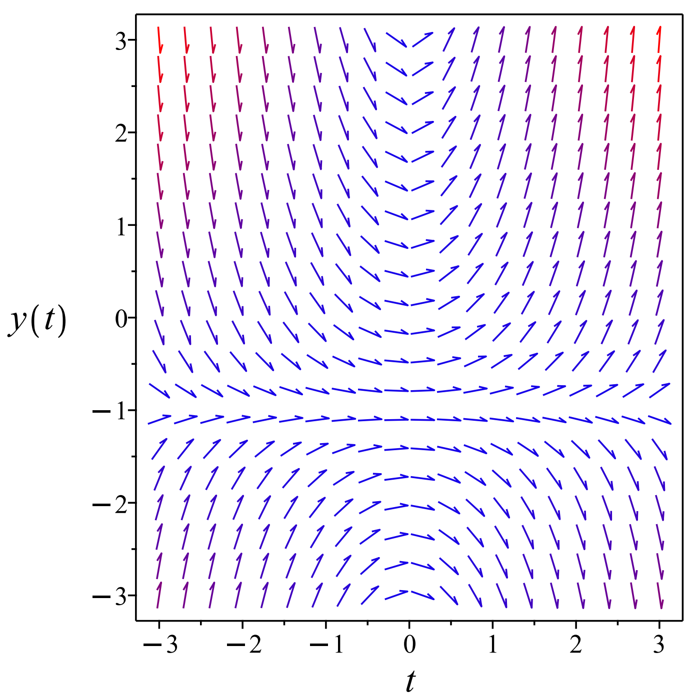


Figure 128: Slope field plot

Verification of solutions

$$y = -1 + e^{\frac{t^2}{2}+c_1}$$

Verified OK.

2.22.5 Maple step by step solution

Let's solve

$y' - ty = t$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1} = t$

- Integrate both sides with respect to t

  $\int \frac{y'}{y+1}dt = \int tdt + c_1$

- Evaluate integral

  $\ln(y+1) = \frac{t^2}{2} + c_1$

- Solve for y

  $y = -1 + e^{\frac{t^2}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=t+t*y(t),y(t), singsol=all)
```

$$y(t) = -1 + e^{\frac{t^2}{2}}c_1$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 24

```
DSolve[y'[t]==t+t*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -1 + c_1 e^{\frac{t^2}{2}}$$
$$y(t) \to -1$$

2.23 problem 16 (viii)

Internal problem ID [12922]
Internal file name [`OUTPUT/11574_Tuesday_November_07_2023_11_27_16_PM_16944620/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 16 (viii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = t^2 - 2}$$

2.23.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int t^2 - 2 \, \mathrm{d}t \\ &= \frac{1}{3}t^3 - 2t + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{3}t^3 - 2t + c_1 \tag{1}$$

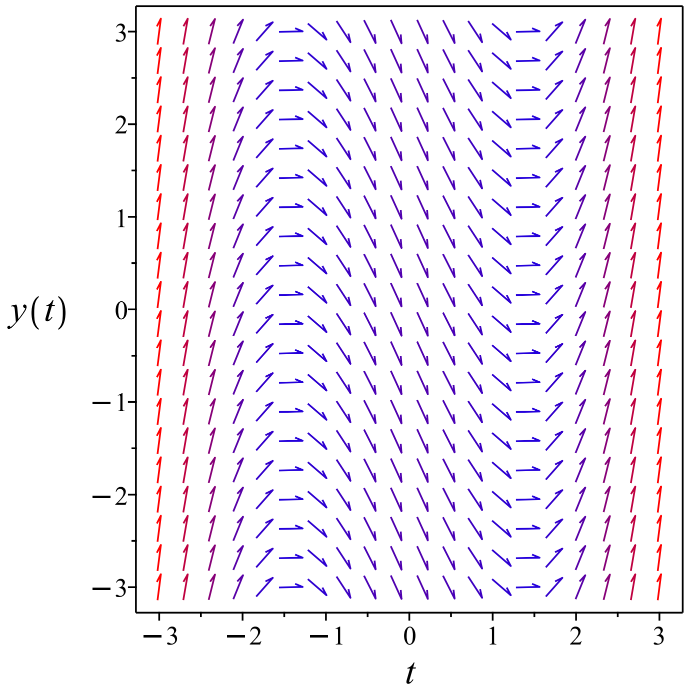


Figure 129: Slope field plot

Verification of solutions

$$y = \frac{1}{3}t^3 - 2t + c_1$$

Verified OK.

2.23.2 Maple step by step solution

Let's solve

$y' = t^2 - 2$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to t

  $\int y' dt = \int (t^2 - 2)\, dt + c_1$

- Evaluate integral

  $y = \frac{1}{3}t^3 - 2t + c_1$

- Solve for y

$$y = \frac{1}{3}t^3 - 2t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=t^2-2,y(t), singsol=all)
```

$$y(t) = \frac{1}{3}t^3 - 2t + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 18

```
DSolve[y'[t]==t^2-2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{t^3}{3} - 2t + c_1$$

2.24 problem 19 a(i)

Internal problem ID [12923]
Internal file name [`OUTPUT/11575_Tuesday_November_07_2023_11_27_16_PM_12198609/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 19 a(i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\theta' + \frac{11\cos(\theta)}{10} = \frac{9}{10}}$$

2.24.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\frac{9}{10} - \frac{11\cos(\theta)}{10}} d\theta = t + c_1$$

$$-\sqrt{10}\,\operatorname{arctanh}\left(\sqrt{10}\,\tan\left(\frac{\theta}{2}\right)\right) = t + c_1$$

Solving for θ gives these solutions

$$\theta_1 = -2\arctan\left(\frac{\tanh\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

Summary
The solution(s) found are the following

$$\theta = -2\arctan\left(\frac{\tanh\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right) \tag{1}$$

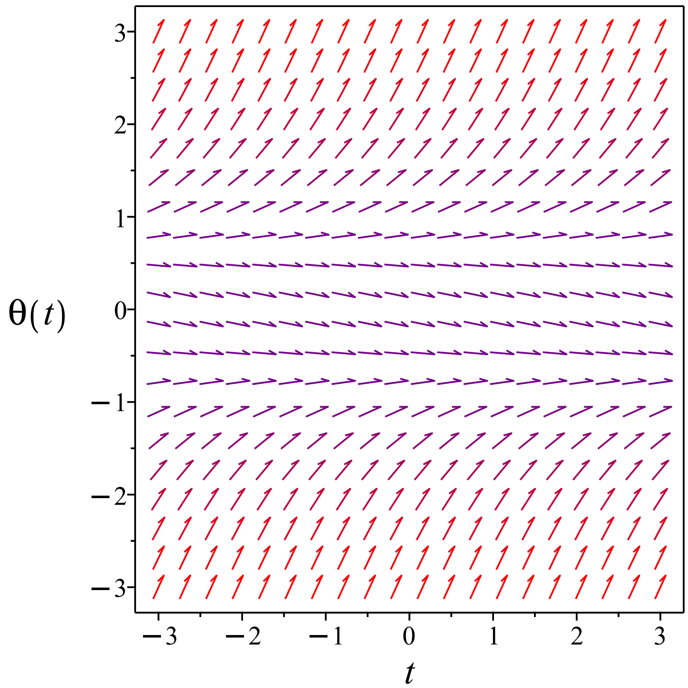


Figure 130: Slope field plot

Verification of solutions

$$\theta = -2\arctan\left(\frac{\tanh\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

Verified OK.

2.24.2 Maple step by step solution

Let's solve

$\theta' + \frac{11\cos(\theta)}{10} = \frac{9}{10}$

- Highest derivative means the order of the ODE is 1

  θ'

- Separate variables

  $\frac{\theta'}{\frac{9}{10}-\frac{11\cos(\theta)}{10}} = 1$

- Integrate both sides with respect to t

  $\int \frac{\theta'}{\frac{9}{10}-\frac{11\cos(\theta)}{10}}dt = \int 1dt + c_1$

- Evaluate integral

  $-\sqrt{10}\,\operatorname{arctanh}\left(\tan\left(\frac{\theta}{2}\right)\sqrt{10}\right) = t + c_1$

- Solve for θ

  $\theta = -2\arctan\left(\frac{\tanh\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(theta(t),t)=1-cos(theta(t))+(1+cos(theta(t)))*(-1/10),theta(t), singsol=all)
```

$$\theta(t) = -2\arctan\left(\frac{\tanh\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

✓ Solution by Mathematica

Time used: 1.026 (sec). Leaf size: 69

```
DSolve[theta'[t]==1-Cos[theta[t]]+(1+Cos[theta[t]])*(-1/10),theta[t],t,IncludeSingularSoluti
```

$$\theta(t) \to -2\arctan\left(\frac{\tanh\left(\frac{t-10c_1}{\sqrt{10}}\right)}{\sqrt{10}}\right)$$
$$\theta(t) \to -\arccos\left(\frac{9}{11}\right)$$
$$\theta(t) \to \arccos\left(\frac{9}{11}\right)$$
$$\theta(t) \to -2\arctan\left(\frac{1}{\sqrt{10}}\right)$$
$$\theta(t) \to 2\arctan\left(\frac{1}{\sqrt{10}}\right)$$

2.25 problem 19 a(ii)

Internal problem ID [12924]
Internal file name [`OUTPUT/11576_Tuesday_November_07_2023_11_27_16_PM_37495739/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 19 a(ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\theta' = 2}$$

2.25.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\theta &= \int 2 \, \mathrm{d}t \\ &= 2t + c_1\end{aligned}$$

Summary
The solution(s) found are the following

$$\theta = 2t + c_1 \tag{1}$$

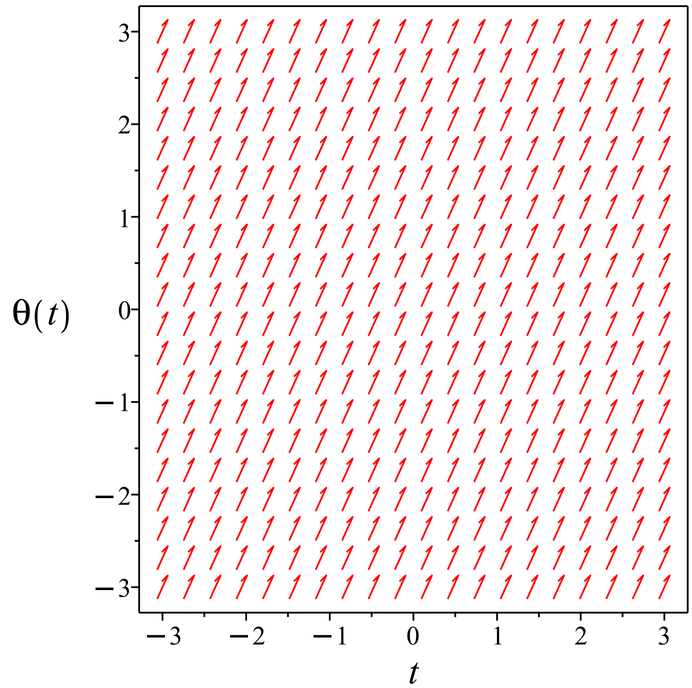


Figure 131: Slope field plot

Verification of solutions

$$\theta = 2t + c_1$$

Verified OK.

2.25.2 Maple step by step solution

Let's solve

$\theta' = 2$

- Highest derivative means the order of the ODE is 1

  θ'

- Integrate both sides with respect to t

  $\int \theta' dt = \int 2dt + c_1$

- Evaluate integral

  $\theta = 2t + c_1$

- Solve for θ

  $\theta = 2t + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(theta(t),t)=1-cos(theta(t))+(1+cos(theta(t))),theta(t), singsol=all)
```

$$\theta(t) = 2t + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 11

```
DSolve[theta'[t]==1-Cos[theta[t]]+(1+Cos[theta[t]]),theta[t],t,IncludeSingularSolutions -> T
```

$$\theta(t) \to 2t + c_1$$

2.26 problem 19 a(iii)

Internal problem ID [12925]
Internal file name [`OUTPUT/11577_Tuesday_November_07_2023_11_27_17_PM_45474179/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 19 a(iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\theta' + \frac{9\cos(\theta)}{10} = \frac{11}{10}}$$

2.26.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\frac{11}{10} - \frac{9\cos(\theta)}{10}} d\theta = t + c_1$$

$$\sqrt{10}\, \arctan\left(\sqrt{10}\, \tan\left(\frac{\theta}{2}\right)\right) = t + c_1$$

Solving for θ gives these solutions

$$\theta_1 = 2\arctan\left(\frac{\tan\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

Summary
The solution(s) found are the following

$$\theta = 2\arctan\left(\frac{\tan\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right) \tag{1}$$

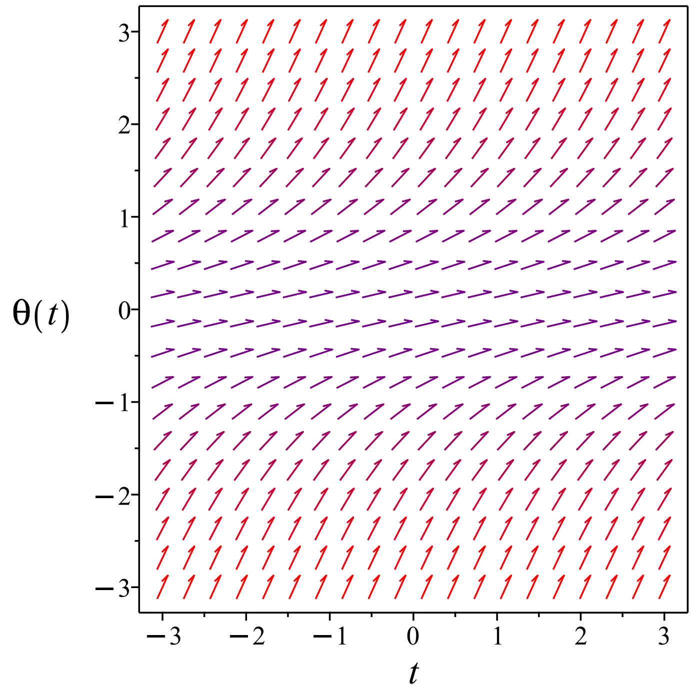


Figure 132: Slope field plot

Verification of solutions

$$\theta = 2\arctan\left(\frac{\tan\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

Verified OK.

2.26.2 Maple step by step solution

Let's solve

$\theta' + \frac{9\cos(\theta)}{10} = \frac{11}{10}$

- Highest derivative means the order of the ODE is 1

  θ'

- Separate variables

  $\frac{\theta'}{\frac{11}{10} - \frac{9\cos(\theta)}{10}} = 1$

- Integrate both sides with respect to t

  $\int \frac{\theta'}{\frac{11}{10} - \frac{9\cos(\theta)}{10}} dt = \int 1 dt + c_1$

- Evaluate integral

  $$\sqrt{10}\, \arctan\left(\tan\left(\frac{\theta}{2}\right)\sqrt{10}\right) = t + c_1$$

- Solve for θ

  $$\theta = 2\arctan\left(\frac{\tan\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

<u>Maple trace</u>

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(theta(t),t)=1-cos(theta(t))+(1+cos(theta(t)))*(1/10),theta(t), singsol=all)
```

$$\theta(t) = 2\arctan\left(\frac{\tan\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

✓ Solution by Mathematica

Time used: 10.277 (sec). Leaf size: 55

```
DSolve[theta'[t]==1-Cos[theta[t]]+(1+Cos[theta[t]])*(1/10),theta[t],t,IncludeSingularSolutio
```

$$\theta(t) \to 2\arctan\left(\frac{\tan\left(\frac{t-10c_1}{\sqrt{10}}\right)}{\sqrt{10}}\right)$$
$$\theta(t) \to -\arccos\left(\frac{11}{9}\right)$$
$$\theta(t) \to \arccos\left(\frac{11}{9}\right)$$
$$\theta(t) \to \text{Interval}[\{-\pi, \pi\}]$$

2.27 problem 20

Internal problem ID [12926]
Internal file name [`OUTPUT/11578_Tuesday_November_07_2023_11_27_17_PM_9986808/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{v' + \frac{v}{RC} = 0}$$

2.27.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{RC}{v} dv = \int dt$$
$$-RC \ln(v) = t + c_1$$

Raising both side to exponential gives

$$e^{-RC \ln(v)} = e^{t+c_1}$$

Which simplifies to

$$v^{-RC} = c_2 e^t$$

<u>Summary</u>
The solution(s) found are the following

$$v = \left(c_2 e^t\right)^{-\frac{1}{RC}} \tag{1}$$

Verification of solutions

$$v = \left(c_2 \mathrm{e}^{t}\right)^{-\frac{1}{RC}}$$

Verified OK.

2.27.2 Maple step by step solution

Let's solve

$v' + \frac{v}{RC} = 0$

- Highest derivative means the order of the ODE is 1

  v'

- Separate variables

  $\frac{v'}{v} = -\frac{1}{RC}$

- Integrate both sides with respect to t

  $\int \frac{v'}{v} dt = \int -\frac{1}{RC} dt + c_1$

- Evaluate integral

  $\ln(v) = -\frac{t}{RC} + c_1$

- Solve for v

  $v = \mathrm{e}^{\frac{c_1 RC - t}{RC}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(v(t),t)=-v(t)/(R*C),v(t), singsol=all)
```

$$v(t) = c_1 e^{-\frac{t}{RC}}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 24

```
DSolve[v'[t]==-v[t]/(r*c),v[t],t,IncludeSingularSolutions -> True]
```

$$v(t) \to c_1 e^{-\frac{t}{cr}}$$
$$v(t) \to 0$$

2.28 problem 21

Internal problem ID [12927]
Internal file name [`OUTPUT/11579_Tuesday_November_07_2023_11_27_18_PM_8602688/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{v' - \frac{K - v}{RC} = 0}$$

2.28.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{RC}{K - v} dv = \int dt$$
$$-RC \ln(K - v) = t + c_1$$

Raising both side to exponential gives

$$e^{-RC \ln(K-v)} = e^{t+c_1}$$

Which simplifies to

$$(K - v)^{-RC} = c_2 e^t$$

Summary
The solution(s) found are the following

$$v = -(c_2 e^t)^{-\frac{1}{RC}} + K \tag{1}$$

Verification of solutions

$$v = -\left(c_2 e^{t}\right)^{-\frac{1}{RC}} + K$$

Verified OK.

2.28.2 Maple step by step solution

Let's solve

$v' - \frac{K-v}{RC} = 0$

- Highest derivative means the order of the ODE is 1

  v'

- Separate variables

  $\frac{v'}{K-v} = \frac{1}{RC}$

- Integrate both sides with respect to t

  $\int \frac{v'}{K-v} dt = \int \frac{1}{RC} dt + c_1$

- Evaluate integral

  $-\ln(K-v) = \frac{t}{RC} + c_1$

- Solve for v

  $v = -e^{-\frac{c_1 RC + t}{RC}} + K$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(v(t),t)=(K-v(t))/(R*C),v(t), singsol=all)
```

$$v(t) = K + c_1 e^{-\frac{t}{RC}}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 26

```
DSolve[v'[t]==(k-v[t])/(r*c),v[t],t,IncludeSingularSolutions -> True]
```

$$v(t) \to k + c_1 e^{-\frac{t}{cr}}$$
$$v(t) \to k$$

2.29 problem 22

Internal problem ID [12928]
Internal file name [`OUTPUT/11580_Tuesday_November_07_2023_11_27_18_PM_28378280/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.3 page 47
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{v' + 2v = 2V(t)}$$

2.29.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$v' + p(t)v = q(t)$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 2V(t) \end{aligned}$$

Hence the ode is

$$v' + 2v = 2V(t)$$

The integrating factor μ is

$$\mu = e^{\int 2dt}$$
$$= e^{2t}$$

The ode becomes

$$\frac{d}{dt}(\mu v) = (\mu)(2V(t))$$
$$\frac{d}{dt}(e^{2t}v) = (e^{2t})(2V(t))$$
$$d(e^{2t}v) = (2V(t)e^{2t})\ dt$$

Integrating gives

$$e^{2t}v = \int 2V(t)e^{2t}\,dt$$
$$e^{2t}v = \int 2V(t)e^{2t}dt + c_1$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$v = e^{-2t}\left(\int 2V(t)e^{2t}dt\right) + c_1e^{-2t}$$

which simplifies to

$$v = e^{-2t}\left(2\left(\int V(t)e^{2t}dt\right) + c_1\right)$$

Summary
The solution(s) found are the following

$$v = e^{-2t}\left(2\left(\int V(t)e^{2t}dt\right) + c_1\right) \tag{1}$$

Verification of solutions

$$v = e^{-2t}\left(2\left(\int V(t)e^{2t}dt\right) + c_1\right)$$

Verified OK.

2.29.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
v' &= 2V(t) - 2v \\
v' &= \omega(t, v)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_v - \xi_t) - \omega^2 \xi_v - \omega_t \xi - \omega_v \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 115: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,v) &= 0 \\
\eta(t,v) &= \mathrm{e}^{-2t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,v) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dv}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial v}\right)S(t,v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-2t}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{2t} v
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,v)S_v}{R_t + \omega(t,v)R_v} \tag{2}
$$

Where in the above R_t, R_v, S_t, S_v are all partial derivatives and $\omega(t,v)$ is the right hand side of the original ode given by

$$
\omega(t,v) = 2V(t) - 2v
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_v &= 0 \\
S_t &= 2\,\mathrm{e}^{2t} v \\
S_v &= \mathrm{e}^{2t}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2V(t)\,\mathrm{e}^{2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2V(R)\,\mathrm{e}^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int 2V(R)\,\mathrm{e}^{2R} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, v coordinates. This results in

$$\mathrm{e}^{2t} v = \int 2V(t)\,\mathrm{e}^{2t} dt + c_1$$

Which simplifies to

$$\mathrm{e}^{2t} v = \int 2V(t)\,\mathrm{e}^{2t} dt + c_1$$

Which gives

$$v = \left(\int 2V(t)\,\mathrm{e}^{2t} dt + c_1\right) \mathrm{e}^{-2t}$$

Summary

The solution(s) found are the following

$$v = \left(\int 2V(t)\,\mathrm{e}^{2t} dt + c_1\right) \mathrm{e}^{-2t} \tag{1}$$

Verification of solutions

$$v = \left(\int 2V(t)\,\mathrm{e}^{2t} dt + c_1\right) \mathrm{e}^{-2t}$$

Verified OK.

2.29.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,v)\,\mathrm{d}t + N(t,v)\,\mathrm{d}v = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}v &= (2V(t) - 2v)\,\mathrm{d}t\\ (-2V(t) + 2v)\,\mathrm{d}t + \mathrm{d}v &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,v) &= -2V(t) + 2v\\ N(t,v) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}(-2V(t) + 2v)\\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial v} - \frac{\partial N}{\partial t}\right)\\ &= 1((2) - (0))\\ &= 2\end{aligned}$$

Since A does not depend on v, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}t}\\ &= e^{\int 2 \,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2t}\\ &= \mathrm{e}^{2t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{2t}(-2V(t) + 2v)\\ &= -2(V(t) - v)\,\mathrm{e}^{2t}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= e^{2t}(1) \\
&= e^{2t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{dv}{dt} &= 0 \\
\left(-2(V(t) - v)\, e^{2t}\right) + \left(e^{2t}\right) \frac{dv}{dt} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, v)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial v} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\, dt &= \int \overline{M}\, dt \\
\int \frac{\partial \phi}{\partial t}\, dt &= \int -2(V(t) - v)\, e^{2t}\, dt
\end{aligned}
$$

$$
\phi = \int^{t} -2(V(\_a) - v)\, e^{2-a} d\_a + f(v) \tag{3}
$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both t and v. Taking derivative of equation (3) w.r.t v gives

$$
\frac{\partial \phi}{\partial v} = e^{2t} + f'(v) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = e^{2t}$. Therefore equation (4) becomes

$$
e^{2t} = e^{2t} + f'(v) \tag{5}
$$

Solving equation (5) for $f'(v)$ gives

$$
f'(v) = 0
$$

Therefore

$$f(v) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(v)$ into equation (3) gives ϕ

$$\phi = \int^t -2(V(\_a) - v)\, e^{2\_a} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^t -2(V(\_a) - v)\, e^{2\_a} d\_a$$

Summary

The solution(s) found are the following

$$\int^t -2(V(\_a) - v)\, e^{2\_a} d\_a = c_1 \tag{1}$$

Verification of solutions

$$\int^t -2(V(\_a) - v)\, e^{2\_a} d\_a = c_1$$

Verified OK.

2.29.4 Maple step by step solution

Let's solve

$v' + 2v = 2V(t)$

- Highest derivative means the order of the ODE is 1

  v'

- Isolate the derivative

  $v' = 2V(t) - 2v$

- Group terms with v on the lhs of the ODE and the rest on the rhs of the ODE

  $v' + 2v = 2V(t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(v' + 2v) = 2\mu(t)\,V(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, v)$

  $\mu(t)\,(v' + 2v) = \mu'(t)\, v + \mu(t)\, v'$

- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{2t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, v)\right) dt = \int 2\mu(t)\, V(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, v = \int 2\mu(t)\, V(t)\, dt + c_1$

- Solve for v

  $v = \frac{\int 2\mu(t)V(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{2t}$

  $v = \frac{\int 2V(t)\mathrm{e}^{2t}dt + c_1}{\mathrm{e}^{2t}}$

- Simplify

  $v = \mathrm{e}^{-2t}\left(2\left(\int V(t)\, \mathrm{e}^{2t}dt\right) + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(v(t),t)=(V(t)-v(t))/(1/2*1),v(t), singsol=all)
```

$$v(t) = \left(2\left(\int V(t)\, \mathrm{e}^{2t}dt\right) + c_1\right)\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 32

```
DSolve[v'[t]==(V[t]-v[t])/(1/2*1),v[t],t,IncludeSingularSolutions -> True]
```

$$v(t) \to e^{-2t} \left(\int_1^t 2e^{2K[1]} V(K[1]) dK[1] + c_1 \right)$$

3 Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61

3.1 problem 1

Internal problem ID [12929]
Internal file name [`OUTPUT/11581_Tuesday_November_07_2023_11_27_19_PM_97849223/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 2y = 1}$$

With initial conditions

$$[y(0) = 3]$$

3.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 1 \end{aligned}$$

Hence the ode is

$$y' - 2y = 1$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.1.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{2y+1} dy = \int dt$$
$$\frac{\ln(2y+1)}{2} = t + c_1$$

Raising both side to exponential gives

$$\sqrt{2y+1} = e^{t+c_1}$$

Which simplifies to

$$\sqrt{2y+1} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{c_2^2}{2} - \frac{1}{2}$$

$$c_2 = -\sqrt{7}$$

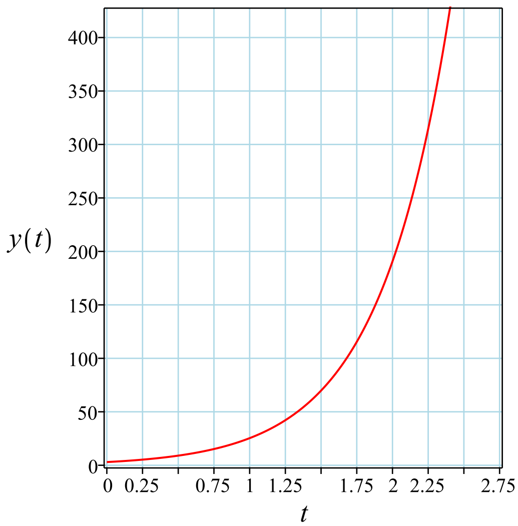
Substituting c_2 found above in the general solution gives

$$y = \frac{7\,e^{2t}}{2} - \frac{1}{2}$$

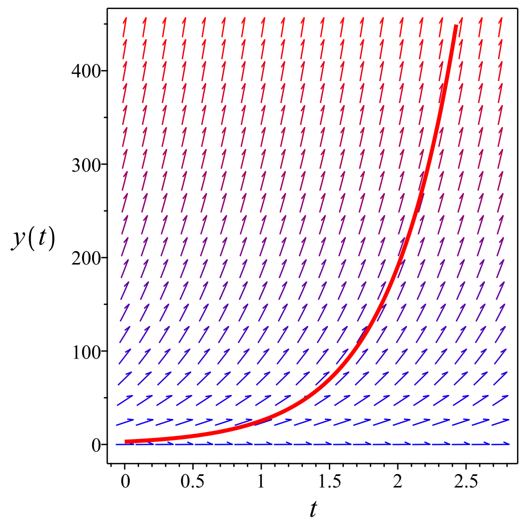
Summary

The solution(s) found are the following

$$y = \frac{7\,e^{2t}}{2} - \frac{1}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{7\,\mathrm{e}^{2t}}{2} - \frac{1}{2}$$

Verified OK.

3.1.3 Maple step by step solution

Let's solve

$[y' - 2y = 1, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y+1} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{2y+1} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{\ln(2y+1)}{2} = t + c_1$

- Solve for y

$y = -\frac{1}{2} + \frac{e^{2t+2c_1}}{2}$

- Use initial condition $y(0) = 3$

  $3 = -\frac{1}{2} + \frac{e^{2c_1}}{2}$

- Solve for c_1

  $c_1 = \frac{\ln(7)}{2}$

- Substitute $c_1 = \frac{\ln(7)}{2}$ into general solution and simplify

  $y = \frac{7\,e^{2t}}{2} - \frac{1}{2}$

- Solution to the IVP

  $y = \frac{7\,e^{2t}}{2} - \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([diff(y(t),t)=2*y(t)+1,y(0) = 3],y(t), singsol=all)
```

$$y(t) = -\frac{1}{2} + \frac{7\,e^{2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 18

```
DSolve[{y'[t]==2*y[t]+1,{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}\left(7e^{2t} - 1\right)$$

3.2 problem 2

Internal problem ID [12930]
Internal file name [`OUTPUT/11582_Tuesday_November_07_2023_11_27_20_PM_36050666/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{y' + y^2 = t}$$

With initial conditions

$$[y(0) = 1]$$

3.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= -y^2 + t \end{aligned}$$

The t domain of $f(t, y)$ when $y = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-y^2 + t) \\ &= -2y\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.2.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t, y) \\ &= -y^2 + t\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2 + t$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = t$, $f_1(t) = 0$ and $f_2(t) = -1$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned}f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= t\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(t) + tu(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 \operatorname{AiryAi}(t) + c_2 \operatorname{AiryBi}(t)$$

The above shows that

$$u'(t) = c_1 \operatorname{AiryAi}(1, t) + c_2 \operatorname{AiryBi}(1, t)$$

Using the above in (1) gives the solution

$$y = \frac{c_1 \operatorname{AiryAi}(1, t) + c_2 \operatorname{AiryBi}(1, t)}{c_1 \operatorname{AiryAi}(t) + c_2 \operatorname{AiryBi}(t)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 \operatorname{AiryAi}(1, t) + \operatorname{AiryBi}(1, t)}{c_3 \operatorname{AiryAi}(t) + \operatorname{AiryBi}(t)}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{3\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} - 3\Gamma\left(\frac{2}{3}\right)^2 c_3 3^{\frac{1}{6}}}{2\, 3^{\frac{5}{6}}\pi + 2\pi c_3 3^{\frac{1}{3}}}$$

$$c_3 = \frac{-2\, 3^{\frac{5}{6}}\pi + 3\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}}}{3\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} + 2\pi 3^{\frac{1}{3}}}$$

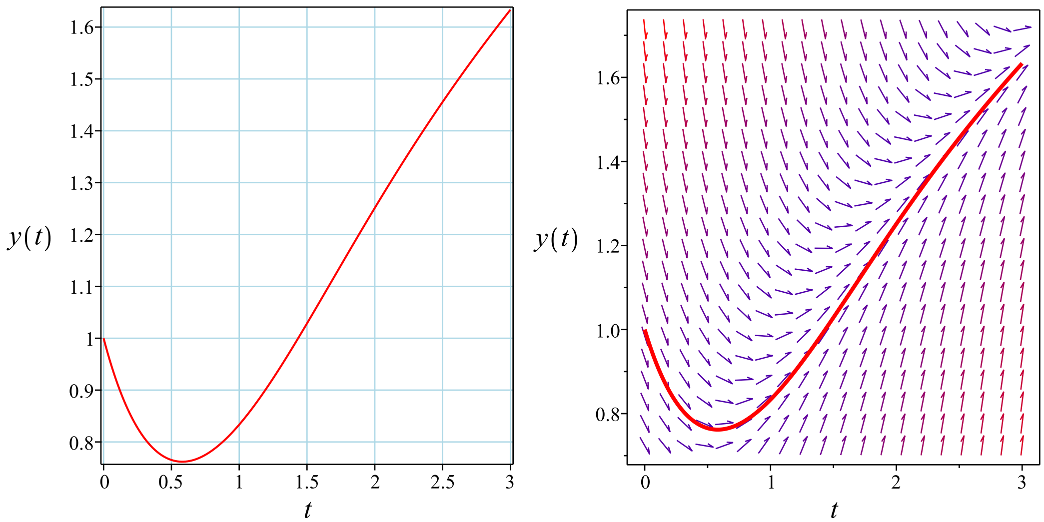
Substituting c_3 found above in the general solution gives

$$y = \frac{-2 \operatorname{AiryAi}(1, t)\, \pi 3^{\frac{5}{6}} + 3 \operatorname{AiryAi}(1, t)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} + 3 \operatorname{AiryBi}(1, t)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} + 2 \operatorname{AiryBi}(1, t)\, \pi 3^{\frac{1}{3}}}{-2 \operatorname{AiryAi}(t)\, \pi 3^{\frac{5}{6}} + 3 \operatorname{AiryAi}(t)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} + 3 \operatorname{AiryBi}(t)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} + 2 \operatorname{AiryBi}(t)\, \pi 3^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{-2 \operatorname{AiryAi}(1, t)\, \pi 3^{\frac{5}{6}} + 3 \operatorname{AiryAi}(1, t)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} + 3 \operatorname{AiryBi}(1, t)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} + 2 \operatorname{AiryBi}(1, t)\, \pi 3^{\frac{1}{3}}}{-2 \operatorname{AiryAi}(t)\, \pi 3^{\frac{5}{6}} + 3 \operatorname{AiryAi}(t)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} + 3 \operatorname{AiryBi}(t)\, \Gamma\left(\frac{2}{3}\right)^2 3^{\frac{1}{6}} + 2 \operatorname{AiryBi}(t)\, \pi 3^{\frac{1}{3}}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-2\,\mathrm{AiryAi}\left(1,t\right)\pi 3^{\frac{5}{6}}+3\,\mathrm{AiryAi}\left(1,t\right)\Gamma\left(\frac{2}{3}\right)^{2}3^{\frac{2}{3}}+3\,\mathrm{AiryBi}\left(1,t\right)\Gamma\left(\frac{2}{3}\right)^{2}3^{\frac{1}{6}}+2\,\mathrm{AiryBi}\left(1,t\right)\pi 3^{\frac{1}{3}}}{-2\,\mathrm{AiryAi}\left(t\right)\pi 3^{\frac{5}{6}}+3\,\mathrm{AiryAi}\left(t\right)\Gamma\left(\frac{2}{3}\right)^{2}3^{\frac{2}{3}}+3\,\mathrm{AiryBi}\left(t\right)\Gamma\left(\frac{2}{3}\right)^{2}3^{\frac{1}{6}}+2\,\mathrm{AiryBi}\left(t\right)\pi 3^{\frac{1}{3}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 89

```
dsolve([diff(y(t),t)=t-y(t)^2,y(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{2\,\mathrm{AiryAi}\,(1,t)\,\pi 3^{\frac{5}{6}} - 3\,\mathrm{AiryAi}\,(1,t)\,\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} - 3\,\mathrm{AiryBi}\,(1,t)\,3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)^2 - 2\,\mathrm{AiryBi}\,(1,t)\,\pi 3^{\frac{1}{3}}}{2\,\mathrm{AiryAi}\,(t)\,\pi 3^{\frac{5}{6}} - 3\,\mathrm{AiryAi}\,(t)\,\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} - 3\,\mathrm{AiryBi}\,(t)\,3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)^2 - 2\,\mathrm{AiryBi}\,(t)\,\pi 3^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 11.27 (sec). Leaf size: 163

```
DSolve[{y'[t]==t-y[t]^2,{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{2it^{3/2}\,\mathrm{Gamma}\left(\frac{1}{3}\right)\mathrm{BesselJ}\left(-\frac{2}{3},\frac{2}{3}it^{3/2}\right) + \sqrt[3]{-3}\,\mathrm{Gamma}\left(\frac{2}{3}\right)\left(it^{3/2}\,\mathrm{BesselJ}\left(-\frac{4}{3},\frac{2}{3}it^{3/2}\right) - it^{3/2}\,\mathrm{BesselJ}\left(\frac{2}{3}\ldots\right.\right.}{2t\left(\sqrt[3]{-3}\,\mathrm{Gamma}\left(\frac{2}{3}\right)\mathrm{BesselJ}\left(-\frac{1}{3},\frac{2}{3}it^{3/2}\right) + \mathrm{Gamma}\left(\frac{1}{3}\right)\mathrm{BesselJ}\left(\frac{1}{3},\frac{2}{3}it^{3/2}\ldots\right.\right.}$$

3.3 problem 3

Internal problem ID [12931]
Internal file name [`OUTPUT/11583_Tuesday_November_07_2023_11_27_21_PM_53982265/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[[_Riccati, _special]]`

$$\boxed{y' - y^2 = -4t}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

3.3.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4t \end{aligned}$$

The t domain of $f(t, y)$ when $y = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - 4t) \\ &= 2y\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

3.3.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t, y) \\ &= y^2 - 4t\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - 4t$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = -4t$, $f_1(t) = 0$ and $f_2(t) = 1$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2) u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned}f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -4t\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(t) - 4tu(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 \operatorname{AiryAi}\left(2^{\frac{2}{3}} t\right) + c_2 \operatorname{AiryBi}\left(2^{\frac{2}{3}} t\right)$$

The above shows that

$$u'(t) = 2^{\frac{2}{3}} \left(\operatorname{AiryBi}\left(1, 2^{\frac{2}{3}} t\right) c_2 + \operatorname{AiryAi}\left(1, 2^{\frac{2}{3}} t\right) c_1\right)$$

Using the above in (1) gives the solution

$$y = -\frac{2^{\frac{2}{3}} \left(\operatorname{AiryBi}\left(1, 2^{\frac{2}{3}} t\right) c_2 + \operatorname{AiryAi}\left(1, 2^{\frac{2}{3}} t\right) c_1\right)}{c_1 \operatorname{AiryAi}\left(2^{\frac{2}{3}} t\right) + c_2 \operatorname{AiryBi}\left(2^{\frac{2}{3}} t\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2^{\frac{2}{3}} \left(\operatorname{AiryBi}\left(1, 2^{\frac{2}{3}} t\right) + \operatorname{AiryAi}\left(1, 2^{\frac{2}{3}} t\right) c_3\right)}{c_3 \operatorname{AiryAi}\left(2^{\frac{2}{3}} t\right) + \operatorname{AiryBi}\left(2^{\frac{2}{3}} t\right)}$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{-3\Gamma\left(\frac{2}{3}\right)^2 2^{\frac{2}{3}} 3^{\frac{2}{3}} + 3\Gamma\left(\frac{2}{3}\right)^2 2^{\frac{2}{3}} c_3 3^{\frac{1}{6}}}{2\, 3^{\frac{5}{6}} \pi + 2\pi c_3 3^{\frac{1}{3}}}$$

$$c_3 = -\frac{3\Gamma\left(\frac{2}{3}\right)^2 2^{\frac{2}{3}} 3^{\frac{2}{3}} + 3^{\frac{5}{6}} \pi}{-3\, 3^{\frac{1}{6}} 2^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^2 + \pi 3^{\frac{1}{3}}}$$

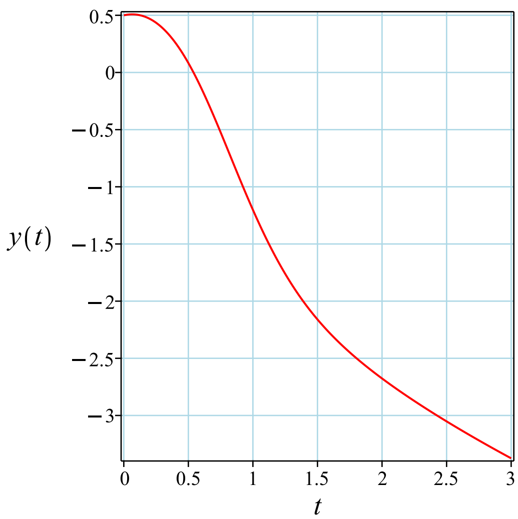
Substituting c_3 found above in the general solution gives

$$y = \frac{-2^{\frac{2}{3}} 3^{\frac{5}{6}} \operatorname{AiryAi}\left(1, 2^{\frac{2}{3}} t\right) \pi - 3\, 2^{\frac{2}{3}} \operatorname{AiryAi}\left(1, 2^{\frac{2}{3}} t\right) 6^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^2 + 2^{\frac{2}{3}} \operatorname{AiryBi}\left(1, 2^{\frac{2}{3}} t\right) 3^{\frac{1}{3}} \pi - 6\, 2^{\frac{1}{3}} \operatorname{AiryBi}\left(1, \right.}{\operatorname{AiryAi}\left(2^{\frac{2}{3}} t\right) 3^{\frac{5}{6}} \pi + 3 \operatorname{AiryBi}\left(2^{\frac{2}{3}} t\right) 3^{\frac{1}{6}} 2^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^2 + 3 \operatorname{AiryAi}\left(2^{\frac{2}{3}} t\right) 6^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^2 - \operatorname{AiryBi}\left(2^{\frac{2}{3}} t\right)}$$

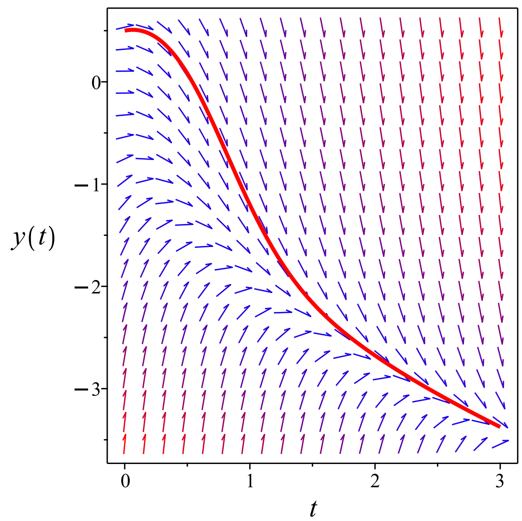
Summary

The solution(s) found are the following

$$y = \frac{-2^{\frac{2}{3}} 3^{\frac{5}{6}} \operatorname{AiryAi}\left(1, 2^{\frac{2}{3}} t\right) \pi - 3\, 2^{\frac{2}{3}} \operatorname{AiryAi}\left(1, 2^{\frac{2}{3}} t\right) 6^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^{2} + 2^{\frac{2}{3}} \operatorname{AiryBi}\left(1, 2^{\frac{2}{3}} t\right) 3^{\frac{1}{3}} \pi - 6\, 2^{\frac{1}{3}} \operatorname{AiryBi}\left(1, 2^{\frac{2}{3}}\right.}{\operatorname{AiryAi}\left(2^{\frac{2}{3}} t\right) 3^{\frac{5}{6}} \pi + 3 \operatorname{AiryBi}\left(2^{\frac{2}{3}} t\right) 3^{\frac{1}{6}} 2^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^{2} + 3 \operatorname{AiryAi}\left(2^{\frac{2}{3}} t\right) 6^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^{2} - \operatorname{AiryBi}\left(2^{\frac{2}{3}} t\right) 3^{\frac{1}{3}}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-2^{\frac{2}{3}} 3^{\frac{5}{6}} \operatorname{AiryAi}\left(1, 2^{\frac{2}{3}} t\right) \pi - 3\, 2^{\frac{2}{3}} \operatorname{AiryAi}\left(1, 2^{\frac{2}{3}} t\right) 6^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^{2} + 2^{\frac{2}{3}} \operatorname{AiryBi}\left(1, 2^{\frac{2}{3}} t\right) 3^{\frac{1}{3}} \pi - 6\, 2^{\frac{1}{3}} \operatorname{AiryBi}\left(1, 2^{\frac{2}{3}}\right.}{\operatorname{AiryAi}\left(2^{\frac{2}{3}} t\right) 3^{\frac{5}{6}} \pi + 3 \operatorname{AiryBi}\left(2^{\frac{2}{3}} t\right) 3^{\frac{1}{6}} 2^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^{2} + 3 \operatorname{AiryAi}\left(2^{\frac{2}{3}} t\right) 6^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)^{2} - \operatorname{AiryBi}\left(2^{\frac{2}{3}} t\right) 3^{\frac{1}{3}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 115

```
dsolve([diff(y(t),t)=y(t)^2-4*t,y(0) = 1/2],y(t), singsol=all)
```

$$y(t) = \frac{2^{\frac{2}{3}}\left(\left(3\,2^{\frac{2}{3}}3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)^{2}-\pi 3^{\frac{1}{3}}\right)\operatorname{AiryBi}\left(1,2^{\frac{2}{3}}t\right)+\operatorname{AiryAi}\left(1,2^{\frac{2}{3}}t\right)\left(3\Gamma\left(\frac{2}{3}\right)^{2}6^{\frac{2}{3}}+3^{\frac{5}{6}}\pi\right)\right)}{\left(-3\Gamma\left(\frac{2}{3}\right)^{2}6^{\frac{2}{3}}-3^{\frac{5}{6}}\pi\right)\operatorname{AiryAi}\left(2^{\frac{2}{3}}t\right)+\operatorname{AiryBi}\left(2^{\frac{2}{3}}t\right)\left(-3\,2^{\frac{2}{3}}3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)^{2}+\pi 3^{\frac{1}{3}}\right)}$$

✓ Solution by Mathematica

Time used: 10.151 (sec). Leaf size: 193

```
DSolve[{y'[t]==y[t]^2-4*t,{y[0]==1/2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{4it^{3/2}\,\text{Gamma}\left(\frac{1}{3}\right)\text{BesselJ}\left(-\frac{2}{3},\frac{4}{3}it^{3/2}\right)+2^{2/3}\sqrt[3]{3}(\sqrt{3}-i)\,\text{Gamma}\left(\frac{2}{3}\right)\left(2t^{3/2}\,\text{BesselJ}\left(-\frac{4}{3},\frac{4}{3}it^{3/2}\right)-2\cdots\right.}{2t\left(2^{2/3}\sqrt[3]{3}\left(-1-i\sqrt{3}\right)\text{Gamma}\left(\frac{2}{3}\right)\text{BesselJ}\left(-\frac{1}{3},\frac{4}{3}it^{3/2}\right)+\text{Gamma}\left(\frac{1}{3}\right)\cdots\right.}$$

3.4 problem 4

Internal problem ID [12932]
Internal file name [`OUTPUT/11584_Tuesday_November_07_2023_11_27_22_PM_23130949/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sin(y) = 0}$$

With initial conditions

$$[y(0) = 1]$$

3.4.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \sin(y) \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(\sin(y)) \\ &= \cos(y) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.4.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\sin(y)} dy = \int dt$$
$$\ln(\csc(y) - \cot(y)) = t + c_1$$

Raising both side to exponential gives

$$\csc(y) - \cot(y) = e^{t+c_1}$$

Which simplifies to

$$\csc(y) - \cot(y) = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{-\cot(1)\sin(1) + 1}{\sin(1)} = c_2$$

$$c_2 = -\cot(1) + \csc(1)$$

Substituting c_2 found above in the general solution gives

$$\csc(y) - \cot(y) = e^t \csc(1) - e^t \cot(1)$$

Summary
The solution(s) found are the following

$$\csc(y) - \cot(y) = (-\cot(1) + \csc(1)) e^t \tag{1}$$

Verification of solutions

$$\csc(y) - \cot(y) = (-\cot(1) + \csc(1)) e^t$$

Verified OK.

3.4.3 Maple step by step solution

Let's solve

$[y' - \sin(y) = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sin(y)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\sin(y)} dt = \int 1 dt + c_1$

- Evaluate integral

  $\ln(\csc(y) - \cot(y)) = t + c_1$

- Solve for y

  $y = \arctan\left(\frac{2\,e^{t+c_1}}{(e^{t+c_1})^2+1}, -\frac{(e^{t+c_1})^2-1}{(e^{t+c_1})^2+1}\right)$

- Use initial condition $y(0) = 1$

  $1 = \arctan\left(\frac{2\,e^{c_1}}{(e^{c_1})^2+1}, -\frac{(e^{c_1})^2-1}{(e^{c_1})^2+1}\right)$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 1.594 (sec). Leaf size: 63

```
dsolve([diff(y(t),t)=sin(y(t)),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \arctan\left(-\frac{2\,\mathrm{e}^{t}\sin(1)}{(-1+\cos(1))\,\mathrm{e}^{2t}-\cos(1)-1}, \frac{(1-\cos(1))\,\mathrm{e}^{2t}-\cos(1)-1}{(-1+\cos(1))\,\mathrm{e}^{2t}-\cos(1)-1}\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 16

```
DSolve[{y'[t]==Sin[y[t]],{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \arccos(-\tanh(t - \operatorname{arctanh}(\cos(1))))$$

3.5 problem 5

Internal problem ID [12933]
Internal file name [`OUTPUT/11585_Tuesday_November_07_2023_11_27_29_PM_60020797/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{w' - (3 - w)(w + 1) = 0}$$

With initial conditions

$$[w(0) = 4]$$

3.5.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} w' &= f(t, w) \\ &= -(1 + w)(w - 3) \end{aligned}$$

The w domain of $f(t, w)$ when $t = 0$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = 4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial w} &= \frac{\partial}{\partial w}(-(1 + w)(w - 3)) \\ &= 2 - 2w \end{aligned}$$

The w domain of $\frac{\partial f}{\partial w}$ when $t=0$ is

$$
\{-\infty < w < \infty\}
$$

And the point $w_0 = 4$ is inside this domain. Therefore solution exists and is unique.

3.5.2 Solving as quadrature ode

Integrating both sides gives

$$
\begin{aligned}
\int -\frac{1}{(1+w)(w-3)} dw &= \int dt \\
-\frac{\ln(w-3)}{4} + \frac{\ln(1+w)}{4} &= t + c_1
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\left(-\frac{1}{4}\right)(\ln(w-3) - \ln(1+w)) &= t + c_1 \\
\ln(w-3) - \ln(1+w) &= (-4)(t + c_1) \\
&= -4t - 4c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(w-3)-\ln(1+w)} = -4c_1 e^{-4t}
$$

Which simplifies to

$$
\frac{w-3}{1+w} = c_2 e^{-4t}
$$

Initial conditions are used to solve for c_2. Substituting $t=0$ and $w=4$ in the above solution gives an equation to solve for the constant of integration.

$$
4 = \frac{-c_2 - 3}{-1 + c_2}
$$

$$
c_2 = \frac{1}{5}
$$

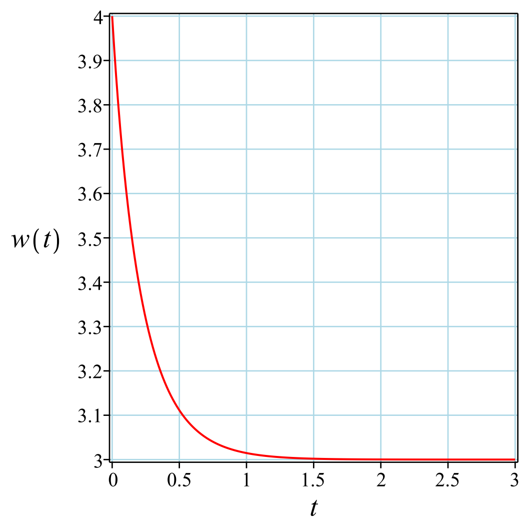
Substituting c_2 found above in the general solution gives

$$
w = \frac{-e^{-4t} - 15}{e^{-4t} - 5}
$$

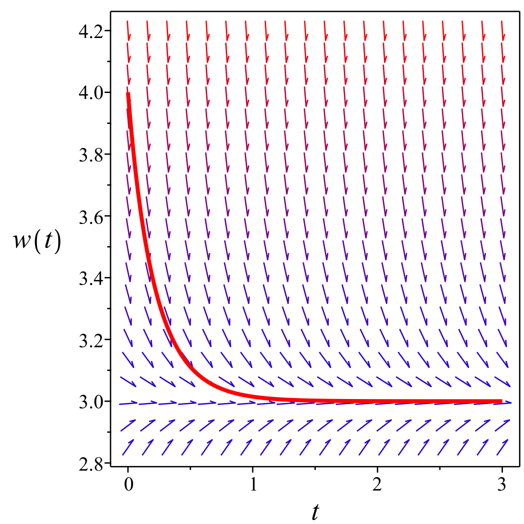
Summary

The solution(s) found are the following

$$w = \frac{-e^{-4t} - 15}{e^{-4t} - 5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$w = \frac{-e^{-4t} - 15}{e^{-4t} - 5}$$

Verified OK.

3.5.3 Maple step by step solution

Let's solve

$[w' - (3 - w)(w + 1) = 0, w(0) = 4]$

- Highest derivative means the order of the ODE is 1

  w'

- Separate variables

  $\frac{w'}{(3-w)(w+1)} = 1$

- Integrate both sides with respect to t

  $\int \frac{w'}{(3-w)(w+1)} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{\ln(w-3)}{4}+\frac{\ln(w+1)}{4}=t+c_1$
- Solve for w

  $w=\frac{3\,\mathrm{e}^{4t+4c_1}+1}{\mathrm{e}^{4t+4c_1}-1}$
- Use initial condition $w(0)=4$

  $4=\frac{3\,\mathrm{e}^{4c_1}+1}{\mathrm{e}^{4c_1}-1}$
- Solve for c_1

  $c_1=\frac{\ln(5)}{4}$
- Substitute $c_1=\frac{\ln(5)}{4}$ into general solution and simplify

  $w=\frac{15\,\mathrm{e}^{4t}+1}{5\,\mathrm{e}^{4t}-1}$
- Solution to the IVP

  $w=\frac{15\,\mathrm{e}^{4t}+1}{5\,\mathrm{e}^{4t}-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 23

```
dsolve([diff(w(t),t)=(3-w(t))*(w(t)+1),w(0) = 4],w(t), singsol=all)
```

$$w(t) = \frac{15\,e^{4t}+1}{-1+5\,e^{4t}}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 26

```
DSolve[{w'[t]==(3-w[t])*(w[t]+1),{w[0]==4}},w[t],t,IncludeSingularSolutions -> True]
```

$$w(t)\to \frac{15e^{4t}+1}{5e^{4t}-1}$$

3.6 problem 6

Internal problem ID [12934]
Internal file name [`OUTPUT/11586_Tuesday_November_07_2023_11_27_30_PM_64553380/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{w' - (3-w)(w+1) = 0}$$

With initial conditions

$$[w(0) = 0]$$

3.6.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} w' &= f(t,w) \\ &= -(1+w)(w-3) \end{aligned}$$

The w domain of $f(t,w)$ when $t=0$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial w} &= \frac{\partial}{\partial w}(-(1+w)(w-3)) \\ &= 2-2w \end{aligned}$$

The w domain of $\frac{\partial f}{\partial w}$ when $t = 0$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.6.2 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{(1+w)(w-3)} dw = \int dt$$
$$-\frac{\ln(w-3)}{4} + \frac{\ln(1+w)}{4} = t + c_1$$

The above can be written as

$$\left(-\frac{1}{4}\right)(\ln(w-3) - \ln(1+w)) = t + c_1$$
$$\ln(w-3) - \ln(1+w) = (-4)(t + c_1)$$
$$= -4t - 4c_1$$

Raising both side to exponential gives

$$e^{\ln(w-3)-\ln(1+w)} = -4c_1 e^{-4t}$$

Which simplifies to

$$\frac{w-3}{1+w} = c_2 e^{-4t}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $w = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-c_2 - 3}{-1 + c_2}$$

$$c_2 = -3$$

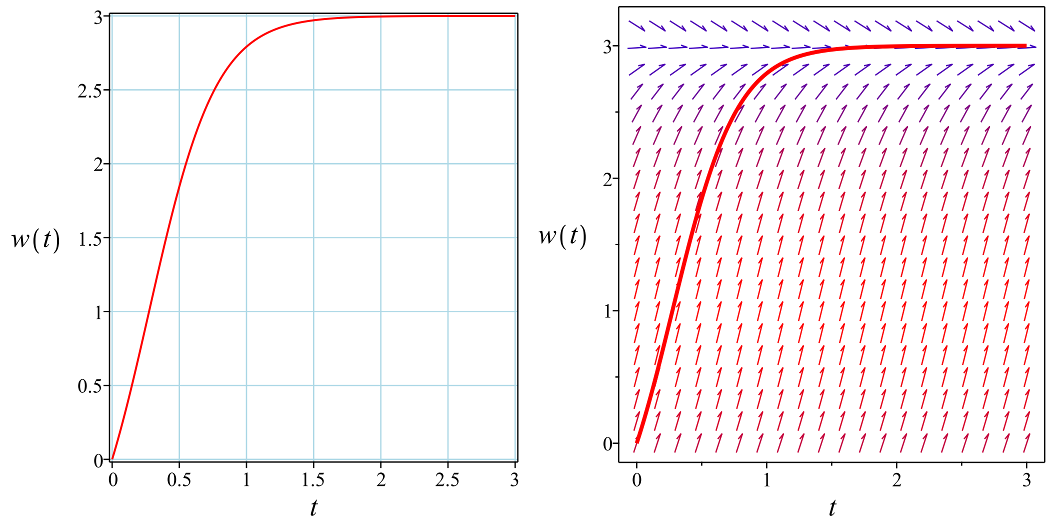
Substituting c_2 found above in the general solution gives

$$w = \frac{-3\,e^{-4t} + 3}{3\,e^{-4t} + 1}$$

Summary

The solution(s) found are the following

$$w = \frac{-3\,\mathrm{e}^{-4t}+3}{3\,\mathrm{e}^{-4t}+1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$w = \frac{-3\,\mathrm{e}^{-4t}+3}{3\,\mathrm{e}^{-4t}+1}$$

Verified OK.

3.6.3 Maple step by step solution

Let's solve

$[w' - (3-w)(w+1) = 0, w(0) = 0]$

- Highest derivative means the order of the ODE is 1

  w'

- Separate variables

  $\frac{w'}{(3-w)(w+1)} = 1$

- Integrate both sides with respect to t

  $\int \frac{w'}{(3-w)(w+1)}dt = \int 1dt + c_1$

- Evaluate integral

  $-\frac{\ln(w-3)}{4} + \frac{\ln(w+1)}{4} = t + c_1$
- Solve for w

  $w = \frac{3\,e^{4t+4c_1}+1}{e^{4t+4c_1}-1}$
- Use initial condition $w(0) = 0$

  $0 = \frac{3\,e^{4c_1}+1}{e^{4c_1}-1}$
- Solve for c_1

  $c_1 = -\frac{\ln(3)}{4} + \frac{I\pi}{4}$
- Substitute $c_1 = -\frac{\ln(3)}{4} + \frac{I\pi}{4}$ into general solution and simplify

  $w = \frac{3\,e^{4t}-3}{e^{4t}+3}$
- Solution to the IVP

  $w = \frac{3\,e^{4t}-3}{e^{4t}+3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 21

```
dsolve([diff(w(t),t)=(3-w(t))*(w(t)+1),w(0) = 0],w(t), singsol=all)
```

$$w(t) = \frac{3\,\mathrm{e}^{4t} - 3}{3 + \mathrm{e}^{4t}}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 23

```
DSolve[{w'[t]==(3-w[t])*(w[t]+1),{w[0]==0}},w[t],t,IncludeSingularSolutions -> True]
```

$$w(t) \to \frac{3(e^{4t} - 1)}{e^{4t} + 3}$$

3.7 problem 7

Internal problem ID [12935]
Internal file name [`OUTPUT/11587_Tuesday_November_07_2023_11_27_31_PM_81315994/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - e^{\frac{2}{y}} = 0}$$

With initial conditions

$$[y(0) = 2]$$

3.7.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= e^{\frac{2}{y}} \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^{\frac{2}{y}}\right) = -\frac{2\,e^{\frac{2}{y}}}{y^2}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

3.7.2 Solving as quadrature ode

Integrating both sides gives

$$\int e^{-\frac{2}{y}} dy = \int dt$$
$$\int^{y} e^{-\frac{2}{\_a}} d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{2} e^{-\frac{2}{\_a}} d\_a = c_1$$

$$c_1 = \int^{2} e^{-\frac{2}{\_a}} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^{y} e^{-\frac{2}{\_a}} d\_a = t + \int^{2} e^{-\frac{2}{\_a}} d\_a$$

Solving for y from the above gives

$$y = \operatorname{RootOf}\left(-\left(\int^{\_Z} e^{-\frac{2}{\_a}} d\_a\right) + t + \int^{2} e^{-\frac{2}{\_a}} d\_a\right)$$

Summary

The solution(s) found are the following

$$y = \operatorname{RootOf}\left(-\left(\int^{\_Z} e^{-\frac{2}{\_a}} d\_a\right) + t + \int^{2} e^{-\frac{2}{\_a}} d\_a\right) \tag{1}$$

Verification of solutions

$$y = \operatorname{RootOf}\left(-\left(\int^{\_Z} \mathrm{e}^{-\frac{2}{\_a}} d\_a\right) + t + \int^{2} \mathrm{e}^{-\frac{2}{\_a}} d\_a\right)$$

Verified OK.

3.7.3 Maple step by step solution

Let's solve

$\left[y' - \mathrm{e}^{\frac{2}{y}} = 0, y(0) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\mathrm{e}^{\frac{2}{y}}} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\mathrm{e}^{\frac{2}{y}}} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{y}{\mathrm{e}^{\frac{2}{y}}} - 2\,\mathrm{Ei}_1\left(\frac{2}{y}\right) = t + c_1$

- Use initial condition $y(0) = 2$

  $\frac{2}{\mathrm{e}} - 2\,\mathrm{Ei}_1(1) = c_1$

- Solve for c_1

  $c_1 = -\frac{2(\mathrm{e}\,\mathrm{Ei}_1(1)-1)}{\mathrm{e}}$

- Substitute $c_1 = -\frac{2(\mathrm{e}\mathrm{Ei}_1(1)-1)}{\mathrm{e}}$ into general solution and simplify

  $y\,\mathrm{e}^{-\frac{2}{y}} - 2\,\mathrm{Ei}_1\left(\frac{2}{y}\right) = -2\,\mathrm{Ei}_1(1) + 2\,\mathrm{e}^{-1} + t$

- Solution to the IVP

  $y\,\mathrm{e}^{-\frac{2}{y}} - 2\,\mathrm{Ei}_1\left(\frac{2}{y}\right) = -2\,\mathrm{Ei}_1(1) + 2\,\mathrm{e}^{-1} + t$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.281 (sec). Leaf size: 37

```
dsolve([diff(y(t),t)=exp(2/y(t)),y(0) = 2],y(t), singsol=all)
```

$$y(t) = -\frac{2}{\mathrm{RootOf}\left(2\_Z\,\mathrm{expIntegral}_1\left(1\right)-2\_Z\,\mathrm{expIntegral}_1\left(-\_Z\right)-2\_Z\,\mathrm{e}^{-1}-t\_Z-2\,\mathrm{e}^{-\_Z}\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y'[t]==Exp[2/y[t]],{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

{}

3.8 problem 8

Internal problem ID [12936]
Internal file name [`OUTPUT/11588_Tuesday_November_07_2023_11_27_32_PM_34093401/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - e^{\frac{2}{y}} = 0}$$

With initial conditions

$$[y(1) = 2]$$

3.8.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= e^{\frac{2}{y}} \end{aligned}$$

The y domain of $f(t, y)$ when $t = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{\frac{2}{y}}\right) \\ &= -\frac{2\,\mathrm{e}^{\frac{2}{y}}}{y^2}\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

3.8.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \mathrm{e}^{-\frac{2}{y}} dy &= \int dt \\ \int^{y} \mathrm{e}^{-\frac{2}{\_a}} d\_a &= t + c_1\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{2} \mathrm{e}^{-\frac{2}{\_a}} d\_a = 1 + c_1$$

$$c_1 = -1 + \int^{2} \mathrm{e}^{-\frac{2}{\_a}} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^{y} \mathrm{e}^{-\frac{2}{\_a}} d\_a = t - 1 + \int^{2} \mathrm{e}^{-\frac{2}{\_a}} d\_a$$

Solving for y from the above gives

$$y = \mathrm{RootOf}\left(-\left(\int^{\_Z} \mathrm{e}^{-\frac{2}{\_a}} d\_a\right) + t - 1 + \int^{2} \mathrm{e}^{-\frac{2}{\_a}} d\_a\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{RootOf}\left(-\left(\int^{\_Z} \mathrm{e}^{-\frac{2}{\_a}} d\_a\right) + t - 1 + \int^{2} \mathrm{e}^{-\frac{2}{\_a}} d\_a\right) \tag{1}$$

Verification of solutions

$$y = \operatorname{RootOf}\left(-\left(\int^{\_Z} e^{-\frac{2}{\_a}} d\_a\right) + t - 1 + \int^{2} e^{-\frac{2}{\_a}} d\_a\right)$$

Verified OK.

3.8.3 Maple step by step solution

Let's solve

$\left[y' - e^{\frac{2}{y}} = 0, y(1) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{e^{\frac{2}{y}}} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{e^{\frac{2}{y}}} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{y}{e^{\frac{2}{y}}} - 2\,\mathrm{Ei}_1\left(\frac{2}{y}\right) = t + c_1$

- Use initial condition $y(1) = 2$

  $\frac{2}{e} - 2\,\mathrm{Ei}_1(1) = 1 + c_1$

- Solve for c_1

  $c_1 = -\frac{2\,e\,\mathrm{Ei}_1(1) + e - 2}{e}$

- Substitute $c_1 = -\frac{2e\mathrm{Ei}_1(1) + e - 2}{e}$ into general solution and simplify

  $y\,e^{-\frac{2}{y}} - 2\,\mathrm{Ei}_1\left(\frac{2}{y}\right) = -2\,\mathrm{Ei}_1(1) - 1 + 2\,e^{-1} + t$

- Solution to the IVP

  $y\,e^{-\frac{2}{y}} - 2\,\mathrm{Ei}_1\left(\frac{2}{y}\right) = -2\,\mathrm{Ei}_1(1) - 1 + 2\,e^{-1} + t$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 38

```
dsolve([diff(y(t),t)=exp(2/y(t)),y(1) = 2],y(t), singsol=all)
```

$$y(t) = -\frac{2}{\operatorname{RootOf}\left(2\_Z \operatorname{expIntegral}_1(1) - 2\_Z \operatorname{expIntegral}_1(-\_Z) - 2\_Z \mathrm{e}^{-1} - t\_Z - 2\,\mathrm{e}^{\_Z} + \_Z\right)}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y'[t]==Exp[2/y[t]],{y[1]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$\{\}$$

3.9 problem 9

Internal problem ID [12937]
Internal file name [`OUTPUT/11589_Tuesday_November_07_2023_11_27_33_PM_74052029/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + y^3 = 0}$$

With initial conditions

$$\left[y(0) = \frac{1}{5}\right]$$

3.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= -y^3 + y^2 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{5}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-y^3 + y^2) \\ &= -3y^2 + 2y \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{5}$ is inside this domain. Therefore solution exists and is unique.

3.9.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^3 + y^2} dy = \int dt$$
$$\int^y \frac{1}{-\_a^3 + \_a^2} d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = \frac{1}{5}$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{\frac{1}{5}} -\frac{1}{\_a^2 (\_a - 1)} d\_a = c_1$$

$$c_1 = -\left(\int^{\frac{1}{5}} \frac{1}{\_a^2 (\_a - 1)} d\_a\right)$$

Substituting c_1 found above in the general solution gives

$$\int^y \frac{1}{-\_a^3 + \_a^2} d\_a = t - \left(\int^{\frac{1}{5}} \frac{1}{\_a^2 (\_a - 1)} d\_a\right)$$

Solving for y from the above gives

$$y = \operatorname{RootOf}\left(\int^{\_Z} \frac{1}{\_a^2 (\_a - 1)} d\_a + t - \left(\int^{\frac{1}{5}} \frac{1}{\_a^2 (\_a - 1)} d\_a\right)\right)$$

Summary

The solution(s) found are the following

$$y = \operatorname{RootOf}\left(\int^{\_Z} \frac{1}{\_a^2 (\_a - 1)} d\_a + t - \left(\int^{\frac{1}{5}} \frac{1}{\_a^2 (\_a - 1)} d\_a\right)\right) \tag{1}$$

Verification of solutions

$$y = \operatorname{RootOf}\left(\int^{\_Z} \frac{1}{\_a^2 (\_a - 1)} d\_a + t - \left(\int^{\frac{1}{5}} \frac{1}{\_a^2 (\_a - 1)} d\_a\right)\right)$$

Verified OK.

3.9.3 Maple step by step solution

Let's solve

$\left[y' - y^2 + y^3 = 0, y(0) = \frac{1}{5}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2 - y^3} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2 - y^3} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\ln(y-1) - \frac{1}{y} + \ln(y) = t + c_1$

- Use initial condition $y(0) = \frac{1}{5}$

  $-\ln\left(\frac{4}{5}\right) - I\pi - 5 - \ln(5) = c_1$

- Solve for c_1

  $c_1 = -\ln\left(\frac{4}{5}\right) - I\pi - 5 - \ln(5)$

- Substitute $c_1 = -\ln\left(\frac{4}{5}\right) - I\pi - 5 - \ln(5)$ into general solution and simplify

  $-\ln(y-1) - \frac{1}{y} + \ln(y) = t - 2\ln(2) - I\pi - 5$

- Solution to the IVP

  $-\ln(y-1) - \frac{1}{y} + \ln(y) = t - 2\ln(2) - I\pi - 5$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 1.594 (sec). Leaf size: 21

```
dsolve([diff(y(t),t)=y(t)^2-y(t)^3,y(0) = 1/5],y(t), singsol=all)
```

$$y(t) = \frac{1}{\text{LambertW}\left(4\,e^{4-t}\right)+1}$$

✓ Solution by Mathematica

Time used: 0.495 (sec). Leaf size: 31

```
DSolve[{y'[t]==y[t]^2-y[t]^3,{y[0]==2/10}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \text{InverseFunction}\left[\frac{1}{\#1}+\log(1-\#1)-\log(\#1)\&\right][-t+5+\log(4)]$$

3.10 problem 10

Internal problem ID [12938]
Internal file name [`OUTPUT/11590_Tuesday_November_07_2023_11_27_34_PM_29147239/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind"**

Maple gives the following as the ode type

`[_Abel]`

Unable to solve or complete the solution.

$$\boxed{y' - 2y^3 = t^2}$$

With initial conditions

$$\left[y(0) = -\frac{1}{2}\right]$$

3.10.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= 2y^3 + t^2 \end{aligned}$$

The t domain of $f(t, y)$ when $y = -\frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(2y^3 + t^2) \\ &= 6y^2\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

3.10.2 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2 + f_3(t)y^3$$

Comparing the above to given ODE which is

$$y' = 2y^3 + t^2 \tag{1}$$

Therefore

$$\begin{aligned}f_0(t) &= t^2 \\ f_1(t) &= 0 \\ f_2(t) &= 0 \\ f_3(t) &= 2\end{aligned}$$

Since $f_2(t) = 0$ then we check the Abel invariant to see if it depends on t or not. The Abel invariant is given by

$$-\frac{f_1^3}{f_0^2 f_3}$$

Which when evaluating gives

$$\frac{4}{27t^7}$$

Since the Abel invariant depends on t then unable to solve this ode at this time.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve([diff(y(t),t)=2*y(t)^3+t^2,y(0) = -1/2],y(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y'[t]==2*y[t]^3+t^2,{y[0]==-1/2}},y[t],t,IncludeSingularSolutions -> True]
```

Not solved

3.11 problem 15

Internal problem ID [12939]
Internal file name [`OUTPUT/11591_Tuesday_November_07_2023_11_27_34_PM_56039897/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sqrt{y} = 0}$$

With initial conditions

$$[y(0) = 1]$$

3.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \sqrt{y} \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(\sqrt{y}) \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.11.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\sqrt{y}} dy = \int dt$$
$$2\sqrt{y} = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

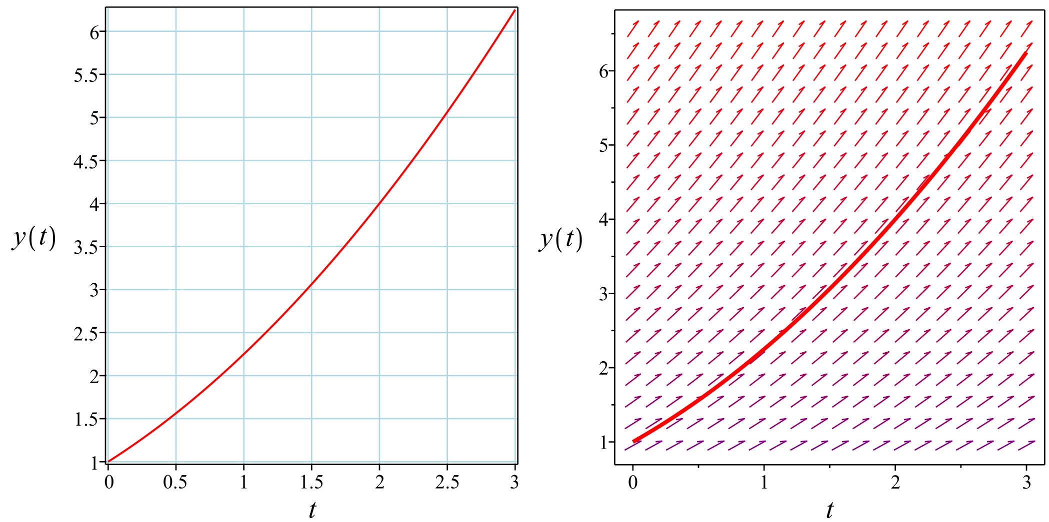
$$2\sqrt{y} = t + 2$$

Solving for y from the above gives

$$y = \frac{(t + 2)^2}{4}$$

Summary
The solution(s) found are the following

$$y = \frac{(t + 2)^2}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{(t+2)^2}{4}$$

Verified OK.

3.11.3 Maple step by step solution

Let's solve

$\left[y' - \sqrt{y} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\sqrt{y}} dt = \int 1 dt + c_1$

- Evaluate integral

  $2\sqrt{y} = t + c_1$

- Solve for y

$y = \frac{1}{4}t^2 + \frac{1}{2}c_1 t + \frac{1}{4}c_1^2$

- Use initial condition $y(0) = 1$

  $1 = \frac{c_1^2}{4}$

- Solve for c_1

  $c_1 = (-2, 2)$

- Substitute $c_1 = (-2, 2)$ into general solution and simplify

  $y = \frac{(-2+t)^2}{4}$

- Solution to the IVP

  $y = \frac{(-2+t)^2}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 11

```
dsolve([diff(y(t),t)=sqrt( y(t)),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{(t+2)^2}{4}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 14

```
DSolve[{y'[t]==Sqrt[ y[t] ],{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{4}(t+2)^2$$

3.12 problem 16

Internal problem ID [12940]
Internal file name [`OUTPUT/11592_Tuesday_November_07_2023_11_27_35_PM_70610284/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y = 2}$$

With initial conditions

$$[y(0) = 1]$$

3.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 1$$
$$q(t) = 2$$

Hence the ode is

$$y' + y = 2$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.12.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y+2} dy = \int dt$$
$$-\ln(-y+2) = t + c_1$$

Raising both side to exponential gives

$$\frac{1}{-y+2} = e^{t+c_1}$$

Which simplifies to

$$\frac{1}{-y+2} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-1 + 2c_2}{c_2}$$

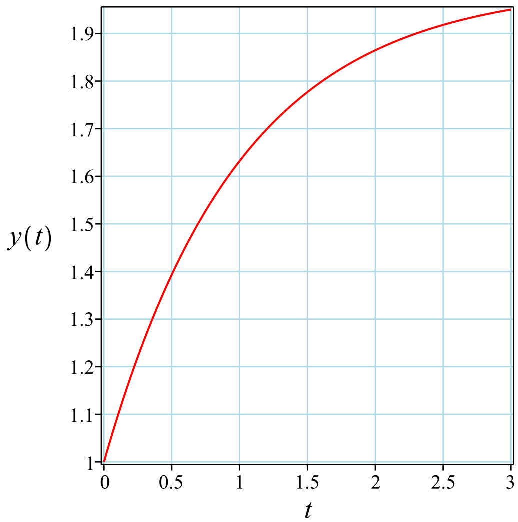
$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

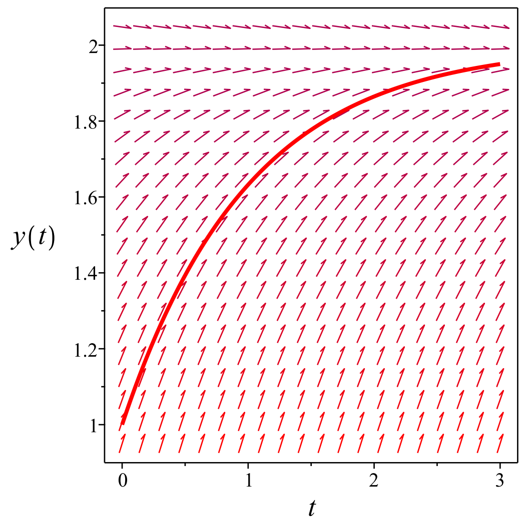
$$y = -e^{-t} + 2$$

Summary
The solution(s) found are the following

$$y = -e^{-t} + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\mathrm{e}^{-t} + 2$$

Verified OK.

3.12.3 Maple step by step solution

Let's solve

$[y' + y = 2, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2-y} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{2-y} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\ln(2-y) = t + c_1$

- Solve for y

  $y = -\mathrm{e}^{-t-c_1} + 2$

- Use initial condition $y(0) = 1$

  $1 = -e^{-c_1} + 2$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = -e^{-t} + 2$
- Solution to the IVP

  $y = -e^{-t} + 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(t),t)=2-y(t),y(0) = 1],y(t), singsol=all)
```

$$y(t) = 2 - e^{-t}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 14

```
DSolve[{y'[t]==2-y[t],{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2 - e^{-t}$$

3.13 problem 17

Internal problem ID [12941]
Internal file name [`OUTPUT/11593_Tuesday_November_07_2023_11_27_36_PM_36577844/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.4 page 61
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\theta' + \frac{11\cos(\theta)}{10} = \frac{9}{10}}$$

With initial conditions

$$[\theta(0) = 1]$$

3.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}\theta' &= f(t,\theta)\\ &= \frac{9}{10} - \frac{11\cos(\theta)}{10}\end{aligned}$$

The θ domain of $f(t,\theta)$ when $t = 0$ is

$$\{-\infty < \theta < \infty\}$$

And the point $\theta_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(\frac{9}{10} - \frac{11\cos(\theta)}{10}\right)\\ &= \frac{11\sin(\theta)}{10}\end{aligned}$$

The θ domain of $\frac{\partial f}{\partial \theta}$ when $t = 0$ is

$$\{-\infty < \theta < \infty\}$$

And the point $\theta_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.13.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{\frac{9}{10} - \frac{11\cos(\theta)}{10}} d\theta &= t + c_1\\ -\sqrt{10}\, \operatorname{arctanh}\left(\sqrt{10}\, \tan\left(\frac{\theta}{2}\right)\right) &= t + c_1\end{aligned}$$

Solving for θ gives these solutions

$$\theta_1 = -2\arctan\left(\frac{\tanh\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $\theta = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -2\arctan\left(\frac{\left(e^{\frac{c_1\sqrt{10}}{5}} - 1\right)\sqrt{10}}{10 + 10\, e^{\frac{c_1\sqrt{10}}{5}}}\right)$$

$$c_1 = \frac{\sqrt{10}\, \ln\left(\frac{\sqrt{10}-10\tan\left(\frac{1}{2}\right)}{\sqrt{10}+10\tan\left(\frac{1}{2}\right)}\right)}{2}$$

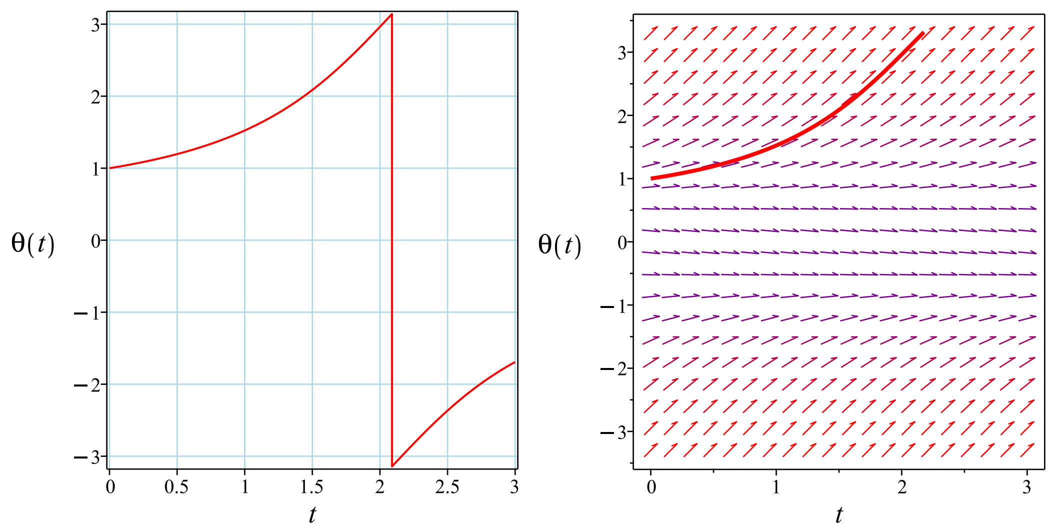
Substituting c_1 found above in the general solution gives

$$\theta = -2\arctan\left(\frac{\left(e^{\frac{\sqrt{10}\,t}{5}}\sqrt{10} - 10\, e^{\frac{\sqrt{10}\,t}{5}}\tan\left(\frac{1}{2}\right) - \sqrt{10} - 10\tan\left(\frac{1}{2}\right)\right)\sqrt{10}}{10\, e^{\frac{\sqrt{10}\,t}{5}}\sqrt{10} - 100\, e^{\frac{\sqrt{10}\,t}{5}}\tan\left(\frac{1}{2}\right) + 10\sqrt{10} + 100\tan\left(\frac{1}{2}\right)}\right)$$

Summary

The solution(s) found are the following

$$\theta = -2 \arctan\left(\frac{\left(\mathrm{e}^{\frac{\sqrt{10}\, t}{5}} \sqrt{10}-10\, \mathrm{e}^{\frac{\sqrt{10}\, t}{5}} \tan\left(\frac{1}{2}\right)-\sqrt{10}-10 \tan\left(\frac{1}{2}\right)\right) \sqrt{10}}{10\, \mathrm{e}^{\frac{\sqrt{10}\, t}{5}} \sqrt{10}-100\, \mathrm{e}^{\frac{\sqrt{10}\, t}{5}} \tan\left(\frac{1}{2}\right)+10 \sqrt{10}+100 \tan\left(\frac{1}{2}\right)}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$\theta = -2 \arctan\left(\frac{\left(\mathrm{e}^{\frac{\sqrt{10}\, t}{5}} \sqrt{10}-10\, \mathrm{e}^{\frac{\sqrt{10}\, t}{5}} \tan\left(\frac{1}{2}\right)-\sqrt{10}-10 \tan\left(\frac{1}{2}\right)\right) \sqrt{10}}{10\, \mathrm{e}^{\frac{\sqrt{10}\, t}{5}} \sqrt{10}-100\, \mathrm{e}^{\frac{\sqrt{10}\, t}{5}} \tan\left(\frac{1}{2}\right)+10 \sqrt{10}+100 \tan\left(\frac{1}{2}\right)}\right)$$

Verified OK.

3.13.3 Maple step by step solution

Let's solve

$\left[\theta' + \frac{11\cos(\theta)}{10} = \frac{9}{10}, \theta(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  θ'

- Separate variables

$$\frac{\theta'}{\frac{9}{10}-\frac{11\cos(\theta)}{10}} = 1$$

- Integrate both sides with respect to t

$$\int \frac{\theta'}{\frac{9}{10}-\frac{11\cos(\theta)}{10}}dt = \int 1dt + c_1$$

- Evaluate integral

$$-\sqrt{10}\,\mathrm{arctanh}\left(\tan\left(\frac{\theta}{2}\right)\sqrt{10}\right) = t + c_1$$

- Solve for θ

$$\theta = -2\arctan\left(\frac{\tanh\left(\frac{(t+c_1)\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

- Use initial condition $\theta(0) = 1$

$$1 = -2\arctan\left(\frac{\tanh\left(\frac{c_1\sqrt{10}}{10}\right)\sqrt{10}}{10}\right)$$

- Solve for c_1

$$c_1 = -\mathrm{arctanh}\left(\sqrt{10}\,\tan\left(\frac{1}{2}\right)\right)\sqrt{10}$$

- Substitute $c_1 = -\mathrm{arctanh}\left(\sqrt{10}\,\tan\left(\frac{1}{2}\right)\right)\sqrt{10}$ into general solution and simplify

$$\theta = -2\arctan\left(\frac{\tanh\left(\frac{\sqrt{10}\,t}{10}-\mathrm{arctanh}\left(\sqrt{10}\,\tan\left(\frac{1}{2}\right)\right)\right)\sqrt{10}}{10}\right)$$

- Solution to the IVP

$$\theta = -2\arctan\left(\frac{\tanh\left(\frac{\sqrt{10}\,t}{10}-\mathrm{arctanh}\left(\sqrt{10}\,\tan\left(\frac{1}{2}\right)\right)\right)\sqrt{10}}{10}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 29

```
dsolve([diff(theta(t),t)=1-cos( theta(t)) + (1+cos(theta(t)))*(-1/10),theta(0) = 1],theta(t)
```

$$\theta(t) = -2 \arctan \left(\frac{\tanh \left(-\operatorname{arctanh} \left(\tan \left(\frac{1}{2} \right) \sqrt{10} \right) + \frac{\sqrt{10}\, t}{10} \right) \sqrt{10}}{10} \right)$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 36

```
DSolve[{theta'[t]==1-Cos[ theta[t]] + (1+Cos[theta[t]])*(-1/10),{theta[0]==1}},theta[t],t,In
```

$$\theta(t) \to -2 \arctan \left(\frac{\tanh \left(\frac{t}{\sqrt{10}} - \operatorname{arctanh}\left(\sqrt{10} \tan \left(\frac{1}{2} \right) \right) \right)}{\sqrt{10}} \right)$$

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4.1 problem 5

Internal problem ID [12942]
Internal file name [`OUTPUT/11594_Tuesday_November_07_2023_11_27_52_PM_60285105/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.5 page 71
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y(y-1)(y-3) = 0}$$

With initial conditions

$$[y(0) = 4]$$

4.1.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y(y-1)(y-3) \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y(y-1)(y-3)) \\ &= (y-1)(y-3) + y(y-3) + y(y-1) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Therefore solution exists and is unique.

4.1.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y(y-1)(y-3)} dy = \int dt$$

$$\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3} = t + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3}} = e^{t+c_1}$$

Which simplifies to

$$\frac{(y-3)^{\frac{1}{6}} y^{\frac{1}{3}}}{\sqrt{y-1}} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\sqrt{3}\, 4^{\frac{1}{3}}}{3} = c_2$$

$$c_2 = \frac{2^{\frac{2}{3}} \sqrt{3}}{3}$$

Substituting c_2 found above in the general solution gives

$$\frac{(y-3)^{\frac{1}{6}} y^{\frac{1}{3}}}{\sqrt{y-1}} = \frac{2^{\frac{2}{3}} \sqrt{3}\, e^t}{3}$$

The above simplifies to

$$-2^{\frac{2}{3}} \sqrt{3}\, e^t \sqrt{y-1} + 3(y-3)^{\frac{1}{6}} y^{\frac{1}{3}} = 0$$

Summary

The solution(s) found are the following

$$-2^{\frac{2}{3}} \sqrt{3}\, e^t \sqrt{y-1} + 3(y-3)^{\frac{1}{6}} y^{\frac{1}{3}} = 0 \tag{1}$$

Verification of solutions

$$-2^{\frac{2}{3}} \sqrt{3}\, e^t \sqrt{y-1} + 3(y-3)^{\frac{1}{6}} y^{\frac{1}{3}} = 0$$

Verified OK.

4.1.3 Maple step by step solution

Let's solve

$[y' - y(y-1)(y-3) = 0, y(0) = 4]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-1)(y-3)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(y-1)(y-3)} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3} = t + c_1$

- Solve for y

  $$y = -\frac{2\left(\frac{\left(1-2\,e^{6t+6c_1}+2\sqrt{-e^{6t+6c_1}+\left(e^{6t+6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}+\frac{1}{2\left(1-2\,e^{6t+6c_1}+2\sqrt{-e^{6t+6c_1}+\left(e^{6t+6c_1}\right)^2}\right)^{\frac{1}{3}}}+\frac{1}{2}\right)^2-\frac{\left(1-2\,e^{6t+6c_1}+2\sqrt{-e^{6t+6c_1}\right.}{2}}{e^{6t+6c_1}-1}$$

- Use initial condition $y(0) = 4$

  $$4 = -\frac{2\left(\frac{\left(1-2\,e^{6c_1}+2\sqrt{-e^{6c_1}+\left(e^{6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}+\frac{1}{2\left(1-2\,e^{6c_1}+2\sqrt{-e^{6c_1}+\left(e^{6c_1}\right)^2}\right)^{\frac{1}{3}}}+\frac{1}{2}\right)^2-\frac{\left(1-2\,e^{6c_1}+2\sqrt{-e^{6c_1}+\left(e^{6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}-\frac{}{2\left(1-2\,e^{6c_1}+\right.}}{e^{6c_1}-1}$$

- Solve for c_1

  $$c_1 = \frac{\ln\left(RootOf\left(\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{4}{3}}+2-2\_Z+2\sqrt{\_Z^2-\_Z}+6\_Z\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{2}{3}}-6\left(1-2\_Z+2\sqrt{}\right.\right.\right.}{6}$$

- Substitute $c_1 = \frac{\ln\left(RootOf\left(\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{4}{3}}+2-2\_Z+2\sqrt{\_Z^2-\_Z}+6\_Z\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{2}{3}}-6\right.\right.}{6}$

- Solution to the IVP

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 1.828 (sec). Leaf size: 133

```
dsolve([diff(y(t),t)=y(t)*(y(t)-1)*(y(t)-3),y(0) = 4],y(t), singsol=all)
```

$$y(t) = \frac{48\left(\frac{e^{6t}}{3}-\frac{9}{16}\right)\left(27-32\,e^{6t}+8\sqrt{16\,e^{12t}-27\,e^{6t}}\right)^{\frac{2}{3}}+48\left(\left(27-32\,e^{6t}+8\sqrt{16\,e^{12t}-27\,e^{6t}}\right)^{\frac{1}{3}}+3\right)\left(e^{6t}-\right.}{\left(27-32\,e^{6t}+8\sqrt{16\,e^{12t}-27\,e^{6t}}\right)^{\frac{2}{3}}\left(16\,e^{6t}-27\right)}$$

✓ Solution by Mathematica

Time used: 0.172 (sec). Leaf size: 132

```
DSolve[{y'[t]==y[t]*(y[t]-1)*(y[t]-3),{y[0]==4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{3i\left(\sqrt{3}+i\right)\sqrt[3]{4\sqrt{e^{6t}\left(16e^{6t}-27\right)^3}+864e^{6t}-256e^{12t}-729}}{32e^{6t}-54} + \frac{9\left(1+i\sqrt{3}\right)}{2\sqrt[3]{4\sqrt{e^{6t}\left(16e^{6t}-27\right)^3}+864e^{6t}-256e^{12t}-729}}+1$$

4.2 problem 6

Internal problem ID [12943]
Internal file name [`OUTPUT/11595_Tuesday_November_07_2023_11_31_55_PM_94325409/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.5 page 71
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y(y-1)(y-3) = 0}$$

With initial conditions

$$[y(0) = 0]$$

4.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y(y-1)(y-3) \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y(y-1)(y-3)) \\ &= (y-1)(y-3) + y(y-3) + y(y-1) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

4.2.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y(y-1)(y-3)}dy = \int dt$$
$$\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3} = t + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3}} = e^{t+c_1}$$

Which simplifies to

$$\frac{(y-3)^{\frac{1}{6}} y^{\frac{1}{3}}}{\sqrt{y-1}} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t=0$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$\frac{(y-3)^{\frac{1}{6}} y^{\frac{1}{3}}}{\sqrt{y-1}} = 0$$

The above simplifies to

$$(y-3)^{\frac{1}{6}} y^{\frac{1}{3}} = 0$$

Summary
The solution(s) found are the following

$$(y-3)^{\frac{1}{6}} y^{\frac{1}{3}} = 0 \tag{1}$$

Verification of solutions

$$(y-3)^{\frac{1}{6}} y^{\frac{1}{3}} = 0$$

Verified OK.

4.2.3 Maple step by step solution

Let's solve

$[y' - y(y-1)(y-3) = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-1)(y-3)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(y-1)(y-3)}dt = \int 1dt + c_1$

- Evaluate integral

  $\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3} = t + c_1$

- Solve for y

  $$y = \frac{-2\left(\frac{\left(1-2\,e^{6t+6c_1}+2\sqrt{-e^{6t+6c_1}+\left(e^{6t+6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}+\frac{1}{2\left(1-2\,e^{6t+6c_1}+2\sqrt{-e^{6t+6c_1}+\left(e^{6t+6c_1}\right)^2}\right)^{\frac{1}{3}}}+\frac{1}{2}\right)^2+e^{6t+6c_1}+\frac{\left(1-2\,e^{6t+6c_1}+2\sqrt{\cdots}\right.}{\cdots}\cdots}{e^{6t+6c_1}-1}$$

- Use initial condition $y(0) = 0$

  $$0 = \frac{-2\left(\frac{\left(1-2\,e^{6c_1}+2\sqrt{-e^{6c_1}+\left(e^{6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}+\frac{1}{2\left(1-2\,e^{6c_1}+2\sqrt{-e^{6c_1}+\left(e^{6c_1}\right)^2}\right)^{\frac{1}{3}}}+\frac{1}{2}\right)^2+e^{6c_1}+\frac{\left(1-2\,e^{6c_1}+2\sqrt{-e^{6c_1}+\left(e^{6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}+\frac{\cdots}{2\left(1-2\cdots\right.}\cdots}{e^{6c_1}-1}$$

- Solve for c_1

  $$c_1 = \frac{\ln\left(RootOf\left(-\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{4}{3}}-2+2\_Z-2\sqrt{\_Z^2-\_Z}+2\_Z\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{2}{3}}-2\left(1-2\_Z+2\cdots\right.\right.\right.}{6}$$

- Substitute $c_1 = \frac{\ln\left(RootOf\left(-\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{4}{3}}-2+2\_Z-2\sqrt{\_Z^2-\_Z}+2\_Z\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{2}{3}}\cdots\right.\right.}{6}$

- Solution to the IVP

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(t),t)=y(t)*(y(t)-1)*(y(t)-3),y(0) = 0],y(t), singsol=all)
```

$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[t]==y[t]*(y[t]-1)*(y[t]-3),{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 0$$

4.3 problem 7

Internal problem ID [12944]
Internal file name [`OUTPUT/11596_Tuesday_November_07_2023_11_32_27_PM_640812/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.5 page 71
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y(y-1)(y-3) = 0}$$

With initial conditions

$$[y(0) = 2]$$

4.3.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y(y-1)(y-3) \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y(y-1)(y-3)) \\ &= (y-1)(y-3) + y(y-3) + y(y-1) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

4.3.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y\,(y-1)\,(y-3)} dy = \int dt$$
$$\frac{\ln\,(y-3)}{6} - \frac{\ln\,(y-1)}{2} + \frac{\ln\,(y)}{3} = t + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3}} = e^{t+c_1}$$

Which simplifies to

$$\frac{(y-3)^{\frac{1}{6}}\, y^{\frac{1}{3}}}{\sqrt{y-1}} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{2^{\frac{1}{3}}\sqrt{3}}{2} + \frac{i2^{\frac{1}{3}}}{2} = c_2$$

$$c_2 = \frac{2^{\frac{1}{3}}\sqrt{3}}{2} + \frac{i2^{\frac{1}{3}}}{2}$$

Substituting c_2 found above in the general solution gives

$$\frac{(y-3)^{\frac{1}{6}}\, y^{\frac{1}{3}}}{\sqrt{y-1}} = \frac{2^{\frac{1}{3}} e^t \sqrt{3}}{2} + \frac{i2^{\frac{1}{3}} e^t}{2}$$

The above simplifies to

$$-2^{\frac{1}{3}} e^t \sqrt{3}\,\sqrt{y-1} - i2^{\frac{1}{3}} e^t \sqrt{y-1} + 2(y-3)^{\frac{1}{6}}\, y^{\frac{1}{3}} = 0$$

Summary

The solution(s) found are the following

$$-\left(\sqrt{3}+i\right) e^t 2^{\frac{1}{3}} \sqrt{y-1} + 2(y-3)^{\frac{1}{6}}\, y^{\frac{1}{3}} = 0 \tag{1}$$

Verification of solutions

$$-\left(\sqrt{3}+i\right) e^t 2^{\frac{1}{3}} \sqrt{y-1} + 2(y-3)^{\frac{1}{6}}\, y^{\frac{1}{3}} = 0$$

Verified OK.

4.3.3 Maple step by step solution

Let's solve

$[y' - y(y-1)(y-3) = 0, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $$\frac{y'}{y(y-1)(y-3)} = 1$$

- Integrate both sides with respect to t

  $$\int \frac{y'}{y(y-1)(y-3)}dt = \int 1dt + c_1$$

- Evaluate integral

  $$\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3} = t + c_1$$

- Solve for y

  $$y = -\frac{2\left(\frac{\left(1-2\,\mathrm{e}^{6t+6c_1}+2\sqrt{-\mathrm{e}^{6t+6c_1}+\left(\mathrm{e}^{6t+6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}+\frac{1}{2\left(1-2\,\mathrm{e}^{6t+6c_1}+2\sqrt{-\mathrm{e}^{6t+6c_1}+\left(\mathrm{e}^{6t+6c_1}\right)^2}\right)^{\frac{1}{3}}}+\frac{1}{2}\right)^2-\frac{\left(1-2\,\mathrm{e}^{6t+6c_1}+2\sqrt{-\mathrm{e}^{6t+6c_1}}\right.}{2}}{\mathrm{e}^{6t+6c_1}-1}$$

- Use initial condition $y(0) = 2$

  $$2 = -\frac{2\left(\frac{\left(1-2\,\mathrm{e}^{6c_1}+2\sqrt{-\mathrm{e}^{6c_1}+\left(\mathrm{e}^{6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}+\frac{1}{2\left(1-2\,\mathrm{e}^{6c_1}+2\sqrt{-\mathrm{e}^{6c_1}+\left(\mathrm{e}^{6c_1}\right)^2}\right)^{\frac{1}{3}}}+\frac{1}{2}\right)^2-\frac{\left(1-2\,\mathrm{e}^{6c_1}+2\sqrt{-\mathrm{e}^{6c_1}+\left(\mathrm{e}^{6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}-\frac{}{2\left(1-2\,\mathrm{e}^{6c_1}+\right.}}{\mathrm{e}^{6c_1}-1}$$

- Solve for c_1

  $$c_1 = \frac{\ln\left(RootOf\left(\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{4}{3}}+2-2\_Z+2\sqrt{\_Z^2-\_Z}+2\_Z\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{2}{3}}-2\left(1-2\_Z+2\sqrt{}\right.\right.\right.}{6}$$

- Substitute $c_1 = \frac{\ln\left(RootOf\left(\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{4}{3}}+2-2\_Z+2\sqrt{\_Z^2-\_Z}+2\_Z\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{2}{3}}-2\right.\right.}{6}$

- Solution to the IVP

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 4.344 (sec). Leaf size: 147

```
dsolve([diff(y(t),t)=y(t)*(y(t)-1)*(y(t)-3),y(0) = 2],y(t), singsol=all)
```

$$y(t) = \frac{(16\,\mathrm{e}^{6t}+9)\left(1+8\,\mathrm{e}^{6t}+4\sqrt{\mathrm{e}^{6t}+4\,\mathrm{e}^{12t}}\right)^{\frac{2}{3}}+\left(24\,\mathrm{e}^{6t}+12\sqrt{\mathrm{e}^{6t}+4\,\mathrm{e}^{12t}}+9\right)\left(1+8\,\mathrm{e}^{6t}+4\sqrt{\mathrm{e}^{6t}+4\,\mathrm{e}^{12t}}\right)^{\frac{1}{3}}+}{(16\,\mathrm{e}^{6t}+3)\left(1+8\,\mathrm{e}^{6t}+4\sqrt{\mathrm{e}^{6t}+4\,\mathrm{e}^{12t}}\right)^{\frac{2}{3}}+\left(8\,\mathrm{e}^{6t}+4\sqrt{\mathrm{e}^{6t}+4\,\mathrm{e}^{12t}}+3\right)\left(1+8\,\mathrm{e}^{6t}+4\sqrt{\mathrm{e}^{6t}+4\,\mathrm{e}^{12t}}\right)^{\frac{1}{3}}+}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 105

```
DSolve[{y'[t]==y[t]*(y[t]-1)*(y[t]-3),{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\sqrt[3]{2\sqrt{e^{6t}\left(4e^{6t}+1\right)^3}+8e^{6t}+16e^{12t}+1}}{4e^{6t}+1} + \frac{1}{\sqrt[3]{2\sqrt{e^{6t}\left(4e^{6t}+1\right)^3}+8e^{6t}+16e^{12t}+1}}+1$$

4.4 problem 8

Internal problem ID [12945]
Internal file name [`OUTPUT/11597_Tuesday_November_07_2023_11_34_05_PM_68301495/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.5 page 71
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y(y-1)(y-3) = 0}$$

With initial conditions

$$[y(0) = -1]$$

4.4.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y(y-1)(y-3) \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y(y-1)(y-3)) \\ &= (y-1)(y-3) + y(y-3) + y(y-1) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

4.4.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y(y-1)(y-3)} dy = \int dt$$

$$\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3} = t + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3}} = e^{t+c_1}$$

Which simplifies to

$$\frac{(y-3)^{\frac{1}{6}} y^{\frac{1}{3}}}{\sqrt{y-1}} = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{2^{\frac{5}{6}}}{2} = c_2$$

$$c_2 = \frac{2^{\frac{5}{6}}}{2}$$

Substituting c_2 found above in the general solution gives

$$\frac{(y-3)^{\frac{1}{6}} y^{\frac{1}{3}}}{\sqrt{y-1}} = \frac{2^{\frac{5}{6}} e^t}{2}$$

The above simplifies to

$$-2^{\frac{5}{6}} e^t \sqrt{y-1} + 2(y-3)^{\frac{1}{6}} y^{\frac{1}{3}} = 0$$

Summary

The solution(s) found are the following

$$-2^{\frac{5}{6}} e^t \sqrt{y-1} + 2(y-3)^{\frac{1}{6}} y^{\frac{1}{3}} = 0 \tag{1}$$

Verification of solutions

$$-2^{\frac{5}{6}} e^t \sqrt{y-1} + 2(y-3)^{\frac{1}{6}} y^{\frac{1}{3}} = 0$$

Verified OK.

4.4.3 Maple step by step solution

Let's solve

$[y' - y(y-1)(y-3) = 0, y(0) = -1]$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'}{y(y-1)(y-3)} = 1$

- Integrate both sides with respect to t

$\int \frac{y'}{y(y-1)(y-3)}dt = \int 1dt + c_1$

- Evaluate integral

$\frac{\ln(y-3)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y)}{3} = t + c_1$

- Solve for y

$$y = -\frac{2\left(\frac{\left(1-2\,e^{6t+6c_1}+2\sqrt{-e^{6t+6c_1}+\left(e^{6t+6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}+\frac{1}{2\left(1-2\,e^{6t+6c_1}+2\sqrt{-e^{6t+6c_1}+\left(e^{6t+6c_1}\right)^2}\right)^{\frac{1}{3}}}+\frac{1}{2}\right)^2-\frac{\left(1-2\,e^{6t+6c_1}+2\sqrt{-e^{6t+6c_1}\cdots}\right.}{2}\cdots}{e^{6t+6c_1}-1}$$

- Use initial condition $y(0) = -1$

$$-1 = -\frac{2\left(\frac{\left(1-2\,e^{6c_1}+2\sqrt{-e^{6c_1}+\left(e^{6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}+\frac{1}{2\left(1-2\,e^{6c_1}+2\sqrt{-e^{6c_1}+\left(e^{6c_1}\right)^2}\right)^{\frac{1}{3}}}+\frac{1}{2}\right)^2-\frac{\left(1-2\,e^{6c_1}+2\sqrt{-e^{6c_1}+\left(e^{6c_1}\right)^2}\right)^{\frac{1}{3}}}{2}-\frac{\cdots}{2\left(1-2\,e^{6c_1}\cdots\right.}\cdots}{e^{6c_1}-1}$$

- Solve for c_1

$$c_1 = \frac{\ln\left(RootOf\left(-\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{4}{3}}-2+2\_Z-2\sqrt{\_Z^2-\_Z}+4\_Z\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{2}{3}}-4\left(1-2\_Z+2\cdots\right.\right.\right.}{6}$$

- Substitute $c_1 = \frac{\ln\left(RootOf\left(-\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{4}{3}}-2+2\_Z-2\sqrt{\_Z^2-\_Z}+4\_Z\left(1-2\_Z+2\sqrt{\_Z^2-\_Z}\right)^{\frac{2}{3}}\cdots\right.\right.}{6}$
- Solution to the IVP

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 1.703 (sec). Leaf size: 133

```
dsolve([diff(y(t),t)=y(t)*(y(t)-1)*(y(t)-3),y(0) = -1],y(t), singsol=all)
```

$$y(t) = \frac{\left(2\,\mathrm{e}^{6t}-4\right)\left(1-\mathrm{e}^{6t}+\sqrt{\mathrm{e}^{6t}\left(\mathrm{e}^{6t}-2\right)}\right)^{\frac{2}{3}}+\left(\left(i\sqrt{3}-1\right)\left(1-\mathrm{e}^{6t}+\sqrt{\mathrm{e}^{6t}\left(\mathrm{e}^{6t}-2\right)}\right)^{\frac{1}{3}}-i\sqrt{3}-1\right)\left(\mathrm{e}^{6t}-\sqrt{\cdots}\right.}{\left(1-\mathrm{e}^{6t}+\sqrt{\mathrm{e}^{6t}\left(\mathrm{e}^{6t}-2\right)}\right)^{\frac{2}{3}}\left(2\,\mathrm{e}^{6t}-4\right)}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 104

```
DSolve[{y'[t]==y[t]*(y[t]-1)*(y[t]-3),{y[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\sqrt[3]{2\sqrt{e^{6t}\left(e^{6t}-2\right)^3}+8e^{6t}-2e^{12t}-8}}{e^{6t}-2}-\frac{2^{2/3}}{\sqrt[3]{\sqrt{e^{6t}\left(e^{6t}-2\right)^3}+4e^{6t}-e^{12t}-4}}+1$$

4.5 problem 12

Internal problem ID [12946]
Internal file name [`OUTPUT/11598_Tuesday_November_07_2023_11_51_50_PM_63682322/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.5 page 71
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 = 0}$$

4.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y^2} dy = t + c_1$$
$$\frac{1}{y} = t + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{1}{t + c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{t + c_1} \tag{1}$$

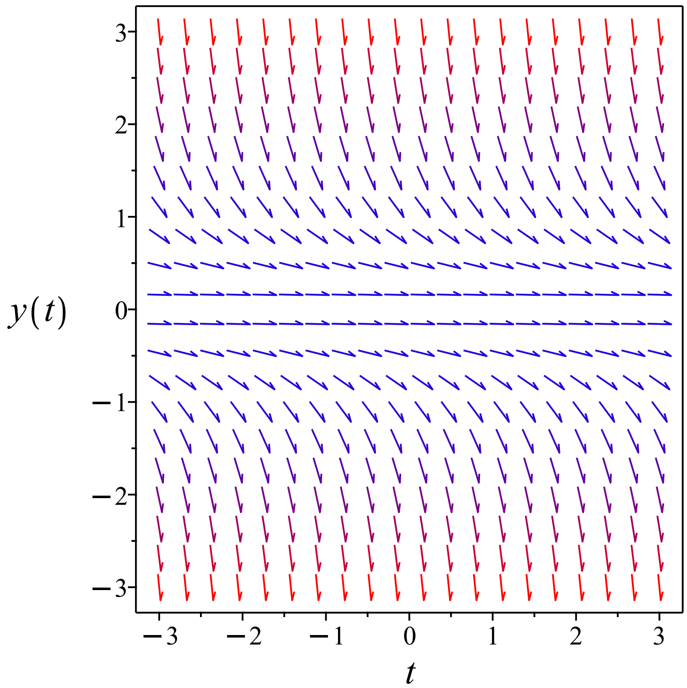


Figure 141: Slope field plot

Verification of solutions

$$y = \frac{1}{t + c_1}$$

Verified OK.

4.5.2 Maple step by step solution

Let's solve

$y' + y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = -1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int (-1)\, dt + c_1$

- Evaluate integral

$$-\frac{1}{y} = -t + c_1$$

- Solve for y

$$y = -\frac{1}{-t + c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve(diff(y(t),t)=-y(t)^2,y(t), singsol=all)
```

$$y(t) = \frac{1}{t + c_1}$$

✓ Solution by Mathematica

Time used: 0.156 (sec). Leaf size: 18

```
DSolve[y'[t]==-y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{t - c_1}$$
$$y(t) \to 0$$

4.6 problem 13

Internal problem ID [12947]
Internal file name [`OUTPUT/11599_Tuesday_November_07_2023_11_51_50_PM_48642611/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.5 page 71
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^3 = 0}$$

With initial conditions

$$[y(0) = 1]$$

4.6.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^3 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^3) \\ &= 3y^2 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.6.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{y^3} dy &= t + c_1 \\ -\frac{1}{2y^2} &= t + c_1\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}y_1 &= \frac{1}{\sqrt{-2t - 2c_1}} \\ y_2 &= -\frac{1}{\sqrt{-2t - 2c_1}}\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{\sqrt{-2c_1}}$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{\sqrt{-2c_1}}$$

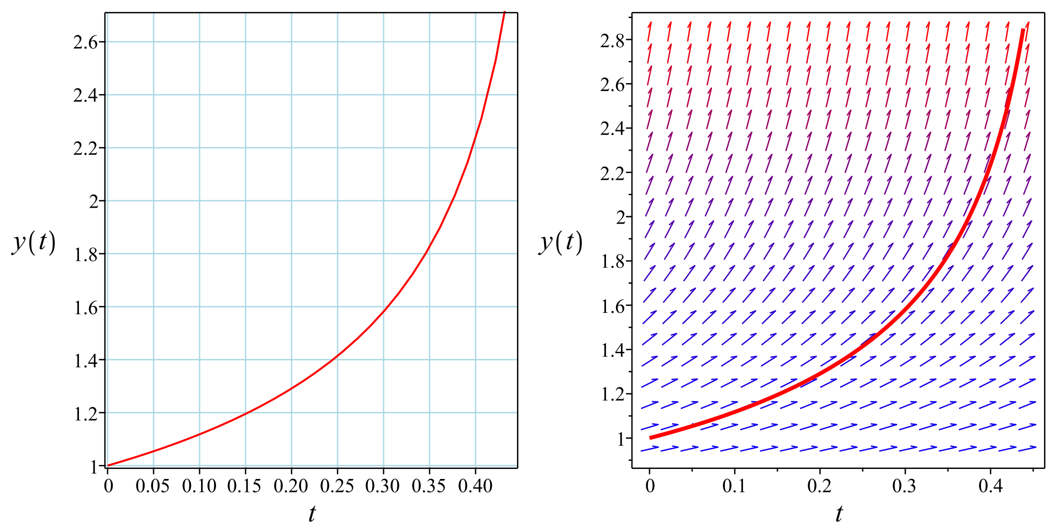
$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{\sqrt{1 - 2t}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{\sqrt{1 - 2t}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{1 - 2t}}$$

Verified OK.

4.6.3 Maple step by step solution

Let's solve

$[y' - y^3 = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^3} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = t + c_1$
- Solve for y

  $\left\{y = \frac{1}{\sqrt{-2t-2c_1}}, y = -\frac{1}{\sqrt{-2t-2c_1}}\right\}$
- Use initial condition $y(0) = 1$

  $1 = \frac{1}{\sqrt{-2c_1}}$
- Solve for c_1

  $c_1 = -\frac{1}{2}$
- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $y = \frac{1}{\sqrt{1-2t}}$
- Use initial condition $y(0) = 1$

  $1 = -\frac{1}{\sqrt{-2c_1}}$
- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \frac{1}{\sqrt{1-2t}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([diff(y(t),t)=y(t)^3,y(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{1}{\sqrt{-2t+1}}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 14

```
DSolve[{y'[t]==y[t]^3,{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{\sqrt{1-2t}}$$

4.7 problem 14

Internal problem ID [12948]
Internal file name [`OUTPUT/11600_Tuesday_November_07_2023_11_51_51_PM_51908698/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.5 page 71
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{1}{(y+1)(-2+t)} = 0}$$

With initial conditions

$$[y(0) = 0]$$

4.7.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \frac{1}{(y+1)(-2+t)} \end{aligned}$$

The t domain of $f(t, y)$ when $y = 0$ is

$$\{t < 2 \lor 2 < t\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{y < -1 \vee -1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{(y+1)(-2+t)}\right) \\ &= -\frac{1}{(y+1)^2(-2+t)}\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{t < 2 \vee 2 < t\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{y < -1 \vee -1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

4.7.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t, y) \\ &= f(t)g(y) \\ &= \frac{1}{(y+1)(-2+t)}\end{aligned}$$

Where $f(t) = \frac{1}{-2+t}$ and $g(y) = \frac{1}{y+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y+1}}\,dy &= \frac{1}{-2+t}\,dt \\ \int \frac{1}{\frac{1}{y+1}}\,dy &= \int \frac{1}{-2+t}\,dt \\ \frac{1}{2}y^2 + y &= \ln(-2+t) + c_1\end{aligned}$$

Which results in

$$y = -1 + \sqrt{1 + 2\ln(-2+t) + 2c_1}$$
$$y = -1 - \sqrt{1 + 2\ln(-2+t) + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -1 - \sqrt{1 + 2\ln(2) + 2i\pi + 2c_1}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -1 + \sqrt{1 + 2\ln(2) + 2i\pi + 2c_1}$$

$$c_1 = -\ln(2) - i\pi$$

Substituting c_1 found above in the general solution gives

$$y = -1 + \sqrt{1 + 2\ln(-2+t) - 2\ln(2) - 2i\pi}$$

Summary

The solution(s) found are the following

$$y = -1 + \sqrt{1 + 2\ln(-2+t) - 2\ln(2) - 2i\pi} \tag{1}$$

Verification of solutions

$$y = -1 + \sqrt{1 + 2\ln(-2+t) - 2\ln(2) - 2i\pi}$$

Verified OK.

4.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{1}{(y+1)(-2+t)}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 134: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= -2 + t \\
\eta(t, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{-2+t} dt \end{aligned}$$

Which results in

$$S = \ln(-2+t)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{1}{(y+1)(-2+t)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= \frac{1}{-2+t} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y+1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R+1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 + R + c_1 \tag{4}$$

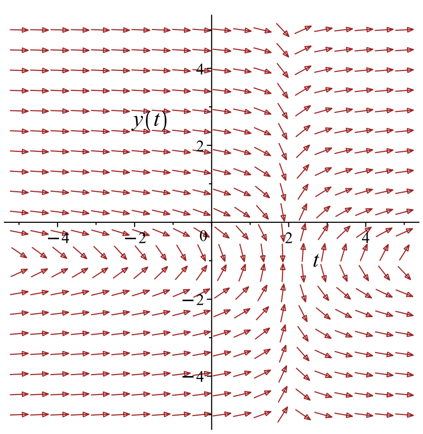
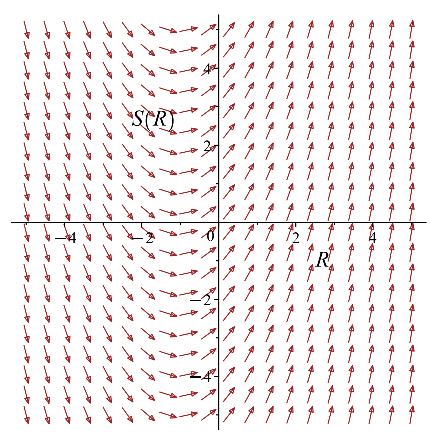
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\ln(-2+t) = \frac{y^2}{2} + y + c_1$$

Which simplifies to

$$\ln(-2+t) = \frac{y^2}{2} + y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{1}{(y+1)(-2+t)}$ 	$R = y$ $S = \ln(-2+t)$	$\frac{dS}{dR} = R + 1$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) + i\pi = c_1$$

$$c_1 = \ln(2) + i\pi$$

Substituting c_1 found above in the general solution gives

$$\ln(-2+t) = \frac{y^2}{2} + y + \ln(2) + i\pi$$

Summary

The solution(s) found are the following

$$\ln(-2+t) = \frac{y^2}{2} + y + \ln(2) + i\pi \tag{1}$$

Verification of solutions

$$\ln(-2+t) = \frac{y^2}{2} + y + \ln(2) + i\pi$$

Verified OK.

4.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$(y+1)\,\mathrm{d}y=\left(\frac{1}{-2+t}\right)\mathrm{d}t$$

$$\left(-\frac{1}{-2+t}\right)\mathrm{d}t+(y+1)\,\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{1}{-2+t}\\ N(t,y)&=y+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{-2+t}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t}&=\frac{\partial}{\partial t}(y+1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial\phi}{\partial t}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -\frac{1}{-2+t} \, \mathrm{d}t$$
$$\phi = -\ln(-2+t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y + 1$. Therefore equation (4) becomes

$$y + 1 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y + 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y+1) \, \mathrm{d}y$$
$$f(y) = \frac{1}{2}y^2 + y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^2}{2} - \ln(-2+t) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^2}{2} - \ln(-2+t) + y$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\ln(2) - i\pi = c_1$$

$$c_1 = -\ln(2) - i\pi$$

Substituting c_1 found above in the general solution gives

$$\frac{y^2}{2} - \ln(-2 + t) + y = -\ln(2) - i\pi$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - \ln(-2 + t) + y = -\ln(2) - i\pi \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - \ln(-2 + t) + y = -\ln(2) - i\pi$$

Verified OK.

4.7.5 Maple step by step solution

Let's solve

$\left[y' - \frac{1}{(y+1)(-2+t)} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(y + 1) = \frac{1}{-2+t}$

- Integrate both sides with respect to t

  $\int y'(y + 1)\, dt = \int \frac{1}{-2+t} dt + c_1$

- Evaluate integral

  $\frac{y^2}{2} + y = \ln(-2 + t) + c_1$

- Solve for y

$$\left\{y = -1 - \sqrt{1 + 2\ln(-2+t) + 2c_1}, y = -1 + \sqrt{1 + 2\ln(-2+t) + 2c_1}\right\}$$

- Use initial condition $y(0) = 0$

  $0 = -1 - \sqrt{1 + 2\ln(2) + 2\,\mathrm{I}\pi + 2c_1}$
- Solution does not satisfy initial condition
- Use initial condition $y(0) = 0$

  $0 = -1 + \sqrt{1 + 2\ln(2) + 2\,\mathrm{I}\pi + 2c_1}$
- Solve for c_1

  $c_1 = -\ln(2) - \mathrm{I}\pi$
- Substitute $c_1 = -\ln(2) - \mathrm{I}\pi$ into general solution and simplify

  $y = -1 + \sqrt{1 + 2\ln(-2+t) - 2\ln(2) - 2\,\mathrm{I}\pi}$
- Solution to the IVP

  $y = -1 + \sqrt{1 + 2\ln(-2+t) - 2\ln(2) - 2\,\mathrm{I}\pi}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 24

```
dsolve([diff(y(t),t)=1/( (y(t)+1)*(t-2)),y(0) = 0],y(t), singsol=all)
```

$$y(t) = -1 + \sqrt{1 - 2i\pi + 2\ln(t-2) - 2\ln(2)}$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 28

```
DSolve[{y'[t]==1/( (y[t]+1)*(t-2)),{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -1 + \sqrt{2\log(t-2) - 2i\pi + 1 - \log(4)}$$

4.8 problem 15

Internal problem ID [12949]
Internal file name [`OUTPUT/11601_Tuesday_November_07_2023_11_51_52_PM_90524771/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.5 page 71
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \frac{1}{(y+2)^2} = 0}$$

With initial conditions

$$[y(0) = 1]$$

4.8.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= \frac{1}{(y+2)^2} \end{aligned}$$

The y domain of $f(t,y)$ when $t=0$ is

$$\{y < -2 \lor -2 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{(y+2)^2}\right) \\ &= -\frac{2}{(y+2)^3}\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{y < -2 \lor -2 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.8.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int (y+2)^2 dy &= t + c_1 \\ \frac{(y+2)^3}{3} &= t + c_1\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}y_1 &= (3t+3c_1)^{\frac{1}{3}} - 2 \\ y_2 &= -\frac{(3t+3c_1)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(3t+3c_1)^{\frac{1}{3}}}{2} - 2 \\ y_3 &= -\frac{(3t+3c_1)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(3t+3c_1)^{\frac{1}{3}}}{2} - 2\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{c_1^{\frac{1}{3}} 3^{\frac{1}{3}}}{2} + \frac{i 3^{\frac{5}{6}} c_1^{\frac{1}{3}}}{2} - 2$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{c_1^{\frac{1}{3}} 3^{\frac{1}{3}}}{2} - \frac{i 3^{\frac{5}{6}} c_1^{\frac{1}{3}}}{2} - 2$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1^{\frac{1}{3}} 3^{\frac{1}{3}} - 2$$

$$c_1 = 9$$

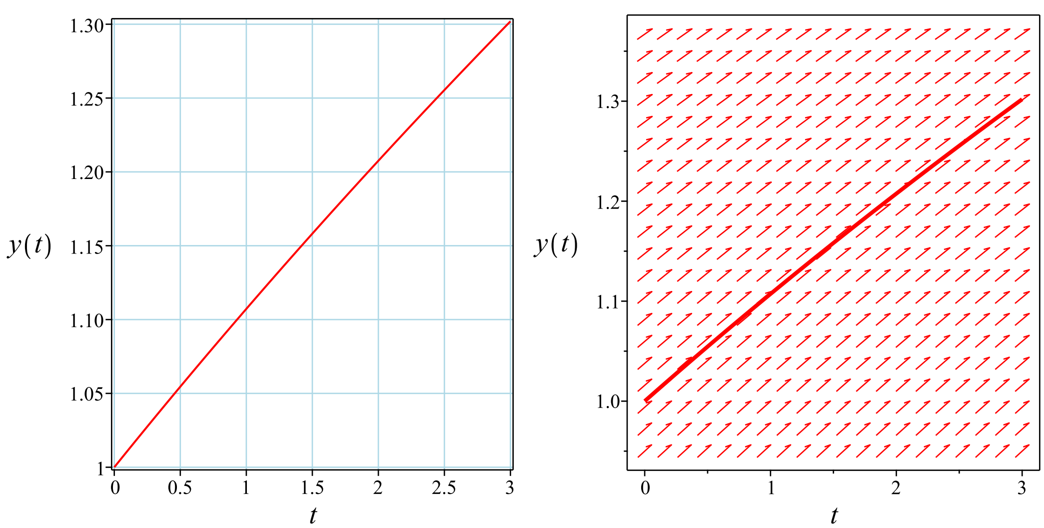
Substituting c_1 found above in the general solution gives

$$y = (3t + 27)^{\frac{1}{3}} - 2$$

Summary

The solution(s) found are the following

$$y = (3t + 27)^{\frac{1}{3}} - 2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (3t + 27)^{\frac{1}{3}} - 2$$

Verified OK.

4.8.3 Maple step by step solution

Let's solve

$\left[y' - \frac{1}{(y+2)^2} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(y+2)^2 = 1$

- Integrate both sides with respect to t

  $\int y'(y+2)^2\, dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{(y+2)^3}{3} = t + c_1$

- Solve for y

  $y = (3t + 3c_1)^{\frac{1}{3}} - 2$

- Use initial condition $y(0) = 1$

  $1 = c_1^{\frac{1}{3}} 3^{\frac{1}{3}} - 2$

- Solve for c_1

  $c_1 = 9$

- Substitute $c_1 = 9$ into general solution and simplify

  $y = (3t + 27)^{\frac{1}{3}} - 2$

- Solution to the IVP

  $y = (3t + 27)^{\frac{1}{3}} - 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([diff(y(t),t)=1/(y(t)+2)^2,y(0) = 1],y(t), singsol=all)
```

$$y(t) = (3t + 27)^{\frac{1}{3}} - 2$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[{y'[t]==1/(y[t]+2)^2,{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \sqrt[3]{3}\sqrt[3]{t+9} - 2$$

4.9 problem 16

Internal problem ID [12950]
Internal file name [`OUTPUT/11602_Tuesday_November_07_2023_11_51_53_PM_77268850/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.5 page 71
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{t}{y-2} = 0}$$

With initial conditions

$$[y(-1) = 0]$$

4.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(t, y) \\
&= \frac{t}{y-2}
\end{aligned}
$$

The t domain of $f(t, y)$ when $y = 0$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = -1$ is inside this domain. The y domain of $f(t, y)$ when $t = -1$ is

$$
\{y < 2 \lor 2 < y\}
$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{t}{y-2}\right) \\
&= -\frac{t}{(y-2)^2}
\end{aligned}
$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$
\{-\infty < t < \infty\}
$$

And the point $t_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = -1$ is

$$
\{y < 2 \lor 2 < y\}
$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

4.9.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(t, y) \\
&= f(t)g(y) \\
&= \frac{t}{y-2}
\end{aligned}
$$

Where $f(t) = t$ and $g(y) = \frac{1}{y-2}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y-2}}\,dy = t\,dt$$

$$\int \frac{1}{\frac{1}{y-2}}\,dy = \int t\,dt$$

$$\frac{1}{2}y^2 - 2y = \frac{t^2}{2} + c_1$$

Which results in

$$y = 2 + \sqrt{t^2 + 2c_1 + 4}$$

$$y = 2 - \sqrt{t^2 + 2c_1 + 4}$$

Initial conditions are used to solve for c_1. Substituting $t = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 2 - \sqrt{5 + 2c_1}$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = 2 - \sqrt{t^2 + 3}$$

Initial conditions are used to solve for c_1. Substituting $t = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

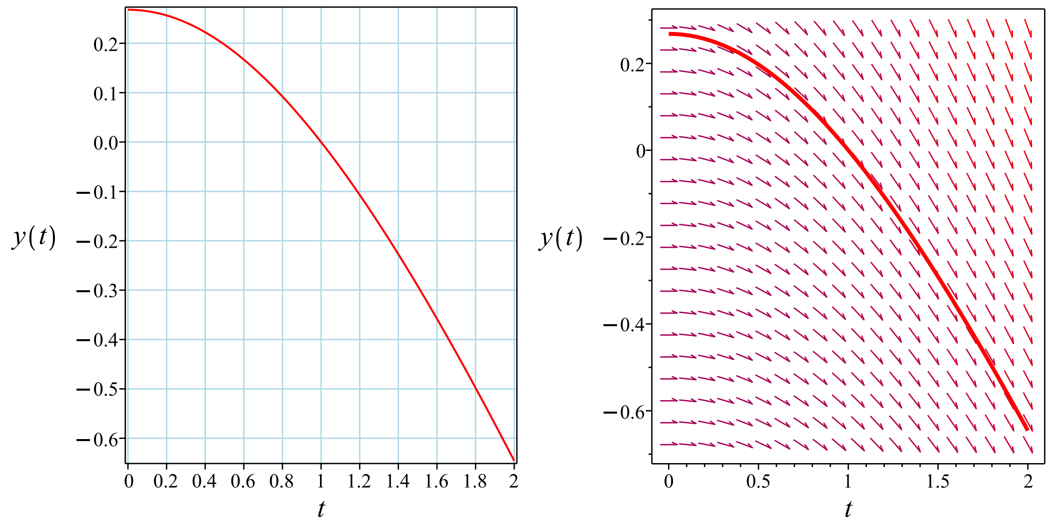
$$0 = 2 + \sqrt{5 + 2c_1}$$

Warning: Unable to solve for constant of integration.

Summary

The solution(s) found are the following

$$y = 2 - \sqrt{t^2 + 3}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2 - \sqrt{t^2 + 3}$$

Verified OK.

4.9.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{t}{y - 2} \tag{1}$$

Which becomes

$$(y - 2)\, dy = (t)\, dt \tag{2}$$

But the RHS is complete differential because

$$(t)\, dt = d\left(\frac{t^2}{2}\right)$$

Hence (2) becomes

$$(y - 2)\, dy = d\left(\frac{t^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y = 2 + \sqrt{t^2 + 2c_1 + 4} + c_1$$
$$y = 2 - \sqrt{t^2 + 2c_1 + 4} + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 2 - \sqrt{5 + 2c_1} + c_1$$

$$c_1 = \sqrt{2} - 1$$

Substituting c_1 found above in the general solution gives

$$y = 1 - \sqrt{t^2 + 2\sqrt{2} + 2} + \sqrt{2}$$

Initial conditions are used to solve for c_1. Substituting $t = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 2 + \sqrt{5 + 2c_1} + c_1$$

$$c_1 = -\sqrt{2} - 1$$

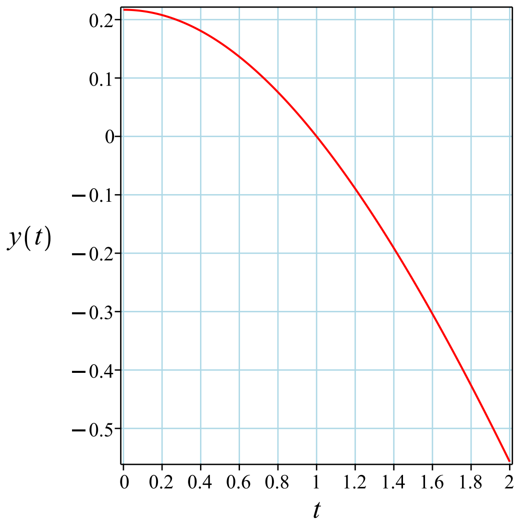
Substituting c_1 found above in the general solution gives

$$y = 1 + \sqrt{t^2 - 2\sqrt{2} + 2} - \sqrt{2}$$

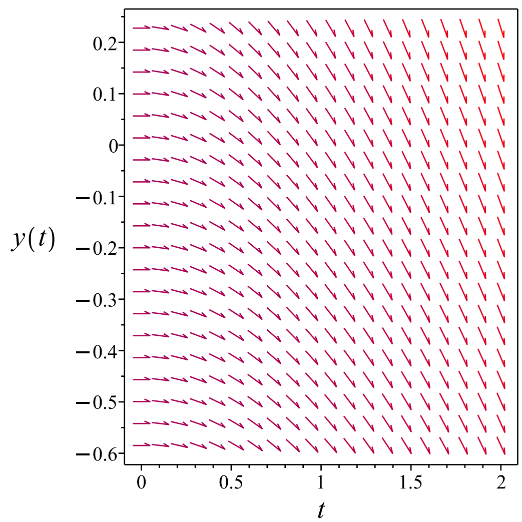
Summary

The solution(s) found are the following

$$y = 1 + \sqrt{t^2 - 2\sqrt{2} + 2} - \sqrt{2} \tag{1}$$
$$y = 1 - \sqrt{t^2 + 2\sqrt{2} + 2} + \sqrt{2} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + \sqrt{t^2 - 2\sqrt{2} + 2} - \sqrt{2}$$

Verified OK.

$$y = 1 - \sqrt{t^2 + 2\sqrt{2} + 2} + \sqrt{2}$$

Verified OK.

4.9.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = t + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{X + x_0}{Y(X) + y_0 - 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{1}{u} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{1}{u(X)} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) u(X)\, X + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2 - 1}{uX} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\,du = -\frac{1}{X}\,dX$$
$$\int \frac{1}{\frac{u^2-1}{u}}\,du = \int -\frac{1}{X}\,dX$$
$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} = -\ln(X) + c_2$$

The above can be written as

$$\left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) = -\ln(X) + 2c_2$$
$$\ln(u-1) + \ln(u+1) = (2)(-\ln(X) + 2c_2)$$
$$= -2\ln(X) + 4c_2$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-2\ln(X)+2c_2}$$

Which simplifies to

$$u^2 - 1 = \frac{2c_2}{X^2}$$
$$= \frac{c_3}{X^2}$$

The solution is

$$u(X)^2 - 1 = \frac{c_3}{X^2}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{Y(X)^2}{X^2} - 1 = \frac{c_3}{X^2}$$

Which simplifies to

$$-(X - Y(X))(X + Y(X)) = c_3$$

Using the solution for $Y(X)$

$$-(X - Y(X))(X + Y(X)) = c_3$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= t + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y + 2 \\
X &= t
\end{aligned}
$$

Then the solution in y becomes

$$
-(t - y + 2)(t + y - 2) = c_3
$$

Initial conditions are used to solve for c_3. Substituting $t = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
3 = c_3
$$

$$
c_3 = 3
$$

Substituting c_3 found above in the general solution gives

$$
-(t - y + 2)(t + y - 2) = 3
$$

Summary

The solution(s) found are the following

$$
-(t - y + 2)(t + y - 2) = 3 \tag{1}
$$

Verification of solutions

$$
-(t - y + 2)(t + y - 2) = 3
$$

Verified OK.

4.9.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{t}{y - 2} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 138: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= \frac{1}{t} \\ \eta(t, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{t}} dt \end{aligned}$$

Which results in

$$S = \frac{t^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{t}{y - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= t \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y - 2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R - 2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 - 2R + c_1 \tag{4}$$

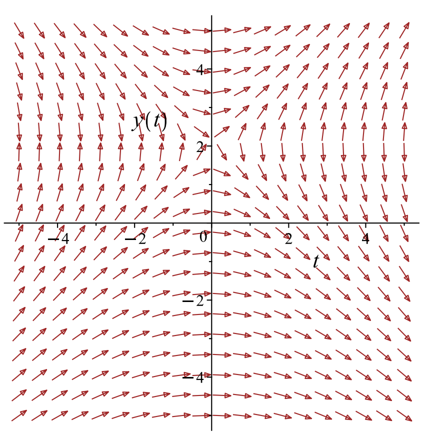
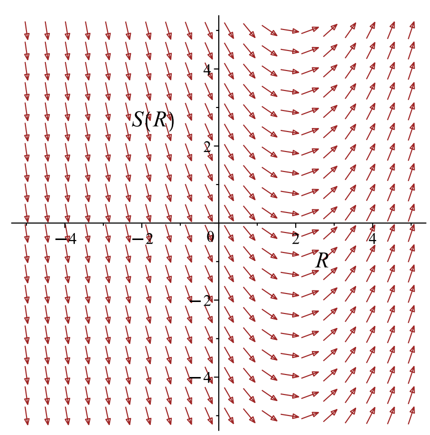
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{t^2}{2} = \frac{y^2}{2} - 2y + c_1$$

Which simplifies to

$$\frac{t^2}{2} = \frac{y^2}{2} - 2y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{t}{y-2}$ 	$R = y$ $S = \frac{t^2}{2}$	$\frac{dS}{dR} = R - 2$ 

Initial conditions are used to solve for c_1. Substituting $t = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{t^2}{2} = \frac{1}{2}y^2 - 2y + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$\frac{t^2}{2} = \frac{y^2}{2} - 2y + \frac{1}{2} \tag{1}$$

Verification of solutions

$$\frac{t^2}{2} = \frac{y^2}{2} - 2y + \frac{1}{2}$$

Verified OK.

4.9.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y-2)\,\mathrm{d}y &= (t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t + (y-2)\,\mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -t\\ N(t, y) &= y-2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(y-2)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial\phi}{\partial t} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, dt = \int M \, dt$$
$$\int \frac{\partial \phi}{\partial t} \, dt = \int -t \, dt$$
$$\phi = -\frac{t^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - 2$. Therefore equation (4) becomes

$$y - 2 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y - 2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (y - 2) \, dy$$
$$f(y) = \frac{1}{2}y^2 - 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{2}t^2 + \frac{1}{2}y^2 - 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{2}t^2 + \frac{1}{2}y^2 - 2y$$

Initial conditions are used to solve for c_1. Substituting $t=-1$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2}=c_1$$

$$c_1=-\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{2}t^2+\frac{1}{2}y^2-2y=-\frac{1}{2}$$

Summary
The solution(s) found are the following

$$-\frac{t^2}{2}+\frac{y^2}{2}-2y=-\frac{1}{2} \tag{1}$$

Verification of solutions

$$-\frac{t^2}{2}+\frac{y^2}{2}-2y=-\frac{1}{2}$$

Verified OK.

4.9.7 Maple step by step solution

Let's solve

$\left[y'-\frac{t}{y-2}=0, y(-1)=0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(y-2)=t$

- Integrate both sides with respect to t

  $\int y'(y-2)\,dt=\int t dt+c_1$

- Evaluate integral

  $\frac{y^2}{2}-2y=\frac{t^2}{2}+c_1$

- Solve for y

$$\left\{y=2-\sqrt{t^2+2c_1+4}, y=2+\sqrt{t^2+2c_1+4}\right\}$$

- Use initial condition $y(-1)=0$

$$0=2-\sqrt{5+2c_1}$$

- Solve for c_1

$$c_1=-\frac{1}{2}$$

- Substitute $c_1=-\frac{1}{2}$ into general solution and simplify

$$y=2-\sqrt{t^2+3}$$

- Use initial condition $y(-1)=0$

$$0=2+\sqrt{5+2c_1}$$

- Solution does not satisfy initial condition
- Solution to the IVP

$$y=2-\sqrt{t^2+3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)=t/(y(t)-2),y(-1) = 0],y(t), singsol=all)
```

$$y(t)=2-\sqrt{t^2+3}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 21

```
DSolve[{y'[t]==1/(y[t]-2),{y[-1]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2 - \sqrt{2}\sqrt{t+3}$$

5 Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89

5.1 problem 1 and 13 (i)

Internal problem ID [12951]
Internal file name [`OUTPUT/11603_Tuesday_November_07_2023_11_51_54_PM_78927745/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 1 and 13 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 3y(y-2) = 0}$$

With initial conditions

$$[y(0) = 1]$$

5.1.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= 3y(y-2) \end{aligned}$$

The y domain of $f(t,y)$ when $t=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(3y(y-2)) \\ &= 6y - 6 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.1.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{3y(y-2)} dy = \int dt$$
$$\frac{\ln(y-2)}{6} - \frac{\ln(y)}{6} = t + c_1$$

The above can be written as

$$\left(\frac{1}{6}\right)(\ln(y-2) - \ln(y)) = t + c_1$$
$$\ln(y-2) - \ln(y) = (6)(t + c_1)$$
$$= 6t + 6c_1$$

Raising both side to exponential gives

$$e^{\ln(y-2)-\ln(y)} = 6c_1 e^{6t}$$

Which simplifies to

$$\frac{y-2}{y} = c_2 e^{6t}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{2}{-1 + c_2}$$

$$c_2 = -1$$

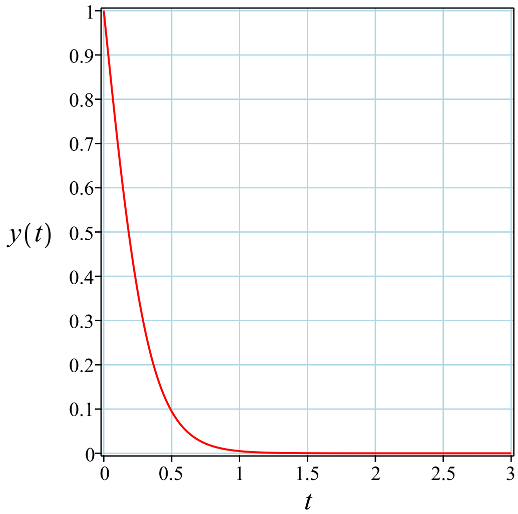
Substituting c_2 found above in the general solution gives

$$y = \frac{2}{e^{6t} + 1}$$

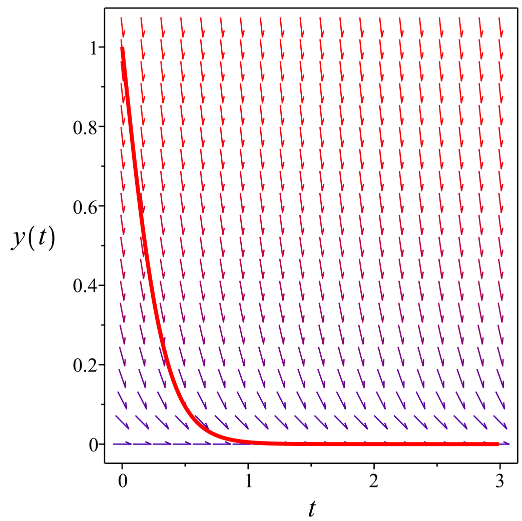
Summary

The solution(s) found are the following

$$y = \frac{2}{e^{6t} + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{e^{6t} + 1}$$

Verified OK.

5.1.3 Maple step by step solution

Let's solve

$[y' - 3y(y-2) = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-2)} = 3$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(y-2)} dt = \int 3dt + c_1$

- Evaluate integral

  $\frac{\ln(y-2)}{2} - \frac{\ln(y)}{2} = 3t + c_1$
- Solve for y

  $y = -\frac{2}{e^{6t+2c_1}-1}$
- Use initial condition $y(0) = 1$

  $1 = -\frac{2}{e^{2c_1}-1}$
- Solve for c_1

  $c_1 = \frac{I}{2}\pi$
- Substitute $c_1 = \frac{I}{2}\pi$ into general solution and simplify

  $y = \frac{2}{e^{6t}+1}$
- Solution to the IVP

  $y = \frac{2}{e^{6t}+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(t),t)=3*y(t)*(y(t)-2),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{2}{1 + e^{6t}}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 16

```
DSolve[{y'[t]==3*y[t]*(y[t]-2),{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{2}{e^{6t} + 1}$$

5.2 problem 1 and 13 (ii)

Internal problem ID [12952]
Internal file name [`OUTPUT/11604_Tuesday_November_07_2023_11_51_55_PM_71168806/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 1 and 13 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 3y(y-2) = 0}$$

With initial conditions

$$[y(-2) = -1]$$

5.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= 3y(y-2) \end{aligned}$$

The y domain of $f(t, y)$ when $t = -2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(3y(y-2)) \\ &= 6y - 6 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = -2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

5.2.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{3y(y-2)} dy = \int dt$$
$$\frac{\ln(y-2)}{6} - \frac{\ln(y)}{6} = t + c_1$$

The above can be written as

$$\left(\frac{1}{6}\right)(\ln(y-2) - \ln(y)) = t + c_1$$
$$\ln(y-2) - \ln(y) = (6)(t + c_1)$$
$$= 6t + 6c_1$$

Raising both side to exponential gives

$$e^{\ln(y-2) - \ln(y)} = 6c_1 e^{6t}$$

Which simplifies to

$$\frac{y-2}{y} = c_2 e^{6t}$$

Initial conditions are used to solve for c_2. Substituting $t = -2$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{2}{-1 + c_2 e^{-12}}$$

$$c_2 = 3\,e^{12}$$

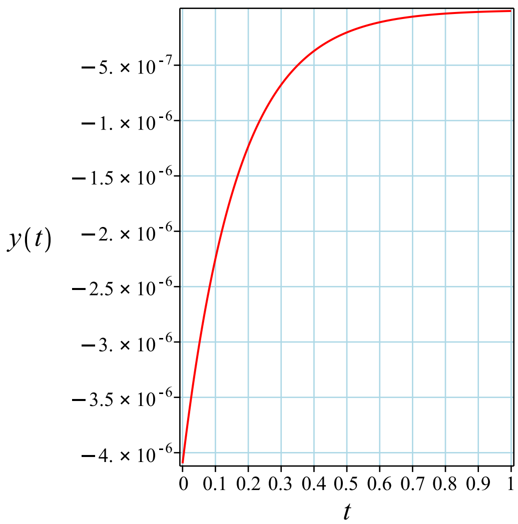
Substituting c_2 found above in the general solution gives

$$y = -\frac{2}{-1 + 3\,e^{12+6t}}$$

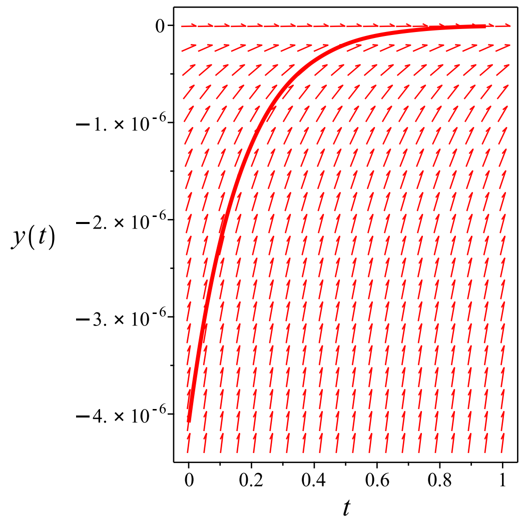
Summary

The solution(s) found are the following

$$y = -\frac{2}{-1 + 3\,e^{12+6t}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2}{-1 + 3\,e^{12+6t}}$$

Verified OK.

5.2.3 Maple step by step solution

Let's solve

$[y' - 3y(y-2) = 0, y(-2) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-2)} = 3$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(y-2)} dt = \int 3dt + c_1$

- Evaluate integral

  $\frac{\ln(y-2)}{2} - \frac{\ln(y)}{2} = 3t + c_1$
- Solve for y

  $y = -\frac{2}{e^{6t+2c_1}-1}$
- Use initial condition $y(-2) = -1$

  $-1 = -\frac{2}{e^{-12+2c_1}-1}$
- Solve for c_1

  $c_1 = 6 + \frac{\ln(3)}{2}$
- Substitute $c_1 = 6 + \frac{\ln(3)}{2}$ into general solution and simplify

  $y = -\frac{2}{-1+3\,e^{12+6t}}$
- Solution to the IVP

  $y = -\frac{2}{-1+3\,e^{12+6t}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 18

```
dsolve([diff(y(t),t)=3*y(t)*(y(t)-2),y(-2) = -1],y(t), singsol=all)
```

$$y(t) = -\frac{2}{3\,e^{6t+12} - 1}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[{y'[t]==3*y[t]*(y[t]-2),{y[-2]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{2}{1 - 3e^{6(t+2)}}$$

5.3 problem 1 and 13 (iii)

Internal problem ID [12953]
Internal file name [`OUTPUT/11605_Tuesday_November_07_2023_11_51_56_PM_10796876/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 1 and 13 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 3y(y-2) = 0}$$

With initial conditions

$$[y(0) = 3]$$

5.3.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= 3y(y-2) \end{aligned}$$

The y domain of $f(t,y)$ when $t=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(3y(y-2)) \\ &= 6y - 6 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 3$ is inside this domain. Therefore solution exists and is unique.

5.3.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{3y(y-2)} dy = \int dt$$
$$\frac{\ln(y-2)}{6} - \frac{\ln(y)}{6} = t + c_1$$

The above can be written as

$$\left(\frac{1}{6}\right)(\ln(y-2) - \ln(y)) = t + c_1$$
$$\ln(y-2) - \ln(y) = (6)(t + c_1)$$
$$= 6t + 6c_1$$

Raising both side to exponential gives

$$e^{\ln(y-2)-\ln(y)} = 6c_1 e^{6t}$$

Which simplifies to

$$\frac{y-2}{y} = c_2 e^{6t}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -\frac{2}{-1 + c_2}$$

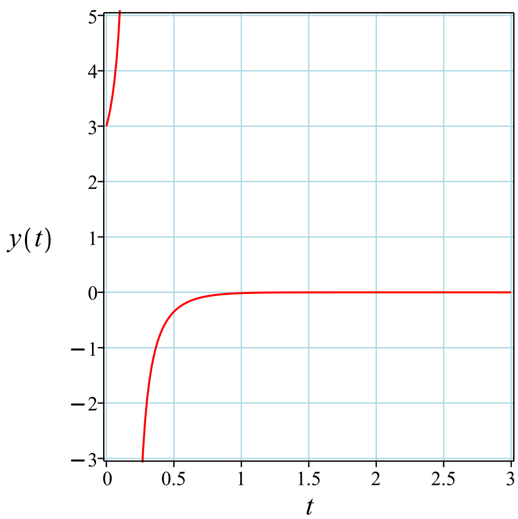
$$c_2 = \frac{1}{3}$$

Substituting c_2 found above in the general solution gives

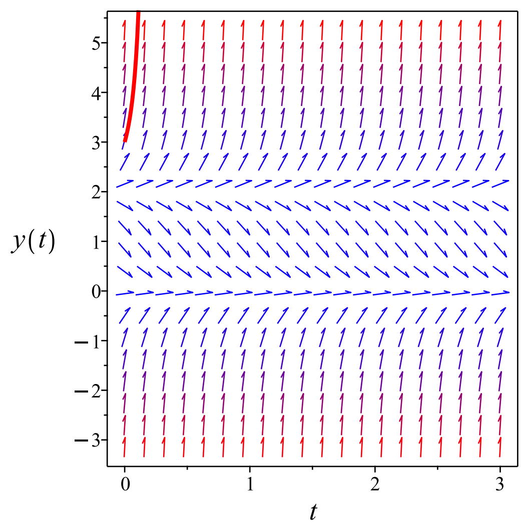
$$y = -\frac{6}{e^{6t} - 3}$$

Summary
The solution(s) found are the following

$$y = -\frac{6}{e^{6t} - 3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{6}{e^{6t} - 3}$$

Verified OK.

5.3.3 Maple step by step solution

Let's solve

$[y' - 3y(y - 2) = 0, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-2)} = 3$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(y-2)}dt = \int 3dt + c_1$

- Evaluate integral

  $\frac{\ln(y-2)}{2} - \frac{\ln(y)}{2} = 3t + c_1$

- Solve for y

  $y = -\frac{2}{e^{6t+2c_1}-1}$

- Use initial condition $y(0) = 3$

  $3 = -\frac{2}{e^{2c_1}-1}$

- Solve for c_1

  $c_1 = -\frac{\ln(3)}{2}$

- Substitute $c_1 = -\frac{\ln(3)}{2}$ into general solution and simplify

  $y = -\frac{6}{e^{6t}-3}$

- Solution to the IVP

  $y = -\frac{6}{e^{6t}-3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(y(t),t)=3*y(t)*(y(t)-2),y(0) = 3],y(t), singsol=all)
```

$$y(t) = -\frac{6}{e^{6t} - 3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 16

```
DSolve[{y'[t]==3*y[t]*(y[t]-2),{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{6}{e^{6t} - 3}$$

5.4 problem 1 and 13 (iv)

Internal problem ID [12954]
Internal file name [`OUTPUT/11606_Tuesday_November_07_2023_11_51_56_PM_11603891/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 1 and 13 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 3y(y-2) = 0}$$

With initial conditions

$$[y(0) = 2]$$

5.4.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= 3y(y-2) \end{aligned}$$

The y domain of $f(t,y)$ when $t=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(3y(y-2)) \\ &= 6y - 6 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

5.4.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{3y(y-2)} dy = \int dt$$
$$\frac{\ln(y-2)}{6} - \frac{\ln(y)}{6} = t + c_1$$

The above can be written as

$$\left(\frac{1}{6}\right)(\ln(y-2) - \ln(y)) = t + c_1$$
$$\ln(y-2) - \ln(y) = (6)(t + c_1)$$
$$= 6t + 6c_1$$

Raising both side to exponential gives

$$e^{\ln(y-2)-\ln(y)} = 6c_1 e^{6t}$$

Which simplifies to

$$\frac{y-2}{y} = c_2 e^{6t}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{2}{-1 + c_2}$$

$$c_2 = 0$$

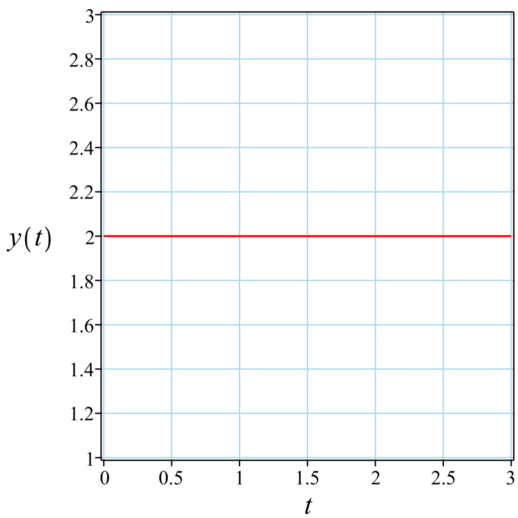
Substituting c_2 found above in the general solution gives

$$y = 2$$

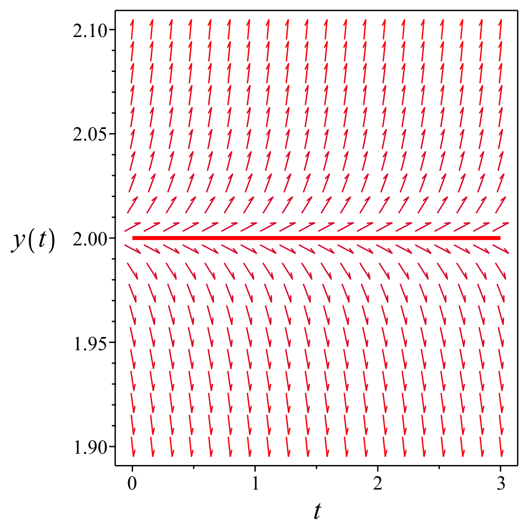
Summary

The solution(s) found are the following

$$y = 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2$$

Verified OK.

5.4.3 Maple step by step solution

Let's solve

$[y' - 3y(y - 2) = 0, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-2)} = 3$

- Integrate both sides with respect to t

  $\int \frac{y'}{y(y-2)} dt = \int 3dt + c_1$

- Evaluate integral

$$\frac{\ln(y-2)}{2} - \frac{\ln(y)}{2} = 3t + c_1$$

- Solve for y

$$y = -\frac{2}{e^{6t+2c_1} - 1}$$

- Use initial condition $y(0) = 2$

$$2 = -\frac{2}{e^{2c_1} - 1}$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(t),t)=3*y(t)*(y(t)-2),y(0) = 2],y(t), singsol=all)
```

$$y(t) = 2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[t]==3*y[t]*(y[t]-2),{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2$$

5.5 problem 2 and 14(i)

Internal problem ID [12955]
Internal file name [`OUTPUT/11607_Tuesday_November_07_2023_11_51_57_PM_65364348/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 2 and 14(i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 4y = -12}$$

With initial conditions

$$[y(0) = 1]$$

5.5.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y - 12 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - 4y - 12\right) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.5.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4y - 12} dy = \int dt$$
$$\frac{\ln (y - 6)}{8} - \frac{\ln (y + 2)}{8} = t + c_1$$

The above can be written as

$$\left(\frac{1}{8}\right) (\ln (y - 6) - \ln (y + 2)) = t + c_1$$
$$\ln (y - 6) - \ln (y + 2) = (8) (t + c_1)$$
$$= 8t + 8c_1$$

Raising both side to exponential gives

$$e^{\ln(y-6)-\ln(y+2)} = 8c_1 e^{8t}$$

Which simplifies to

$$\frac{y - 6}{y + 2} = c_2 e^{8t}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-2c_2 - 6}{-1 + c_2}$$

$$c_2 = -\frac{5}{3}$$

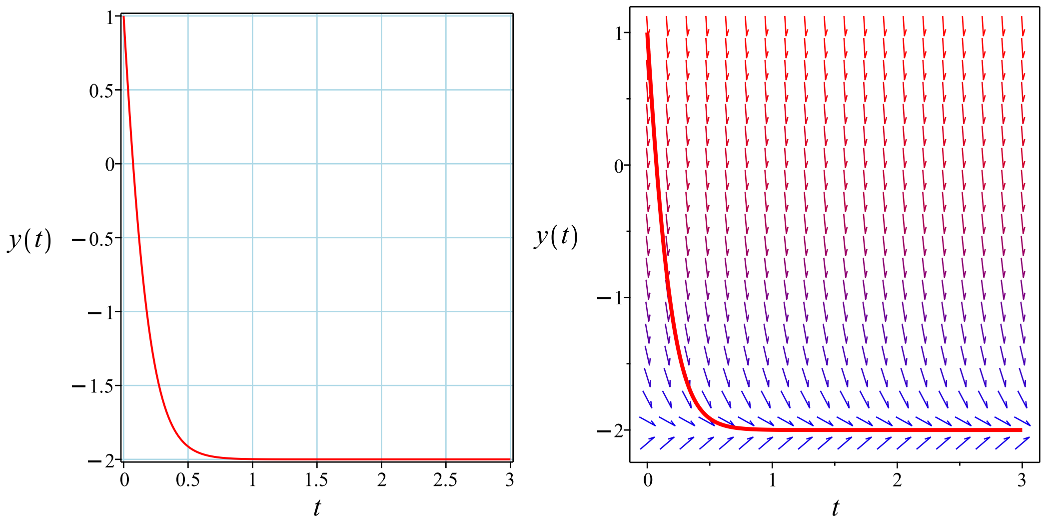
Substituting c_2 found above in the general solution gives

$$y = \frac{-10\, e^{8t} + 18}{5\, e^{8t} + 3}$$

Summary

The solution(s) found are the following

$$y = \frac{-10\,e^{8t} + 18}{5\,e^{8t} + 3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-10\,e^{8t} + 18}{5\,e^{8t} + 3}$$

Verified OK.

5.5.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = -12, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y-12} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y-12}dt = \int 1dt + c_1$

- Evaluate integral

  $\frac{\ln(y-6)}{8}-\frac{\ln(y+2)}{8}=t+c_1$

- Solve for y

  $y=-\frac{2(3+\mathrm{e}^{8t+8c_1})}{\mathrm{e}^{8t+8c_1}-1}$

- Use initial condition $y(0)=1$

  $1=-\frac{2(3+\mathrm{e}^{8c_1})}{\mathrm{e}^{8c_1}-1}$

- Solve for c_1

  $c_1=\frac{\ln\left(\frac{5}{3}\right)}{8}+\frac{\mathrm{I}\pi}{8}$

- Substitute $c_1=\frac{\ln\left(\frac{5}{3}\right)}{8}+\frac{\mathrm{I}\pi}{8}$ into general solution and simplify

  $y=\frac{-10\,\mathrm{e}^{8t}+18}{5\,\mathrm{e}^{8t}+3}$

- Solution to the IVP

  $y=\frac{-10\,\mathrm{e}^{8t}+18}{5\,\mathrm{e}^{8t}+3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 23

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)-12,y(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{18 - 10\,e^{8t}}{5\,e^{8t} + 3}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 26

```
DSolve[{y'[t]==y[t]^2-4*y[t]-12,{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{18 - 10e^{8t}}{5e^{8t} + 3}$$

5.6 problem 2 and 14(ii)

Internal problem ID [12956]
Internal file name [`OUTPUT/11608_Tuesday_November_07_2023_11_51_58_PM_29876747/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 2 and 14(ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 4y = -12}$$

With initial conditions

$$[y(1) = 0]$$

5.6.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y - 12 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - 4y - 12\right) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

5.6.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4y - 12} dy = \int dt$$
$$\frac{\ln(y-6)}{8} - \frac{\ln(y+2)}{8} = t + c_1$$

The above can be written as

$$\left(\frac{1}{8}\right)(\ln(y-6) - \ln(y+2)) = t + c_1$$
$$\ln(y-6) - \ln(y+2) = (8)(t + c_1)$$
$$= 8t + 8c_1$$

Raising both side to exponential gives

$$e^{\ln(y-6)-\ln(y+2)} = 8c_1 e^{8t}$$

Which simplifies to

$$\frac{y-6}{y+2} = c_2 e^{8t}$$

Initial conditions are used to solve for c_2. Substituting $t = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-2c_2 e^8 - 6}{-1 + c_2 e^8}$$

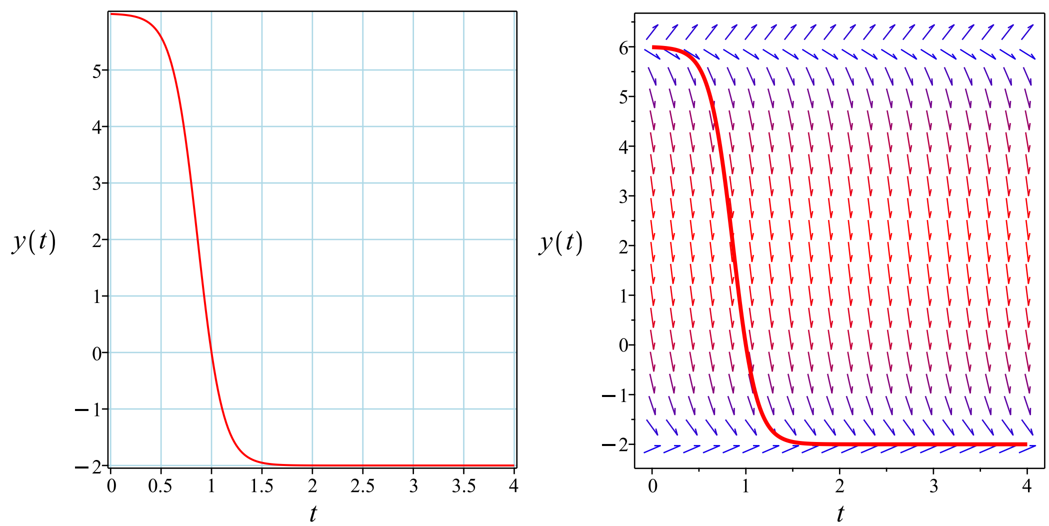
$$c_2 = -3\,e^{-8}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{-6\,e^{-8+8t} + 6}{3\,e^{-8+8t} + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{-6\,\mathrm{e}^{-8+8t}+6}{3\,\mathrm{e}^{-8+8t}+1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-6\,\mathrm{e}^{-8+8t}+6}{3\,\mathrm{e}^{-8+8t}+1}$$

Verified OK.

5.6.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = -12, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y-12} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y-12}dt = \int 1dt + c_1$

- Evaluate integral

  $\frac{\ln(y-6)}{8} - \frac{\ln(y+2)}{8} = t + c_1$

- Solve for y

  $y = -\frac{2(3+e^{8t+8c_1})}{e^{8t+8c_1}-1}$

- Use initial condition $y(1) = 0$

  $0 = -\frac{2(3+e^{8+8c_1})}{e^{8+8c_1}-1}$

- Solve for c_1

  $c_1 = -1 + \frac{\ln(3)}{8} + \frac{I\pi}{8}$

- Substitute $c_1 = -1 + \frac{\ln(3)}{8} + \frac{I\pi}{8}$ into general solution and simplify

  $y = \frac{-6\,e^{-8+8t}+6}{3\,e^{-8+8t}+1}$

- Solution to the IVP

  $y = \frac{-6\,e^{-8+8t}+6}{3\,e^{-8+8t}+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 26

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)-12,y(1) = 0],y(t), singsol=all)
```

$$y(t) = \frac{6 - 6\,\mathrm{e}^{-8+8t}}{3\,\mathrm{e}^{-8+8t} + 1}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 32

```
DSolve[{y'[t]==y[t]^2-4*y[t]-12,{y[1]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{6e^8 - 6e^{8t}}{3e^{8t} + e^8}$$

5.7 problem 2 and 14(iii)

Internal problem ID [12957]
Internal file name [`OUTPUT/11609_Tuesday_November_07_2023_11_51_59_PM_96989353/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 2 and 14(iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 4y = -12}$$

With initial conditions

$$[y(0) = 6]$$

5.7.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y - 12 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 6$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - 4y - 12\right) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 6$ is inside this domain. Therefore solution exists and is unique.

5.7.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4y - 12} dy = \int dt$$
$$\frac{\ln(y-6)}{8} - \frac{\ln(y+2)}{8} = t + c_1$$

The above can be written as

$$\left(\frac{1}{8}\right)(\ln(y-6) - \ln(y+2)) = t + c_1$$
$$\ln(y-6) - \ln(y+2) = (8)(t + c_1)$$
$$= 8t + 8c_1$$

Raising both side to exponential gives

$$e^{\ln(y-6) - \ln(y+2)} = 8c_1 e^{8t}$$

Which simplifies to

$$\frac{y-6}{y+2} = c_2 e^{8t}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$6 = \frac{-2c_2 - 6}{-1 + c_2}$$

$$c_2 = 0$$

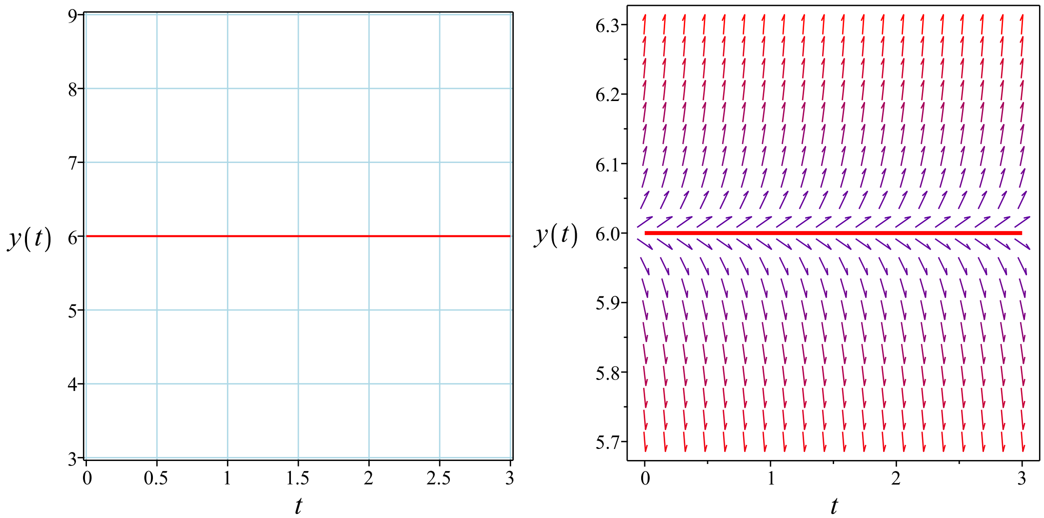
Substituting c_2 found above in the general solution gives

$$y = 6$$

Summary

The solution(s) found are the following

$$y = 6 \tag{1}$$



Verification of solutions

$$y = 6$$

Verified OK.

5.7.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = -12, y(0) = 6]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y-12} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y-12} dt = \int 1 dt + c_1$

- Evaluate integral

$$\frac{\ln(y-6)}{8} - \frac{\ln(y+2)}{8} = t + c_1$$

- Solve for y

$$y = -\frac{2(3 + e^{8t+8c_1})}{e^{8t+8c_1} - 1}$$

- Use initial condition $y(0) = 6$

$$6 = -\frac{2(3 + e^{8c_1})}{e^{8c_1} - 1}$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 5

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)-12,y(0) = 6],y(t), singsol=all)
```

$$y(t) = 6$$

✓ Solution by Mathematica
Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[t]==y[t]^2-4*y[t]-12,{y[0]==6}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 6$$

5.8 problem 2 and 14(iv)

Internal problem ID [12958]
Internal file name [`OUTPUT/11610_Tuesday_November_07_2023_11_52_00_PM_8019104/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 2 and 14(iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 4y = -12}$$

With initial conditions

$$[y(0) = 5]$$

5.8.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y - 12 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 5$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - 4y - 12\right) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 5$ is inside this domain. Therefore solution exists and is unique.

5.8.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{y^2 - 4y - 12} dy &= \int dt \\ \frac{\ln(y-6)}{8} - \frac{\ln(y+2)}{8} &= t + c_1 \end{aligned}$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{8}\right)(\ln(y-6) - \ln(y+2)) &= t + c_1 \\ \ln(y-6) - \ln(y+2) &= (8)(t + c_1) \\ &= 8t + 8c_1 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(y-6) - \ln(y+2)} = 8c_1 e^{8t}$$

Which simplifies to

$$\frac{y-6}{y+2} = c_2 e^{8t}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \frac{-2c_2 - 6}{-1 + c_2}$$

$$c_2 = -\frac{1}{7}$$

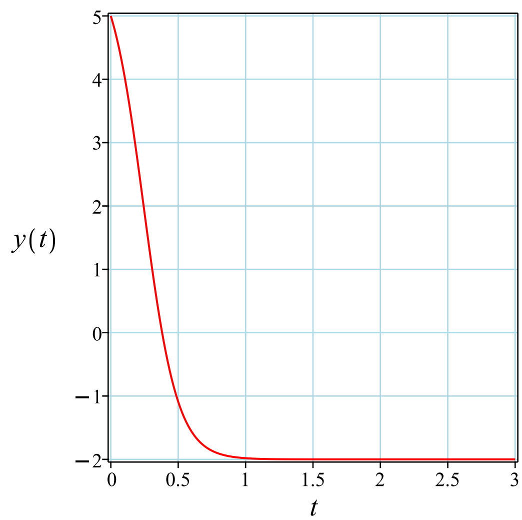
Substituting c_2 found above in the general solution gives

$$y = \frac{-2\,e^{8t} + 42}{e^{8t} + 7}$$

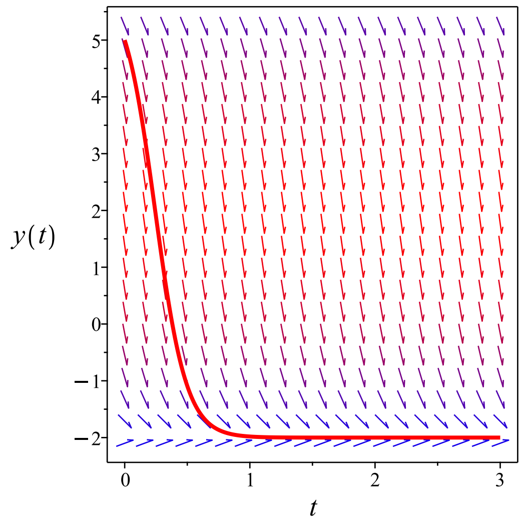
Summary

The solution(s) found are the following

$$y = \frac{-2\,e^{8t} + 42}{e^{8t} + 7} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-2\,e^{8t} + 42}{e^{8t} + 7}$$

Verified OK.

5.8.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = -12, y(0) = 5]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y-12} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y-12}dt = \int 1dt + c_1$

- Evaluate integral

  $\frac{\ln(y-6)}{8} - \frac{\ln(y+2)}{8} = t + c_1$
- Solve for y

  $y = -\frac{2(3+\mathrm{e}^{8t+8c_1})}{\mathrm{e}^{8t+8c_1}-1}$
- Use initial condition $y(0) = 5$

  $5 = -\frac{2(3+\mathrm{e}^{8c_1})}{\mathrm{e}^{8c_1}-1}$
- Solve for c_1

  $c_1 = -\frac{\ln(7)}{8} + \frac{\mathrm{I}\pi}{8}$
- Substitute $c_1 = -\frac{\ln(7)}{8} + \frac{\mathrm{I}\pi}{8}$ into general solution and simplify

  $y = \frac{-2\,\mathrm{e}^{8t}+42}{\mathrm{e}^{8t}+7}$
- Solution to the IVP

  $y = \frac{-2\,\mathrm{e}^{8t}+42}{\mathrm{e}^{8t}+7}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 20

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)-12,y(0) = 5],y(t), singsol=all)
```

$$y(t) = \frac{42 - 2\,e^{8t}}{e^{8t} + 7}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 24

```
DSolve[{y'[t]==y[t]^2-4*y[t]-12,{y[0]==5}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{42 - 2e^{8t}}{e^{8t} + 7}$$

5.9 problem 3 and 15(i)

Internal problem ID [12959]
Internal file name [`OUTPUT/11611_Tuesday_November_07_2023_11_52_01_PM_45048317/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 3 and 15(i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \cos(y) = 0}$$

With initial conditions

$$[y(0) = 0]$$

5.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \cos(y) \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(\cos(y)) \\ &= -\sin(y) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

5.9.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\cos(y)} dy = \int dt$$
$$\ln(\sec(y) + \tan(y)) = t + c_1$$

Raising both side to exponential gives

$$\sec(y) + \tan(y) = e^{t+c_1}$$

Which simplifies to

$$\sec(y) + \tan(y) = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2$$

$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$\sec(y) + \tan(y) = e^t$$

Summary
The solution(s) found are the following

$$\sec(y) + \tan(y) = e^t \tag{1}$$

Verification of solutions

$$\sec(y) + \tan(y) = e^t$$

Verified OK.

5.9.3 Maple step by step solution

Let's solve

$[y' - \cos(y) = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\cos(y)} dt = \int 1 dt + c_1$

- Evaluate integral

  $\ln(\sec(y) + \tan(y)) = t + c_1$

- Solve for y

  $y = \arctan\left(\frac{(e^{t+c_1})^2 - 1}{(e^{t+c_1})^2 + 1}, \frac{2\,e^{t+c_1}}{(e^{t+c_1})^2 + 1}\right)$

- Use initial condition $y(0) = 0$

  $0 = \arctan\left(\frac{(e^{c_1})^2 - 1}{(e^{c_1})^2 + 1}, \frac{2\,e^{c_1}}{(e^{c_1})^2 + 1}\right)$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 32

```
dsolve([diff(y(t),t)=cos( y(t)),y(0) = 0],y(t), singsol=all)
```

$$y(t) = \arctan\left(\frac{e^{2t} - 1}{e^{2t} + 1}, \frac{2\,e^{t}}{e^{2t} + 1}\right)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 8

```
DSolve[{y'[t]==Cos[ y[t]],{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \arcsin(\tanh(t))$$

5.10 problem 3 and 15(ii)

Internal problem ID [12960]
Internal file name [`OUTPUT/11612_Tuesday_November_07_2023_11_52_02_PM_74617560/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 3 and 15(ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \cos(y) = 0}$$

With initial conditions

$$[y(-1) = 1]$$

5.10.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \cos(y) \end{aligned}$$

The y domain of $f(t, y)$ when $t = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(\cos(y)) \\ &= -\sin(y) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.10.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\cos(y)} dy = \int dt$$
$$\ln(\sec(y) + \tan(y)) = t + c_1$$

Raising both side to exponential gives

$$\sec(y) + \tan(y) = e^{t+c_1}$$

Which simplifies to

$$\sec(y) + \tan(y) = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\sec(1) + \tan(1) = c_2 e^{-1}$$

$$c_2 = (\sec(1) + \tan(1)) e$$

Substituting c_2 found above in the general solution gives

$$\sec(y) + \tan(y) = \sec(1) e^{1+t} + \tan(1) e^{1+t}$$

Summary
The solution(s) found are the following

$$\sec(y) + \tan(y) = e^{1+t}(\sec(1) + \tan(1)) \tag{1}$$

Verification of solutions

$$\sec(y) + \tan(y) = e^{1+t}(\sec(1) + \tan(1))$$

Verified OK.

5.10.3 Maple step by step solution

Let's solve

$[y' - \cos(y) = 0, y(-1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\cos(y)} dt = \int 1 dt + c_1$

- Evaluate integral

  $\ln(\sec(y) + \tan(y)) = t + c_1$

- Solve for y

  $y = \arctan\left(\frac{(e^{t+c_1})^2 - 1}{(e^{t+c_1})^2 + 1}, \frac{2\,e^{t+c_1}}{(e^{t+c_1})^2 + 1}\right)$

- Use initial condition $y(-1) = 1$

  $1 = \arctan\left(\frac{(e^{-1+c_1})^2 - 1}{(e^{-1+c_1})^2 + 1}, \frac{2\,e^{-1+c_1}}{(e^{-1+c_1})^2 + 1}\right)$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 1.672 (sec). Leaf size: 79

```
dsolve([diff(y(t),t)=cos( y(t)),y(-1) = 1],y(t), singsol=all)
```

$$y(t) = \arctan\left(\frac{\sin(1)\,e^{2+2t} + e^{2+2t} + \sin(1) - 1}{\sin(1)\,e^{2+2t} + e^{2+2t} - \sin(1) + 1}, \frac{2\,e^{t+1}\cos(1)}{\sin(1)\,e^{2+2t} + e^{2+2t} - \sin(1) + 1}\right)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 13

```
DSolve[{y'[t]==Cos[ y[t]],{y[-1]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \arcsin\left(\coth\left(t + 1 + \coth^{-1}(\sin(1))\right)\right)$$

5.11 problem 3 and 15(iii)

Internal problem ID [12961]
Internal file name [`OUTPUT/11613_Tuesday_November_07_2023_11_52_04_PM_67352258/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 3 and 15(iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \cos(y) = 0}$$

With initial conditions

$$\left[y(0) = -\frac{\pi}{2}\right]$$

5.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \cos(y) \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{\pi}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(\cos(y)) \\ &= -\sin(y) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{\pi}{2}$ is inside this domain. Therefore solution exists and is unique.

5.11.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\cos(y)} dy = \int dt$$
$$\ln(\sec(y) + \tan(y)) = t + c_1$$

Raising both side to exponential gives

$$\sec(y) + \tan(y) = e^{t+c_1}$$

Which simplifies to

$$\sec(y) + \tan(y) = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = -\frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$\sec(y) + \tan(y) = 0$$

Summary

The solution(s) found are the following

$$\sec(y) + \tan(y) = 0 \tag{1}$$

Verification of solutions

$$\sec(y) + \tan(y) = 0$$

Verified OK.

5.11.3 Maple step by step solution

Let's solve

$\left[y' - \cos\left(y\right) = 0, y(0) = -\frac{\pi}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\cos(y)}dt = \int 1dt + c_1$

- Evaluate integral

  $\ln\left(\sec\left(y\right) + \tan\left(y\right)\right) = t + c_1$

- Solve for y

  $y = \arctan\left(\frac{\left(\mathrm{e}^{t+c_1}\right)^2-1}{\left(\mathrm{e}^{t+c_1}\right)^2+1}, \frac{2\,\mathrm{e}^{t+c_1}}{\left(\mathrm{e}^{t+c_1}\right)^2+1}\right)$

- Use initial condition $y(0) = -\frac{\pi}{2}$

  $-\frac{\pi}{2} = \arctan\left(\frac{\left(\mathrm{e}^{c_1}\right)^2-1}{\left(\mathrm{e}^{c_1}\right)^2+1}, \frac{2\,\mathrm{e}^{c_1}}{\left(\mathrm{e}^{c_1}\right)^2+1}\right)$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 7

```
dsolve([diff(y(t),t)=cos( y(t)),y(0) = -1/2*Pi],y(t), singsol=all)
```

$$y(t) = -\frac{\pi}{2}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 10

```
DSolve[{y'[t]==Cos[ y[t]],{y[0]==-Pi/2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{\pi}{2}$$

5.12 problem 3 and 15(iv)

Internal problem ID [12962]
Internal file name [`OUTPUT/11614_Tuesday_November_07_2023_11_52_05_PM_58954603/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 3 and 15(iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \cos(y) = 0}$$

With initial conditions

$$[y(0) = \pi]$$

5.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= \cos(y) \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \pi$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(\cos(y)) \\ &= -\sin(y) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \pi$ is inside this domain. Therefore solution exists and is unique.

5.12.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\cos(y)} dy = \int dt$$
$$\ln(\sec(y) + \tan(y)) = t + c_1$$

Raising both side to exponential gives

$$\sec(y) + \tan(y) = e^{t+c_1}$$

Which simplifies to

$$\sec(y) + \tan(y) = c_2 e^t$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = \pi$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_2$$

$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

$$\sec(y) + \tan(y) = -e^t$$

Summary
The solution(s) found are the following

$$\sec(y) + \tan(y) = -e^t \tag{1}$$

Verification of solutions

$$\sec(y) + \tan(y) = -e^t$$

Verified OK.

5.12.3 Maple step by step solution

Let's solve

$[y' - \cos(y) = 0, y(0) = \pi]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\cos(y)} dt = \int 1 dt + c_1$

- Evaluate integral

  $\ln(\sec(y) + \tan(y)) = t + c_1$

- Solve for y

  $y = \arctan\left(\frac{(e^{t+c_1})^2 - 1}{(e^{t+c_1})^2 + 1}, \frac{2\,e^{t+c_1}}{(e^{t+c_1})^2 + 1}\right)$

- Use initial condition $y(0) = \pi$

  $\pi = \arctan\left(\frac{(e^{c_1})^2 - 1}{(e^{c_1})^2 + 1}, \frac{2\,e^{c_1}}{(e^{c_1})^2 + 1}\right)$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve([diff(y(t),t)=cos( y(t)),y(0) = Pi],y(t), singsol=all)
```

$$y(t) = \arctan\left(\frac{e^{2t}-1}{e^{2t}+1}, -\frac{2\,e^{t}}{e^{2t}+1}\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y'[t]==Cos[ y[t]],{y[0]==Pi}},y[t],t,IncludeSingularSolutions -> True]
```

{}

5.13 problem 4

Internal problem ID [12963]
Internal file name [`OUTPUT/11615_Tuesday_November_07_2023_11_52_06_PM_48665625/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{w' - w\cos(w) = 0}$$

5.13.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{w\cos(w)}dw = \int dt$$
$$\int^{w} \frac{1}{\_a\cos(\_a)}d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^{w} \frac{1}{\_a\cos(\_a)}d\_a = t + c_1 \tag{1}$$

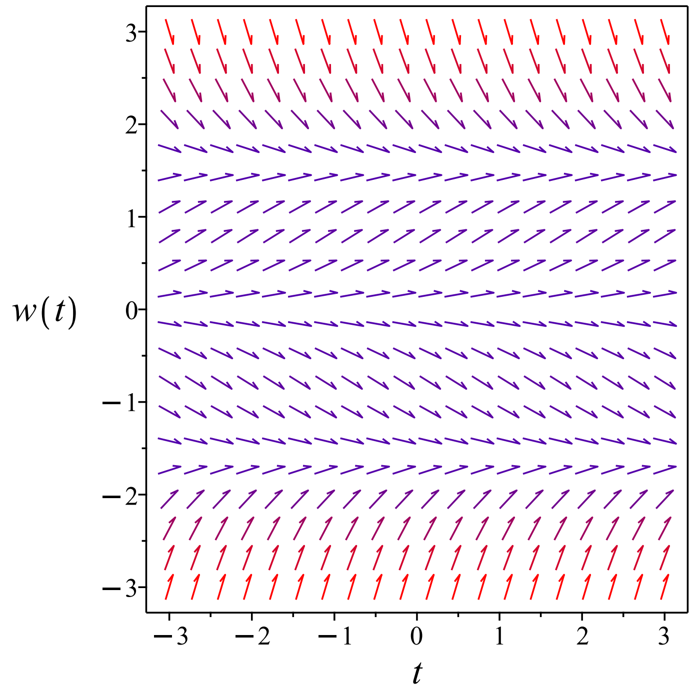


Figure 154: Slope field plot

Verification of solutions

$$\int^{w} \frac{1}{\_a \cos(\_a)} d\_a = t + c_1$$

Verified OK.

5.13.2 Maple step by step solution

Let's solve

$w' - w\cos(w) = 0$

- Highest derivative means the order of the ODE is 1

  w'

- Separate variables

  $\frac{w'}{w\cos(w)} = 1$

- Integrate both sides with respect to t

  $\int \frac{w'}{w\cos(w)} dt = \int 1 dt + c_1$

- Cannot compute integral

$$\int \frac{w'}{w\cos(w)}dt = t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(w(t),t)=w(t)*cos( w(t)),w(t), singsol=all)
```

$$t - \left(\int^{w(t)} \frac{\sec(\_a)}{\_a} d\_a\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 7.857 (sec). Leaf size: 50

```
DSolve[w'[t]==w[t]*Cos[ w[t]],w[t],t,IncludeSingularSolutions -> True]
```

$$\begin{aligned}
w(t) &\to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sec(K[1])}{K[1]} dK[1]\&\right][t + c_1] \\
w(t) &\to 0 \\
w(t) &\to -\frac{\pi}{2} \\
w(t) &\to \frac{\pi}{2}
\end{aligned}$$

5.14 problem 4 and 16(i)

Internal problem ID [12964]
Internal file name [`OUTPUT/11616_Tuesday_November_07_2023_11_52_06_PM_51224347/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 4 and 16(i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{w' - w\cos(w) = 0}$$

With initial conditions

$$[w(0) = 0]$$

5.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} w' &= f(t, w) \\ &= w\cos(w) \end{aligned}$$

The w domain of $f(t, w)$ when $t = 0$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial w} &= \frac{\partial}{\partial w}(w\cos(w)) \\ &= \cos(w) - w\sin(w) \end{aligned}$$

The w domain of $\frac{\partial f}{\partial w}$ when $t = 0$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = 0$ is inside this domain. Therefore solution exists and is unique.

5.14.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{w \cos(w)} dw = \int dt$$
$$\int^{w} \frac{1}{\_a \cos(\_a)} d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $w = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{0} \frac{1}{\_a \cos(\_a)} d\_a = c_1$$

$$c_1 = \int^{0} \frac{\sec(\_a)}{\_a} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^{w} \frac{1}{\_a \cos(\_a)} d\_a = t + \int^{0} \frac{\sec(\_a)}{\_a} d\_a$$

Solving for w from the above gives

$$w = \text{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a} d\_a\right) + t + \int^{0} \frac{\sec(\_a)}{\_a} d\_a\right)$$

Summary

The solution(s) found are the following

$$w = \text{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a} d\_a\right) + t + \int^{0} \frac{\sec(\_a)}{\_a} d\_a\right) \tag{1}$$

Verification of solutions

$$w = \text{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a} d\_a\right) + t + \int^{0} \frac{\sec(\_a)}{\_a} d\_a\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(w(t),t)=w(t)*cos( w(t)),w(0) = 0],w(t), singsol=all)
```

$$w(t) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{w'[t]==w[t]*Cos[ w[t]],{w[0]==0}},w[t],t,IncludeSingularSolutions -> True]
```

$$w(t) \to 0$$

5.15 problem 4 and 16(ii)

Internal problem ID [12965]
Internal file name [`OUTPUT/11617_Tuesday_November_07_2023_11_52_07_PM_93951882/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 4 and 16(ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{w' - w\cos(w) = 0}$$

With initial conditions

$$[w(3) = 1]$$

5.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} w' &= f(t, w) \\ &= w\cos(w) \end{aligned}$$

The w domain of $f(t, w)$ when $t = 3$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial w} &= \frac{\partial}{\partial w}(w\cos(w)) \\ &= \cos(w) - w\sin(w) \end{aligned}$$

The w domain of $\frac{\partial f}{\partial w}$ when $t=3$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.15.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{w\cos(w)}dw = \int dt$$
$$\int^{w} \frac{1}{\_a\cos(\_a)}d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t=3$ and $w=1$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{1} \frac{1}{\_a\cos(\_a)}d\_a = 3 + c_1$$

$$c_1 = -3 + \int^{1} \frac{\sec(\_a)}{\_a}d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^{w} \frac{1}{\_a\cos(\_a)}d\_a = t - 3 + \int^{1} \frac{\sec(\_a)}{\_a}d\_a$$

Solving for w from the above gives

$$w = \mathrm{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a}d\_a\right) + t - 3 + \int^{1} \frac{\sec(\_a)}{\_a}d\_a\right)$$

Summary

The solution(s) found are the following

$$w = \mathrm{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a}d\_a\right) + t - 3 + \int^{1} \frac{\sec(\_a)}{\_a}d\_a\right) \tag{1}$$

Verification of solutions

$$w = \mathrm{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a}d\_a\right) + t - 3 + \int^{1} \frac{\sec(\_a)}{\_a}d\_a\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.407 (sec). Leaf size: 20

```
dsolve([diff(w(t),t)=w(t)*cos( w(t)),w(3) = 1],w(t), singsol=all)
```

$$w(t) = \mathrm{RootOf}\left(\int_{\_Z}^{1} \frac{\sec(\_a)}{\_a} d\_a + t - 3\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{w'[t]==w[t]*Cos[ w[t]],{w[3]==1}},w[t],t,IncludeSingularSolutions -> True]
```

{}

5.16 problem 4 and 16(iii)

Internal problem ID [12966]
Internal file name [`OUTPUT/11618_Tuesday_November_07_2023_11_52_08_PM_34366214/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 4 and 16(iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{w' - w\cos(w) = 0}$$

With initial conditions

$$[w(0) = 2]$$

5.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} w' &= f(t, w) \\ &= w\cos(w) \end{aligned}$$

The w domain of $f(t, w)$ when $t = 0$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial w} &= \frac{\partial}{\partial w}(w\cos(w)) \\ &= \cos(w) - w\sin(w) \end{aligned}$$

The w domain of $\frac{\partial f}{\partial w}$ when $t = 0$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = 2$ is inside this domain. Therefore solution exists and is unique.

5.16.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{w\cos(w)}dw = \int dt$$
$$\int^{w} \frac{1}{\_a\cos(\_a)}d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $w = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{2} \frac{1}{\_a\cos(\_a)}d\_a = c_1$$

$$c_1 = \int^{2} \frac{\sec(\_a)}{\_a}d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^{w} \frac{1}{\_a\cos(\_a)}d\_a = t + \int^{2} \frac{\sec(\_a)}{\_a}d\_a$$

Solving for w from the above gives

$$w = \text{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a}d\_a\right) + t + \int^{2} \frac{\sec(\_a)}{\_a}d\_a\right)$$

Summary

The solution(s) found are the following

$$w = \text{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a}d\_a\right) + t + \int^{2} \frac{\sec(\_a)}{\_a}d\_a\right) \tag{1}$$

Verification of solutions

$$w = \text{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a}d\_a\right) + t + \int^{2} \frac{\sec(\_a)}{\_a}d\_a\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 19

```
dsolve([diff(w(t),t)=w(t)*cos( w(t)),w(0) = 2],w(t), singsol=all)
```

$$w(t) = \text{RootOf}\left(\int_{\_Z}^{2} \frac{\sec(\_a)}{\_a} d\_a + t\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{w'[t]==w[t]*Cos[ w[t]],{w[0]==2}},w[t],t,IncludeSingularSolutions -> True]
```

{}

5.17 problem 4 and 16(iv)

Internal problem ID [12967]
Internal file name [`OUTPUT/11619_Tuesday_November_07_2023_11_52_09_PM_98692280/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 4 and 16(iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{w' - w\cos(w) = 0}$$

With initial conditions

$$[w(0) = -1]$$

5.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} w' &= f(t, w) \\ &= w\cos(w) \end{aligned}$$

The w domain of $f(t, w)$ when $t = 0$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial w} &= \frac{\partial}{\partial w}(w\cos(w)) \\ &= \cos(w) - w\sin(w) \end{aligned}$$

The w domain of $\frac{\partial f}{\partial w}$ when $t = 0$ is

$$\{-\infty < w < \infty\}$$

And the point $w_0 = -1$ is inside this domain. Therefore solution exists and is unique.

5.17.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{w \cos(w)} dw = \int dt$$
$$\int^{w} \frac{1}{\_a \cos(\_a)} d\_a = t + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $w = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{-1} \frac{1}{\_a \cos(\_a)} d\_a = c_1$$

$$c_1 = \int^{-1} \frac{\sec(\_a)}{\_a} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^{w} \frac{1}{\_a \cos(\_a)} d\_a = t + \int^{-1} \frac{\sec(\_a)}{\_a} d\_a$$

Solving for w from the above gives

$$w = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a} d\_a\right) + t + \int^{-1} \frac{\sec(\_a)}{\_a} d\_a\right)$$

Summary
The solution(s) found are the following

$$w = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a} d\_a\right) + t + \int^{-1} \frac{\sec(\_a)}{\_a} d\_a\right) \tag{1}$$

Verification of solutions

$$w = \operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{\sec(\_a)}{\_a} d\_a\right) + t + \int^{-1} \frac{\sec(\_a)}{\_a} d\_a\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 19

```
dsolve([diff(w(t),t)=w(t)*cos( w(t)),w(0) = -1],w(t), singsol=all)
```

$$w(t) = \mathrm{RootOf}\left(\int_{\_Z}^{-1} \frac{\sec(\_a)}{\_a} d\_a + t\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{w'[t]==w[t]*Cos[ w[t]],{w[0]==-1}},w[t],t,IncludeSingularSolutions -> True]
```

{}

5.18 problem 5

Internal problem ID [12968]
Internal file name [`OUTPUT/11620_Tuesday_November_07_2023_11_52_11_PM_7664455/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{w' - (1-w)\sin(w) = 0}$$

5.18.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{(-1+w)\sin(w)}dw = \int dt$$
$$\int^{w} -\frac{1}{(-1+\_a)\sin(\_a)}d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^{w} -\frac{1}{(-1+\_a)\sin(\_a)}d\_a = t + c_1 \tag{1}$$

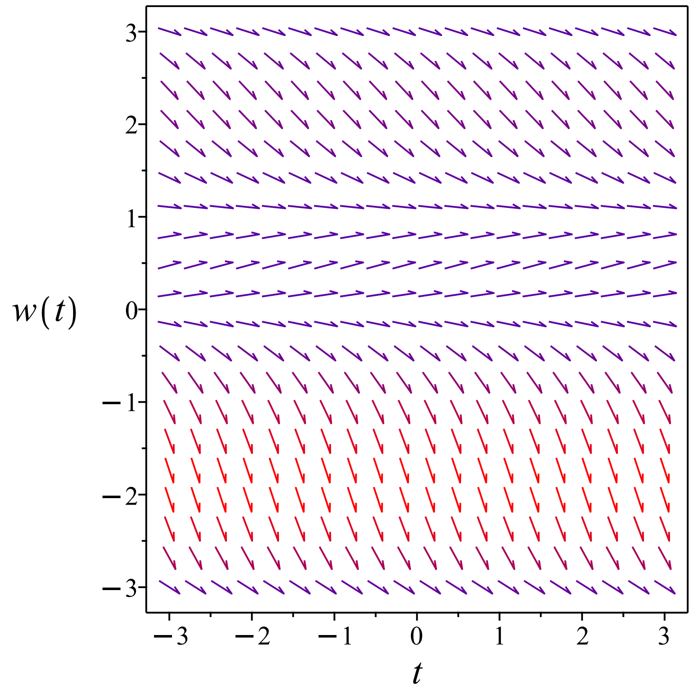


Figure 155: Slope field plot

Verification of solutions

$$\int^{w} -\frac{1}{(-1+\_a)\sin(\_a)}d\_a = t + c_1$$

Verified OK.

5.18.2 Maple step by step solution

Let's solve

$w' - (1-w)\sin(w) = 0$

- Highest derivative means the order of the ODE is 1

  w'

- Separate variables

  $\frac{w'}{(1-w)\sin(w)} = 1$

- Integrate both sides with respect to t

  $\int \frac{w'}{(1-w)\sin(w)}dt = \int 1dt + c_1$

- Cannot compute integral

$$\int \frac{w'}{(1-w)\sin(w)} dt = t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(w(t),t)=(1-w(t))*sin( w(t)),w(t), singsol=all)
```

$$t + \int^{w(t)} \frac{\csc(\_a)}{\_a - 1} d\_a + c_1 = 0$$

✓ Solution by Mathematica

Time used: 12.825 (sec). Leaf size: 41

```
DSolve[w'[t]==(1-w[t])*Sin[ w[t]],w[t],t,IncludeSingularSolutions -> True]
```

$$w(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\csc(K[1])}{K[1]-1} dK[1]\&\right][-t + c_1]$$
$$w(t) \to 0$$
$$w(t) \to 1$$

5.19 problem 6

Internal problem ID [12969]
Internal file name [OUTPUT/11621_Tuesday_November_07_2023_11_52_12_PM_45223112/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - \frac{1}{y-2} = 0}$$

5.19.1 Solving as quadrature ode

Integrating both sides gives

$$\int (y-2)\,dy = t + c_1$$
$$\frac{1}{2}y^2 - 2y = t + c_1$$

Solving for y gives these solutions

$$y_1 = 2 - \sqrt{4 + 2t + 2c_1}$$
$$y_2 = 2 + \sqrt{4 + 2t + 2c_1}$$

Summary
The solution(s) found are the following

$$y = 2 - \sqrt{4 + 2t + 2c_1} \tag{1}$$
$$y = 2 + \sqrt{4 + 2t + 2c_1} \tag{2}$$

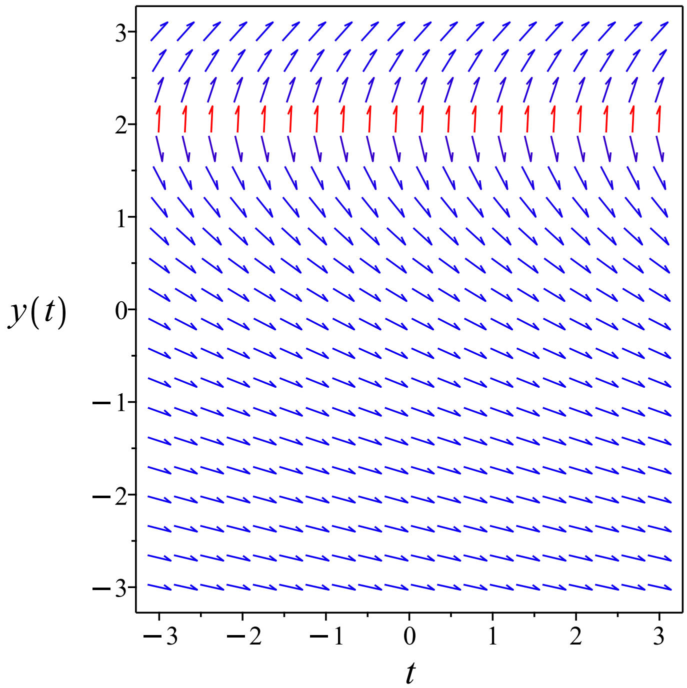


Figure 156: Slope field plot

Verification of solutions

$$y = 2 - \sqrt{4 + 2t + 2c_1}$$

Verified OK.

$$y = 2 + \sqrt{4 + 2t + 2c_1}$$

Verified OK.

5.19.2 Maple step by step solution

Let's solve

$y' - \frac{1}{y-2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(y-2) = 1$

- Integrate both sides with respect to t

$\int y'(y-2)\,dt = \int 1dt + c_1$

- Evaluate integral

  $\frac{y^2}{2} - 2y = t + c_1$

- Solve for y

  $\{y = 2 - \sqrt{4+2t+2c_1}, y = 2 + \sqrt{4+2t+2c_1}\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(y(t),t)=1/(y(t)-2),y(t), singsol=all)
```

$$y(t) = 2 - \sqrt{4+2t+2c_1}$$
$$y(t) = 2 + \sqrt{4+2t+2c_1}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 44

```
DSolve[y'[t]==1/(y[t]-2),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2 - \sqrt{2}\sqrt{t+2+c_1}$$
$$y(t) \to 2 + \sqrt{2}\sqrt{t+2+c_1}$$

5.20 problem 7

Internal problem ID [12970]
Internal file name [`OUTPUT/11622_Tuesday_November_07_2023_11_52_13_PM_76106352/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{v' + v^2 + 2v = -2}$$

5.20.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-v^2 - 2v - 2} dv = t + c_1$$
$$-\arctan(v+1) = t + c_1$$

Solving for v gives these solutions

$$v_1 = -1 - \tan(t + c_1)$$

Summary
The solution(s) found are the following

$$v = -1 - \tan(t + c_1) \tag{1}$$

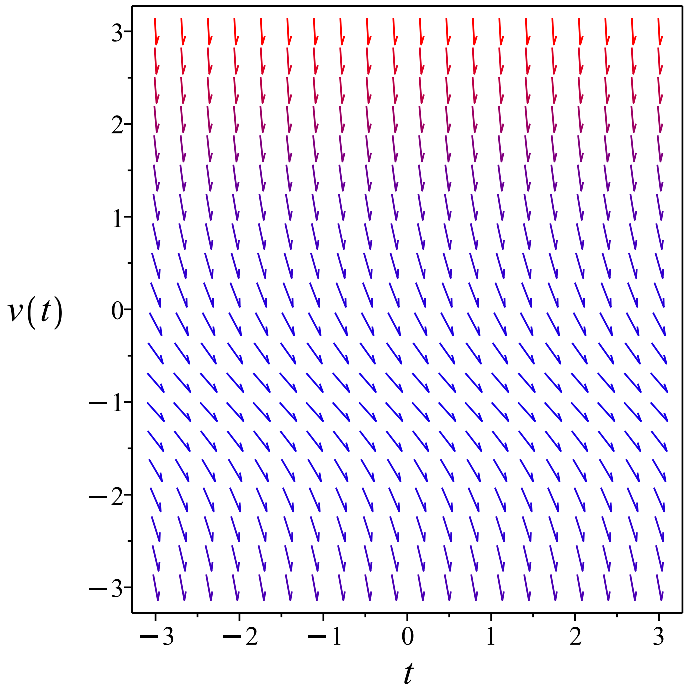


Figure 157: Slope field plot

Verification of solutions

$$v = -1 - \tan(t + c_1)$$

Verified OK.

5.20.2 Maple step by step solution

Let's solve

$v' + v^2 + 2v = -2$

- Highest derivative means the order of the ODE is 1

  v'

- Separate variables

  $\frac{v'}{-v^2-2v-2} = 1$

- Integrate both sides with respect to t

  $\int \frac{v'}{-v^2-2v-2} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\arctan(v+1) = t + c_1$$

- Solve for v

$$v = -1 - \tan(t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(v(t),t)=-v(t)^2-2*v(t)-2,v(t), singsol=all)
```

$$v(t) = -1 - \tan(t + c_1)$$

✓ Solution by Mathematica

Time used: 0.699 (sec). Leaf size: 30

```
DSolve[v'[t]==-v[t]^2-2*v[t]-2,v[t],t,IncludeSingularSolutions -> True]
```

$$v(t) \to -1 - \tan(t - c_1)$$
$$v(t) \to -1 - i$$
$$v(t) \to -1 + i$$

5.21 problem 8

Internal problem ID [12971]
Internal file name [`OUTPUT/11623_Tuesday_November_07_2023_11_52_14_PM_25586871/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{w' - 3w^3 + 12w^2 = 0}$$

5.21.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{3w^3 - 12w^2} dw = \int dt$$
$$\int^w \frac{1}{3\_a^3 - 12\_a^2} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^w \frac{1}{3\_a^3 - 12\_a^2} d\_a = t + c_1 \tag{1}$$

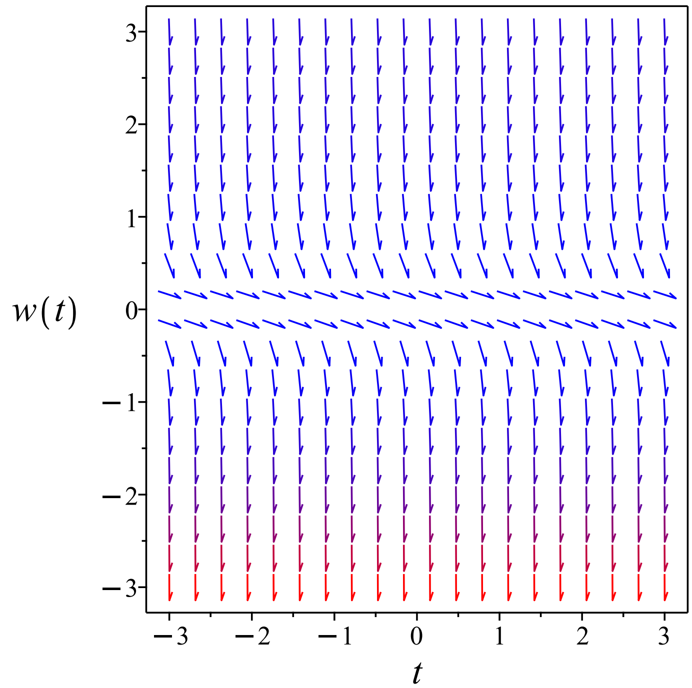


Figure 158: Slope field plot

Verification of solutions

$$\int^{w} \frac{1}{3\_a^{3} - 12\_a^{2}} d\_a = t + c_1$$

Verified OK.

5.21.2 Maple step by step solution

Let's solve

$w' - 3w^3 + 12w^2 = 0$

- Highest derivative means the order of the ODE is 1

  w'

- Separate variables

  $\frac{w'}{3w^3 - 12w^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{w'}{3w^3 - 12w^2} dt = \int 1 dt + c_1$

- Evaluate integral

$$\frac{\ln(w-4)}{48} + \frac{1}{12w} - \frac{\ln(w)}{48} = t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 49

```
dsolve(diff(w(t),t)=3*w(t)^3-12*w(t)^2,w(t), singsol=all)
```

$$w(t) = e^{\operatorname{RootOf}\left(\ln(e^{\_Z}+4)e^{\_Z}+48c_1e^{\_Z}-\_Z\,e^{\_Z}+48t\,e^{\_Z}+4\ln(e^{\_Z}+4)+192c_1-4\_Z+192t-4\right)} + 4$$

✓ Solution by Mathematica

Time used: 0.392 (sec). Leaf size: 50

```
DSolve[w'[t]==3*w[t]^3-12*w[t]^2,w[t],t,IncludeSingularSolutions -> True]
```

$$w(t) \to \text{InverseFunction}\left[\frac{1}{4\#1} + \frac{1}{16}\log(4-\#1) - \frac{\log(\#1)}{16}\&\right][3t + c_1]$$
$$w(t) \to 0$$
$$w(t) \to 4$$

5.22 problem 9

Internal problem ID [12972]
Internal file name [`OUTPUT/11624_Tuesday_November_07_2023_11_52_15_PM_92452613/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - \cos(y) = 1}$$

5.22.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{1+\cos(y)}dy = t + c_1$$
$$\tan\left(\frac{y}{2}\right) = t + c_1$$

Solving for y gives these solutions

$$y_1 = 2\arctan(t + c_1)$$

Summary
The solution(s) found are the following

$$y = 2\arctan(t + c_1) \tag{1}$$

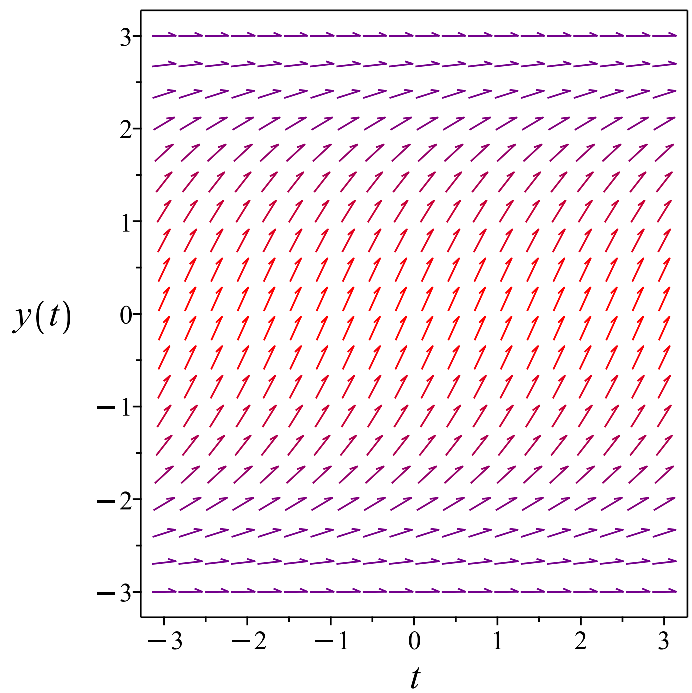


Figure 159: Slope field plot

Verification of solutions

$$y = 2\arctan\left(t + c_1\right)$$

Verified OK.

5.22.2 Maple step by step solution

Let's solve

$y' - \cos\left(y\right) = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+\cos(y)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{1+\cos(y)}dt = \int 1dt + c_1$

- Evaluate integral

$$\tan\left(\frac{y}{2}\right) = t + c_1$$

- Solve for y

$$y = 2\arctan\left(t + c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve(diff(y(t),t)=1+cos(y(t)),y(t), singsol=all)
```

$$y(t) = 2\arctan\left(t + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.462 (sec). Leaf size: 35

```
DSolve[y'[t]==1+cos[y[t]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{\cos(K[1]) + 1} dK[1]\&\right][t + c_1]$$
$$y(t) \to \cos^{(-1)}(-1)$$

5.23 problem 10

Internal problem ID [12973]
Internal file name [`OUTPUT/11625_Tuesday_November_07_2023_11_52_15_PM_5645639/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - \tan(y) = 0}$$

5.23.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\tan(y)} dy = \int dt$$
$$\ln(\sin(y)) = t + c_1$$

Raising both side to exponential gives

$$\sin(y) = e^{t+c_1}$$

Which simplifies to

$$\sin(y) = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = \arcsin(c_2 e^t) \tag{1}$$

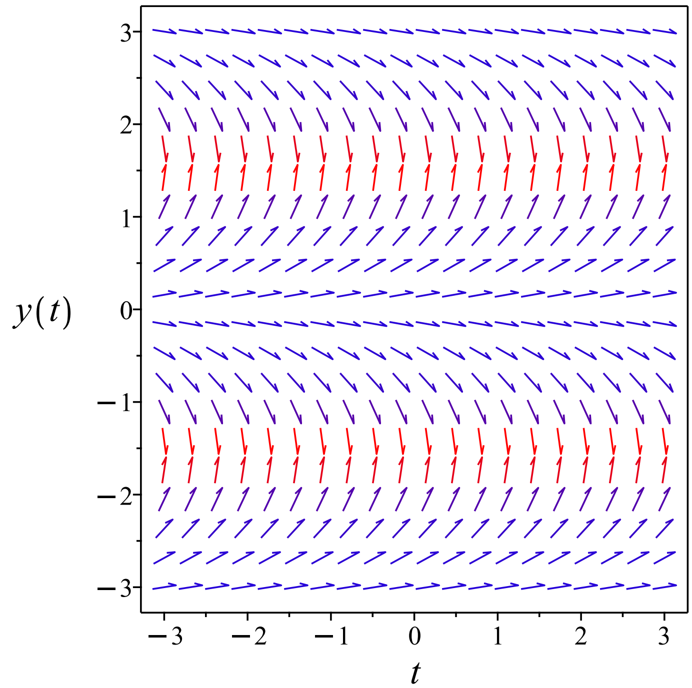


Figure 160: Slope field plot

Verification of solutions

$$y = \arcsin\left(c_2 e^t\right)$$

Verified OK.

5.23.2 Maple step by step solution

Let's solve

$y' - \tan(y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\tan(y)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\tan(y)} dt = \int 1 dt + c_1$

- Evaluate integral

$$\ln(\sin(y)) = t + c_1$$

- Solve for y

$$y = \arcsin\left(e^{t+c_1}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve(diff(y(t),t)=tan( y(t)),y(t), singsol=all)
```

$$y(t) = \arcsin\left(c_1 e^t\right)$$

✓ Solution by Mathematica

Time used: 50.012 (sec). Leaf size: 17

```
DSolve[y'[t]==Tan[y[t]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \arcsin\left(e^{t+c_1}\right)$$
$$y(t) \to 0$$

5.24 problem 11

Internal problem ID [12974]
Internal file name [`OUTPUT/11626_Tuesday_November_07_2023_11_52_16_PM_91685053/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y\ln(|y|) = 0}$$

5.24.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y\ln(|y|)}dy = t + c_1$$

$$\left\{\begin{array}{cc} \ln(-\ln(-y)) & y < 0 \\ \text{undefined} & y = 0 \\ \ln(-\ln(y)) & 0 < y \end{array}\right. = t + c_1$$

Solving for y gives these solutions

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-\mathrm{e}^{t+c_1}} \tag{1}$$

$$y = -\mathrm{e}^{-\mathrm{e}^{t+c_1}} \tag{2}$$

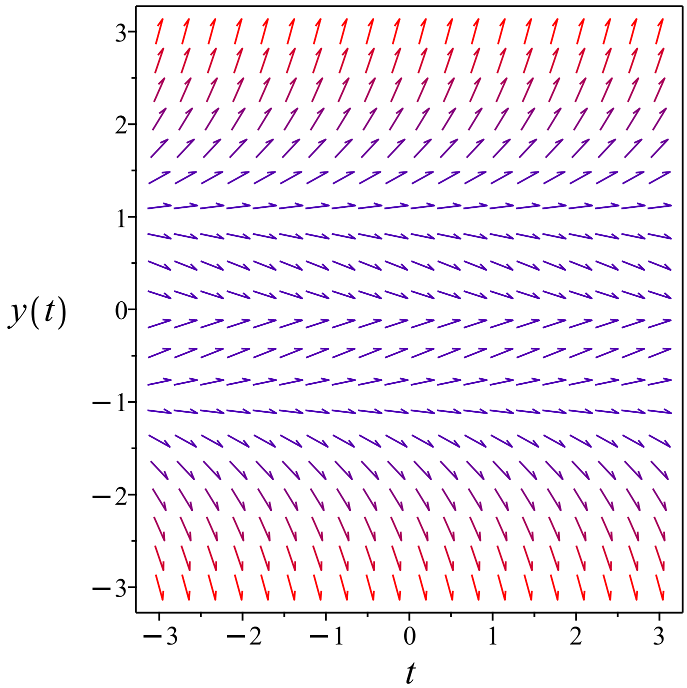


Figure 161: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\mathrm{e}^{t+c_1}}$$

Verified OK.

$$y = -\mathrm{e}^{-\mathrm{e}^{t+c_1}}$$

Verified OK.

5.24.2 Maple step by step solution

Let's solve

$y' - y\ln(|y|) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y\ln(|y|)} = 1$

- Integrate both sides with respect to t

$$\int \frac{y'}{y \ln(|y|)} dt = \int 1 dt + c_1$$

- Cannot compute integral

$$\int \frac{y'}{y \ln(|y|)} dt = t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 21

```
dsolve(diff(y(t),t)=y(t)*ln(abs(y(t))),y(t), singsol=all)
```

$$y(t) = e^{-c_1 e^t}$$
$$y(t) = -e^{-c_1 e^t}$$

✓ Solution by Mathematica

Time used: 0.321 (sec). Leaf size: 35

```
DSolve[y'[t]==y[t]*Log[Abs[y[t]]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{K[1] \log(|K[1]|)} dK[1]\&\right][t + c_1]$$
$$y(t) \to 1$$

5.25 problem 12

Internal problem ID [12975]
Internal file name [`OUTPUT/11627_Tuesday_November_07_2023_11_52_33_PM_86942080/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{w' - \left(w^2 - 2\right) \arctan\left(w\right) = 0}$$

5.25.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\left(w^2 - 2\right) \arctan\left(w\right)} dw = \int dt$$
$$\int^{w} \frac{1}{\left(\_a^2 - 2\right) \arctan\left(\_a\right)} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^{w} \frac{1}{\left(\_a^2 - 2\right) \arctan\left(\_a\right)} d\_a = t + c_1 \tag{1}$$

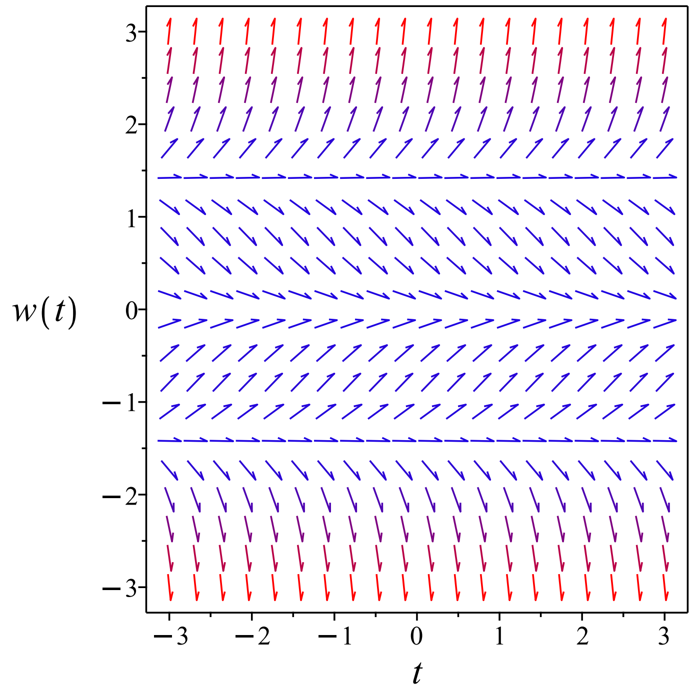


Figure 162: Slope field plot

Verification of solutions

$$\int^{w} \frac{1}{(\_a^2 - 2)\arctan(\_a)} d\_a = t + c_1$$

Verified OK.

5.25.2 Maple step by step solution

Let's solve

$w' - (w^2 - 2)\arctan(w) = 0$

- Highest derivative means the order of the ODE is 1

  w'

- Separate variables

  $\frac{w'}{(w^2-2)\arctan(w)} = 1$

- Integrate both sides with respect to t

  $\int \frac{w'}{(w^2-2)\arctan(w)} dt = \int 1 dt + c_1$

- Cannot compute integral

$$\int \frac{w'}{(w^2-2)\arctan(w)}dt = t + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(w(t),t)=(w(t)^2-2)*arctan( w(t) ),w(t), singsol=all)
```

$$t - \left(\int^{w(t)} \frac{1}{(\_a^2-2)\arctan(\_a)}d\_a\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.909 (sec). Leaf size: 62

```
DSolve[w'[t]==(w[t]^2-2)*Arctan[ w[t]],w[t],t,IncludeSingularSolutions -> True]
```

$$w(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{\text{Arctan}(K[1])\,(K[1]^2-2)}dK[1]\&\right][t + c_1]$$
$$w(t) \to -\sqrt{2}$$
$$w(t) \to \sqrt{2}$$
$$w(t) \to \text{Arctan}^{(-1)}(0)$$

5.26 problem 22

Internal problem ID [12976]
Internal file name [`OUTPUT/11628_Tuesday_November_07_2023_11_52_34_PM_74422193/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 4y = 2}$$

With initial conditions

$$[y(0) = -1]$$

5.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y + 2 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - 4y + 2) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

5.26.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4y + 2} dy = t + c_1$$
$$-\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$$

Solving for y gives these solutions

$$y_1 = -\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{-\sqrt{2}\, e^{2\sqrt{2}\, c_1} + \sqrt{2} + 2\, e^{2\sqrt{2}\, c_1} + 2}{e^{2\sqrt{2}\, c_1} + 1}$$

$$c_1 = \frac{\ln\left(\frac{3+\sqrt{2}}{\sqrt{2}-3}\right)\sqrt{2}}{4}$$

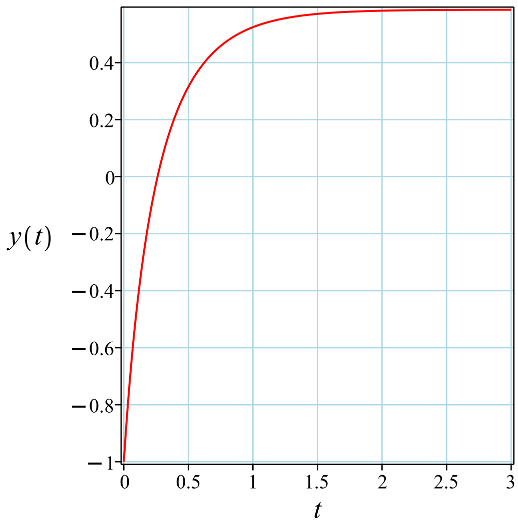
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{2}\, e^{2\sqrt{2}\, t} + 10\, e^{2\sqrt{2}\, t} - 7\sqrt{2} - 14}{6\sqrt{2}\, e^{2\sqrt{2}\, t} + 11\, e^{2\sqrt{2}\, t} - 7}$$

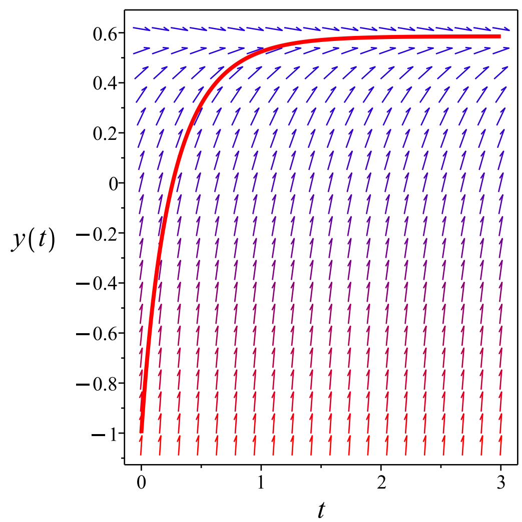
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\, e^{2\sqrt{2}\, t} + 10\, e^{2\sqrt{2}\, t} - 7\sqrt{2} - 14}{6\sqrt{2}\, e^{2\sqrt{2}\, t} + 11\, e^{2\sqrt{2}\, t} - 7} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} + 10\,\mathrm{e}^{2\sqrt{2}\,t} - 7\sqrt{2} - 14}{6\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} + 11\,\mathrm{e}^{2\sqrt{2}\,t} - 7}$$

Verified OK.

5.26.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = 2, y(0) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y+2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y+2} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$

- Solve for y

$$y = -\left(-\sqrt{2} + \tanh\left(\left(t + c_1\right)\sqrt{2}\right)\right)\sqrt{2}$$

- Use initial condition $y(0) = -1$

$$-1 = -\left(-\sqrt{2} + \tanh\left(\sqrt{2}\, c_1\right)\right)\sqrt{2}$$

- Solve for c_1

$$c_1 = \frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{3\sqrt{2}}{2}\right)}{2}$$

- Substitute $c_1 = \frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{3\sqrt{2}}{2}\right)}{2}$ into general solution and simplify

$$y = 2 - \sqrt{2}\,\tanh\left(\mathrm{arctanh}\left(\frac{3\sqrt{2}}{2}\right) + \sqrt{2}\, t\right)$$

- Solution to the IVP

$$y = 2 - \sqrt{2}\,\tanh\left(\mathrm{arctanh}\left(\frac{3\sqrt{2}}{2}\right) + \sqrt{2}\, t\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 24

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)+2,y(0) = -1],y(t), singsol=all)
```

$$y(t) = -\sqrt{2}\,\tanh\left(\operatorname{arctanh}\left(\frac{3\sqrt{2}}{2}\right)+\sqrt{2}\,t\right)+2$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 59

```
DSolve[{y'[t]==y[t]^2-4*y[t]+2,{y[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{\left(\sqrt{2}-4\right)e^{2\sqrt{2}t}+4+\sqrt{2}}{\left(3+\sqrt{2}\right)e^{2\sqrt{2}t}-3+\sqrt{2}}$$

5.27 problem 23

Internal problem ID [12977]
Internal file name [`OUTPUT/11629_Tuesday_November_07_2023_11_52_50_PM_18408871/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 4y = 2}$$

With initial conditions

$$[y(0) = 2]$$

5.27.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y + 2 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - 4y + 2) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

5.27.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4y + 2} dy = t + c_1$$
$$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$$

Solving for y gives these solutions

$$y_1 = -\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{-\sqrt{2}\,e^{2\sqrt{2}\,c_1} + \sqrt{2} + 2\,e^{2\sqrt{2}\,c_1} + 2}{e^{2\sqrt{2}\,c_1} + 1}$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \lim_{c_1 \to 0} \left(-\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}\right)$$

Summary

The solution(s) found are the following

$$y = \lim_{c_1 \to 0} \left(-\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}\right) \tag{1}$$

Verification of solutions

$$y = \lim_{c_1 \to 0} \left(-\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}\right)$$

Verified OK.

5.27.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = 2, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y+2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y+2}dt = \int 1dt + c_1$

- Evaluate integral

  $-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$

- Solve for y

  $y = -\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}$

- Use initial condition $y(0) = 2$

  $2 = -\left(-\sqrt{2} + \tanh\left(\sqrt{2}\,c_1\right)\right)\sqrt{2}$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = 2 - \sqrt{2}\,\tanh\left(\sqrt{2}\,t\right)$

- Solution to the IVP

  $y = 2 - \sqrt{2}\,\tanh\left(\sqrt{2}\,t\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)+2,y(0) = 2],y(t), singsol=all)
```

$$y(t) = -\sqrt{2}\,\tanh\left(\sqrt{2}\,t\right) + 2$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 46

```
DSolve[{y'[t]==y[t]^2-4*y[t]+2,{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{-\left(\sqrt{2}-2\right)e^{2\sqrt{2}t} + 2 + \sqrt{2}}{e^{2\sqrt{2}t} + 1}$$

5.28 problem 24

Internal problem ID [12978]
Internal file name [`OUTPUT/11630_Tuesday_November_07_2023_11_52_51_PM_41929186/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 4y = 2}$$

With initial conditions

$$[y(0) = -2]$$

5.28.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y + 2 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - 4y + 2) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

5.28.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4y + 2} dy = t + c_1$$

$$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$$

Solving for y gives these solutions

$$y_1 = -\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \frac{-\sqrt{2}\,e^{2\sqrt{2}\,c_1} + \sqrt{2} + 2\,e^{2\sqrt{2}\,c_1} + 2}{e^{2\sqrt{2}\,c_1} + 1}$$

$$c_1 = \frac{\ln\left(\frac{4+\sqrt{2}}{\sqrt{2}-4}\right)\sqrt{2}}{4}$$

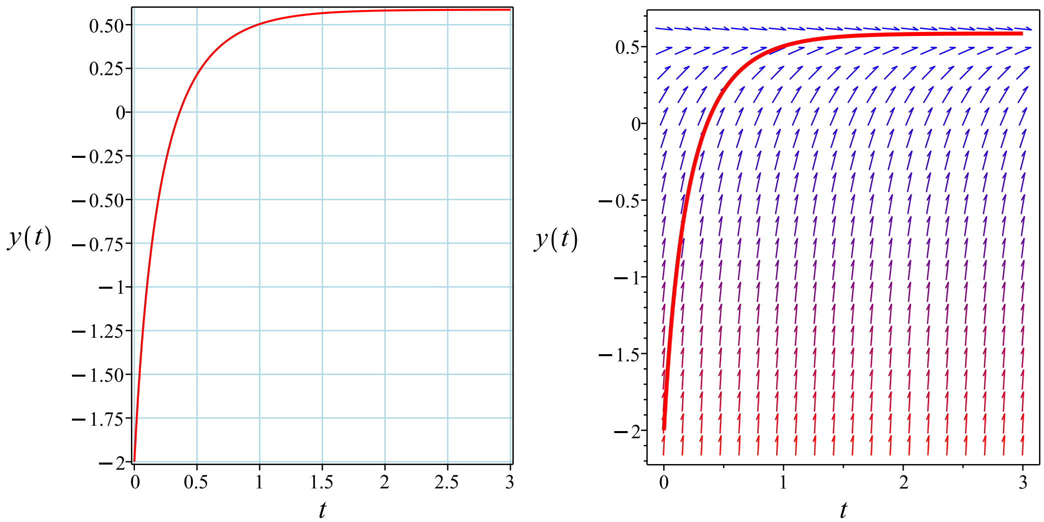
Substituting c_1 found above in the general solution gives

$$y = \frac{10\,e^{2\sqrt{2}\,t} - \sqrt{2}\,e^{2\sqrt{2}\,t} - 7\sqrt{2} - 14}{4\sqrt{2}\,e^{2\sqrt{2}\,t} + 9\,e^{2\sqrt{2}\,t} - 7}$$

Summary

The solution(s) found are the following

$$y = \frac{10\,e^{2\sqrt{2}\,t} - \sqrt{2}\,e^{2\sqrt{2}\,t} - 7\sqrt{2} - 14}{4\sqrt{2}\,e^{2\sqrt{2}\,t} + 9\,e^{2\sqrt{2}\,t} - 7} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{10\,e^{2\sqrt{2}\,t} - \sqrt{2}\,e^{2\sqrt{2}\,t} - 7\sqrt{2} - 14}{4\sqrt{2}\,e^{2\sqrt{2}\,t} + 9\,e^{2\sqrt{2}\,t} - 7}$$

Verified OK.

5.28.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = 2, y(0) = -2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y+2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y+2}dt = \int 1dt + c_1$

- Evaluate integral

  $-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$

- Solve for y

$$y = -\left(-\sqrt{2}+\tanh\left(\left(t+c_1\right)\sqrt{2}\right)\right)\sqrt{2}$$

- Use initial condition $y(0) = -2$

$$-2 = -\left(-\sqrt{2}+\tanh\left(\sqrt{2}\, c_1\right)\right)\sqrt{2}$$

- Solve for c_1

$$c_1 = \frac{\sqrt{2}\,\mathrm{arctanh}\left(2\sqrt{2}\right)}{2}$$

- Substitute $c_1 = \frac{\sqrt{2}\,\mathrm{arctanh}\left(2\sqrt{2}\right)}{2}$ into general solution and simplify

$$y = 2-\sqrt{2}\,\tanh\left(\mathrm{arctanh}\left(2\sqrt{2}\right)+\sqrt{2}\, t\right)$$

- Solution to the IVP

$$y = 2-\sqrt{2}\,\tanh\left(\mathrm{arctanh}\left(2\sqrt{2}\right)+\sqrt{2}\, t\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 24

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)+2,y(0) = -2],y(t), singsol=all)
```

$$y(t) = -\sqrt{2}\, \tanh\left(\operatorname{arctanh}\left(2\sqrt{2}\right) + \sqrt{2}\, t\right) + 2$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 59

```
DSolve[{y'[t]==y[t]^2-4*y[t]+2,{y[0]==-2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{2\left(\left(\sqrt{2}-3\right) e^{2\sqrt{2}t} + 3 + \sqrt{2}\right)}{\left(4+\sqrt{2}\right) e^{2\sqrt{2}t} - 4 + \sqrt{2}}$$

5.29 problem 25

Internal problem ID [12979]
Internal file name [`OUTPUT/11631_Tuesday_November_07_2023_11_53_08_PM_97639020/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y^2 + 4y = 2}$$

With initial conditions

$$[y(0) = -4]$$

5.29.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y + 2 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - 4y + 2) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -4$ is inside this domain. Therefore solution exists and is unique.

5.29.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4y + 2} dy = t + c_1$$
$$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$$

Solving for y gives these solutions

$$y_1 = -\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-4 = \frac{-\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,c_1} + \sqrt{2} + 2\,\mathrm{e}^{2\sqrt{2}\,c_1} + 2}{\mathrm{e}^{2\sqrt{2}\,c_1} + 1}$$

$$c_1 = \frac{\ln\left(\frac{6+\sqrt{2}}{\sqrt{2}-6}\right)\sqrt{2}}{4}$$

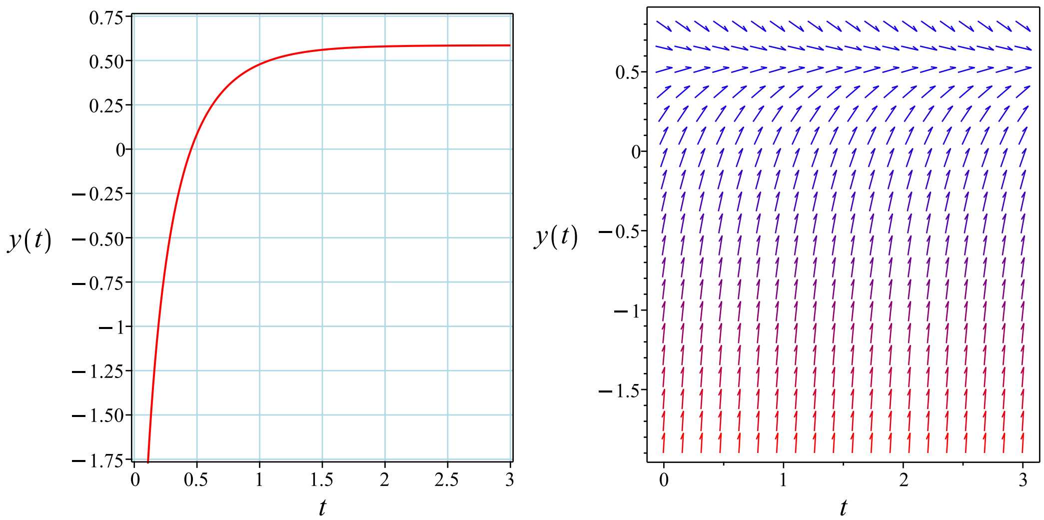
Substituting c_1 found above in the general solution gives

$$y = \frac{26\,\mathrm{e}^{2\sqrt{2}\,t} - 7\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} - 17\sqrt{2} - 34}{6\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} + 19\,\mathrm{e}^{2\sqrt{2}\,t} - 17}$$

Summary

The solution(s) found are the following

$$y = \frac{26\,\mathrm{e}^{2\sqrt{2}\,t} - 7\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} - 17\sqrt{2} - 34}{6\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} + 19\,\mathrm{e}^{2\sqrt{2}\,t} - 17} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{26\,e^{2\sqrt{2}\,t} - 7\sqrt{2}\,e^{2\sqrt{2}\,t} - 17\sqrt{2} - 34}{6\sqrt{2}\,e^{2\sqrt{2}\,t} + 19\,e^{2\sqrt{2}\,t} - 17}$$

Verified OK.

5.29.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = 2, y(0) = -4]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y+2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y+2} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$

- Solve for y

$$y = -\left(-\sqrt{2}+\tanh\left(\left(t+c_1\right)\sqrt{2}\right)\right)\sqrt{2}$$

- Use initial condition $y(0) = -4$

$$-4 = -\left(-\sqrt{2}+\tanh\left(\sqrt{2}\,c_1\right)\right)\sqrt{2}$$

- Solve for c_1

$$c_1 = \frac{\sqrt{2}\,\mathrm{arctanh}\left(3\sqrt{2}\right)}{2}$$

- Substitute $c_1 = \frac{\sqrt{2}\,\mathrm{arctanh}\left(3\sqrt{2}\right)}{2}$ into general solution and simplify

$$y = 2-\sqrt{2}\,\tanh\left(\mathrm{arctanh}\left(3\sqrt{2}\right)+\sqrt{2}\,t\right)$$

- Solution to the IVP

$$y = 2-\sqrt{2}\,\tanh\left(\mathrm{arctanh}\left(3\sqrt{2}\right)+\sqrt{2}\,t\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 24

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)+2,y(0) = -4],y(t), singsol=all)
```

$$y(t) = -\sqrt{2}\,\tanh\left(\operatorname{arctanh}\left(3\sqrt{2}\right)+\sqrt{2}\,t\right)+2$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 63

```
DSolve[{y'[t]==y[t]^2-4*y[t]+2,{y[0]==-4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{2\left(\left(2\sqrt{2}-5\right)e^{2\sqrt{2}t}+5+2\sqrt{2}\right)}{\left(6+\sqrt{2}\right)e^{2\sqrt{2}t}-6+\sqrt{2}}$$

5.30 problem 26

Internal problem ID [12980]
Internal file name [`OUTPUT/11632_Tuesday_November_07_2023_11_53_25_PM_2702011/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 4y = 2}$$

With initial conditions

$$[y(0) = 4]$$

5.30.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y + 2 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - 4y + 2) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Therefore solution exists and is unique.

5.30.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4y + 2} dy = t + c_1$$
$$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$$

Solving for y gives these solutions

$$y_1 = -\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{-\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,c_1} + \sqrt{2} + 2\,\mathrm{e}^{2\sqrt{2}\,c_1} + 2}{\mathrm{e}^{2\sqrt{2}\,c_1} + 1}$$

$$c_1 = \frac{\ln\left(\frac{\sqrt{2}-2}{2+\sqrt{2}}\right)\sqrt{2}}{4}$$

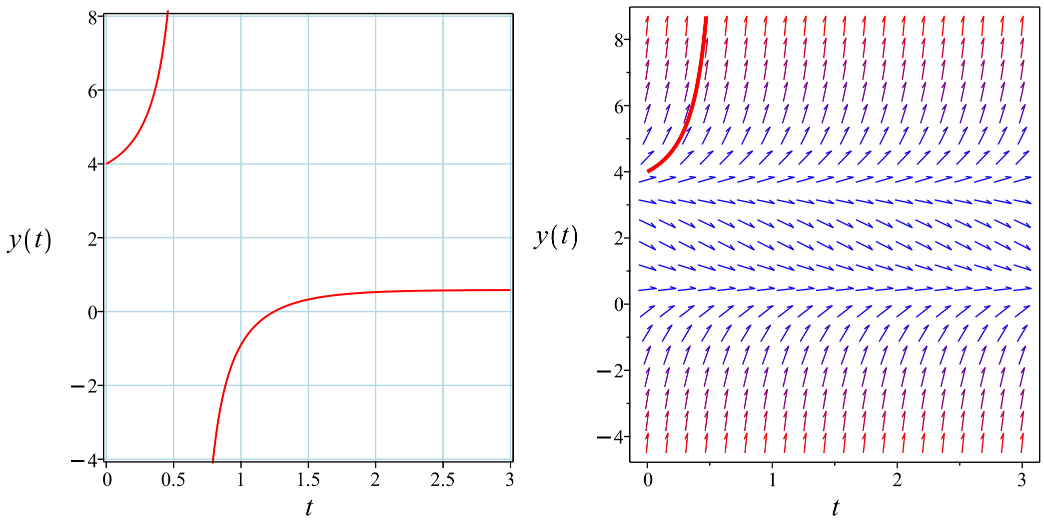
Substituting c_1 found above in the general solution gives

$$y = \frac{7\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} + \sqrt{2} - 10\,\mathrm{e}^{2\sqrt{2}\,t} + 2}{2\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} - 3\,\mathrm{e}^{2\sqrt{2}\,t} + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{7\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} + \sqrt{2} - 10\,\mathrm{e}^{2\sqrt{2}\,t} + 2}{2\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,t} - 3\,\mathrm{e}^{2\sqrt{2}\,t} + 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{7\sqrt{2}\, e^{2\sqrt{2}\, t} + \sqrt{2} - 10\, e^{2\sqrt{2}\, t} + 2}{2\sqrt{2}\, e^{2\sqrt{2}\, t} - 3\, e^{2\sqrt{2}\, t} + 1}$$

Verified OK.

5.30.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = 2, y(0) = 4]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y+2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y+2}dt = \int 1dt + c_1$

- Evaluate integral

  $-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$

- Solve for y

$$y = -\left(-\sqrt{2}+\tanh\left(\left(t+c_1\right)\sqrt{2}\right)\right)\sqrt{2}$$

- Use initial condition $y(0) = 4$

$$4 = -\left(-\sqrt{2}+\tanh\left(\sqrt{2}\, c_1\right)\right)\sqrt{2}$$

- Solve for c_1

$$c_1 = -\frac{\sqrt{2}\,\mathrm{arctanh}\left(\sqrt{2}\right)}{2}$$

- Substitute $c_1 = -\frac{\sqrt{2}\,\mathrm{arctanh}\left(\sqrt{2}\right)}{2}$ into general solution and simplify

$$y = 2-\sqrt{2}\,\tanh\left(-\mathrm{arctanh}(\sqrt{2})+\sqrt{2}\, t\right)$$

- Solution to the IVP

$$y = 2-\sqrt{2}\,\tanh\left(-\mathrm{arctanh}(\sqrt{2})+\sqrt{2}\, t\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 24

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)+2,y(0) = 4],y(t), singsol=all)
```

$$y(t) = -\sqrt{2}\,\tanh\left(-\,\mathrm{arctanh}\left(\sqrt{2}\right)+\sqrt{2}\, t\right)+2$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 62

```
DSolve[{y'[t]==y[t]^2-4*y[t]+2,{y[0]==4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\left(4\sqrt{2}-6\right)e^{2\sqrt{2}t}+6+4\sqrt{2}}{\left(\sqrt{2}-2\right)e^{2\sqrt{2}t}+2+\sqrt{2}}$$

5.31 problem 27

Internal problem ID [12981]
Internal file name [`OUTPUT/11633_Tuesday_November_07_2023_11_53_44_PM_62460508/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 4y = 2}$$

With initial conditions

$$[y(3) = 1]$$

5.31.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 4y + 2 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 3$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - 4y + 2) \\ &= 2y - 4 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 3$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.31.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4y + 2} dy = t + c_1$$
$$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$$

Solving for y gives these solutions

$$y_1 = -\left(-\sqrt{2} + \tanh\left((t + c_1)\sqrt{2}\right)\right)\sqrt{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-\sqrt{2}\,\mathrm{e}^{2(3+c_1)\sqrt{2}} + \sqrt{2} + 2\,\mathrm{e}^{2(3+c_1)\sqrt{2}} + 2}{\mathrm{e}^{2(3+c_1)\sqrt{2}} + 1}$$

$$c_1 = \frac{\left(-6\sqrt{2} + \ln\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right)\right)\sqrt{2}}{4}$$

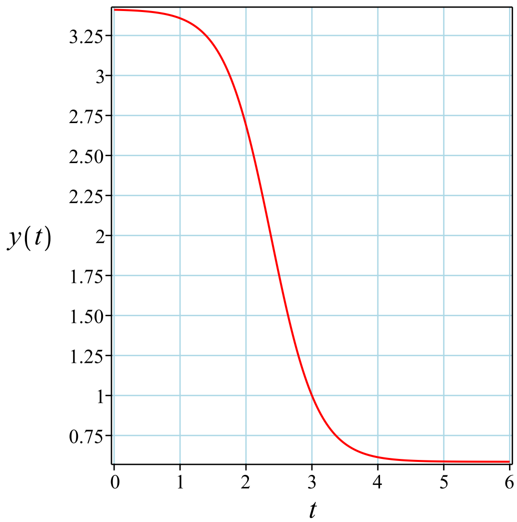
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}^{2\sqrt{2}\,(-3+t)}\sqrt{2} + 2\,\mathrm{e}^{2\sqrt{2}\,(-3+t)} + \sqrt{2} + 2}{3\,\mathrm{e}^{2\sqrt{2}\,(-3+t)} + 2\,\mathrm{e}^{2\sqrt{2}\,(-3+t)}\sqrt{2} + 1}$$

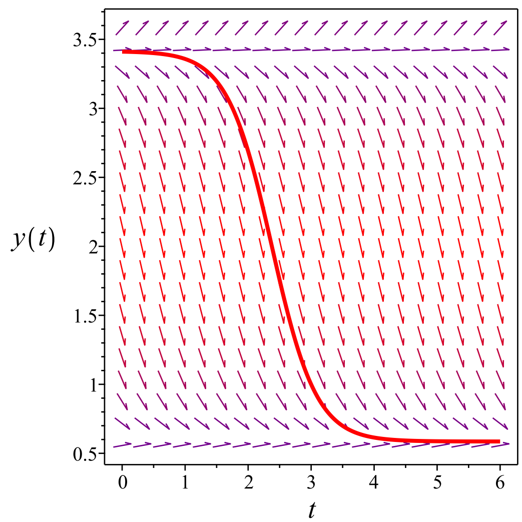
Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{2\sqrt{2}\,(-3+t)}\sqrt{2} + 2\,\mathrm{e}^{2\sqrt{2}\,(-3+t)} + \sqrt{2} + 2}{3\,\mathrm{e}^{2\sqrt{2}\,(-3+t)} + 2\,\mathrm{e}^{2\sqrt{2}\,(-3+t)}\sqrt{2} + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{2\sqrt{2}\,(-3+t)}\sqrt{2} + 2\,e^{2\sqrt{2}\,(-3+t)} + \sqrt{2} + 2}{3\,e^{2\sqrt{2}\,(-3+t)} + 2\,e^{2\sqrt{2}\,(-3+t)}\sqrt{2} + 1}$$

Verified OK.

5.31.3 Maple step by step solution

Let's solve

$[y' - y^2 + 4y = 2, y(3) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4y+2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-4y+2}dt = \int 1dt + c_1$

- Evaluate integral

  $-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{(2y-4)\sqrt{2}}{4}\right)}{2} = t + c_1$

- Solve for y

$$y = -\left(-\sqrt{2}+\tanh\left(\left(t+c_1\right)\sqrt{2}\right)\right)\sqrt{2}$$

- Use initial condition $y(3) = 1$

$$1 = -\left(-\sqrt{2}+\tanh\left(\left(3+c_1\right)\sqrt{2}\right)\right)\sqrt{2}$$

- Solve for c_1

$$c_1 = -\frac{\left(3\sqrt{2}-\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}\right)\right)\sqrt{2}}{2}$$

- Substitute $c_1 = -\frac{\left(3\sqrt{2}-\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}\right)\right)\sqrt{2}}{2}$ into general solution and simplify

$$y = 2-\sqrt{2}\tanh\left(\frac{\left(\sqrt{2}\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}\right)+2t-6\right)\sqrt{2}}{2}\right)$$

- Solution to the IVP

$$y = 2-\sqrt{2}\tanh\left(\frac{\left(\sqrt{2}\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}\right)+2t-6\right)\sqrt{2}}{2}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 32

```
dsolve([diff(y(t),t)=y(t)^2-4*y(t)+2,y(3) = 1],y(t), singsol=all)
```

$$y(t) = -\sqrt{2}\,\tanh\left(\frac{\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}\right)+2t-6\right)\sqrt{2}}{2}\right)+2$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 69

```
DSolve[{y'[t]==y[t]^2-4*y[t]+2,{y[3]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\sqrt{2}\left(e^{2\sqrt{2}t}+e^{6\sqrt{2}}\right)}{\left(1+\sqrt{2}\right)e^{2\sqrt{2}t}+\left(\sqrt{2}-1\right)e^{6\sqrt{2}}}$$

5.32 problem 37 (i)

Internal problem ID [12982]
Internal file name [`OUTPUT/11634_Tuesday_November_07_2023_11_53_45_PM_77175554/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 37 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y\cos\left(\frac{\pi y}{2}\right) = 0}$$

5.32.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y\cos\left(\frac{\pi y}{2}\right)}dy = \int dt$$
$$\int^{y} \frac{1}{\_a\cos\left(\frac{\pi\_a}{2}\right)}d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{1}{\_a\cos\left(\frac{\pi\_a}{2}\right)}d\_a = t + c_1 \tag{1}$$

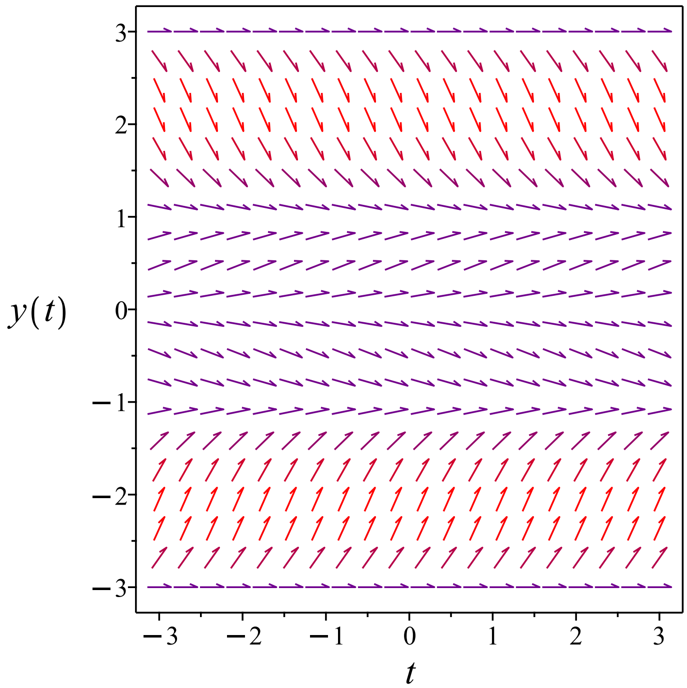


Figure 168: Slope field plot

Verification of solutions

$$\int^{y} \frac{1}{\_a \cos\left(\frac{\pi \_a}{2}\right)} d\_a = t + c_1$$

Verified OK.

5.32.2 Maple step by step solution

Let's solve

$y' - y\cos\left(\frac{\pi y}{2}\right) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y\cos\left(\frac{\pi y}{2}\right)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y\cos\left(\frac{\pi y}{2}\right)} dt = \int 1 dt + c_1$

- Cannot compute integral

  $\int \frac{y'}{y\cos\left(\frac{\pi y}{2}\right)}dt = t + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(diff(y(t),t)=y(t)*cos(Pi/2*y(t)),y(t), singsol=all)
```

$$t - \left(\int^{y(t)} \frac{\sec\left(\frac{\pi\_a}{2}\right)}{\_a} d\_a\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 4.801 (sec). Leaf size: 47

```
DSolve[y'[t]==y[t]*Cos[Pi/2*y[t]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sec\left(\frac{1}{2}\pi K[1]\right)}{K[1]} dK[1]\&\right][t + c_1]$$
$$y(t) \to -1$$
$$y(t) \to 0$$
$$y(t) \to 1$$

5.33 problem 37 (ii)

Internal problem ID [12983]
Internal file name [`OUTPUT/11635_Tuesday_November_07_2023_11_53_46_PM_83788516/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 37 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y + y^2 = 0}$$

5.33.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 + y} dy = \int dt$$
$$-\ln(y-1) + \ln(y) = t + c_1$$

Raising both side to exponential gives

$$e^{-\ln(y-1)+\ln(y)} = e^{t+c_1}$$

Which simplifies to

$$\frac{y}{y-1} = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = \frac{c_2 e^t}{-1 + c_2 e^t} \tag{1}$$

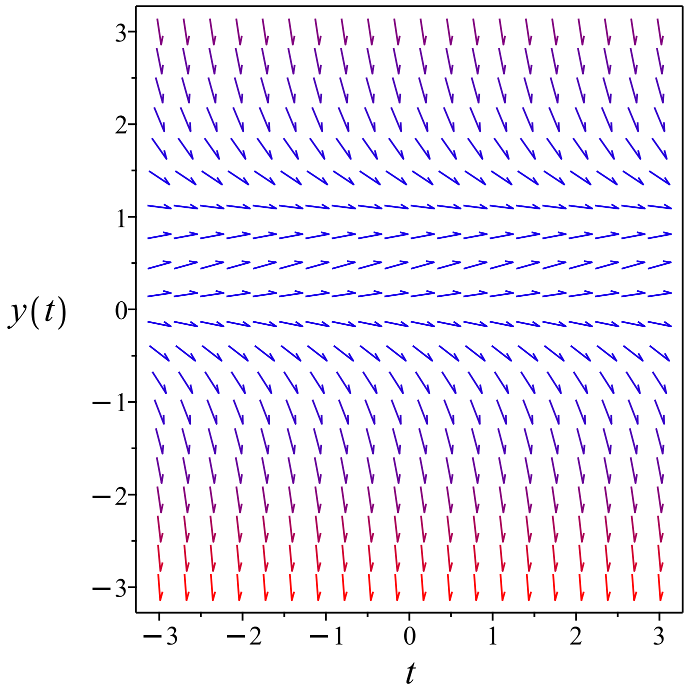


Figure 169: Slope field plot

Verification of solutions

$$y = \frac{c_2 e^t}{-1 + c_2 e^t}$$

Verified OK.

5.33.2 Maple step by step solution

Let's solve

$y' - y + y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-y^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y-y^2} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\ln(y-1)+\ln(y)=t+c_1$$

- Solve for y

$y=\frac{e^{t+c_1}}{-1+e^{t+c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=y(t)-y(t)^2,y(t), singsol=all)
```

$$y(t)=\frac{1}{1+e^{-t}c_1}$$

✓ Solution by Mathematica

Time used: 0.42 (sec). Leaf size: 29

```
DSolve[y'[t]==y[t]-y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to\frac{e^t}{e^t+e^{c_1}}$$
$$y(t)\to 0$$
$$y(t)\to 1$$

5.34 problem 37 (iii)

Internal problem ID [12984]
Internal file name [`OUTPUT/11636_Tuesday_November_07_2023_11_53_48_PM_13625784/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 37 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y \sin\left(\frac{\pi y}{2}\right) = 0}$$

5.34.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y \sin\left(\frac{\pi y}{2}\right)} dy = \int dt$$
$$\int^{y} \frac{1}{\_a \sin\left(\frac{\pi \_a}{2}\right)} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{1}{\_a \sin\left(\frac{\pi \_a}{2}\right)} d\_a = t + c_1 \tag{1}$$

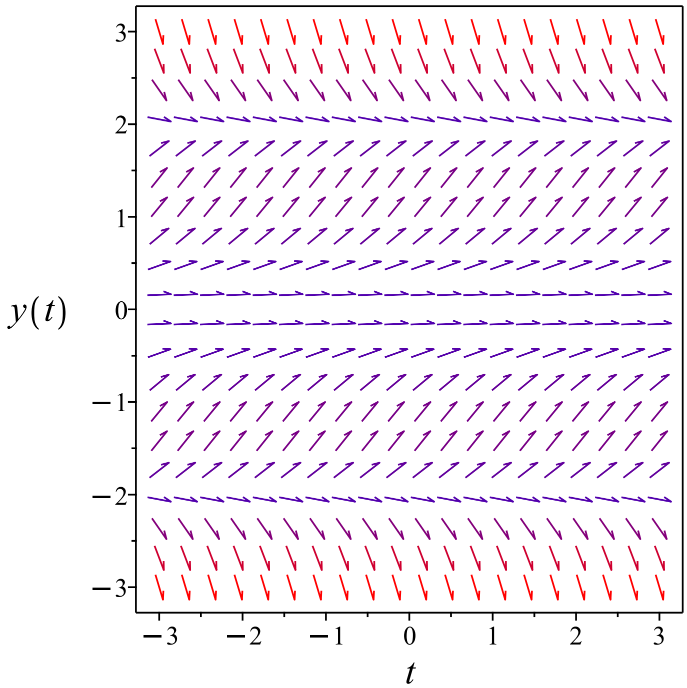


Figure 170: Slope field plot

Verification of solutions

$$\int^{y}\frac{1}{\_a\sin\left(\frac{\pi\_a}{2}\right)}d\_a=t+c_1$$

Verified OK.

5.34.2 Maple step by step solution

Let's solve

$y'-y\sin\left(\frac{\pi y}{2}\right)=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y\sin\left(\frac{\pi y}{2}\right)}=1$

- Integrate both sides with respect to t

  $\int\frac{y'}{y\sin\left(\frac{\pi y}{2}\right)}dt=\int 1dt+c_1$

- Cannot compute integral
  $\int \frac{y'}{y\sin\left(\frac{\pi y}{2}\right)}dt = t + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(t),t)=y(t)*sin(Pi/2*y(t)),y(t), singsol=all)
```

$$t - \left(\int^{y(t)} \frac{\csc\left(\frac{\pi\_a}{2}\right)}{\_a}d\_a\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 7.222 (sec). Leaf size: 37

```
DSolve[y'[t]==y[t]*Sin[Pi/2*y[t]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\csc\left(\frac{1}{2}\pi K[1]\right)}{K[1]}dK[1]\&\right][t + c_1]$$
$$y(t) \to 0$$

5.35 problem 37 (iv)

Internal problem ID [12985]
Internal file name [`OUTPUT/11637_Tuesday_November_07_2023_11_53_49_PM_40171703/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 37 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^3 + y^2 = 0}$$

5.35.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^3 - y^2} dy = \int dt$$
$$\int^y \frac{1}{\_a^3 - \_a^2} d\_a = t + c_1$$

<u>Summary</u>
The solution(s) found are the following

$$\int^y \frac{1}{\_a^3 - \_a^2} d\_a = t + c_1 \tag{1}$$

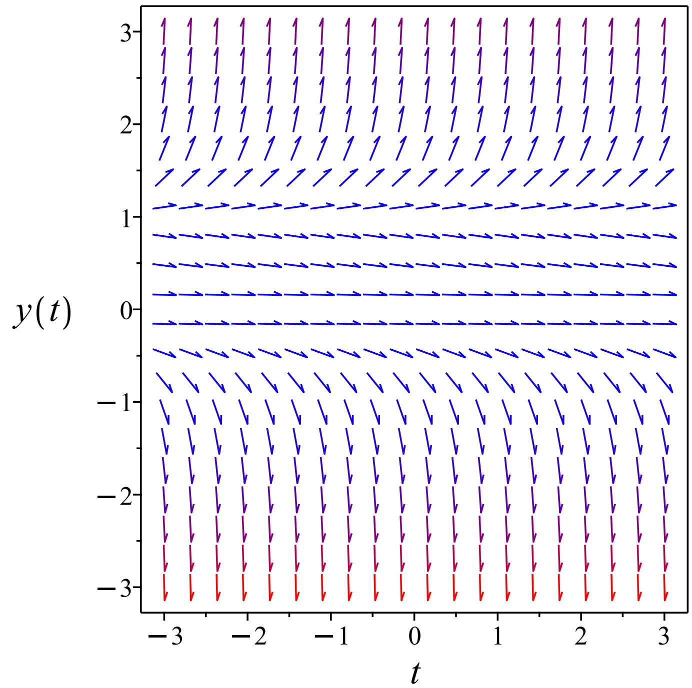


Figure 171: Slope field plot

Verification of solutions

$$\int^y \frac{1}{\_a^3 - \_a^2} d\_a = t + c_1$$

Verified OK.

5.35.2 Maple step by step solution

Let's solve

$y' - y^3 + y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3 - y^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^3 - y^2} dt = \int 1 dt + c_1$

- Evaluate integral

$$\ln(y-1)+\frac{1}{y}-\ln(y)=t+c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)=y(t)^3-y(t)^2,y(t), singsol=all)
```

$$y(t)=\frac{1}{\mathrm{LambertW}\left(-c_1\mathrm{e}^{t-1}\right)+1}$$

✓ Solution by Mathematica

Time used: 0.374 (sec). Leaf size: 38

```
DSolve[y'[t]==y[t]^3-y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to \text{InverseFunction}\left[\frac{1}{\#1}+\log(1-\#1)-\log(\#1)\&\right][t+c_1]$$
$$y(t)\to 0$$
$$y(t)\to 1$$

5.36 problem 37 (v)

Internal problem ID [12986]
Internal file name [`OUTPUT/11638_Tuesday_November_07_2023_11_53_52_PM_77068426/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 37 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - \cos\left(\frac{\pi y}{2}\right) = 0}$$

5.36.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\cos\left(\frac{\pi y}{2}\right)} dy = \int dt$$
$$\frac{2\ln\left(\sec\left(\frac{\pi y}{2}\right) + \tan\left(\frac{\pi y}{2}\right)\right)}{\pi} = t + c_1$$

Raising both side to exponential gives

$$e^{\frac{2\ln\left(\sec\left(\frac{\pi y}{2}\right)+\tan\left(\frac{\pi y}{2}\right)\right)}{\pi}} = e^{t+c_1}$$

Which simplifies to

$$\left(\sec\left(\frac{\pi y}{2}\right) + \tan\left(\frac{\pi y}{2}\right)\right)^{\frac{2}{\pi}} = c_2 e^t$$

Summary

The solution(s) found are the following

$$y = \frac{2\arctan\left(\frac{(c_2 e^t)^{\pi}-1}{(c_2 e^t)^{\pi}+1}, \frac{2(c_2 e^t)^{\frac{\pi}{2}}}{(c_2 e^t)^{\pi}+1}\right)}{\pi} \tag{1}$$

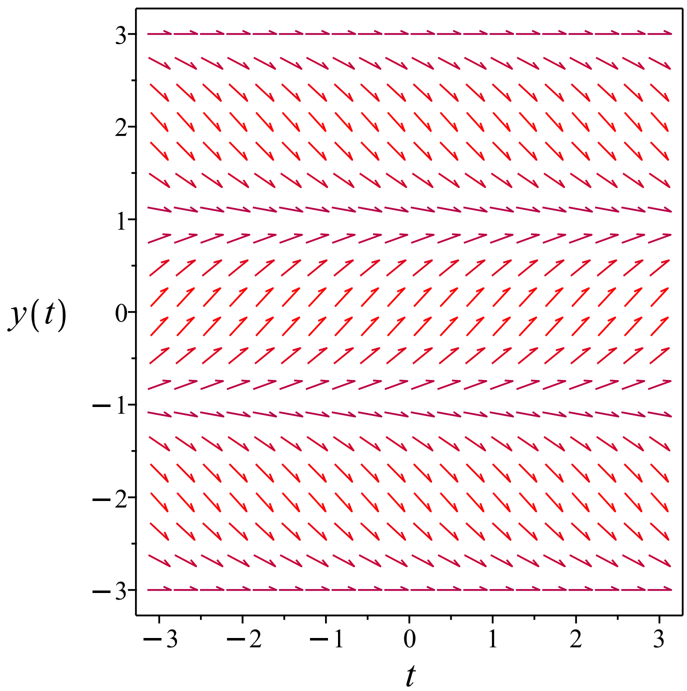


Figure 172: Slope field plot

Verification of solutions

$$y = \frac{2\arctan\left(\frac{(c_2 e^t)^{\pi}-1}{(c_2 e^t)^{\pi}+1}, \frac{2(c_2 e^t)^{\frac{\pi}{2}}}{(c_2 e^t)^{\pi}+1}\right)}{\pi}$$

Verified OK.

5.36.2 Maple step by step solution

Let's solve

$y' - \cos\left(\frac{\pi y}{2}\right) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos\left(\frac{\pi y}{2}\right)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\cos\left(\frac{\pi y}{2}\right)} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{2\ln\left(\sec\left(\frac{\pi y}{2}\right)+\tan\left(\frac{\pi y}{2}\right)\right)}{\pi} = t + c_1$

- Solve for y

  $y = \frac{2\arctan\left(\frac{\left(e^{\frac{1}{2}\pi c_1+\frac{1}{2}\pi t}\right)^2-1}{\left(e^{\frac{1}{2}\pi c_1+\frac{1}{2}\pi t}\right)^2+1}, \frac{2\,e^{\frac{1}{2}\pi c_1+\frac{1}{2}\pi t}}{\left(e^{\frac{1}{2}\pi c_1+\frac{1}{2}\pi t}\right)^2+1}\right)}{\pi}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 48

```
dsolve(diff(y(t),t)=cos(Pi/2*y(t)),y(t), singsol=all)
```

$$y(t) = \frac{2 \arctan\left(\frac{e^{\pi(t+c_1)}-1}{e^{\pi(t+c_1)}+1}, \frac{2\,e^{\frac{\pi(t+c_1)}{2}}}{e^{\pi(t+c_1)}+1}\right)}{\pi}$$

✓ Solution by Mathematica

Time used: 0.846 (sec). Leaf size: 31

```
DSolve[y'[t]==Cos[Pi/2*y[t]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{2 \arcsin\left(\coth\left(\frac{1}{2}\pi(t+c_1)\right)\right)}{\pi}$$
$$y(t) \to -1$$
$$y(t) \to 1$$

5.37 problem 37 (vi)

Internal problem ID [12987]
Internal file name [`OUTPUT/11639_Tuesday_November_07_2023_11_53_53_PM_26533520/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 37 (vi).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + y = 0}$$

5.37.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - y} dy = \int dt$$
$$\ln(y-1) - \ln(y) = t + c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(y)} = e^{t+c_1}$$

Which simplifies to

$$\frac{y-1}{y} = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{-1 + c_2 e^t} \tag{1}$$

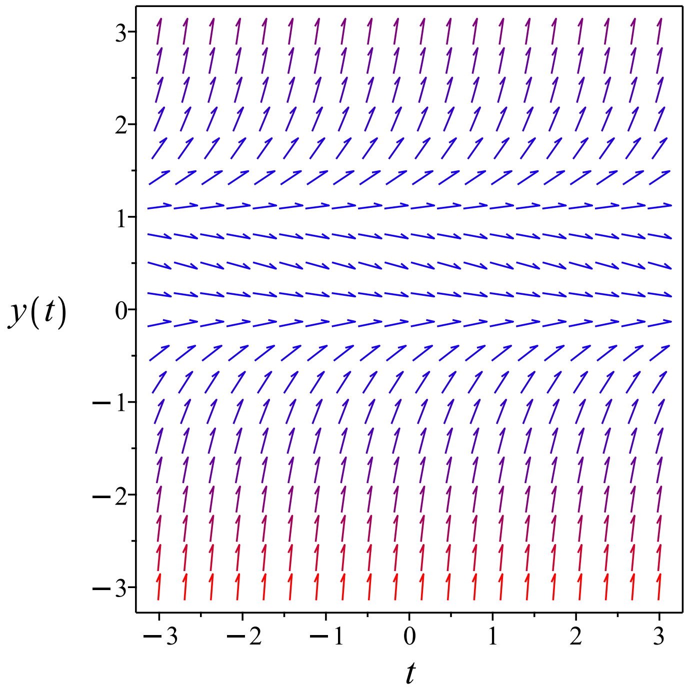


Figure 173: Slope field plot

Verification of solutions

$$y = -\frac{1}{-1 + c_2 e^t}$$

Verified OK.

5.37.2 Maple step by step solution

Let's solve

$y' - y^2 + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-y} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-y} dt = \int 1 dt + c_1$

- Evaluate integral

$$\ln(y-1)-\ln(y)=t+c_1$$

- Solve for y

$$y=-\frac{1}{-1+e^{t+c_1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)=y(t)^2-y(t),y(t), singsol=all)
```

$$y(t)=\frac{1}{1+c_1e^t}$$

✓ Solution by Mathematica

Time used: 0.336 (sec). Leaf size: 25

```
DSolve[y'[t]==y[t]^2-y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to\frac{1}{1+e^{t+c_1}}$$
$$y(t)\to 0$$
$$y(t)\to 1$$

5.38 problem 37 (vii)

Internal problem ID [12988]
Internal file name [`OUTPUT/11640_Tuesday_November_07_2023_11_53_55_PM_18805721/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 37 (vii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y \sin\left(\frac{\pi y}{2}\right) = 0}$$

5.38.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y \sin\left(\frac{\pi y}{2}\right)} dy = \int dt$$
$$\int^{y} \frac{1}{\_a \sin\left(\frac{\pi \_a}{2}\right)} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{1}{\_a \sin\left(\frac{\pi \_a}{2}\right)} d\_a = t + c_1 \tag{1}$$

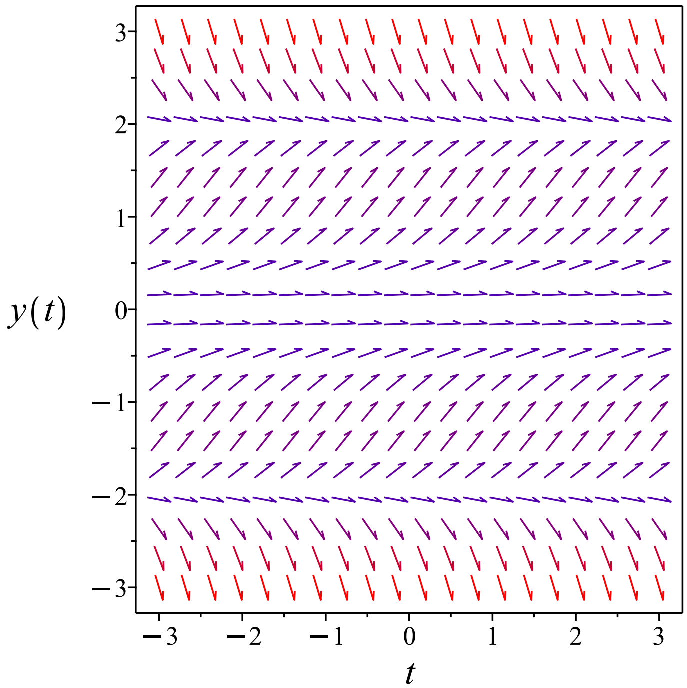


Figure 174: Slope field plot

Verification of solutions

$$\int^{y} \frac{1}{\_a \sin\left(\frac{\pi \_a}{2}\right)} d\_a = t + c_1$$

Verified OK.

5.38.2 Maple step by step solution

Let's solve

$y' - y \sin\left(\frac{\pi y}{2}\right) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y \sin\left(\frac{\pi y}{2}\right)} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y \sin\left(\frac{\pi y}{2}\right)} dt = \int 1 dt + c_1$

- Cannot compute integral

  $\int \frac{y'}{y\sin\left(\frac{\pi y}{2}\right)}dt = t + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(t),t)=y(t)*sin(Pi/2*y(t)),y(t), singsol=all)
```

$$t - \left(\int^{y(t)} \frac{\csc\left(\frac{\pi\_a}{2}\right)}{\_a}d\_a\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.786 (sec). Leaf size: 37

```
DSolve[y'[t]==y[t]*Sin[Pi/2*y[t]],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\csc\left(\frac{1}{2}\pi K[1]\right)}{K[1]}dK[1]\&\right][t + c_1]$$

$$y(t) \to 0$$

5.39 problem 37 (viii)

Internal problem ID [12989]
Internal file name [`OUTPUT/11641_Tuesday_November_07_2023_11_53_56_PM_58776703/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.6 page 89
Problem number: 37 (viii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + y^3 = 0}$$

5.39.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^3 + y^2} dy = \int dt$$
$$\int^y \frac{1}{-\_a^3 + \_a^2} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^y \frac{1}{-\_a^3 + \_a^2} d\_a = t + c_1 \tag{1}$$

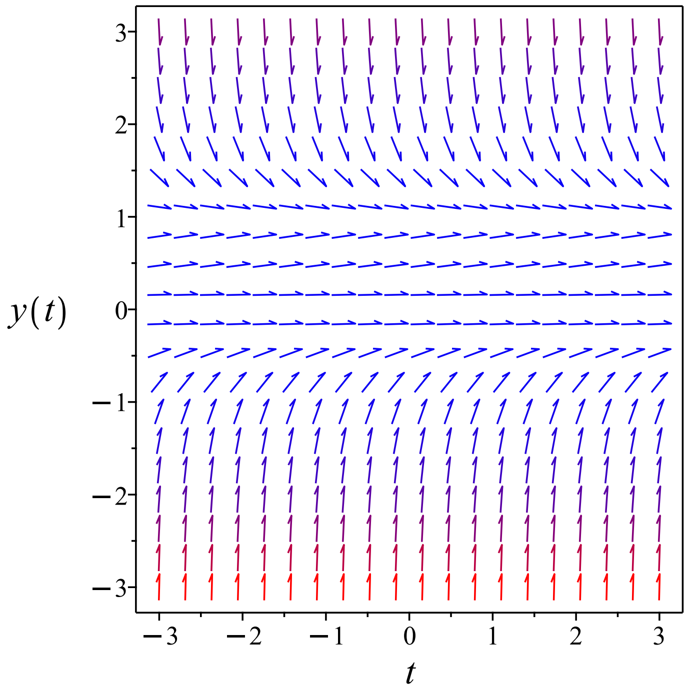


Figure 175: Slope field plot

Verification of solutions

$$\int^{y} \frac{1}{-\_a^{3}+\_a^{2}}d\_a = t + c_1$$

Verified OK.

5.39.2 Maple step by step solution

Let's solve

$y' - y^2 + y^3 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-y^3} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-y^3}dt = \int 1dt + c_1$

- Evaluate integral

$$-\ln(y-1)-\frac{1}{y}+\ln(y)=t+c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 20

```
dsolve(diff(y(t),t)=y(t)^2-y(t)^3,y(t), singsol=all)
```

$$y(t)=\frac{1}{\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-t-1}}{c_1}\right)+1}$$

✓ Solution by Mathematica

Time used: 0.408 (sec). Leaf size: 40

```
DSolve[y'[t]==y[t]^2-y[t]^3,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to \text{InverseFunction}\left[\frac{1}{\#1}+\log(1-\#1)-\log(\#1)\&\right][-t+c_1]$$
$$y(t)\to 0$$
$$y(t)\to 1$$

6 Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121

6.1 problem 1

Internal problem ID [12990]
Internal file name [`OUTPUT/11642_Tuesday_November_07_2023_11_53_58_PM_51670567/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 4y = 9\,\mathrm{e}^{-t}}$$

6.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned}p(t) &= 4\\ q(t) &= 9\,\mathrm{e}^{-t}\end{aligned}$$

Hence the ode is

$$y' + 4y = 9\,\mathrm{e}^{-t}$$

The integrating factor μ is

$$\mu = e^{\int 4dt}$$
$$= e^{4t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(9\,e^{-t}\right)$$
$$\frac{d}{dt}\left(e^{4t}y\right) = \left(e^{4t}\right)\left(9\,e^{-t}\right)$$
$$d\left(e^{4t}y\right) = \left(9\,e^{3t}\right)\,dt$$

Integrating gives

$$e^{4t}y = \int 9\,e^{3t}\,dt$$
$$e^{4t}y = 3\,e^{3t} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{4t}$ results in

$$y = 3\,e^{-4t}e^{3t} + c_1e^{-4t}$$

which simplifies to

$$y = 3\,e^{-t} + c_1e^{-4t}$$

Summary
The solution(s) found are the following

$$y = 3\,e^{-t} + c_1e^{-4t} \tag{1}$$

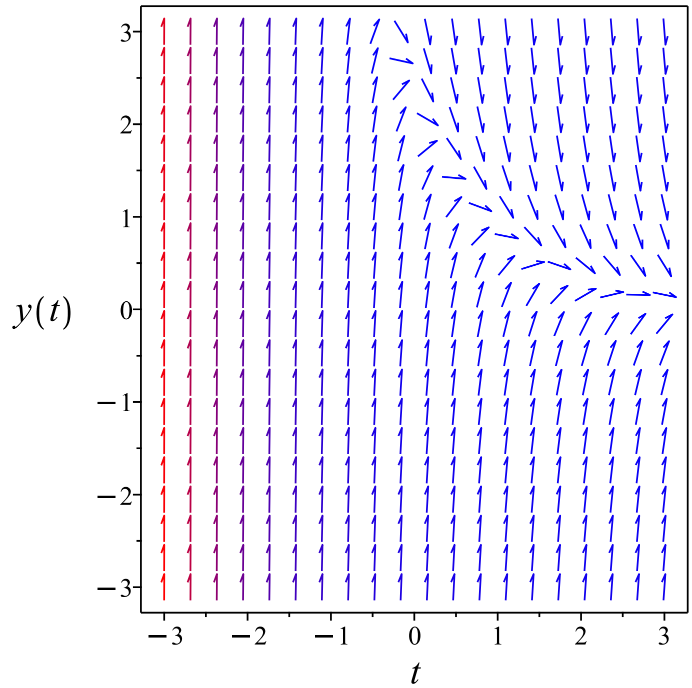


Figure 176: Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{-t} + c_1\mathrm{e}^{-4t}$$

Verified OK.

6.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -4y + 9\,\mathrm{e}^{-t} \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 176: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{-4t}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-4t}} dy \end{aligned}$$

Which results in

$$S = e^{4t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -4y + 9 e^{-t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 4 e^{4t} y \\ S_y &= e^{4t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 9 e^{3t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 9 e^{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3\,e^{3R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{4t}y = 3\,e^{3t} + c_1$$

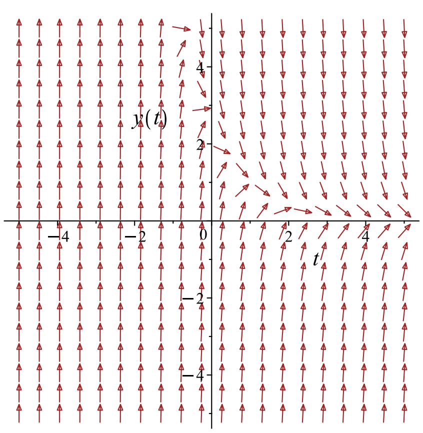
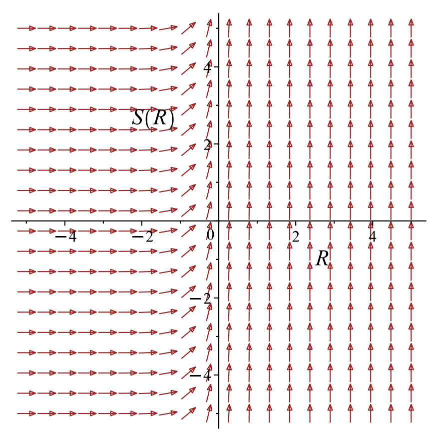
Which simplifies to

$$e^{4t}y = 3\,e^{3t} + c_1$$

Which gives

$$y = \left(3\,e^{3t} + c_1\right)e^{-4t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -4y + 9\,e^{-t}$ 	$R = t$ $S = e^{4t}y$	$\frac{dS}{dR} = 9\,e^{3R}$ 

Summary

The solution(s) found are the following

$$y = \left(3\,e^{3t} + c_1\right)e^{-4t} \tag{1}$$

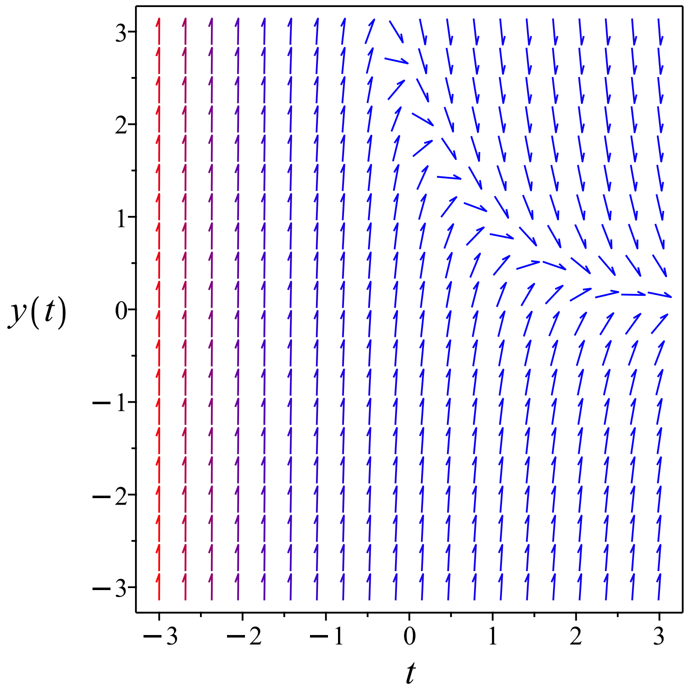


Figure 177: Slope field plot

Verification of solutions

$$y = \left(3\,e^{3t} + c_1\right) e^{-4t}$$

Verified OK.

6.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-4y + 9\,\mathrm{e}^{-t}\right) \mathrm{d}t \\
\left(4y - 9\,\mathrm{e}^{-t}\right) \mathrm{d}t + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= 4y - 9\,\mathrm{e}^{-t} \\
N(t, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4y - 9\,\mathrm{e}^{-t}\right) \\
&= 4
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((4) - (0)) \\ &= 4 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 4 \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{4t} \\ &= \mathrm{e}^{4t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{4t}\left(4y - 9\,\mathrm{e}^{-t}\right) \\ &= \left(4\,\mathrm{e}^{t} y - 9\right)\mathrm{e}^{3t} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{4t}(1) \\ &= \mathrm{e}^{4t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\left(4\,\mathrm{e}^{t} y - 9\right)\mathrm{e}^{3t}\right) + \left(\mathrm{e}^{4t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int \overline{M}\, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int \left(4\, \mathrm{e}^{t} y - 9\right) \mathrm{e}^{3t}\, \mathrm{d}t$$

$$\phi = -3\, \mathrm{e}^{3t} + \mathrm{e}^{4t} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{4t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{4t}$. Therefore equation (4) becomes

$$\mathrm{e}^{4t} = \mathrm{e}^{4t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -3\, \mathrm{e}^{3t} + \mathrm{e}^{4t} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3\, \mathrm{e}^{3t} + \mathrm{e}^{4t} y$$

The solution becomes

$$y = \left(3\, \mathrm{e}^{3t} + c_1\right) \mathrm{e}^{-4t}$$

Summary

The solution(s) found are the following

$$y = \left(3\, \mathrm{e}^{3t} + c_1\right) \mathrm{e}^{-4t} \tag{1}$$

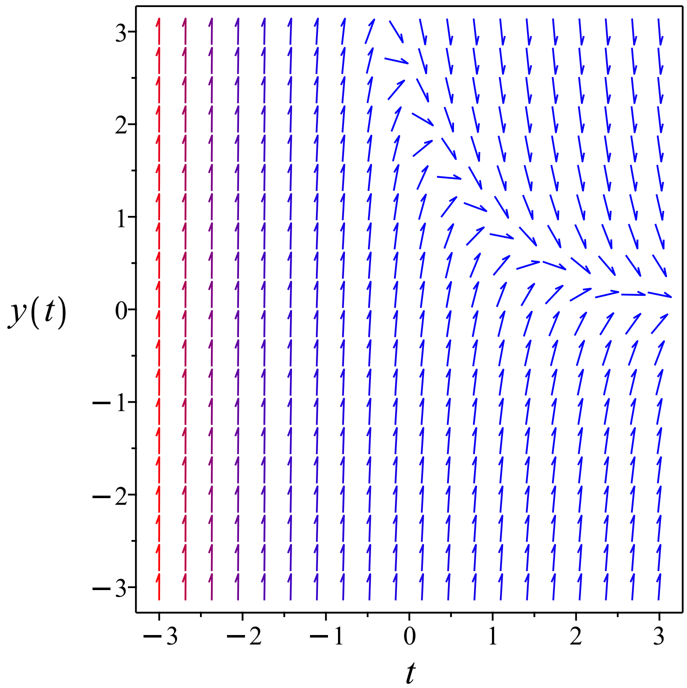


Figure 178: Slope field plot

Verification of solutions

$$y = \left(3\,\mathrm{e}^{3t} + c_1\right)\mathrm{e}^{-4t}$$

Verified OK.

6.1.4 Maple step by step solution

Let's solve

$y' + 4y = 9\,\mathrm{e}^{-t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -4y + 9\,\mathrm{e}^{-t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 4y = 9\,\mathrm{e}^{-t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 4y) = 9\mu(t)\,\mathrm{e}^{-t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y'+4y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = 4\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = e^{4t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 9\mu(t)\,e^{-t}dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 9\mu(t)\,e^{-t}dt + c_1$
- Solve for y

  $y = \frac{\int 9\mu(t)e^{-t}dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = e^{4t}$

  $y = \frac{\int 9\,e^{-t}e^{4t}dt + c_1}{e^{4t}}$
- Evaluate the integrals on the rhs

  $y = \frac{3\,e^{3t} + c_1}{e^{4t}}$
- Simplify

  $y = (3\,e^{3t} + c_1)\,e^{-4t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(t),t)=-4*y(t)+9*exp(-t),y(t), singsol=all)
```

$$y(t) = \left(3\,\mathrm{e}^{3t} + c_1\right)\mathrm{e}^{-4t}$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 21

```
DSolve[y'[t]==-4*y[t]+9*Exp[-t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-4t}\left(3e^{3t} + c_1\right)$$

6.2 problem 2

Internal problem ID [12991]
Internal file name [`OUTPUT/11643_Tuesday_November_07_2023_11_53_59_PM_82349565/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 4y = 3\,\mathrm{e}^{-t}}$$

6.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 4$$
$$q(t) = 3\,\mathrm{e}^{-t}$$

Hence the ode is

$$y' + 4y = 3\,\mathrm{e}^{-t}$$

The integrating factor μ is

$$\mu = e^{\int 4dt}$$
$$= e^{4t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(3\,e^{-t}\right)$$
$$\frac{d}{dt}\left(e^{4t}y\right) = \left(e^{4t}\right)\left(3\,e^{-t}\right)$$
$$d\left(e^{4t}y\right) = \left(3\,e^{3t}\right)\,dt$$

Integrating gives

$$e^{4t}y = \int 3\,e^{3t}\,dt$$
$$e^{4t}y = e^{3t} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{4t}$ results in

$$y = e^{-4t}e^{3t} + c_1 e^{-4t}$$

which simplifies to

$$y = e^{-t} + c_1 e^{-4t}$$

Summary
The solution(s) found are the following

$$y = e^{-t} + c_1 e^{-4t} \tag{1}$$

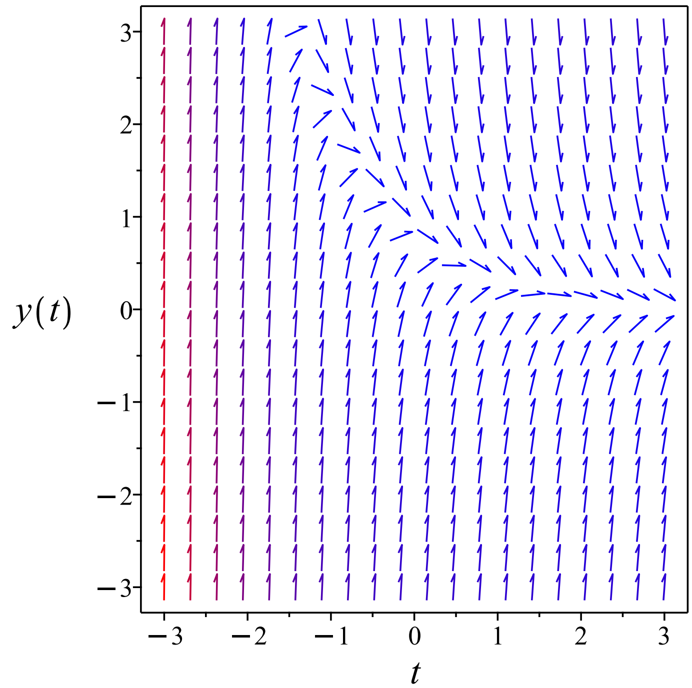


Figure 179: Slope field plot

Verification of solutions

$$y = e^{-t} + c_1 e^{-4t}$$

Verified OK.

6.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -4y + 3\,e^{-t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 179: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-4t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-4t}} dy \end{aligned}$$

Which results in

$$S = e^{4t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -4y + 3 e^{-t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 4 e^{4t} y \\ S_y &= e^{4t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 e^{3t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3 e^{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{3R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{4t} y = e^{3t} + c_1$$

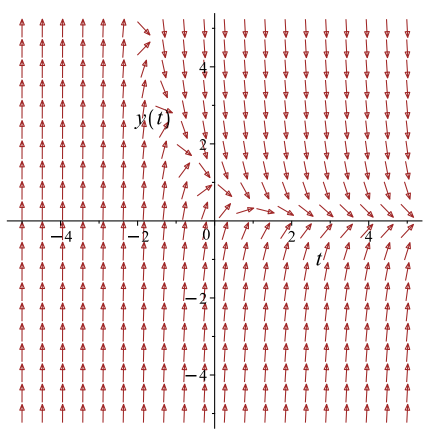
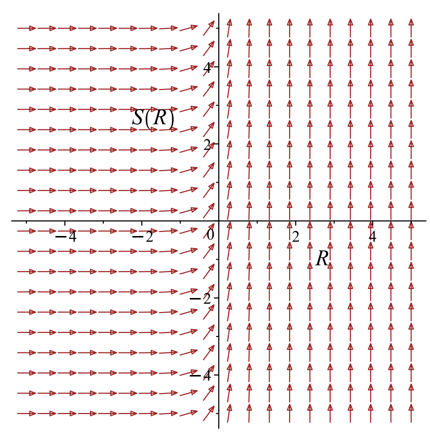
Which simplifies to

$$e^{4t} y = e^{3t} + c_1$$

Which gives

$$y = \left(e^{3t} + c_1\right) e^{-4t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -4y + 3\,e^{-t}$ 	$R = t$ $S = e^{4t} y$	$\frac{dS}{dR} = 3\,e^{3R}$ 

Summary

The solution(s) found are the following

$$y = \left(e^{3t} + c_1\right) e^{-4t} \tag{1}$$

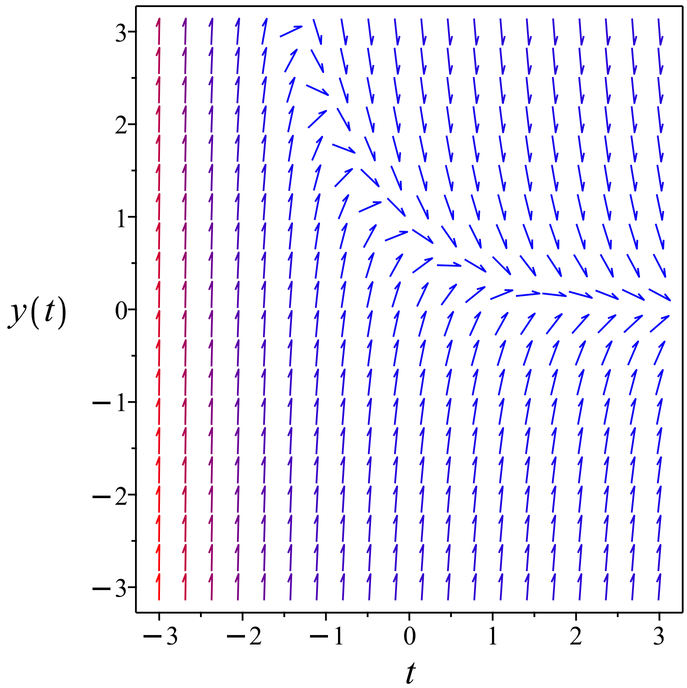


Figure 180: Slope field plot

Verification of solutions

$$y = \left(e^{3t} + c_1\right) e^{-4t}$$

Verified OK.

6.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-4y + 3\,\mathrm{e}^{-t}\right) \mathrm{d}t$$
$$\left(4y - 3\,\mathrm{e}^{-t}\right) \mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = 4y - 3\,\mathrm{e}^{-t}$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4y - 3\,\mathrm{e}^{-t}\right) \\ &= 4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((4) - (0)) \\
&= 4
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int 4 \, \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{4t} \\
&= \mathrm{e}^{4t}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{4t}(4y - 3\,\mathrm{e}^{-t}) \\
&= \left(4\,\mathrm{e}^{t} y - 3\right)\mathrm{e}^{3t}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{4t}(1) \\
&= \mathrm{e}^{4t}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\left(4\,\mathrm{e}^{t} y - 3\right)\mathrm{e}^{3t}\right) + \left(\mathrm{e}^{4t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int (4\,\mathrm{e}^{t}y - 3)\,\mathrm{e}^{3t} \, \mathrm{d}t$$

$$\phi = -\mathrm{e}^{3t} + \mathrm{e}^{4t}y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{4t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{4t}$. Therefore equation (4) becomes

$$\mathrm{e}^{4t} = \mathrm{e}^{4t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{3t} + \mathrm{e}^{4t}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\mathrm{e}^{3t} + \mathrm{e}^{4t}y$$

The solution becomes

$$y = \left(\mathrm{e}^{3t} + c_1\right)\mathrm{e}^{-4t}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(\mathrm{e}^{3t} + c_1\right)\mathrm{e}^{-4t} \tag{1}$$

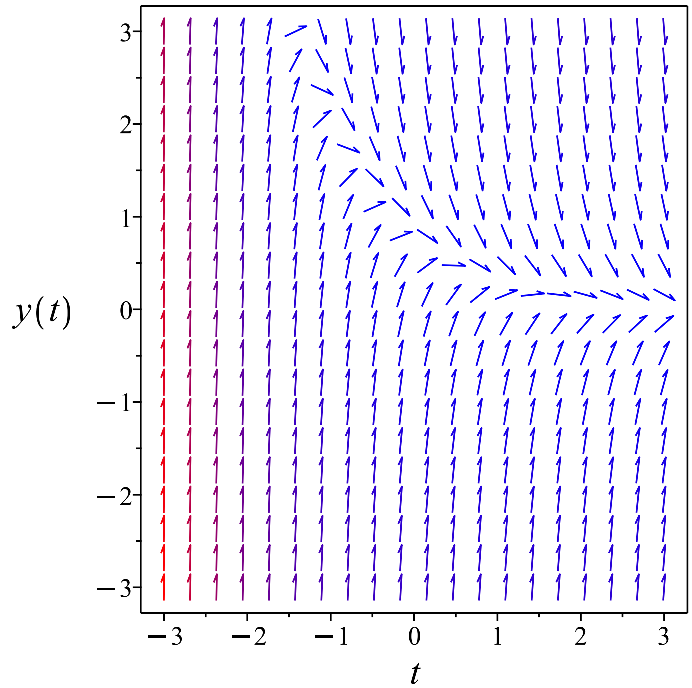


Figure 181: Slope field plot

Verification of solutions

$$y = \left(e^{3t} + c_1\right) e^{-4t}$$

Verified OK.

6.2.4 Maple step by step solution

Let's solve

$y' + 4y = 3\,e^{-t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -4y + 3\,e^{-t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 4y = 3\,e^{-t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + 4y\right) = 3\mu(t)\,e^{-t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\,(y' + 4y) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = 4\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{4t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 3\mu(t)\, \mathrm{e}^{-t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 3\mu(t)\, \mathrm{e}^{-t} dt + c_1$

- Solve for y

  $y = \frac{\int 3\mu(t)\mathrm{e}^{-t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{4t}$

  $y = \frac{\int 3\, \mathrm{e}^{-t} \mathrm{e}^{4t} dt + c_1}{\mathrm{e}^{4t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{3t} + c_1}{\mathrm{e}^{4t}}$

- Simplify

  $y = (\mathrm{e}^{3t} + c_1)\, \mathrm{e}^{-4t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(t),t)=-4*y(t)+3*exp(-t),y(t), singsol=all)
```

$$y(t) = \left(e^{3t} + c_1\right) e^{-4t}$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 19

```
DSolve[y'[t]==-4*y[t]+3*Exp[-t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-4t}\left(e^{3t} + c_1\right)$$

6.3 problem 3

Internal problem ID [12992]
Internal file name [`OUTPUT/11644_Tuesday_November_07_2023_11_54_00_PM_44648510/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 3y = 4\cos(2t)}$$

6.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 3$$
$$q(t) = 4\cos(2t)$$

Hence the ode is

$$y' + 3y = 4\cos(2t)$$

The integrating factor μ is

$$\mu = e^{\int 3dt} \\ = e^{3t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)(4\cos(2t))$$
$$\frac{d}{dt}(e^{3t}y) = (e^{3t})(4\cos(2t))$$
$$d(e^{3t}y) = (4e^{3t}\cos(2t))\,dt$$

Integrating gives

$$e^{3t}y = \int 4e^{3t}\cos(2t)\,dt$$
$$e^{3t}y = \frac{12e^{3t}\cos(2t)}{13} + \frac{8e^{3t}\sin(2t)}{13} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{3t}$ results in

$$y = e^{-3t}\left(\frac{12e^{3t}\cos(2t)}{13} + \frac{8e^{3t}\sin(2t)}{13}\right) + e^{-3t}c_1$$

which simplifies to

$$y = \frac{8\sin(2t)}{13} + \frac{12\cos(2t)}{13} + e^{-3t}c_1$$

Summary

The solution(s) found are the following

$$y = \frac{8\sin(2t)}{13} + \frac{12\cos(2t)}{13} + e^{-3t}c_1 \tag{1}$$

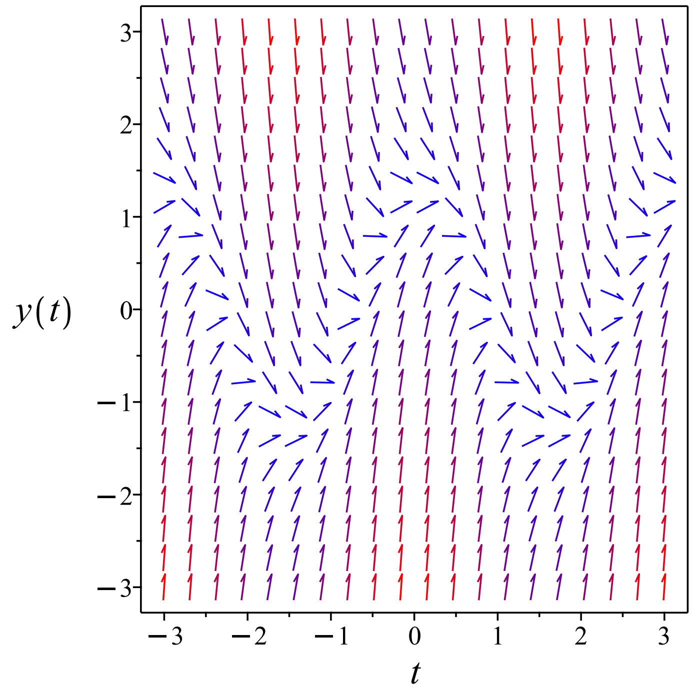


Figure 182: Slope field plot

Verification of solutions

$$y = \frac{8\sin(2t)}{13} + \frac{12\cos(2t)}{13} + e^{-3t}c_1$$

Verified OK.

6.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -3y + 4\cos(2t) \\ y' &= \omega(t,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 182: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{-3t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-3t}} dy \end{aligned}$$

Which results in

$$S = e^{3t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -3y + 4\cos(2t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 3\,e^{3t} y \\ S_y &= e^{3t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4\,e^{3t} \cos(2t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4\,e^{3R} \cos(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + \frac{4\,e^{3R}(3\cos(2R) + 2\sin(2R))}{13} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{3t}y = \frac{4\,e^{3t}(3\cos(2t) + 2\sin(2t))}{13} + c_1$$

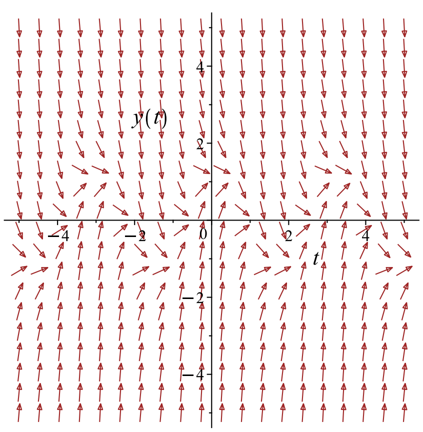
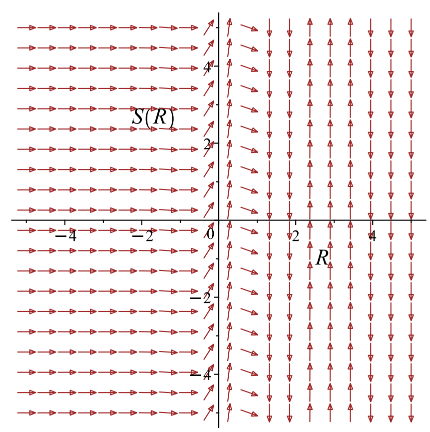
Which simplifies to

$$e^{3t}y = \frac{4\,e^{3t}(3\cos(2t) + 2\sin(2t))}{13} + c_1$$

Which gives

$$y = \frac{e^{-3t}(12\,e^{3t}\cos(2t) + 8\,e^{3t}\sin(2t) + 13c_1)}{13}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -3y + 4\cos(2t)$ 	$R = t$ $S = e^{3t}y$	$\frac{dS}{dR} = 4\,e^{3R}\cos(2R)$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-3t}(12\,e^{3t}\cos(2t) + 8\,e^{3t}\sin(2t) + 13c_1)}{13} \tag{1}$$

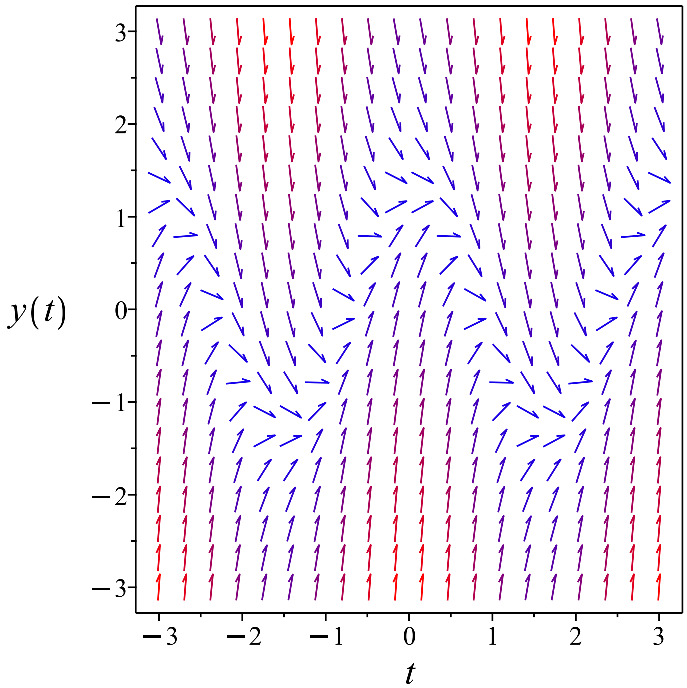


Figure 183: Slope field plot

Verification of solutions

$$y = \frac{e^{-3t}(12\,e^{3t}\cos(2t) + 8\,e^{3t}\sin(2t) + 13c_1)}{13}$$

Verified OK.

6.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-3y + 4\cos(2t))\,\mathrm{d}t$$
$$(3y - 4\cos(2t))\,\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,y) = 3y - 4\cos(2t)$$
$$N(t,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3y - 4\cos(2t)) \\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((3) - (0)) \\
&= 3
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int 3 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{3t} \\
&= \mathrm{e}^{3t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{3t}(3y - 4\cos(2t)) \\
&= (3y - 4\cos(2t))\,\mathrm{e}^{3t}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{3t}(1) \\
&= \mathrm{e}^{3t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left((3y - 4\cos(2t))\,\mathrm{e}^{3t}\right) + \left(\mathrm{e}^{3t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int (3y - 4\cos(2t))\,\mathrm{e}^{3t} \,\mathrm{d}t$$
$$\phi = -\frac{(-13y + 12\cos(2t) + 8\sin(2t))\,\mathrm{e}^{3t}}{13} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{3t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{3t}$. Therefore equation (4) becomes

$$\mathrm{e}^{3t} = \mathrm{e}^{3t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(-13y + 12\cos(2t) + 8\sin(2t))\,\mathrm{e}^{3t}}{13} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(-13y + 12\cos(2t) + 8\sin(2t))\,\mathrm{e}^{3t}}{13}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-3t}(12\,\mathrm{e}^{3t}\cos(2t) + 8\,\mathrm{e}^{3t}\sin(2t) + 13c_1)}{13}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-3t}(12\,e^{3t}\cos(2t) + 8\,e^{3t}\sin(2t) + 13c_1)}{13} \tag{1}$$

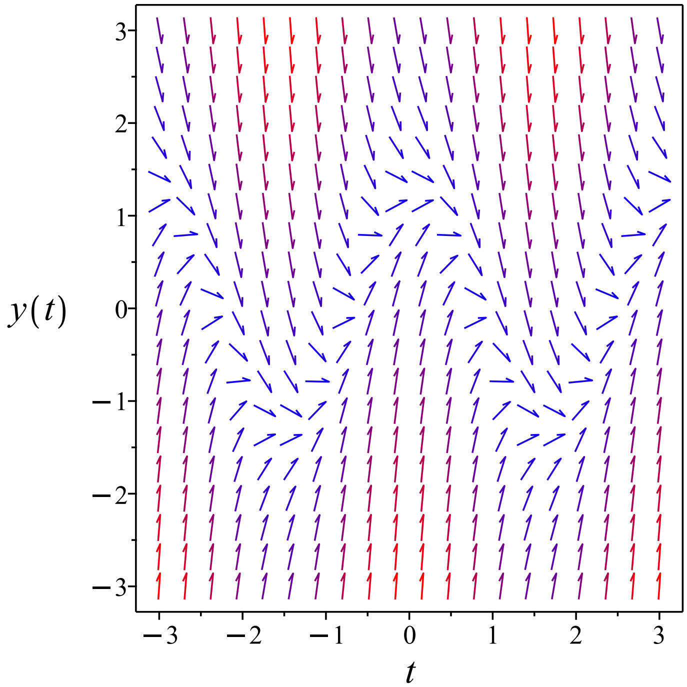


Figure 184: Slope field plot

Verification of solutions

$$y = \frac{e^{-3t}(12\,e^{3t}\cos(2t) + 8\,e^{3t}\sin(2t) + 13c_1)}{13}$$

Verified OK.

6.3.4 Maple step by step solution

Let's solve

$y' + 3y = 4\cos(2t)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -3y + 4\cos(2t)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + 3y = 4\cos(2t)$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)(y' + 3y) = 4\mu(t)\cos(2t)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$$\mu(t)(y' + 3y) = \mu'(t)\, y + \mu(t)\, y'$$

- Isolate $\mu'(t)$

$$\mu'(t) = 3\mu(t)$$

- Solve to find the integrating factor

$$\mu(t) = e^{3t}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 4\mu(t)\cos(2t)\, dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\, y = \int 4\mu(t)\cos(2t)\, dt + c_1$$

- Solve for y

$$y = \frac{\int 4\mu(t)\cos(2t)dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = e^{3t}$

$$y = \frac{\int 4\, e^{3t}\cos(2t)dt + c_1}{e^{3t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{12\, e^{3t}\cos(2t)}{13} + \frac{8\, e^{3t}\sin(2t)}{13} + c_1}{e^{3t}}$$

- Simplify

$$y = \frac{8\sin(2t)}{13} + \frac{12\cos(2t)}{13} + e^{-3t}c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(t),t)=-3*y(t)+4*cos(2*t),y(t), singsol=all)
```

$$y(t) = \frac{12\cos(2t)}{13} + \frac{8\sin(2t)}{13} + c_1 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.155 (sec). Leaf size: 31

```
DSolve[y'[t]==-3*y[t]+4*Cos[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{4}{13}(2\sin(2t) + 3\cos(2t)) + c_1 e^{-3t}$$

6.4 problem 4

Internal problem ID [12993]
Internal file name [`OUTPUT/11645_Tuesday_November_07_2023_11_54_01_PM_88644979/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = \sin (2t)}$$

6.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = \sin (2t)$$

Hence the ode is

$$y' - 2y = \sin (2t)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-2)dt} \\
&= e^{-2t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)(\sin(2t)) \\
\frac{d}{dt}(e^{-2t}y) &= (e^{-2t})(\sin(2t)) \\
d(e^{-2t}y) &= (e^{-2t}\sin(2t))\ dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-2t}y &= \int e^{-2t}\sin(2t)\ dt \\
e^{-2t}y &= -\frac{e^{-2t}\cos(2t)}{4} - \frac{e^{-2t}\sin(2t)}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-2t}$ results in

$$
y = e^{2t}\left(-\frac{e^{-2t}\cos(2t)}{4} - \frac{e^{-2t}\sin(2t)}{4}\right) + c_1 e^{2t}
$$

which simplifies to

$$
y = c_1 e^{2t} - \frac{\sin(2t)}{4} - \frac{\cos(2t)}{4}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 e^{2t} - \frac{\sin(2t)}{4} - \frac{\cos(2t)}{4} \tag{1}
$$

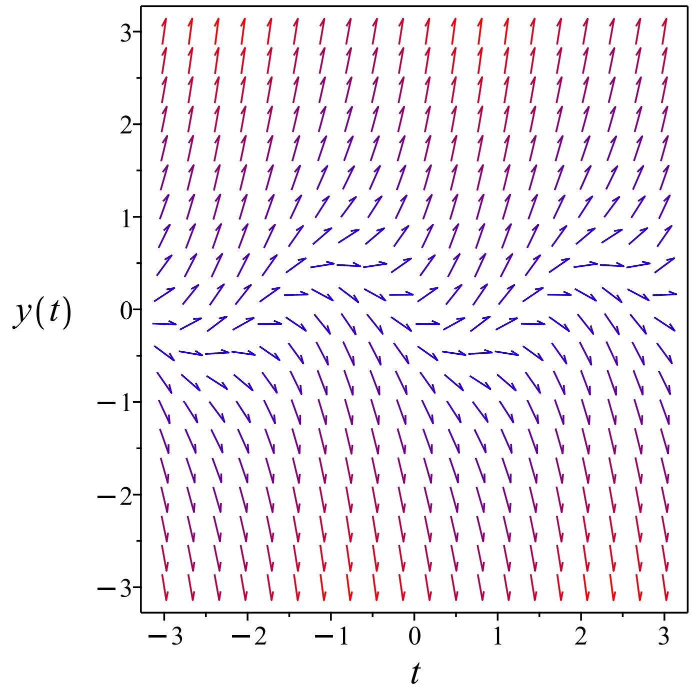


Figure 185: Slope field plot

Verification of solutions

$$y = c_1 e^{2t} - \frac{\sin(2t)}{4} - \frac{\cos(2t)}{4}$$

Verified OK.

6.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2y + \sin(2t)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 185: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, y) &= 0 \\ \eta(t, y) &= e^{2t}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{2t}} dy\end{aligned}$$

Which results in

$$S = e^{-2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = 2y + \sin(2t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1 \\ R_y &= 0 \\ S_t &= -2 e^{-2t} y \\ S_y &= e^{-2t}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-2t} \sin(2t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-2R} \sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{e^{-2R}(\cos(2R) + \sin(2R))}{4} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-2t}y = -\frac{(\sin(2t) + \cos(2t))\, e^{-2t}}{4} + c_1$$

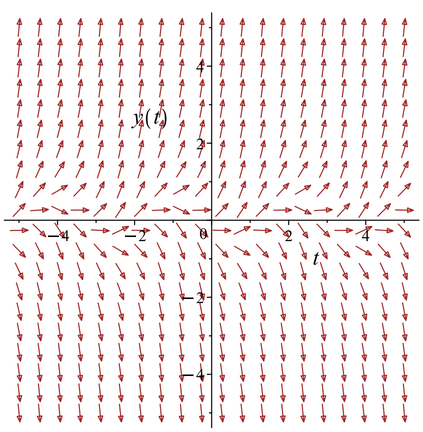
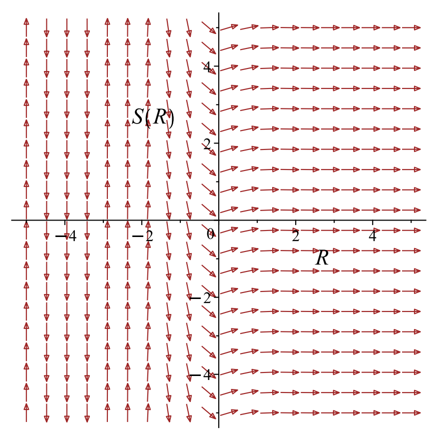
Which simplifies to

$$e^{-2t}y = -\frac{(\sin(2t) + \cos(2t))\, e^{-2t}}{4} + c_1$$

Which gives

$$y = -\frac{e^{2t}(e^{-2t}\sin(2t) + e^{-2t}\cos(2t) - 4c_1)}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2y + \sin(2t)$ 	$R = t$ $S = e^{-2t}y$	$\frac{dS}{dR} = e^{-2R}\sin(2R)$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{2t}(e^{-2t}\sin(2t) + e^{-2t}\cos(2t) - 4c_1)}{4} \tag{1}$$

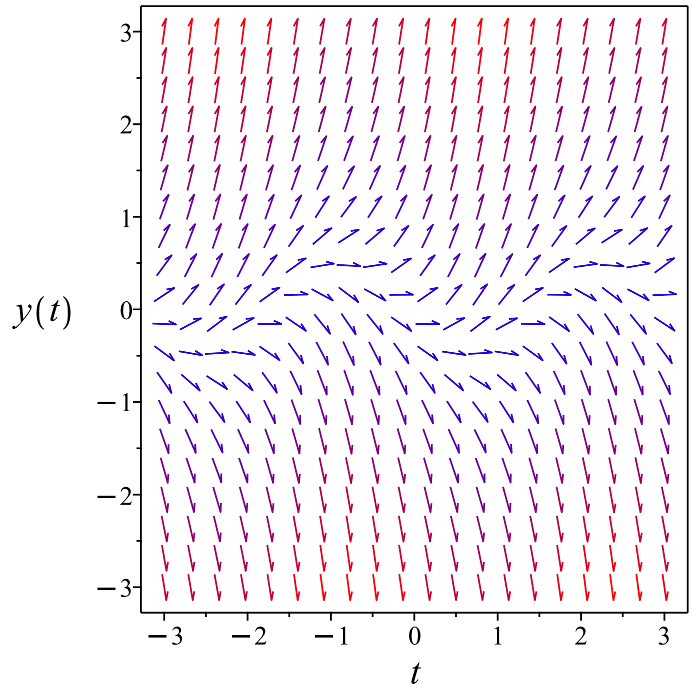


Figure 186: Slope field plot

Verification of solutions

$$y = -\frac{e^{2t}\left(e^{-2t}\sin\left(2t\right)+e^{-2t}\cos\left(2t\right)-4c_1\right)}{4}$$

Verified OK.

6.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (2y + \sin(2t))\, \mathrm{d}t\\ (-2y - \sin(2t))\, \mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -2y - \sin(2t)\\ N(t, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2y - \sin(2t))\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((-2) - (0)) \\ &= -2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, dt} \\ &= e^{\int -2 \, dt} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2t} \\ &= \mathrm{e}^{-2t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-2t}(-2y - \sin(2t)) \\ &= (-2y - \sin(2t))\,\mathrm{e}^{-2t} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-2t}(1) \\ &= \mathrm{e}^{-2t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left((-2y - \sin(2t))\,\mathrm{e}^{-2t}\right) + \left(\mathrm{e}^{-2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int (-2y - \sin(2t))\,\mathrm{e}^{-2t} \,\mathrm{d}t$$

$$\phi = \frac{(4y + \cos(2t) + \sin(2t))\,\mathrm{e}^{-2t}}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-2t} = \mathrm{e}^{-2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(4y + \cos(2t) + \sin(2t))\,\mathrm{e}^{-2t}}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(4y + \cos(2t) + \sin(2t))\,\mathrm{e}^{-2t}}{4}$$

The solution becomes

$$y = -\frac{\mathrm{e}^{2t}(\mathrm{e}^{-2t}\sin(2t) + \mathrm{e}^{-2t}\cos(2t) - 4c_1)}{4}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{2t}(e^{-2t}\sin(2t) + e^{-2t}\cos(2t) - 4c_1)}{4} \tag{1}$$

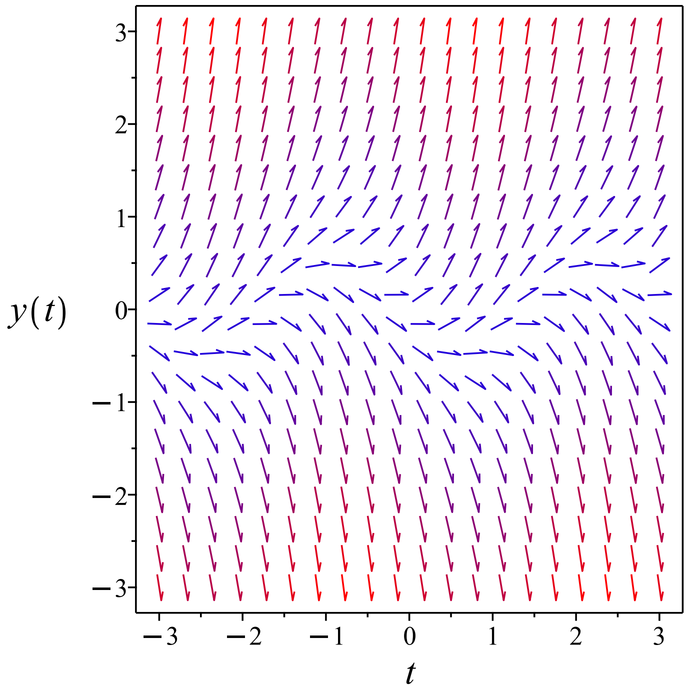


Figure 187: Slope field plot

Verification of solutions

$$y = -\frac{e^{2t}(e^{-2t}\sin(2t) + e^{-2t}\cos(2t) - 4c_1)}{4}$$

Verified OK.

6.4.4 Maple step by step solution

Let's solve

$y' - 2y = \sin(2t)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = 2y + \sin(2t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' - 2y = \sin(2t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)(y' - 2y) = \mu(t)\sin(2t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$\mu(t)(y' - 2y) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

$\mu'(t) = -2\mu(t)$

- Solve to find the integrating factor

$\mu(t) = \mathrm{e}^{-2t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\sin(2t)\, dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\, y = \int \mu(t)\sin(2t)\, dt + c_1$

- Solve for y

$y = \frac{\int \mu(t)\sin(2t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-2t}$

$y = \frac{\int \mathrm{e}^{-2t}\sin(2t)dt + c_1}{\mathrm{e}^{-2t}}$

- Evaluate the integrals on the rhs

$y = \frac{-\frac{\mathrm{e}^{-2t}\sin(2t)}{4} - \frac{\mathrm{e}^{-2t}\cos(2t)}{4} + c_1}{\mathrm{e}^{-2t}}$

- Simplify

$y = c_1\mathrm{e}^{2t} - \frac{\sin(2t)}{4} - \frac{\cos(2t)}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(t),t)=2*y(t)+sin(2*t),y(t), singsol=all)
```

$$y(t) = -\frac{\cos(2t)}{4} - \frac{\sin(2t)}{4} + c_1 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 30

```
DSolve[y'[t]==2*y[t]+Sin[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{4}\sin(2t) - \frac{1}{4}\cos(2t) + c_1 e^{2t}$$

6.5 problem 5

Internal problem ID [12994]
Internal file name [`OUTPUT/11646_Tuesday_November_07_2023_11_54_02_PM_16551769/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = -4\,e^{3t}}$$

6.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= -4\,e^{3t} \end{aligned}$$

Hence the ode is

$$y' - 3y = -4\,e^{3t}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-3)dt} \\
&= e^{-3t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(-4\,e^{3t}\right) \\
\frac{d}{dt}\left(e^{-3t} y\right) &= \left(e^{-3t}\right)\left(-4\,e^{3t}\right) \\
d\left(e^{-3t} y\right) &= -4\,dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-3t} y &= \int -4\,dt \\
e^{-3t} y &= -4t + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-3t}$ results in

$$
y = -4t\,e^{3t} + c_1 e^{3t}
$$

which simplifies to

$$
y = e^{3t}(-4t + c_1)
$$

Summary

The solution(s) found are the following

$$
y = e^{3t}(-4t + c_1) \tag{1}
$$

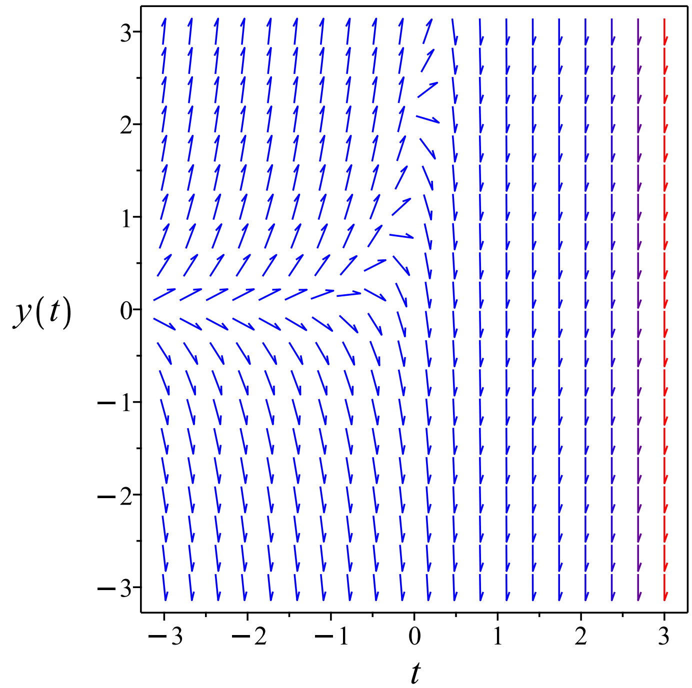


Figure 188: Slope field plot

<u>Verification of solutions</u>

$$y = e^{3t}(-4t + c_1)$$

Verified OK.

6.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3y - 4\,e^{3t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 188: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= e^{3t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{3t}} dy \end{aligned}$$

Which results in

$$S = e^{-3t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = 3y - 4 e^{3t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -3 e^{-3t} y \\ S_y &= e^{-3t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -4 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -4$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -4R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-3t}y = -4t + c_1$$

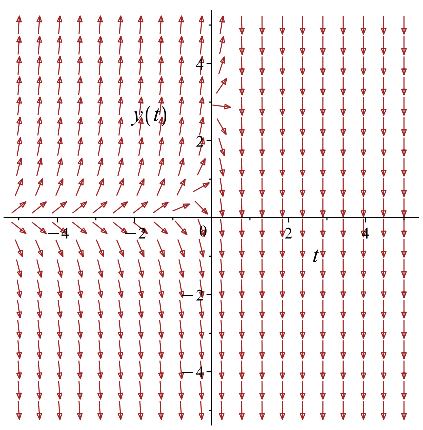
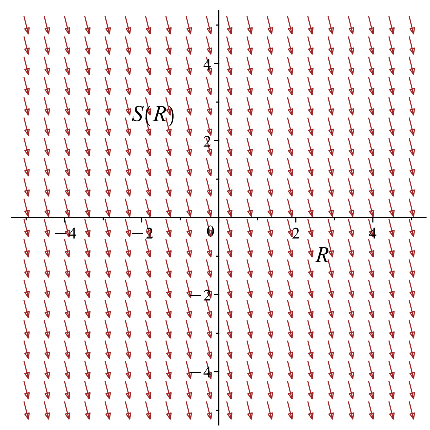
Which simplifies to

$$e^{-3t}y = -4t + c_1$$

Which gives

$$y = e^{3t}(-4t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dt} = 3y - 4e^{3t}$

</td><td>$R = t$
$S = e^{-3t}y$</td><td>$\frac{dS}{dR} = -4$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = e^{3t}(-4t + c_1) \tag{1}$$

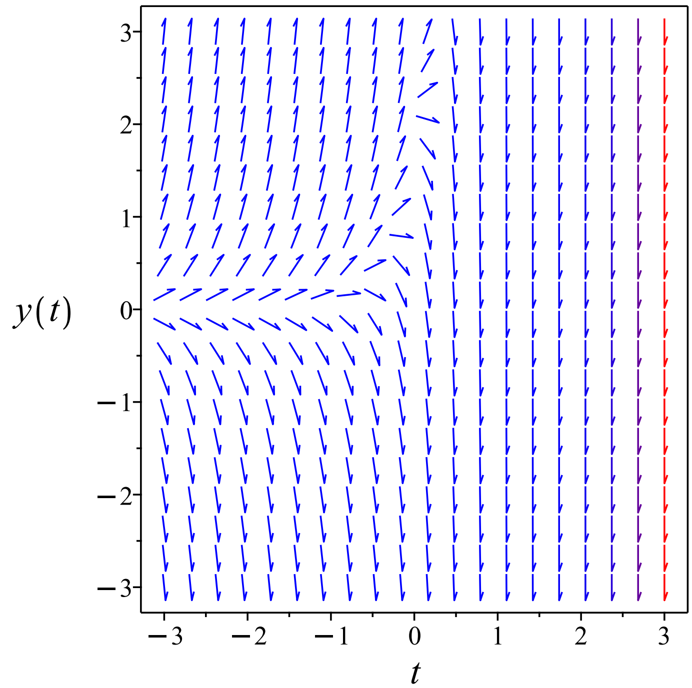


Figure 189: Slope field plot

Verification of solutions

$$y = e^{3t}(-4t + c_1)$$

Verified OK.

6.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(3y - 4\,\mathrm{e}^{3t}\right)\mathrm{d}t$$
$$\left(-3y + 4\,\mathrm{e}^{3t}\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -3y + 4\,\mathrm{e}^{3t}$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-3y + 4\,\mathrm{e}^{3t}\right)$$
$$= -3$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((-3) - (0)) \\ &= -3 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}t} \\ &= e^{\int -3 \,\mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3t} \\ &= \mathrm{e}^{-3t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-3t}\left(-3y + 4\,\mathrm{e}^{3t}\right) \\ &= -3\,\mathrm{e}^{-3t} y + 4 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-3t}(1) \\ &= \mathrm{e}^{-3t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(-3\,\mathrm{e}^{-3t} y + 4\right) + \left(\mathrm{e}^{-3t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -3 \, \mathrm{e}^{-3t} y + 4 \, \mathrm{d}t$$

$$\phi = 4t + \mathrm{e}^{-3t} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-3t} = \mathrm{e}^{-3t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 4t + \mathrm{e}^{-3t} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 4t + \mathrm{e}^{-3t} y$$

The solution becomes

$$y = \mathrm{e}^{3t}(-4t + c_1)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \mathrm{e}^{3t}(-4t + c_1) \tag{1}$$

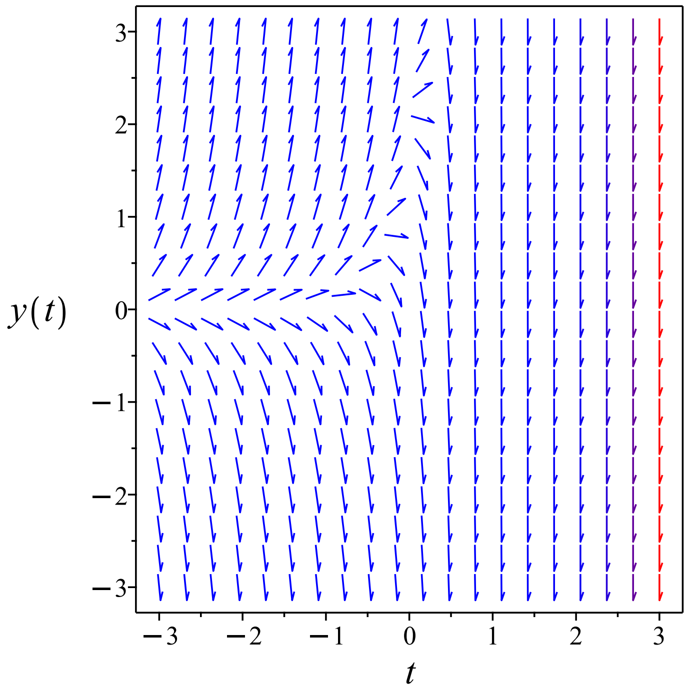


Figure 190: Slope field plot

Verification of solutions

$$y = e^{3t}(-4t + c_1)$$

Verified OK.

6.5.4 Maple step by step solution

Let's solve

$y' - 3y = -4\,e^{3t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3y - 4\,e^{3t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y = -4\,e^{3t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - 3y) = -4\mu(t)\,e^{3t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\,(y' - 3y) = \mu'(t)\, y + \mu(t)\, y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -3\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-3t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int -4\mu(t)\, \mathrm{e}^{3t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int -4\mu(t)\, \mathrm{e}^{3t} dt + c_1$
- Solve for y

  $y = \frac{\int -4\mu(t)\mathrm{e}^{3t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-3t}$

  $y = \frac{\int -4\, \mathrm{e}^{3t}\mathrm{e}^{-3t} dt + c_1}{\mathrm{e}^{-3t}}$
- Evaluate the integrals on the rhs

  $y = \frac{-4t + c_1}{\mathrm{e}^{-3t}}$
- Simplify

  $y = \mathrm{e}^{3t}(-4t + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=3*y(t)-4*exp(3*t),y(t), singsol=all)
```

$$y(t) = (-4t + c_1)\,e^{3t}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 17

```
DSolve[y'[t]==3*y[t]-4*Exp[3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{3t}(-4t + c_1)$$

6.6 problem 6

Internal problem ID [12995]
Internal file name [`OUTPUT/11647_Tuesday_November_07_2023_11_54_02_PM_21596004/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - \frac{y}{2} = 4\,\mathrm{e}^{\frac{t}{2}}}$$

6.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{1}{2}$$
$$q(t) = 4\,\mathrm{e}^{\frac{t}{2}}$$

Hence the ode is

$$y' - \frac{y}{2} = 4\,\mathrm{e}^{\frac{t}{2}}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{2}dt} \\
&= \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(4\,\mathrm{e}^{\frac{t}{2}}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\frac{t}{2}} y\right) &= \left(\mathrm{e}^{-\frac{t}{2}}\right)\left(4\,\mathrm{e}^{\frac{t}{2}}\right) \\
\mathrm{d}\left(\mathrm{e}^{-\frac{t}{2}} y\right) &= 4\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-\frac{t}{2}} y &= \int 4\,\mathrm{d}t \\
\mathrm{e}^{-\frac{t}{2}} y &= 4t + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{t}{2}}$ results in

$$
y = 4t\,\mathrm{e}^{\frac{t}{2}} + c_1 \mathrm{e}^{\frac{t}{2}}
$$

which simplifies to

$$
y = \mathrm{e}^{\frac{t}{2}}(4t + c_1)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{\frac{t}{2}}(4t + c_1) \tag{1}
$$

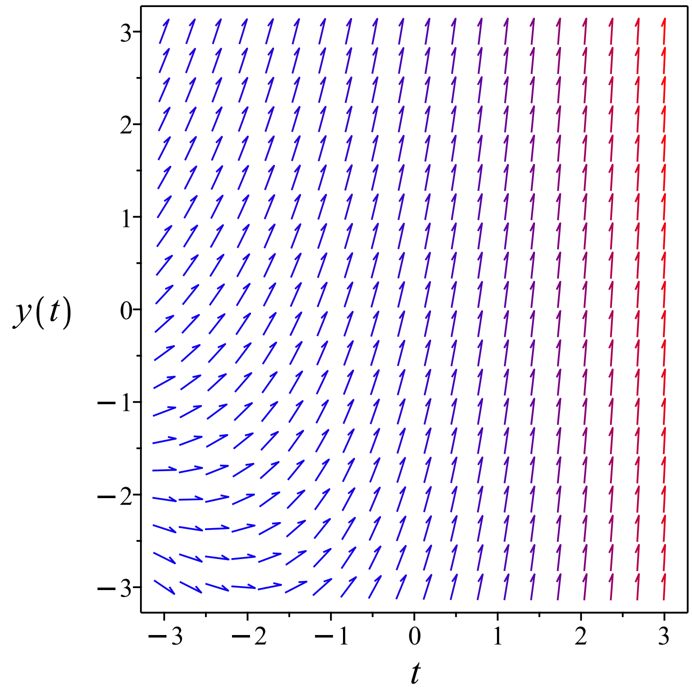


Figure 191: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{t}{2}}(4t + c_1)$$

Verified OK.

6.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{2} + 4\,\mathrm{e}^{\frac{t}{2}}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 191: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= e^{\frac{t}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{t}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{t}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{y}{2} + 4 e^{\frac{t}{2}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{e^{-\frac{t}{2}} y}{2} \\ S_y &= e^{-\frac{t}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 4R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-\frac{t}{2}} y = 4t + c_1$$

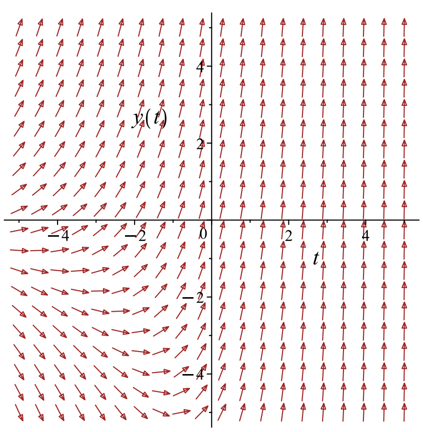
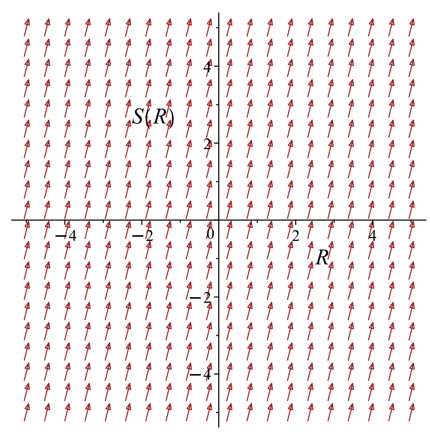
Which simplifies to

$$e^{-\frac{t}{2}} y = 4t + c_1$$

Which gives

$$y = e^{\frac{t}{2}} (4t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{y}{2} + 4\,e^{\frac{t}{2}}$ 	$R = t$ $S = e^{-\frac{t}{2}} y$	$\frac{dS}{dR} = 4$ 

Summary

The solution(s) found are the following

$$y = e^{\frac{t}{2}} (4t + c_1) \tag{1}$$

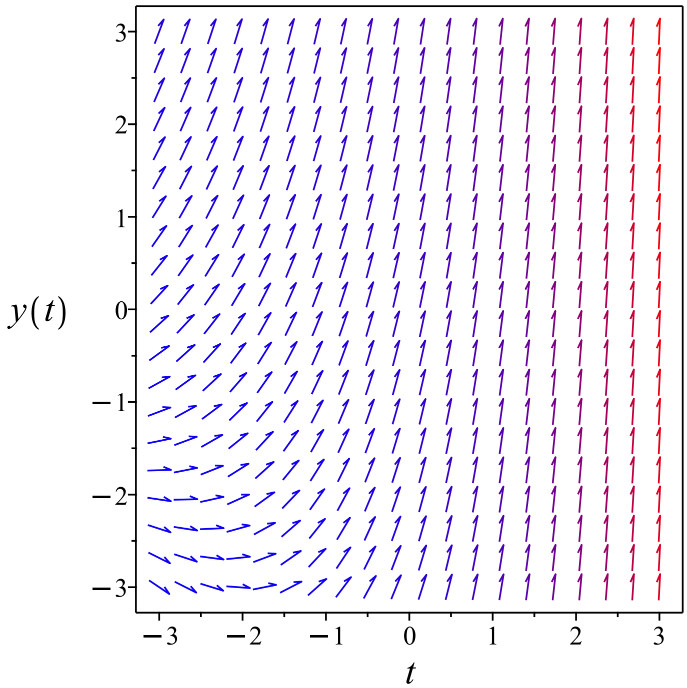


Figure 192: Slope field plot

Verification of solutions

$$y = e^{\frac{t}{2}}(4t + c_1)$$

Verified OK.

6.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(\frac{y}{2} + 4\,\mathrm{e}^{\frac{t}{2}}\right)\mathrm{d}t \\
\left(-\frac{y}{2} - 4\,\mathrm{e}^{\frac{t}{2}}\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -\frac{y}{2} - 4\,\mathrm{e}^{\frac{t}{2}} \\
N(t, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{2} - 4\,\mathrm{e}^{\frac{t}{2}}\right) \\
&= -\frac{1}{2}
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(-\frac{1}{2}\right) - (0)\right)\\ &= -\frac{1}{2}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t}\\ &= e^{\int -\frac{1}{2} \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{t}{2}}\\ &= \mathrm{e}^{-\frac{t}{2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-\frac{t}{2}}\left(-\frac{y}{2} - 4\,\mathrm{e}^{\frac{t}{2}}\right)\\ &= -\frac{\mathrm{e}^{-\frac{t}{2}} y}{2} - 4\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-\frac{t}{2}}(1)\\ &= \mathrm{e}^{-\frac{t}{2}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$
$$\left(-\frac{\mathrm{e}^{-\frac{t}{2}}y}{2} - 4\right) + \left(\mathrm{e}^{-\frac{t}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -\frac{\mathrm{e}^{-\frac{t}{2}}y}{2} - 4\,\mathrm{d}t$$
$$\phi = -4t + \mathrm{e}^{-\frac{t}{2}}y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{t}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{t}{2}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{t}{2}} = \mathrm{e}^{-\frac{t}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -4t + e^{-\frac{t}{2}}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -4t + e^{-\frac{t}{2}}y$$

The solution becomes

$$y = e^{\frac{t}{2}}(4t + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{\frac{t}{2}}(4t + c_1) \tag{1}$$

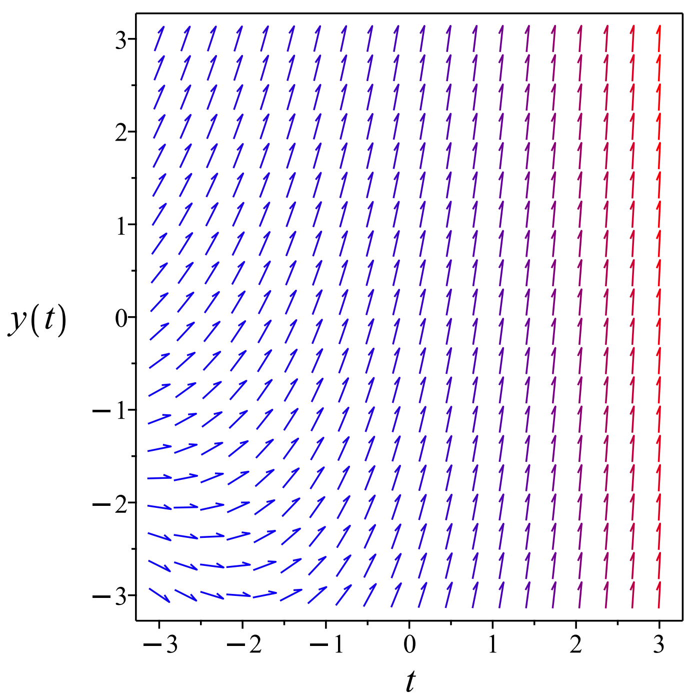


Figure 193: Slope field plot

Verification of solutions

$$y = e^{\frac{t}{2}}(4t + c_1)$$

Verified OK.

6.6.4 Maple step by step solution

Let's solve

$y' - \frac{y}{2} = 4\,\mathrm{e}^{\frac{t}{2}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{2} + 4\,\mathrm{e}^{\frac{t}{2}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{2} = 4\,\mathrm{e}^{\frac{t}{2}}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{y}{2}\right) = 4\mu(t)\,\mathrm{e}^{\frac{t}{2}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y' - \frac{y}{2}\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{\mu(t)}{2}$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-\frac{t}{2}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 4\mu(t)\,\mathrm{e}^{\frac{t}{2}} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 4\mu(t)\,\mathrm{e}^{\frac{t}{2}} dt + c_1$

- Solve for y

  $y = \frac{\int 4\mu(t)\mathrm{e}^{\frac{t}{2}} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-\frac{t}{2}}$

  $y = \frac{\int 4\,\mathrm{e}^{\frac{t}{2}}\,\mathrm{e}^{-\frac{t}{2}} dt + c_1}{\mathrm{e}^{-\frac{t}{2}}}$

- Evaluate the integrals on the rhs

  $y = \frac{4t + c_1}{\mathrm{e}^{-\frac{t}{2}}}$

- Simplify

$$y = e^{\frac{t}{2}}(4t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)=y(t)/2+4*exp(t/2),y(t), singsol=all)
```

$$y(t) = (4t + c_1)\,e^{\frac{t}{2}}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 19

```
DSolve[y'[t]==y[t]/2+4*Exp[t/2],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{t/2}(4t + c_1)$$

6.7 problem 7

Internal problem ID [12996]
Internal file name [`OUTPUT/11648_Tuesday_November_07_2023_11_54_03_PM_64080441/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = e^{\frac{t}{3}}}$$

With initial conditions

$$[y(0) = 1]$$

6.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 2$$
$$q(t) = e^{\frac{t}{3}}$$

Hence the ode is

$$y' + 2y = e^{\frac{t}{3}}$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{\frac{t}{3}}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.7.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dt} \\ &= e^{2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(e^{\frac{t}{3}}\right) \\ \frac{d}{dt}\left(e^{2t} y\right) &= \left(e^{2t}\right)\left(e^{\frac{t}{3}}\right) \\ d\left(e^{2t} y\right) &= e^{\frac{7t}{3}}\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2t} y &= \int e^{\frac{7t}{3}}\, dt \\ e^{2t} y &= \frac{3\, e^{\frac{7t}{3}}}{7} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$y = \frac{3\, e^{-2t} e^{\frac{7t}{3}}}{7} + c_1 e^{-2t}$$

which simplifies to

$$y = \frac{\left(3\, e^{\frac{7t}{3}} + 7c_1\right) e^{-2t}}{7}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{3}{7} + c_1$$

$$c_1 = \frac{4}{7}$$

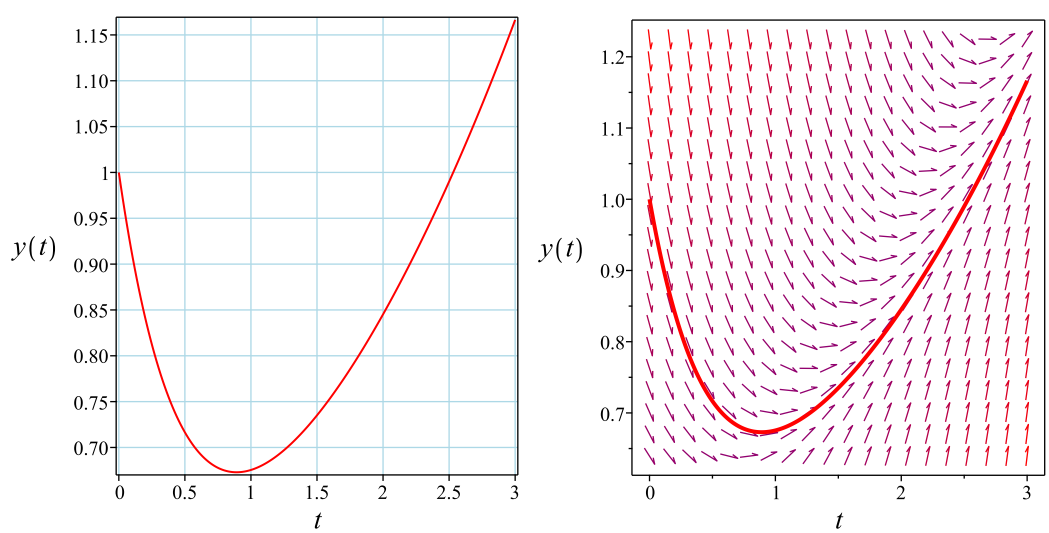
Substituting c_1 found above in the general solution gives

$$y = \frac{\left(3\,\mathrm{e}^{\frac{7t}{3}} + 4\right)\mathrm{e}^{-2t}}{7}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(3\,\mathrm{e}^{\frac{7t}{3}} + 4\right)\mathrm{e}^{-2t}}{7} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\left(3\,\mathrm{e}^{\frac{7t}{3}} + 4\right)\mathrm{e}^{-2t}}{7}$$

Verified OK.

6.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -2y + e^{\frac{t}{3}} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 194: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= 0\\ \eta(t,y) &= \mathrm{e}^{-2t}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\mathrm{e}^{-2t}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -2y + \mathrm{e}^{\frac{t}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1\\ R_y &= 0\\ S_t &= 2\,\mathrm{e}^{2t} y\\ S_y &= \mathrm{e}^{2t}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{\frac{7t}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{\frac{7R}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3\,e^{\frac{7R}{3}}}{7} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{2t} y = \frac{3\,e^{\frac{7t}{3}}}{7} + c_1$$

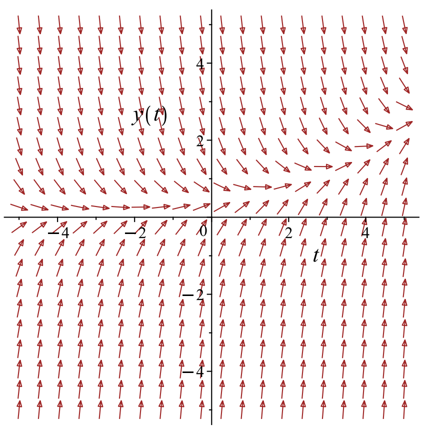
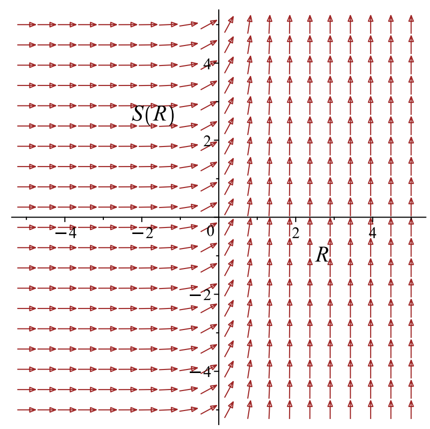
Which simplifies to

$$e^{2t} y = \frac{3\,e^{\frac{7t}{3}}}{7} + c_1$$

Which gives

$$y = \frac{\left(3\,e^{\frac{7t}{3}} + 7c_1\right) e^{-2t}}{7}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -2y + e^{\frac{t}{3}}$ 	$R = t$ $S = e^{2t} y$	$\frac{dS}{dR} = e^{\frac{7R}{3}}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{3}{7} + c_1$$

$$c_1 = \frac{4}{7}$$

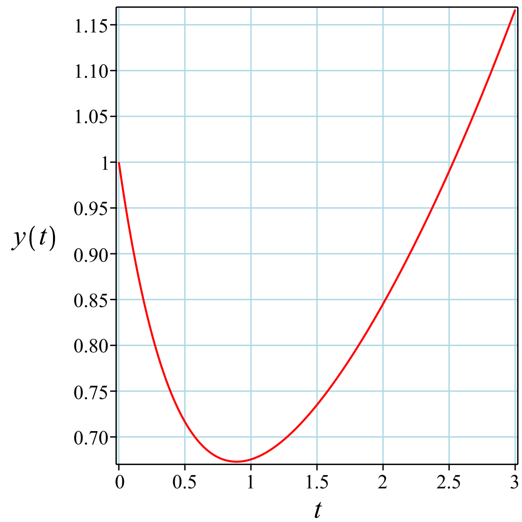
Substituting c_1 found above in the general solution gives

$$y = \frac{3\,e^{\frac{t}{3}}}{7} + \frac{4\,e^{-2t}}{7}$$

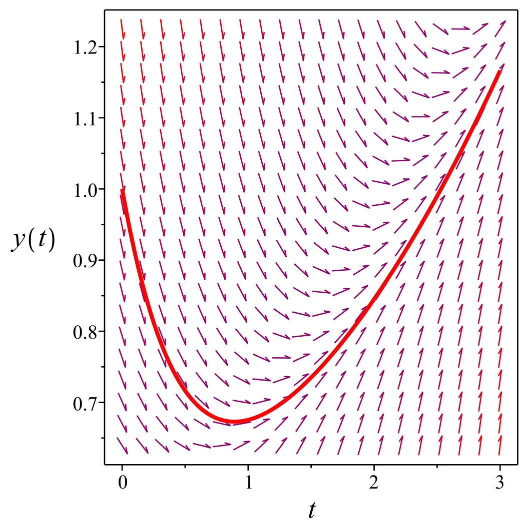
Summary

The solution(s) found are the following

$$y = \frac{3\,e^{\frac{t}{3}}}{7} + \frac{4\,e^{-2t}}{7} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3\,e^{\frac{t}{3}}}{7} + \frac{4\,e^{-2t}}{7}$$

Verified OK.

6.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(-2y+\mathrm{e}^{\frac{t}{3}}\right)\mathrm{d}t\\ \left(2y-\mathrm{e}^{\frac{t}{3}}\right)\mathrm{d}t+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=2y-\mathrm{e}^{\frac{t}{3}}\\ N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(2y-\mathrm{e}^{\frac{t}{3}}\right)\\ &=2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t}&=\frac{\partial}{\partial t}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)\\ &=1((2)-(0))\\ &=2\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t} \\ &= e^{\int 2\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2t} \\ &= \mathrm{e}^{2t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{2t}\left(2y - \mathrm{e}^{\frac{t}{3}}\right) \\ &= \left(2y - \mathrm{e}^{\frac{t}{3}}\right)\mathrm{e}^{2t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{2t}(1) \\ &= \mathrm{e}^{2t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\left(2y - \mathrm{e}^{\frac{t}{3}}\right)\mathrm{e}^{2t}\right) + \left(\mathrm{e}^{2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \left(2y - \mathrm{e}^{\frac{t}{3}}\right)\mathrm{e}^{2t}\,\mathrm{d}t\end{aligned}$$

$$\phi = \mathrm{e}^{2t}y - \frac{3\,\mathrm{e}^{\frac{7t}{3}}}{7} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{2t} = \mathrm{e}^{2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{2t} y - \frac{3\,\mathrm{e}^{\frac{7t}{3}}}{7} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{2t} y - \frac{3\,\mathrm{e}^{\frac{7t}{3}}}{7}$$

The solution becomes

$$y = \frac{\left(3\,\mathrm{e}^{\frac{7t}{3}} + 7c_1\right)\mathrm{e}^{-2t}}{7}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{3}{7} + c_1$$

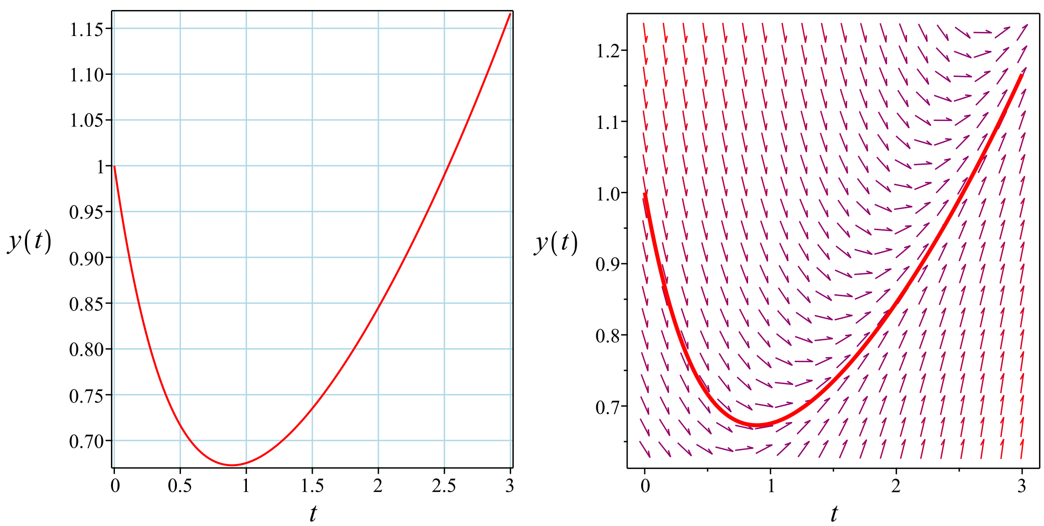
$$c_1 = \frac{4}{7}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\,\mathrm{e}^{\frac{t}{3}}}{7} + \frac{4\,\mathrm{e}^{-2t}}{7}$$

Summary
The solution(s) found are the following

$$y = \frac{3\,\mathrm{e}^{\frac{t}{3}}}{7} + \frac{4\,\mathrm{e}^{-2t}}{7} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{3\,\mathrm{e}^{\frac{t}{3}}}{7} + \frac{4\,\mathrm{e}^{-2t}}{7}$$

Verified OK.

6.7.5 Maple step by step solution

Let's solve

$\left[y' + 2y = \mathrm{e}^{\frac{t}{3}}, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y + e^{\frac{t}{3}}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = e^{\frac{t}{3}}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 2y) = \mu(t)\,e^{\frac{t}{3}}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + 2y) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = e^{2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,e^{\frac{t}{3}}\,dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int \mu(t)\,e^{\frac{t}{3}}\,dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t) e^{\frac{t}{3}}\,dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = e^{2t}$

  $y = \frac{\int e^{\frac{t}{3}} e^{2t}\,dt + c_1}{e^{2t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{3\,e^{\frac{7t}{3}}}{7} + c_1}{e^{2t}}$
- Simplify

  $y = \frac{\left(3\,e^{\frac{7t}{3}} + 7c_1\right) e^{-2t}}{7}$
- Use initial condition $y(0) = 1$

  $1 = \frac{3}{7} + c_1$
- Solve for c_1

  $c_1 = \frac{4}{7}$

- Substitute $c_1 = \frac{4}{7}$ into general solution and simplify

  $$y = \frac{\left(3\,\mathrm{e}^{\frac{7t}{3}}+4\right)\mathrm{e}^{-2t}}{7}$$

- Solution to the IVP

  $$y = \frac{\left(3\,\mathrm{e}^{\frac{7t}{3}}+4\right)\mathrm{e}^{-2t}}{7}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(y(t),t)+2*y(t)=exp(t/3),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{\left(3\,\mathrm{e}^{\frac{7t}{3}}+4\right)\mathrm{e}^{-2t}}{7}$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 25

```
DSolve[{y'[t]+2*y[t]==Exp[t/3],{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{7}e^{-2t}\left(3e^{7t/3}+4\right)$$

6.8 problem 8

Internal problem ID [12997]
Internal file name [`OUTPUT/11649_Tuesday_November_07_2023_11_54_04_PM_44735111/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = 3\,\mathrm{e}^{-2t}}$$

With initial conditions

$$[y(0) = 10]$$

6.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = 3\,\mathrm{e}^{-2t}$$

Hence the ode is

$$y' - 2y = 3\,\mathrm{e}^{-2t}$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3\,\mathrm{e}^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.8.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-2)dt} \\ &= \mathrm{e}^{-2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(3\,\mathrm{e}^{-2t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-2t}y\right) &= \left(\mathrm{e}^{-2t}\right)\left(3\,\mathrm{e}^{-2t}\right) \\ \mathrm{d}\left(\mathrm{e}^{-2t}y\right) &= \left(3\,\mathrm{e}^{-4t}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-2t}y &= \int 3\,\mathrm{e}^{-4t}\,\mathrm{d}t \\ \mathrm{e}^{-2t}y &= -\frac{3\,\mathrm{e}^{-4t}}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2t}$ results in

$$y = -\frac{3\,\mathrm{e}^{2t}\mathrm{e}^{-4t}}{4} + c_1\mathrm{e}^{2t}$$

which simplifies to

$$y = -\frac{3\,\mathrm{e}^{-2t}}{4} + c_1\mathrm{e}^{2t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 10$ in the above solution gives an equation to solve for the constant of integration.

$$10 = -\frac{3}{4} + c_1$$

$$c_1 = \frac{43}{4}$$

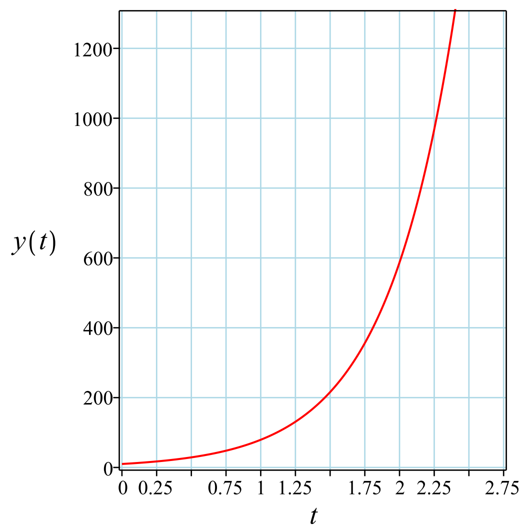
Substituting c_1 found above in the general solution gives

$$y = -\frac{3\,\mathrm{e}^{-2t}}{4} + \frac{43\,\mathrm{e}^{2t}}{4}$$

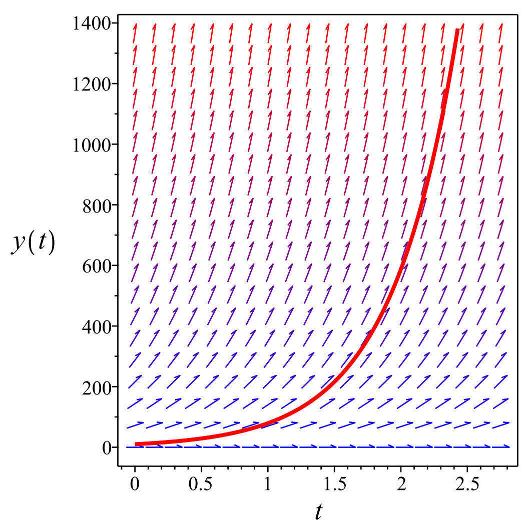
Summary

The solution(s) found are the following

$$y = -\frac{3\,\mathrm{e}^{-2t}}{4} + \frac{43\,\mathrm{e}^{2t}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3\,\mathrm{e}^{-2t}}{4} + \frac{43\,\mathrm{e}^{2t}}{4}$$

Verified OK.

6.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2y + 3\,\mathrm{e}^{-2t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 197: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= 0\\ \eta(t,y) &= e^{2t}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{e^{2t}} dy\end{aligned}$$

Which results in

$$S = e^{-2t} y$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = 2y + 3\,e^{-2t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1\\ R_y &= 0\\ S_t &= -2\,e^{-2t} y\\ S_y &= e^{-2t}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3\,\mathrm{e}^{-4t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3\,\mathrm{e}^{-4R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{3\,\mathrm{e}^{-4R}}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-2t}y = -\frac{3\,\mathrm{e}^{-4t}}{4} + c_1$$

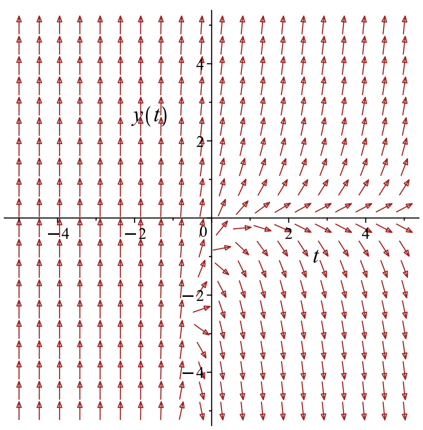
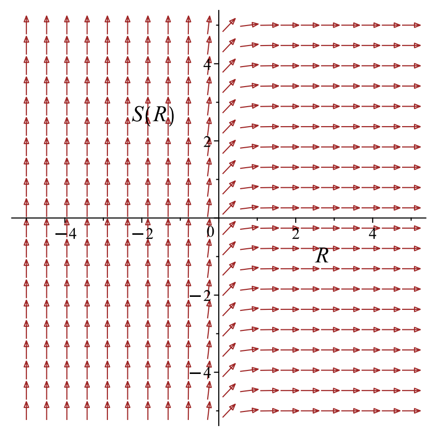
Which simplifies to

$$\mathrm{e}^{-2t}y = -\frac{3\,\mathrm{e}^{-4t}}{4} + c_1$$

Which gives

$$y = -\frac{(3\,\mathrm{e}^{-4t} - 4c_1)\,\mathrm{e}^{2t}}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2y + 3\,\mathrm{e}^{-2t}$ 	$R = t$ $S = \mathrm{e}^{-2t} y$	$\frac{dS}{dR} = 3\,\mathrm{e}^{-4R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 10$ in the above solution gives an equation to solve for the constant of integration.

$$10 = -\frac{3}{4} + c_1$$

$$c_1 = \frac{43}{4}$$

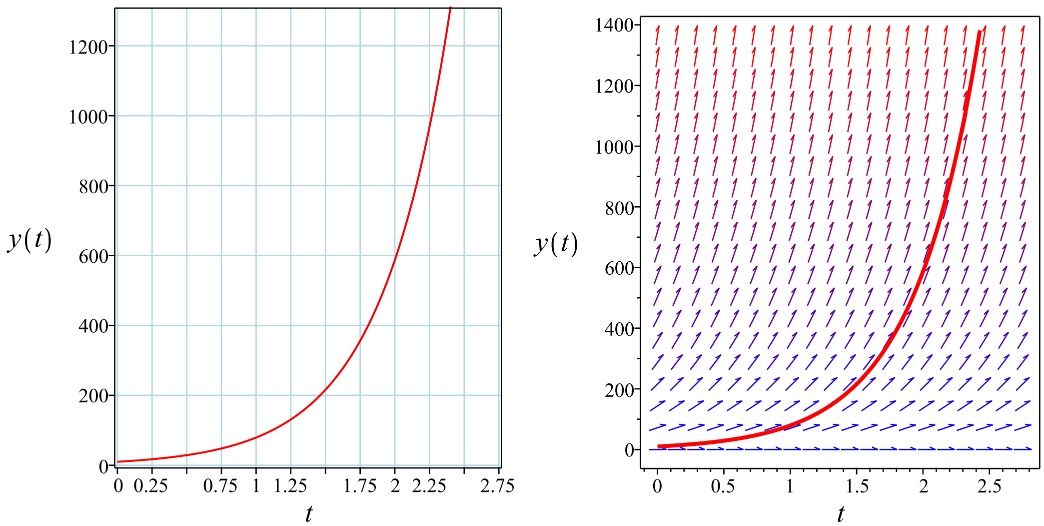
Substituting c_1 found above in the general solution gives

$$y = -\frac{3\,\mathrm{e}^{-2t}}{4} + \frac{43\,\mathrm{e}^{2t}}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{3\,\mathrm{e}^{-2t}}{4} + \frac{43\,\mathrm{e}^{2t}}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{3\,e^{-2t}}{4} + \frac{43\,e^{2t}}{4}$$

Verified OK.

6.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(2y + 3\,\mathrm{e}^{-2t}\right)\mathrm{d}t\\ \left(-2y - 3\,\mathrm{e}^{-2t}\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -2y - 3\,\mathrm{e}^{-2t}\\ N(t,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y - 3\,\mathrm{e}^{-2t}\right)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((-2) - (0))\\ &= -2\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -2 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2t} \\
&= \mathrm{e}^{-2t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-2t} \left(-2y - 3\,\mathrm{e}^{-2t}\right) \\
&= -2\,\mathrm{e}^{-2t} y - 3\,\mathrm{e}^{-4t}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-2t}(1) \\
&= \mathrm{e}^{-2t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-2\,\mathrm{e}^{-2t} y - 3\,\mathrm{e}^{-4t}\right) + \left(\mathrm{e}^{-2t}\right) \frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -2\,\mathrm{e}^{-2t} y - 3\,\mathrm{e}^{-4t} \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{3\,\mathrm{e}^{-4t}}{4} + \mathrm{e}^{-2t} y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-2t} = \mathrm{e}^{-2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3\,\mathrm{e}^{-4t}}{4} + \mathrm{e}^{-2t}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3\,\mathrm{e}^{-4t}}{4} + \mathrm{e}^{-2t}y$$

The solution becomes

$$y = -\frac{(3\,\mathrm{e}^{-4t} - 4c_1)\,\mathrm{e}^{2t}}{4}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 10$ in the above solution gives an equation to solve for the constant of integration.

$$10 = -\frac{3}{4} + c_1$$

$$c_1 = \frac{43}{4}$$

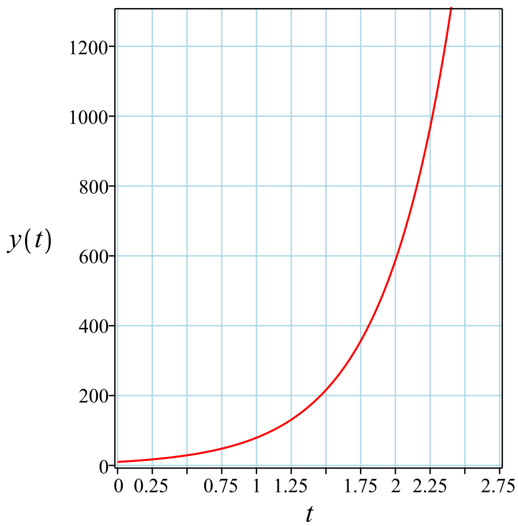
Substituting c_1 found above in the general solution gives

$$y = -\frac{3\,\mathrm{e}^{-2t}}{4} + \frac{43\,\mathrm{e}^{2t}}{4}$$

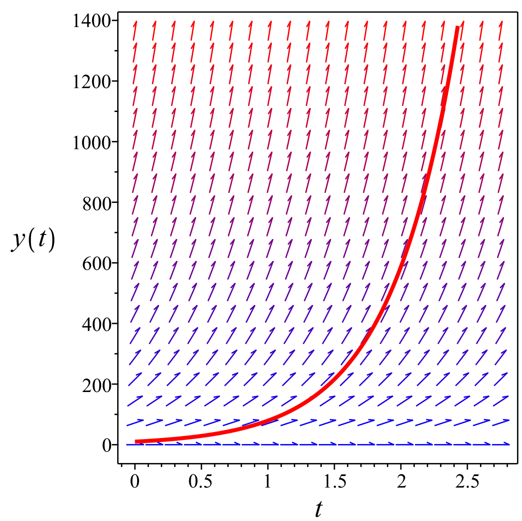
Summary

The solution(s) found are the following

$$y = -\frac{3\,\mathrm{e}^{-2t}}{4} + \frac{43\,\mathrm{e}^{2t}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3\,\mathrm{e}^{-2t}}{4} + \frac{43\,\mathrm{e}^{2t}}{4}$$

Verified OK.

6.8.5 Maple step by step solution

Let's solve

$[y' - 2y = 3\,\mathrm{e}^{-2t}, y(0) = 10]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = 2y + 3\,\mathrm{e}^{-2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = 3\,\mathrm{e}^{-2t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - 2y\right) = 3\mu(t)\,\mathrm{e}^{-2t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y' - 2y\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-2t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 3\mu(t)\,\mathrm{e}^{-2t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 3\mu(t)\,\mathrm{e}^{-2t} dt + c_1$

- Solve for y

  $y = \frac{\int 3\mu(t)\mathrm{e}^{-2t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-2t}$

  $y = \frac{\int 3\left(\mathrm{e}^{-2t}\right)^2 dt + c_1}{\mathrm{e}^{-2t}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{3\left(\mathrm{e}^{-2t}\right)^2}{4} + c_1}{\mathrm{e}^{-2t}}$

- Simplify

  $y = -\frac{3\,\mathrm{e}^{-2t}}{4} + c_1\mathrm{e}^{2t}$

- Use initial condition $y(0) = 10$

  $10 = -\frac{3}{4} + c_1$

- Solve for c_1

  $c_1 = \frac{43}{4}$

- Substitute $c_1 = \frac{43}{4}$ into general solution and simplify

$y = -\frac{3\,e^{-2t}}{4} + \frac{43\,e^{2t}}{4}$

- Solution to the IVP

  $y = -\frac{3\,e^{-2t}}{4} + \frac{43\,e^{2t}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([diff(y(t),t)-2*y(t)=3*exp(-2*t),y(0) = 10],y(t), singsol=all)
```

$$y(t) = \frac{43\,e^{2t}}{4} - \frac{3\,e^{-2t}}{4}$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 23

```
DSolve[{y'[t]-2*y[t]==3*Exp[-2*t],{y[0]==10}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{4}e^{-2t}\left(43e^{4t} - 3\right)$$

6.9 problem 9

Internal problem ID [12998]
Internal file name [`OUTPUT/11650_Tuesday_November_07_2023_11_54_05_PM_64099950/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \cos(2t)}$$

With initial conditions

$$[y(0) = 5]$$

6.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 1$$
$$q(t) = \cos(2t)$$

Hence the ode is

$$y' + y = \cos(2t)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.9.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dt} \\ &= e^t\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)(\cos(2t)) \\ \frac{d}{dt}(e^t y) &= (e^t)(\cos(2t)) \\ d(e^t y) &= (e^t \cos(2t))\ dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^t y &= \int e^t \cos(2t)\ dt \\ e^t y &= \frac{e^t \cos(2t)}{5} + \frac{2\,e^t \sin(2t)}{5} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^t$ results in

$$y = e^{-t}\left(\frac{e^t \cos(2t)}{5} + \frac{2\,e^t \sin(2t)}{5}\right) + c_1 e^{-t}$$

which simplifies to

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + c_1 e^{-t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \frac{1}{5} + c_1$$

$$c_1 = \frac{24}{5}$$

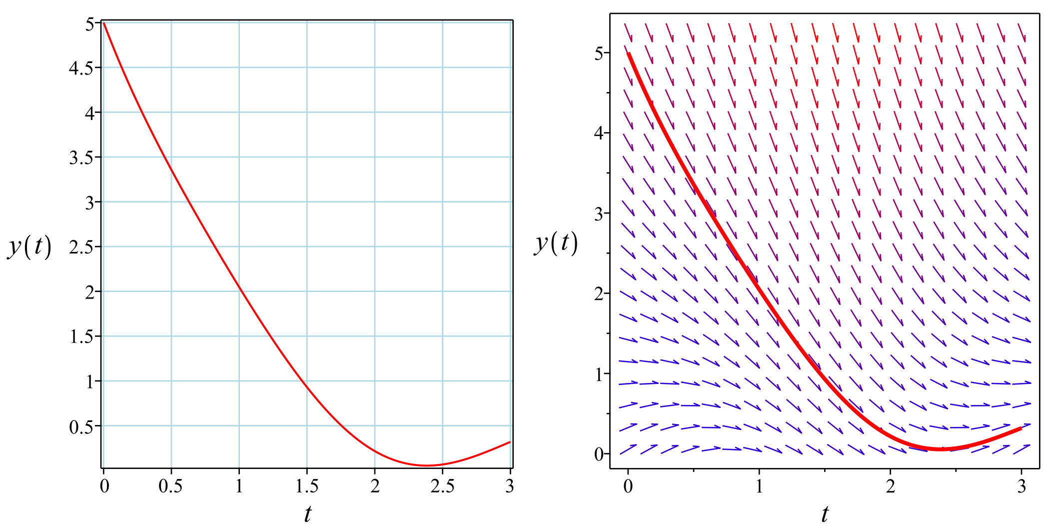
Substituting c_1 found above in the general solution gives

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5}$$

Verified OK.

6.9.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y + \cos(2t) \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 200: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= 0\\ \eta(t,y) &= \mathrm{e}^{-t}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\mathrm{e}^{-t}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -y + \cos(2t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1\\ R_y &= 0\\ S_t &= \mathrm{e}^{t} y\\ S_y &= \mathrm{e}^{t}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{t} \cos(2t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R} \cos(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + \frac{\mathrm{e}^{R}(\cos(2R) + 2\sin(2R))}{5} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{t} y = \frac{\mathrm{e}^{t}(2\sin(2t) + \cos(2t))}{5} + c_1$$

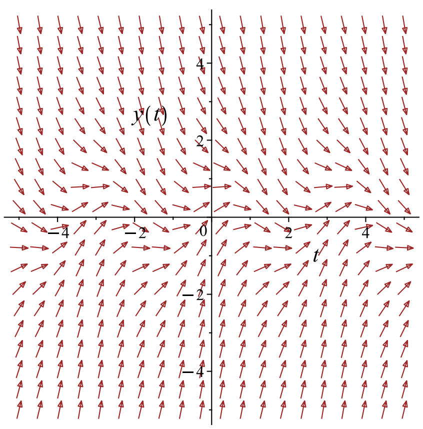
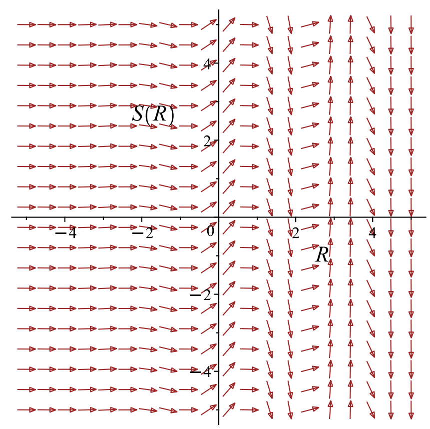
Which simplifies to

$$\mathrm{e}^{t} y = \frac{\mathrm{e}^{t}(2\sin(2t) + \cos(2t))}{5} + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{-t}(\mathrm{e}^{t}\cos(2t) + 2\,\mathrm{e}^{t}\sin(2t) + 5c_1)}{5}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dt} = -y + \cos(2t)$

</td><td>$R = t$
$S = e^t y$</td><td>$\frac{dS}{dR} = e^R \cos(2R)$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \frac{1}{5} + c_1$$

$$c_1 = \frac{24}{5}$$

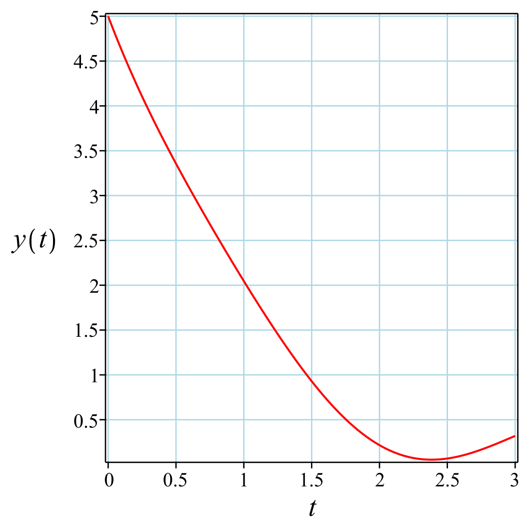
Substituting c_1 found above in the general solution gives

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5}$$

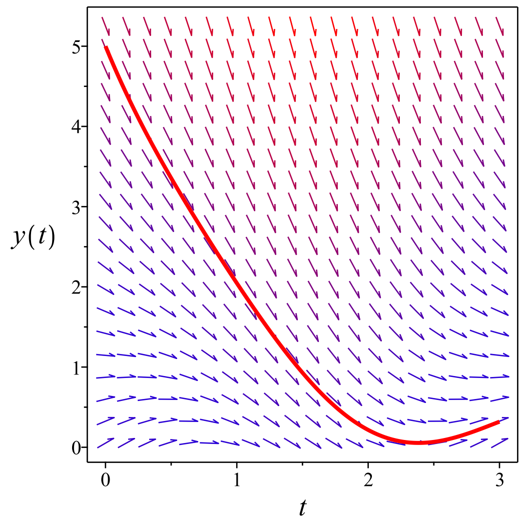
Summary

The solution(s) found are the following

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5}$$

Verified OK.

6.9.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (-y + \cos(2t))\,\mathrm{d}t\\ (y - \cos(2t))\,\mathrm{d}t + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, y) &= y - \cos(2t)\\ N(t, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \cos(2t))\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((1) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dt} \\ &= e^{\int 1 \, dt}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{t} \\ &= \mathrm{e}^{t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{t}(y - \cos(2t)) \\ &= (y - \cos(2t))\,\mathrm{e}^{t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{t}(1) \\ &= \mathrm{e}^{t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left((y - \cos(2t))\,\mathrm{e}^{t}\right) + \left(\mathrm{e}^{t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int (y - \cos(2t))\,\mathrm{e}^{t} \, \mathrm{d}t\end{aligned}$$

$$\phi = -\frac{\mathrm{e}^{t}(-5y + \cos(2t) + 2\sin(2t))}{5} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{t}$. Therefore equation (4) becomes

$$\mathrm{e}^{t} = \mathrm{e}^{t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{t}(-5y + \cos(2t) + 2\sin(2t))}{5} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{t}(-5y + \cos(2t) + 2\sin(2t))}{5}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-t}(\mathrm{e}^{t}\cos(2t) + 2\,\mathrm{e}^{t}\sin(2t) + 5c_1)}{5}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \frac{1}{5} + c_1$$

$$c_1 = \frac{24}{5}$$

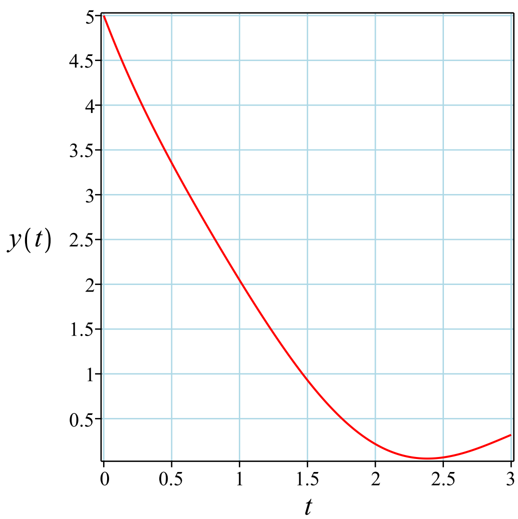
Substituting c_1 found above in the general solution gives

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5}$$

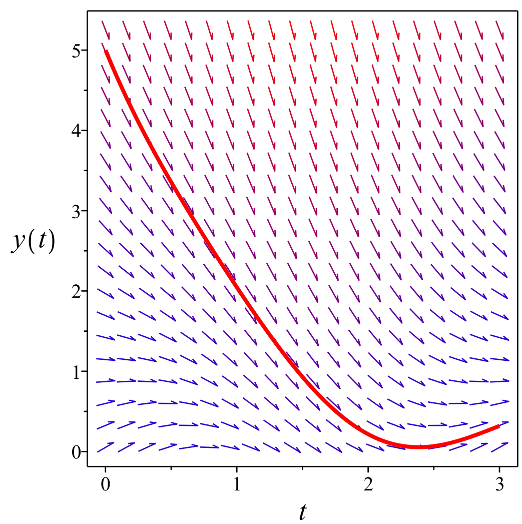
Summary

The solution(s) found are the following

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5}$$

Verified OK.

6.9.5 Maple step by step solution

Let's solve

$[y' + y = \cos(2t), y(0) = 5]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y + \cos(2t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = \cos(2t)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(y' + y) = \mu(t)\cos(2t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)(y' + y) = \mu'(t)\, y + \mu(t)\, y'$
- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\cos(2t)\, dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\cos(2t)\, dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t)\cos(2t)dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{t}$

  $y = \frac{\int \mathrm{e}^{t}\cos(2t)dt + c_1}{\mathrm{e}^{t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{\mathrm{e}^{t}\cos(2t)}{5} + \frac{2\,\mathrm{e}^{t}\sin(2t)}{5} + c_1}{\mathrm{e}^{t}}$
- Simplify

  $y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + c_1 \mathrm{e}^{-t}$
- Use initial condition $y(0) = 5$

  $5 = \frac{1}{5} + c_1$
- Solve for c_1

  $c_1 = \frac{24}{5}$
- Substitute $c_1 = \frac{24}{5}$ into general solution and simplify

  $y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,\mathrm{e}^{-t}}{5}$

- Solution to the IVP

  $y = \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} + \frac{24\,e^{-t}}{5}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(y(t),t)+y(t)=cos(2*t),y(0) = 5],y(t), singsol=all)
```

$$y(t) = \frac{\cos(2t)}{5} + \frac{2\sin(2t)}{5} + \frac{24\,e^{-t}}{5}$$

✓ Solution by Mathematica

Time used: 0.144 (sec). Leaf size: 27

```
DSolve[{y'[t]+y[t]==Cos[2*t],{y[0]==5}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{5}\left(24e^{-t} + 2\sin(2t) + \cos(2t)\right)$$

6.10 problem 10

Internal problem ID [12999]
Internal file name [`OUTPUT/11651_Tuesday_November_07_2023_11_54_06_PM_33112791/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 3y = \cos(2t)}$$

With initial conditions

$$[y(0) = -1]$$

6.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= \cos(2t) \end{aligned}$$

Hence the ode is

$$y' + 3y = \cos(2t)$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.10.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 3dt}\\ &= e^{3t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)(\cos(2t))\\ \frac{d}{dt}(e^{3t}y) &= (e^{3t})(\cos(2t))\\ d(e^{3t}y) &= (e^{3t}\cos(2t))\ dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{3t}y &= \int e^{3t}\cos(2t)\ dt\\ e^{3t}y &= \frac{3\,e^{3t}\cos(2t)}{13} + \frac{2\,e^{3t}\sin(2t)}{13} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3t}$ results in

$$y = e^{-3t}\left(\frac{3\,e^{3t}\cos(2t)}{13} + \frac{2\,e^{3t}\sin(2t)}{13}\right) + e^{-3t}c_1$$

which simplifies to

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} + e^{-3t}c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{3}{13} + c_1$$

$$c_1 = -\frac{16}{13}$$

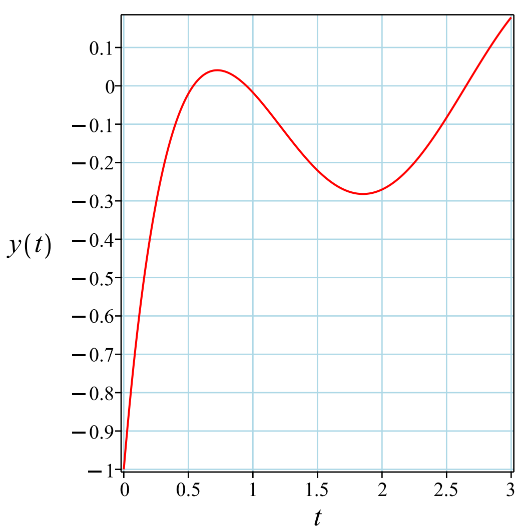
Substituting c_1 found above in the general solution gives

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,\mathrm{e}^{-3t}}{13}$$

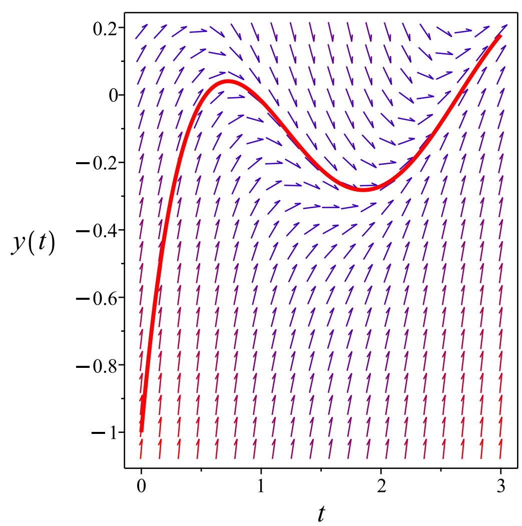
Summary

The solution(s) found are the following

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,\mathrm{e}^{-3t}}{13} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,\mathrm{e}^{-3t}}{13}$$

Verified OK.

6.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -3y + \cos(2t) \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 203: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,y) = 0$$
$$\eta(t,y) = \mathrm{e}^{-3t} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$S = \int \frac{1}{\eta} dy$$
$$= \int \frac{1}{\mathrm{e}^{-3t}} dy$$

Which results in

$$S = \mathrm{e}^{3t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -3y + \cos(2t)$$

Evaluating all the partial derivatives gives

$$R_t = 1$$
$$R_y = 0$$
$$S_t = 3\,\mathrm{e}^{3t} y$$
$$S_y = \mathrm{e}^{3t}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{3t} \cos(2t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{3R} \cos(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + \frac{e^{3R}(3\cos(2R) + 2\sin(2R))}{13} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{3t} y = \frac{e^{3t}(3\cos(2t) + 2\sin(2t))}{13} + c_1$$

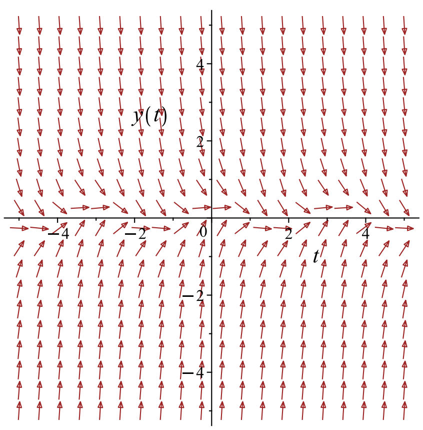
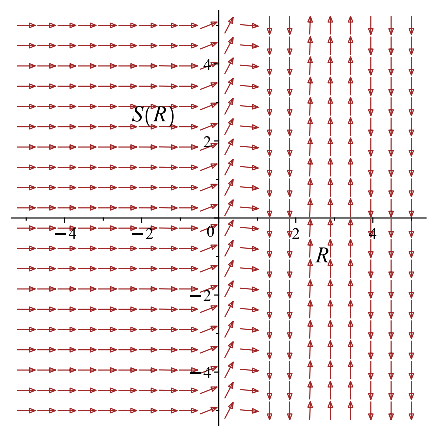
Which simplifies to

$$e^{3t} y = \frac{e^{3t}(3\cos(2t) + 2\sin(2t))}{13} + c_1$$

Which gives

$$y = \frac{e^{-3t}(3\,e^{3t}\cos(2t) + 2\,e^{3t}\sin(2t) + 13c_1)}{13}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -3y + \cos(2t)$ 	$R = t$ $S = e^{3t}y$	$\frac{dS}{dR} = e^{3R}\cos(2R)$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{3}{13} + c_1$$

$$c_1 = -\frac{16}{13}$$

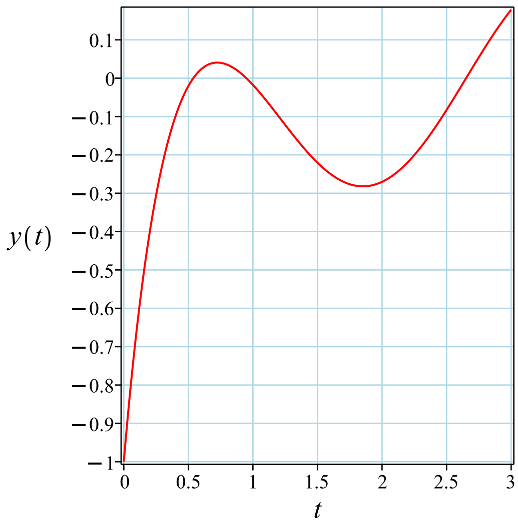
Substituting c_1 found above in the general solution gives

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,e^{-3t}}{13}$$

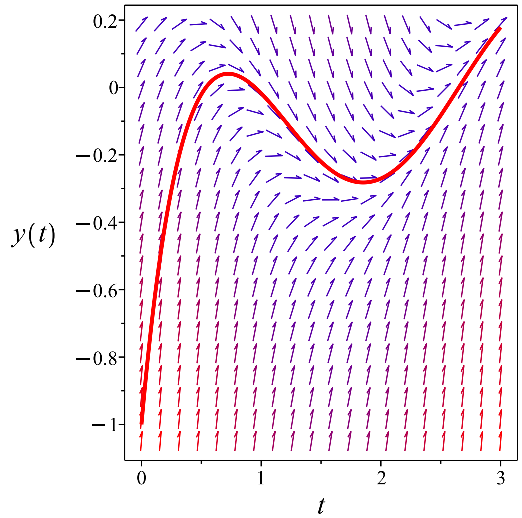
Summary

The solution(s) found are the following

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,e^{-3t}}{13} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,\mathrm{e}^{-3t}}{13}$$

Verified OK.

6.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(-3y+\cos(2t))\,\mathrm{d}t\\(3y-\cos(2t))\,\mathrm{d}t+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=3y-\cos(2t)\\N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(3y-\cos(2t))\\&=3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t}&=\frac{\partial}{\partial t}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)\\&=1((3)-(0))\\&=3\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int 3\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{3t}\\ &= \mathrm{e}^{3t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{3t}(3y - \cos(2t))\\ &= (3y - \cos(2t))\,\mathrm{e}^{3t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{3t}(1)\\ &= \mathrm{e}^{3t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \left((3y - \cos(2t))\,\mathrm{e}^{3t}\right) + \left(\mathrm{e}^{3t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int (3y - \cos(2t))\,\mathrm{e}^{3t}\,\mathrm{d}t\end{aligned}$$

$$\phi = -\frac{(-13y + 3\cos(2t) + 2\sin(2t))\,\mathrm{e}^{3t}}{13} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{3t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{3t}$. Therefore equation (4) becomes

$$e^{3t} = e^{3t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(-13y + 3\cos(2t) + 2\sin(2t))\, e^{3t}}{13} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(-13y + 3\cos(2t) + 2\sin(2t))\, e^{3t}}{13}$$

The solution becomes

$$y = \frac{e^{-3t}(3\, e^{3t} \cos(2t) + 2\, e^{3t} \sin(2t) + 13c_1)}{13}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{3}{13} + c_1$$

$$c_1 = -\frac{16}{13}$$

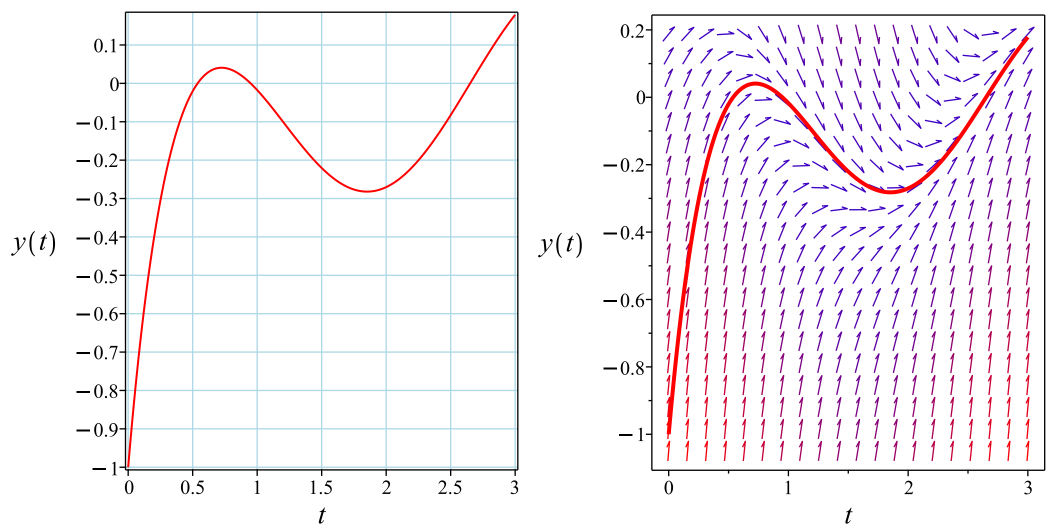
Substituting c_1 found above in the general solution gives

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,e^{-3t}}{13}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,e^{-3t}}{13} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,e^{-3t}}{13}$$

Verified OK.

6.10.5 Maple step by step solution

Let's solve

$[y' + 3y = \cos(2t), y(0) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -3y + \cos(2t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + 3y = \cos(2t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)(y' + 3y) = \mu(t)\cos(2t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$\mu(t)(y' + 3y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

$\mu'(t) = 3\mu(t)$

- Solve to find the integrating factor

$\mu(t) = \mathrm{e}^{3t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\cos(2t)\,dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\,y = \int \mu(t)\cos(2t)\,dt + c_1$

- Solve for y

$y = \frac{\int \mu(t)\cos(2t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{3t}$

$y = \frac{\int \mathrm{e}^{3t}\cos(2t)dt + c_1}{\mathrm{e}^{3t}}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{3\,\mathrm{e}^{3t}\cos(2t)}{13} + \frac{2\,\mathrm{e}^{3t}\sin(2t)}{13} + c_1}{\mathrm{e}^{3t}}$

- Simplify

$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} + \mathrm{e}^{-3t}c_1$

- Use initial condition $y(0) = -1$

$-1 = \frac{3}{13} + c_1$

- Solve for c_1

$c_1 = -\frac{16}{13}$

- Substitute $c_1 = -\frac{16}{13}$ into general solution and simplify

$y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,\mathrm{e}^{-3t}}{13}$

- Solution to the IVP

  $y = \frac{2\sin(2t)}{13} + \frac{3\cos(2t)}{13} - \frac{16\,\mathrm{e}^{-3t}}{13}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(y(t),t)+3*y(t)=cos(2*t),y(0) = -1],y(t), singsol=all)
```

$$y(t) = \frac{3\cos(2t)}{13} + \frac{2\sin(2t)}{13} - \frac{16\,\mathrm{e}^{-3t}}{13}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 30

```
DSolve[{y'[t]+3*y[t]==Cos[2*t],{y[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{13}\left(2\left(\sin(2t) - 8e^{-3t}\right) + 3\cos(2t)\right)$$

6.11 problem 11

Internal problem ID [13000]
Internal file name [`OUTPUT/11652_Tuesday_November_07_2023_11_54_07_PM_17255973/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = 7\,\mathrm{e}^{2t}}$$

With initial conditions

$$[y(0) = 3]$$

6.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = 7\,\mathrm{e}^{2t}$$

Hence the ode is

$$y' - 2y = 7\,e^{2t}$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 7\,e^{2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.11.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-2)dt} \\ &= e^{-2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(7\,e^{2t}\right) \\ \frac{d}{dt}\left(e^{-2t}y\right) &= \left(e^{-2t}\right)\left(7\,e^{2t}\right) \\ d\left(e^{-2t}y\right) &= 7\,dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-2t}y &= \int 7\,dt \\ e^{-2t}y &= 7t + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2t}$ results in

$$y = 7\,e^{2t}t + c_1 e^{2t}$$

which simplifies to

$$y = e^{2t}(7t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

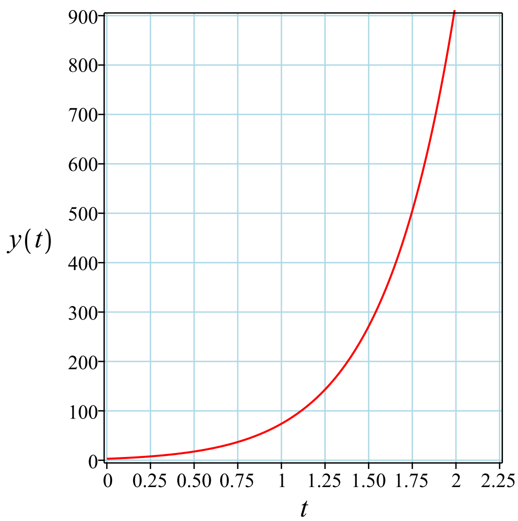
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

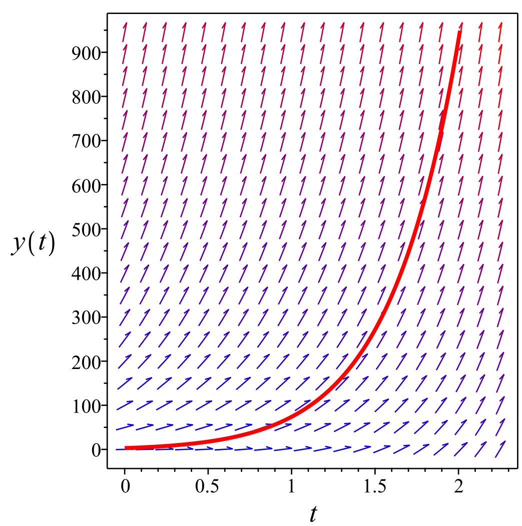
$$y = e^{2t}(3 + 7t)$$

Summary
The solution(s) found are the following

$$y = e^{2t}(3 + 7t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{2t}(3 + 7t)$$

Verified OK.

6.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2y + 7\,\mathrm{e}^{2t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 206: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= e^{2t}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{2t}} dy
\end{aligned}
$$

Which results in

$$
S = e^{-2t} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = 2y + 7\,e^{2t}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= -2\,e^{-2t} y \\
S_y &= e^{-2t}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 7 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 7$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 7R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-2t}y = 7t + c_1$$

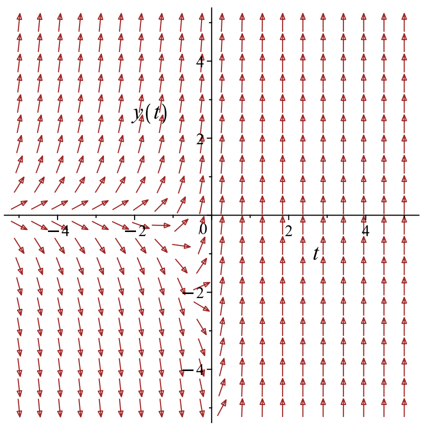
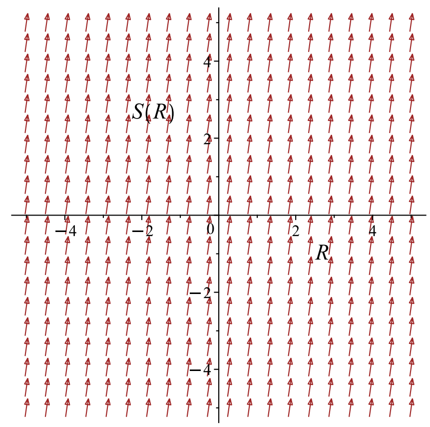
Which simplifies to

$$e^{-2t}y = 7t + c_1$$

Which gives

$$y = e^{2t}(7t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2y + 7\,\mathrm{e}^{2t}$ 	$R = t$ $S = \mathrm{e}^{-2t} y$	$\frac{dS}{dR} = 7$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

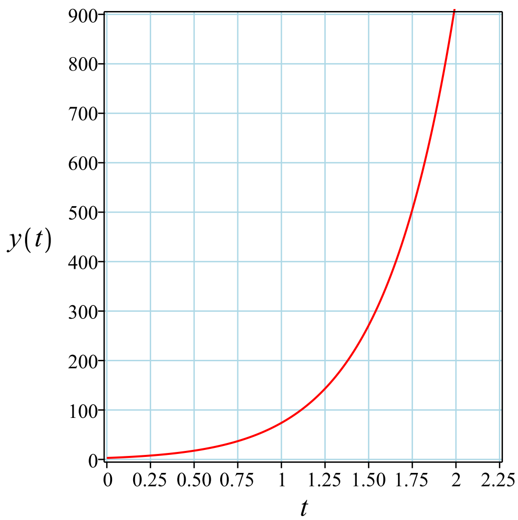
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

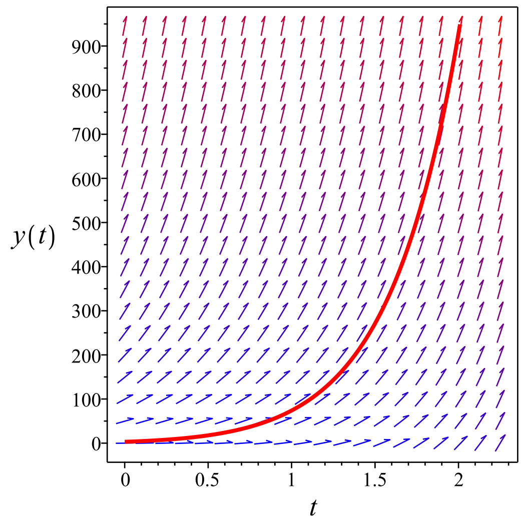
$$y = 7\,\mathrm{e}^{2t} t + 3\,\mathrm{e}^{2t}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 7\,\mathrm{e}^{2t} t + 3\,\mathrm{e}^{2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 7\,\mathrm{e}^{2t}t + 3\,\mathrm{e}^{2t}$$

Verified OK.

6.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(2y + 7\,\mathrm{e}^{2t}\right)\mathrm{d}t\\ \left(-2y - 7\,\mathrm{e}^{2t}\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -2y - 7\,\mathrm{e}^{2t}\\ N(t, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y - 7\,\mathrm{e}^{2t}\right)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((-2) - (0))\\ &= -2\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -2\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2t}\\ &= \mathrm{e}^{-2t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-2t}\left(-2y - 7\,\mathrm{e}^{2t}\right)\\ &= -2\,\mathrm{e}^{-2t}y - 7\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-2t}(1)\\ &= \mathrm{e}^{-2t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \left(-2\,\mathrm{e}^{-2t}y - 7\right) + \left(\mathrm{e}^{-2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -2\,\mathrm{e}^{-2t}y - 7\,\mathrm{d}t\end{aligned}$$

$$\phi = -7t + \mathrm{e}^{-2t}y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-2t} = \mathrm{e}^{-2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -7t + \mathrm{e}^{-2t} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -7t + \mathrm{e}^{-2t} y$$

The solution becomes

$$y = \mathrm{e}^{2t}(7t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

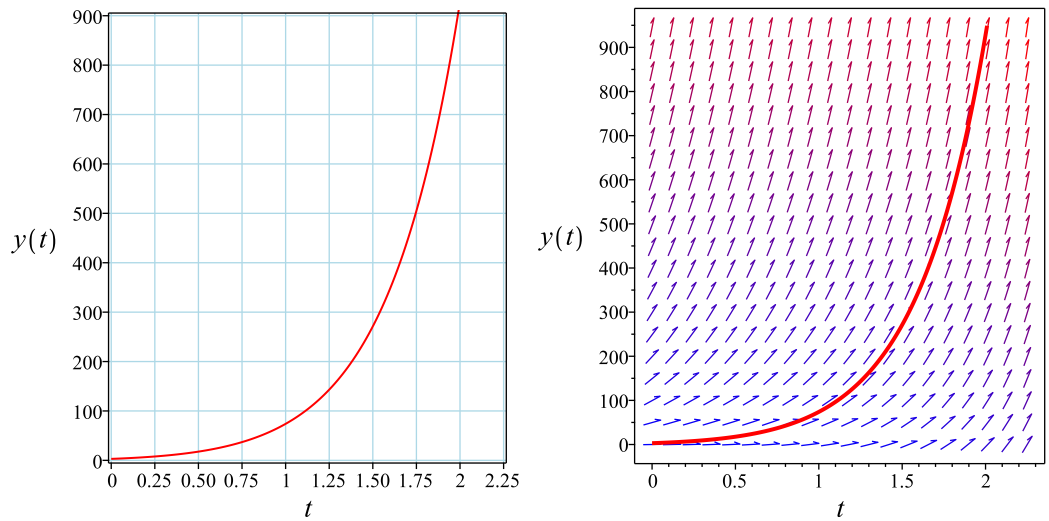
Substituting c_1 found above in the general solution gives

$$y = 7\,\mathrm{e}^{2t} t + 3\,\mathrm{e}^{2t}$$

Summary

The solution(s) found are the following

$$y = 7\,e^{2t}t + 3\,e^{2t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 7\,e^{2t}t + 3\,e^{2t}$$

Verified OK.

6.11.5 Maple step by step solution

Let's solve

$[y' - 2y = 7\,e^{2t}, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2y + 7\,e^{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = 7\,e^{2t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(y'-2y\right)=7\mu(t)\,\mathrm{e}^{2t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}\left(\mu(t)\,y\right)$

$\mu(t)\left(y'-2y\right)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

$\mu'(t)=-2\mu(t)$

- Solve to find the integrating factor

$\mu(t)=\mathrm{e}^{-2t}$

- Integrate both sides with respect to t

$\int\left(\frac{d}{dt}\left(\mu(t)\,y\right)\right)dt=\int 7\mu(t)\,\mathrm{e}^{2t}dt+c_1$

- Evaluate the integral on the lhs

$\mu(t)\,y=\int 7\mu(t)\,\mathrm{e}^{2t}dt+c_1$

- Solve for y

$y=\frac{\int 7\mu(t)\mathrm{e}^{2t}dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=\mathrm{e}^{-2t}$

$y=\frac{\int 7\,\mathrm{e}^{-2t}\mathrm{e}^{2t}dt+c_1}{\mathrm{e}^{-2t}}$

- Evaluate the integrals on the rhs

$y=\frac{7t+c_1}{\mathrm{e}^{-2t}}$

- Simplify

$y=\mathrm{e}^{2t}(7t+c_1)$

- Use initial condition $y(0)=3$

$3=c_1$

- Solve for c_1

$c_1=3$

- Substitute $c_1=3$ into general solution and simplify

$y=\mathrm{e}^{2t}(3+7t)$

- Solution to the IVP

$y=\mathrm{e}^{2t}(3+7t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(t),t)-2*y(t)=7*exp(2*t),y(0) = 3],y(t), singsol=all)
```

$$y(t) = (7t + 3)\,\mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 16

```
DSolve[{y'[t]-2*y[t]==7*Exp[2*t],{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{2t}(7t + 3)$$

6.12 problem 20

Internal problem ID [13001]
Internal file name [`OUTPUT/11653_Tuesday_November_07_2023_11_54_08_PM_48347244/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = 3t^2 + 2t - 1}$$

6.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 3t^2 + 2t - 1 \end{aligned}$$

Hence the ode is

$$y' + 2y = 3t^2 + 2t - 1$$

The integrating factor μ is

$$\mu = e^{\int 2dt}$$
$$= e^{2t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(3t^2+2t-1\right)$$
$$\frac{d}{dt}\left(e^{2t}y\right) = \left(e^{2t}\right)\left(3t^2+2t-1\right)$$
$$d\left(e^{2t}y\right) = \left(\left(3t^2+2t-1\right)e^{2t}\right)\, dt$$

Integrating gives

$$e^{2t}y = \int \left(3t^2+2t-1\right)e^{2t}\, dt$$
$$e^{2t}y = \frac{(6t^2-2t-1)\,e^{2t}}{4} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$y = \frac{e^{-2t}(6t^2-2t-1)\,e^{2t}}{4} + c_1 e^{-2t}$$

which simplifies to

$$y = \frac{3t^2}{2} - \frac{t}{2} - \frac{1}{4} + c_1 e^{-2t}$$

Summary

The solution(s) found are the following

$$y = \frac{3t^2}{2} - \frac{t}{2} - \frac{1}{4} + c_1 e^{-2t} \tag{1}$$

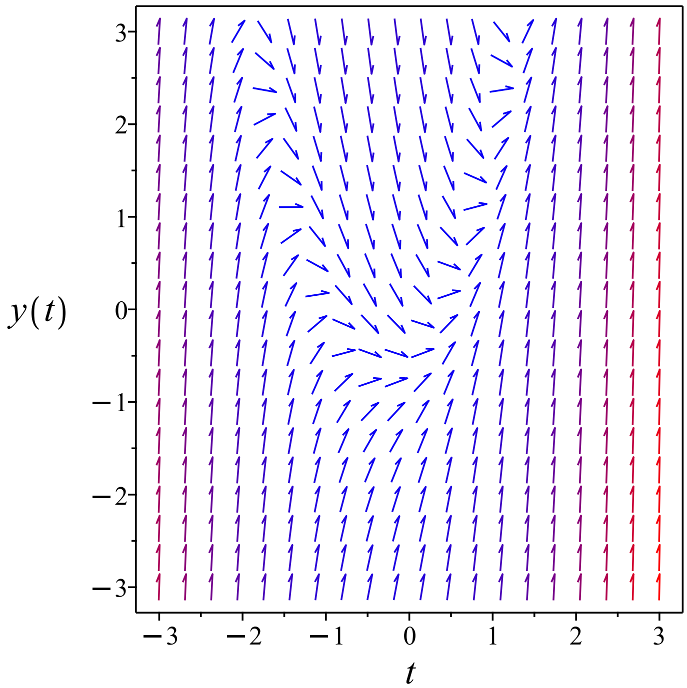


Figure 209: Slope field plot

Verification of solutions

$$y = \frac{3t^2}{2} - \frac{t}{2} - \frac{1}{4} + c_1 e^{-2t}$$

Verified OK.

6.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3t^2 + 2t - 2y - 1$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 209: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= 0 \\ \eta(t, y) &= \mathrm{e}^{-2t} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2t}} dy \end{aligned}$$

Which results in

$$S = e^{2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = 3t^2 + 2t - 2y - 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2\, e^{2t} y \\ S_y &= e^{2t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \left(3t^2 + 2t - 1\right) e^{2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \left(3R^2 + 2R - 1\right) e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(6R^2 - 2R - 1)\,\mathrm{e}^{2R}}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{2t} y = \frac{(6t^2 - 2t - 1)\,\mathrm{e}^{2t}}{4} + c_1$$

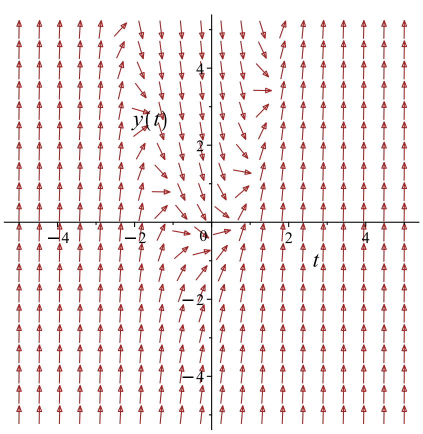
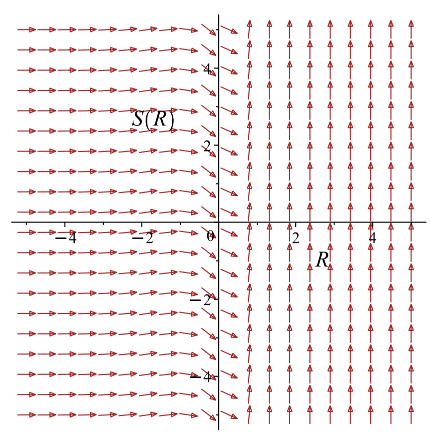
Which simplifies to

$$\mathrm{e}^{2t} y = \frac{(6t^2 - 2t - 1)\,\mathrm{e}^{2t}}{4} + c_1$$

Which gives

$$y = \frac{(6\,\mathrm{e}^{2t} t^2 - 2\,\mathrm{e}^{2t} t - \mathrm{e}^{2t} + 4c_1)\,\mathrm{e}^{-2t}}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 3t^2 + 2t - 2y - 1$ 	$R = t$ $S = \mathrm{e}^{2t} y$	$\frac{dS}{dR} = (3R^2 + 2R - 1)\,\mathrm{e}^{2R}$ 

Summary

The solution(s) found are the following

$$y = \frac{(6\,\mathrm{e}^{2t} t^2 - 2\,\mathrm{e}^{2t} t - \mathrm{e}^{2t} + 4c_1)\,\mathrm{e}^{-2t}}{4} \tag{1}$$

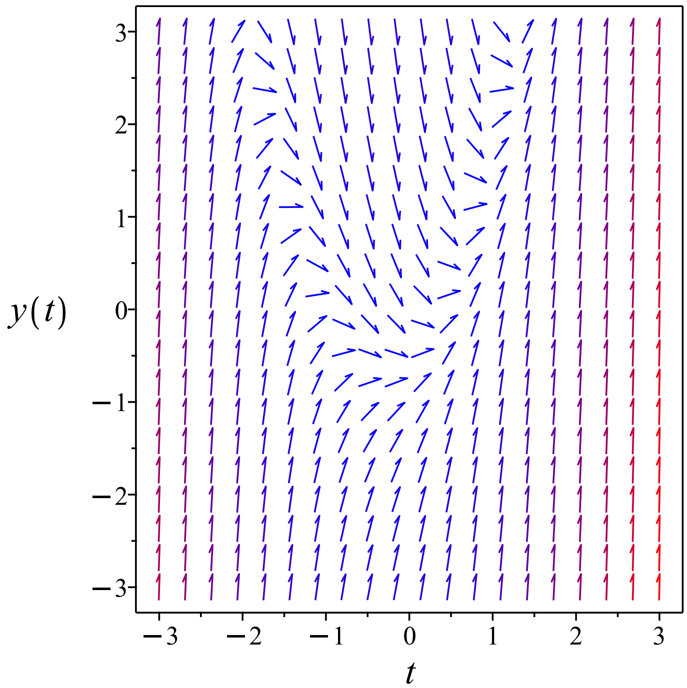


Figure 210: Slope field plot

Verification of solutions

$$y = \frac{(6\,e^{2t}t^2 - 2\,e^{2t}t - e^{2t} + 4c_1)\,e^{-2t}}{4}$$

Verified OK.

6.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(3t^2 + 2t - 2y - 1\right) \mathrm{d}t \\
\left(-3t^2 - 2t + 2y + 1\right) \mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -3t^2 - 2t + 2y + 1 \\
N(t, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3t^2 - 2t + 2y + 1\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((2) - (0)) \\
&= 2
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int 2 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2t} \\
&= \mathrm{e}^{2t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{2t}\left(-3t^2 - 2t + 2y + 1\right) \\
&= \left(-3t^2 - 2t + 2y + 1\right) \mathrm{e}^{2t}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{2t}(1) \\
&= \mathrm{e}^{2t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\left(-3t^2 - 2t + 2y + 1\right) \mathrm{e}^{2t}\right) + \left(\mathrm{e}^{2t}\right) \frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \left(-3t^2 - 2t + 2y + 1\right) \mathrm{e}^{2t} \, \mathrm{d}t$$

$$\phi = -\frac{\mathrm{e}^{2t}(6t^2 - 2t - 4y - 1)}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{2t} = \mathrm{e}^{2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{2t}(6t^2 - 2t - 4y - 1)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{2t}(6t^2 - 2t - 4y - 1)}{4}$$

The solution becomes

$$y = \frac{(6\,\mathrm{e}^{2t}t^2 - 2\,\mathrm{e}^{2t}t - \mathrm{e}^{2t} + 4c_1)\,\mathrm{e}^{-2t}}{4}$$

Summary
The solution(s) found are the following

$$y = \frac{(6\,e^{2t}t^2 - 2\,e^{2t}t - e^{2t} + 4c_1)\,e^{-2t}}{4} \tag{1}$$

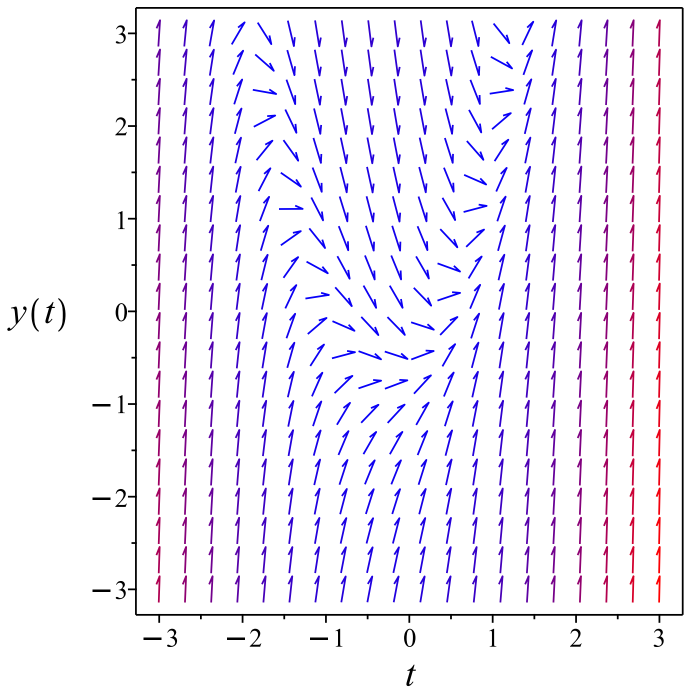


Figure 211: Slope field plot

Verification of solutions

$$y = \frac{(6\,e^{2t}t^2 - 2\,e^{2t}t - e^{2t} + 4c_1)\,e^{-2t}}{4}$$

Verified OK.

6.12.4 Maple step by step solution

Let's solve

$y' + 2y = 3t^2 + 2t - 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -2y + 3t^2 + 2t - 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + 2y = 3t^2 + 2t - 1$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(y' + 2y\right) = \mu(t)\left(3t^2 + 2t - 1\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$\mu(t)\left(y' + 2y\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

$\mu'(t) = 2\mu(t)$

- Solve to find the integrating factor

$\mu(t) = \mathrm{e}^{2t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\left(3t^2 + 2t - 1\right) dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\, y = \int \mu(t)\left(3t^2 + 2t - 1\right) dt + c_1$

- Solve for y

$y = \frac{\int \mu(t)(3t^2+2t-1)dt+c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{2t}$

$y = \frac{\int (3t^2+2t-1)\mathrm{e}^{2t}dt+c_1}{\mathrm{e}^{2t}}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{(6t^2-2t-1)\mathrm{e}^{2t}}{4}+c_1}{\mathrm{e}^{2t}}$

- Simplify

$y = \frac{3t^2}{2} - \frac{t}{2} - \frac{1}{4} + c_1\mathrm{e}^{-2t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(diff(y(t),t)+2*y(t)=3*t^2+2*t-1,y(t), singsol=all)
```

$$y(t) = \frac{3t^2}{2} - \frac{t}{2} - \frac{1}{4} + \mathrm{e}^{-2t} c_1$$

✓ Solution by Mathematica

Time used: 0.193 (sec). Leaf size: 28

```
DSolve[y'[t]+2*y[t]==3*t^2+2*t-1,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{4}\left(6t^2 - 2t - 1\right) + c_1 e^{-2t}$$

6.13 problem 21

Internal problem ID [13002]
Internal file name [`OUTPUT/11654_Tuesday_November_07_2023_11_54_09_PM_23375726/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = t^2 + 2t + 1 + e^{4t}}$$

6.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 2$$
$$q(t) = t^2 + 2t + 1 + e^{4t}$$

Hence the ode is

$$y' + 2y = t^2 + 2t + 1 + e^{4t}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2dt} \\
&= e^{2t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(t^2+2t+1+e^{4t}\right) \\
\frac{d}{dt}\left(e^{2t}y\right) &= \left(e^{2t}\right)\left(t^2+2t+1+e^{4t}\right) \\
d\left(e^{2t}y\right) &= \left(\left(t^2+2t+1+e^{4t}\right)e^{2t}\right) dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{2t}y &= \int \left(t^2+2t+1+e^{4t}\right)e^{2t}\, dt \\
e^{2t}y &= \frac{e^{2t}}{4}+\frac{e^{6t}}{6}+\frac{e^{2t}t^2}{2}+\frac{e^{2t}t}{2}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$
y = e^{-2t}\left(\frac{e^{2t}}{4}+\frac{e^{6t}}{6}+\frac{e^{2t}t^2}{2}+\frac{e^{2t}t}{2}\right)+c_1e^{-2t}
$$

which simplifies to

$$
y = \frac{1}{4}+\frac{e^{4t}}{6}+\frac{t^2}{2}+\frac{t}{2}+c_1e^{-2t}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{4}+\frac{e^{4t}}{6}+\frac{t^2}{2}+\frac{t}{2}+c_1e^{-2t} \tag{1}
$$

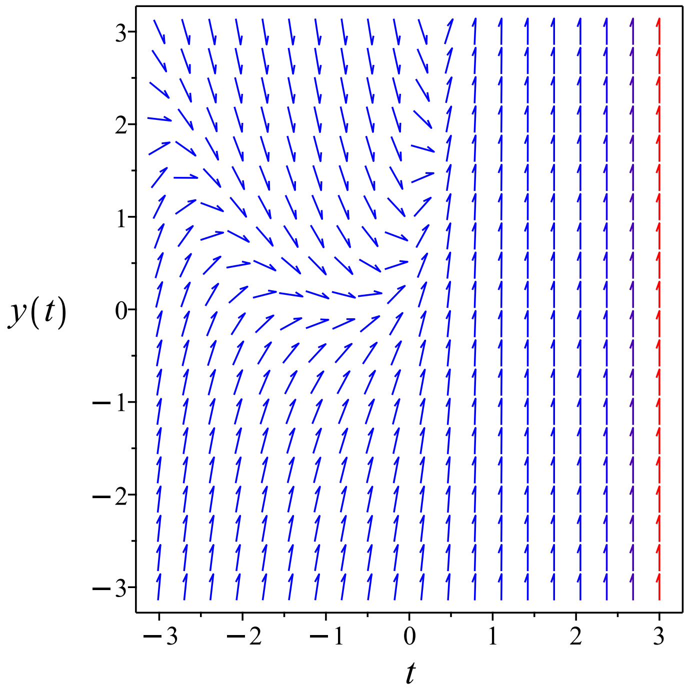


Figure 212: Slope field plot

Verification of solutions

$$y = \frac{1}{4} + \frac{e^{4t}}{6} + \frac{t^2}{2} + \frac{t}{2} + c_1 e^{-2t}$$

Verified OK.

6.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -2y + t^2 + 2t + 1 + e^{4t} \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 212: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-2t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2t}} dy \end{aligned}$$

Which results in

$$S = e^{2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -2y + t^2 + 2t + 1 + e^{4t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2\,e^{2t} y \\ S_y &= e^{2t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (1 + t)^2 e^{2t} + e^{6t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (1 + R)^2 e^{2R} + e^{6R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}R^2}{2} + \frac{e^{2R}R}{2} + \frac{e^{2R}}{4} + \frac{e^{6R}}{6} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{2t}y = \frac{e^{2t}t^2}{2} + \frac{e^{2t}t}{2} + \frac{e^{2t}}{4} + \frac{e^{6t}}{6} + c_1$$

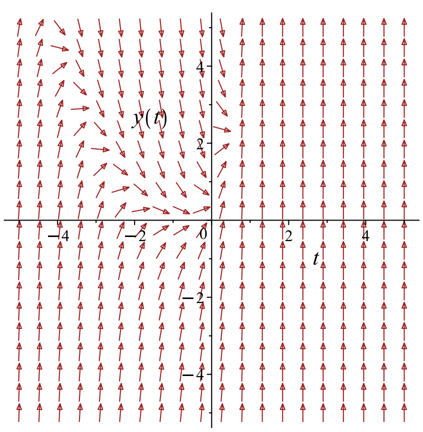
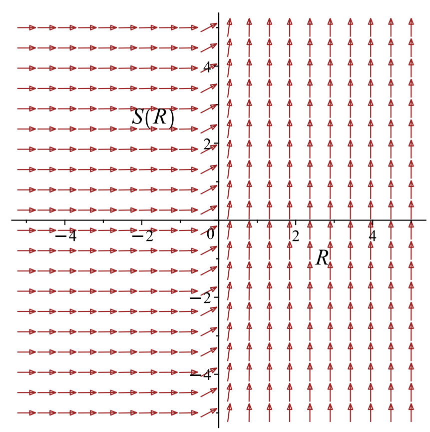
Which simplifies to

$$\frac{(-2t^2 - 2t + 4y - 1)\,e^{2t}}{4} - c_1 - \frac{e^{6t}}{6} = 0$$

Which gives

$$y = \frac{(6\,e^{2t}t^2 + 6\,e^{2t}t + 2\,e^{6t} + 3\,e^{2t} + 12c_1)\,e^{-2t}}{12}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -2y + t^2 + 2t + 1 + e^{4t}$ 	$R = t$ $S = e^{2t}y$	$\frac{dS}{dR} = (1 + R)^2\,e^{2R} + e^{6R}$ 

Summary
The solution(s) found are the following

$$y = \frac{(6\,e^{2t}t^2 + 6\,e^{2t}t + 2\,e^{6t} + 3\,e^{2t} + 12c_1)\,e^{-2t}}{12} \tag{1}$$

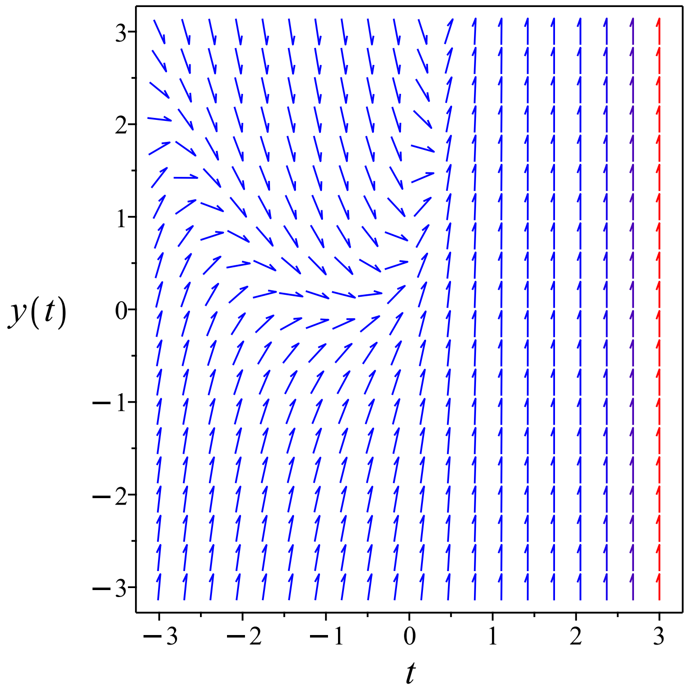


Figure 213: Slope field plot

Verification of solutions

$$
y = \frac{(6\,\mathrm{e}^{2t} t^2 + 6\,\mathrm{e}^{2t} t + 2\,\mathrm{e}^{6t} + 3\,\mathrm{e}^{2t} + 12 c_1)\,\mathrm{e}^{-2t}}{12}
$$

Verified OK.

6.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-2y + t^2 + 2t + 1 + \mathrm{e}^{4t}\right)\mathrm{d}t$$
$$\left(2y - t^2 - 2t - 1 - \mathrm{e}^{4t}\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,y) = 2y - t^2 - 2t - 1 - \mathrm{e}^{4t}$$
$$N(t,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2y - t^2 - 2t - 1 - \mathrm{e}^{4t}\right)$$
$$= 2$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((2) - (0)) \\ &= 2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 2 \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2t} \\ &= \mathrm{e}^{2t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{2t}\left(2y - t^2 - 2t - 1 - \mathrm{e}^{4t}\right) \\ &= -\mathrm{e}^{2t}\left(-2y + t^2 + 2t + 1 + \mathrm{e}^{4t}\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{2t}(1) \\ &= \mathrm{e}^{2t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(-\mathrm{e}^{2t}\left(-2y + t^2 + 2t + 1 + \mathrm{e}^{4t}\right)\right) + \left(\mathrm{e}^{2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, dt = \int \overline{M} \, dt$$

$$\int \frac{\partial \phi}{\partial t} \, dt = \int -e^{2t} \left(-2y + t^2 + 2t + 1 + e^{4t}\right) dt$$

$$\phi = \frac{(-2t^2 - 2t + 4y - 1) e^{2t}}{4} - \frac{e^{6t}}{6} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{2t}$. Therefore equation (4) becomes

$$e^{2t} = e^{2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(-2t^2 - 2t + 4y - 1) e^{2t}}{4} - \frac{e^{6t}}{6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(-2t^2 - 2t + 4y - 1) e^{2t}}{4} - \frac{e^{6t}}{6}$$

The solution becomes

$$y = \frac{(6 e^{2t} t^2 + 6 e^{2t} t + 2 e^{6t} + 3 e^{2t} + 12 c_1) e^{-2t}}{12}$$

Summary
The solution(s) found are the following

$$y = \frac{(6\,e^{2t}t^2 + 6\,e^{2t}t + 2\,e^{6t} + 3\,e^{2t} + 12c_1)\,e^{-2t}}{12} \tag{1}$$

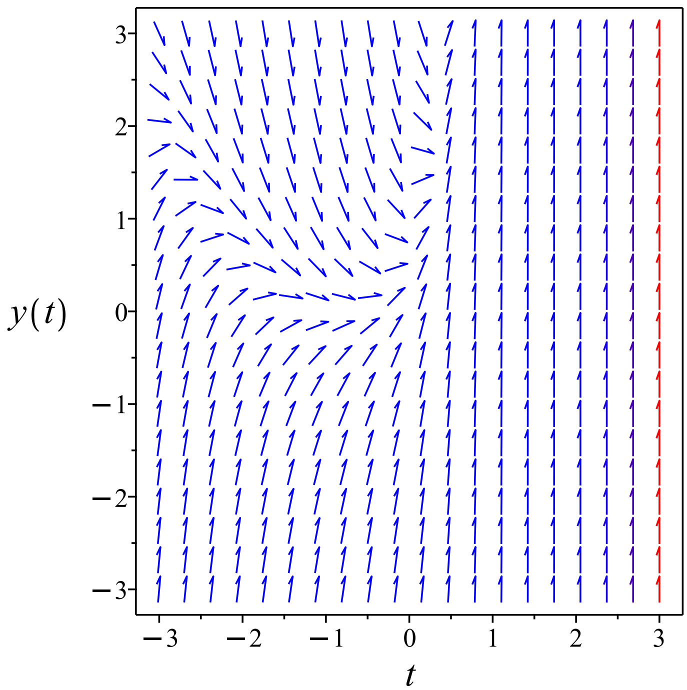


Figure 214: Slope field plot

Verification of solutions

$$y = \frac{(6\,e^{2t}t^2 + 6\,e^{2t}t + 2\,e^{6t} + 3\,e^{2t} + 12c_1)\,e^{-2t}}{12}$$

Verified OK.

6.13.4 Maple step by step solution

Let's solve

$y' + 2y = t^2 + 2t + 1 + e^{4t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -2y + t^2 + 2t + 1 + e^{4t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + 2y = t^2 + 2t + 1 + e^{4t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\,(y' + 2y) = \mu(t)\,(t^2 + 2t + 1 + e^{4t})$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$\mu(t)\,(y' + 2y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

$\mu'(t) = 2\mu(t)$

- Solve to find the integrating factor

$\mu(t) = e^{2t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,(t^2 + 2t + 1 + e^{4t})\,dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\,y = \int \mu(t)\,(t^2 + 2t + 1 + e^{4t})\,dt + c_1$

- Solve for y

$y = \frac{\int \mu(t)(t^2+2t+1+e^{4t})dt+c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{2t}$

$y = \frac{\int (t^2+2t+1+e^{4t})e^{2t}dt+c_1}{e^{2t}}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{(e^t)^2}{4}+\frac{(e^t)^6}{6}+\frac{t^2(e^t)^2}{2}+\frac{(e^t)^2 t}{2}+c_1}{e^{2t}}$

- Simplify

$y = \frac{1}{4} + \frac{e^{4t}}{6} + \frac{t^2}{2} + \frac{t}{2} + c_1 e^{-2t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(t),t)+2*y(t)=t^2+2*t+1+exp(4*t),y(t), singsol=all)
```

$$y(t) = \frac{t^2}{2} + \frac{t}{2} + \frac{1}{4} + \frac{e^{4t}}{6} + e^{-2t}c_1$$

✓ Solution by Mathematica

Time used: 0.557 (sec). Leaf size: 35

```
DSolve[y'[t]+2*y[t]==t^2+2*t+1+Exp[4*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{12}\left(6t^2 + 6t + 2e^{4t} + 3\right) + c_1e^{-2t}$$

6.14 problem 22

Internal problem ID [13003]
Internal file name [`OUTPUT/11655_Tuesday_November_07_2023_11_54_09_PM_13578292/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = t^3 + \sin(3t)}$$

6.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= t^3 + \sin(3t) \end{aligned}$$

Hence the ode is

$$y' + y = t^3 + \sin(3t)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1 dt} \\ &= e^{t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(t^{3}+\sin(3t)\right) \\ \frac{d}{dt}\left(e^{t} y\right) &= \left(e^{t}\right)\left(t^{3}+\sin(3t)\right) \\ d\left(e^{t} y\right) &= \left(\left(t^{3}+\sin(3t)\right) e^{t}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{t} y &= \int \left(t^{3}+\sin(3t)\right) e^{t} dt \\ e^{t} y &= e^{t} t^{3}-3t^{2} e^{t}+6t\,e^{t}-6\,e^{t}-\frac{3\,e^{t} \cos(3t)}{10}+\frac{e^{t} \sin(3t)}{10}+c_{1}\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{t}$ results in

$$y = e^{-t}\left(e^{t} t^{3}-3t^{2} e^{t}+6t\,e^{t}-6\,e^{t}-\frac{3\,e^{t} \cos(3t)}{10}+\frac{e^{t} \sin(3t)}{10}\right)+c_{1} e^{-t}$$

which simplifies to

$$y = t^{3}-3t^{2}+6t+\frac{\sin(3t)}{10}-\frac{3\cos(3t)}{10}-6+c_{1} e^{-t}$$

Summary

The solution(s) found are the following

$$y = t^{3}-3t^{2}+6t+\frac{\sin(3t)}{10}-\frac{3\cos(3t)}{10}-6+c_{1} e^{-t} \tag{1}$$

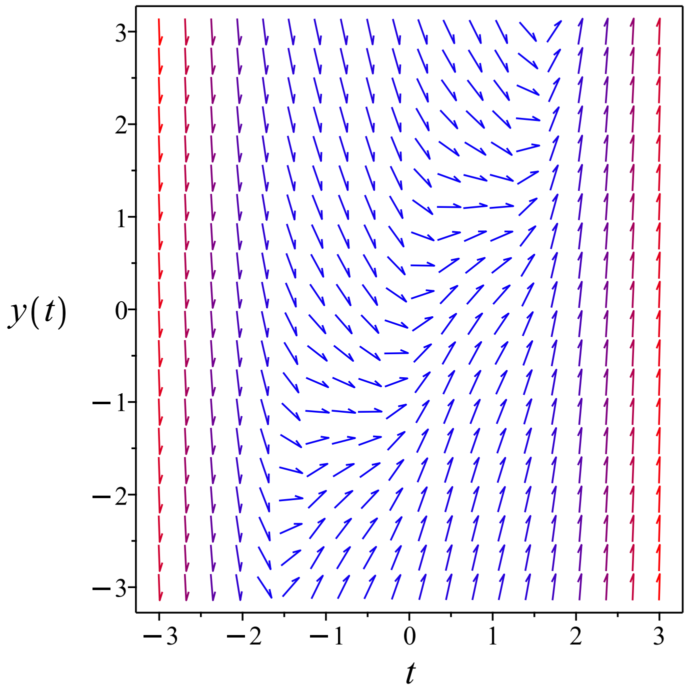


Figure 215: Slope field plot

Verification of solutions

$$y = t^3 - 3t^2 + 6t + \frac{\sin(3t)}{10} - \frac{3\cos(3t)}{10} - 6 + c_1 e^{-t}$$

Verified OK.

6.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + t^3 + \sin(3t)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 215: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= 0 \\ \eta(t, y) &= \mathrm{e}^{-t} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-t}} dy \end{aligned}$$

Which results in

$$S = e^{t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -y + t^3 + \sin(3t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= e^{t} y \\ S_y &= e^{t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \left(t^3 + \sin(3t)\right) e^{t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \left(R^3 + \sin(3R)\right) e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^R R^3 - 3\,e^R R^2 + 6R\,e^R - 6\,e^R + c_1 - \frac{e^R(3\cos(3R) - \sin(3R))}{10} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^t y = e^t t^3 - 3t^2 e^t + 6t\,e^t - 6\,e^t + c_1 - \frac{e^t(3\cos(3t) - \sin(3t))}{10}$$

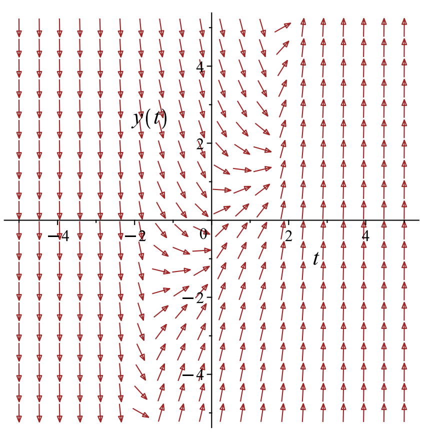
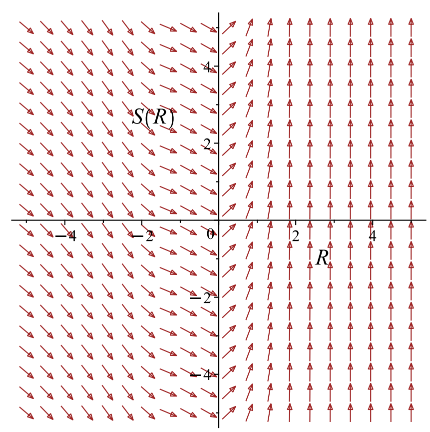
Which simplifies to

$$\frac{3\,e^t\cos(3t)}{10} - \frac{e^t\sin(3t)}{10} + \left(-t^3 + 3t^2 - 6t + y + 6\right)e^t - c_1 = 0$$

Which gives

$$y = \frac{e^{-t}(10\,e^t t^3 - 30t^2 e^t + e^t\sin(3t) - 3\,e^t\cos(3t) + 60t\,e^t - 60\,e^t + 10c_1)}{10}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -y + t^3 + \sin(3t)$ 	$R = t$ $S = e^t y$	$\frac{dS}{dR} = (R^3 + \sin(3R))\,e^R$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-t}(10\,e^t t^3 - 30t^2 e^t + e^t\sin(3t) - 3\,e^t\cos(3t) + 60t\,e^t - 60\,e^t + 10c_1)}{10} \tag{1}$$

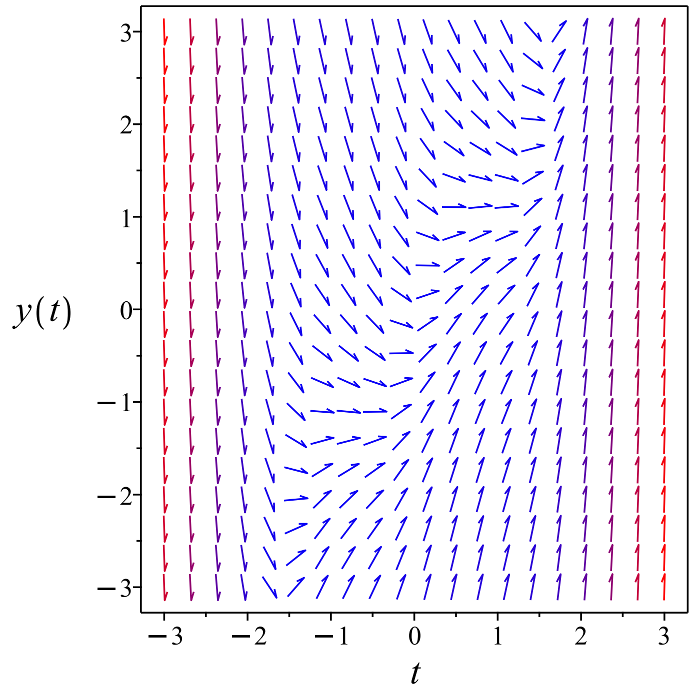


Figure 216: Slope field plot

Verification of solutions

$$y = \frac{e^{-t}(10\,e^{t}t^{3} - 30t^{2}e^{t} + e^{t}\sin(3t) - 3\,e^{t}\cos(3t) + 60t\,e^{t} - 60\,e^{t} + 10c_{1})}{10}$$

Verified OK.

6.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-y + t^3 + \sin(3t)\right) \mathrm{d}t \\
\left(y - t^3 - \sin(3t)\right) \mathrm{d}t + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= y - t^3 - \sin(3t) \\
N(t, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - t^3 - \sin(3t)\right) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int 1 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{t} \\
&= \mathrm{e}^{t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{t}\left(y - t^3 - \sin(3t)\right) \\
&= -\mathrm{e}^{t}\left(-y + t^3 + \sin(3t)\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{t}(1) \\
&= \mathrm{e}^{t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{t}\left(-y + t^3 + \sin(3t)\right)\right) + \left(\mathrm{e}^{t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -\mathrm{e}^{t}\left(-y + t^{3} + \sin(3t)\right) \mathrm{d}t$$

$$\phi = -\frac{\mathrm{e}^{t}(10t^{3} - 30t^{2} + \sin(3t) - 3\cos(3t) + 60t - 10y - 60)}{10} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{t}$. Therefore equation (4) becomes

$$\mathrm{e}^{t} = \mathrm{e}^{t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{t}(10t^{3} - 30t^{2} + \sin(3t) - 3\cos(3t) + 60t - 10y - 60)}{10} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{t}(10t^{3} - 30t^{2} + \sin(3t) - 3\cos(3t) + 60t - 10y - 60)}{10}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-t}(10\,\mathrm{e}^{t} t^{3} - 30t^{2}\mathrm{e}^{t} + \mathrm{e}^{t}\sin(3t) - 3\,\mathrm{e}^{t}\cos(3t) + 60t\,\mathrm{e}^{t} - 60\,\mathrm{e}^{t} + 10c_1)}{10}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-t}\left(10\,e^{t}t^{3}-30t^{2}e^{t}+e^{t}\sin\left(3t\right)-3\,e^{t}\cos\left(3t\right)+60t\,e^{t}-60\,e^{t}+10c_{1}\right)}{10} \tag{1}$$

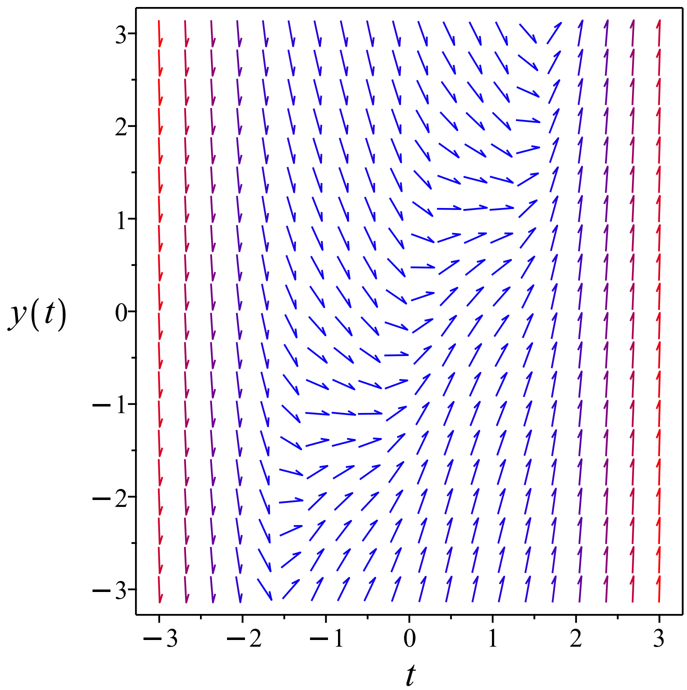


Figure 217: Slope field plot

Verification of solutions

$$y = \frac{e^{-t}\left(10\,e^{t}t^{3}-30t^{2}e^{t}+e^{t}\sin\left(3t\right)-3\,e^{t}\cos\left(3t\right)+60t\,e^{t}-60\,e^{t}+10c_{1}\right)}{10}$$

Verified OK.

6.14.4 Maple step by step solution

Let's solve

$y' + y = t^3 + \sin\left(3t\right)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y + t^3 + \sin(3t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + y = t^3 + \sin(3t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)(y' + y) = \mu(t)(t^3 + \sin(3t))$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) y)$

$\mu(t)(y' + y) = \mu'(t) y + \mu(t) y'$

- Isolate $\mu'(t)$

$\mu'(t) = \mu(t)$

- Solve to find the integrating factor

$\mu(t) = e^t$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t) y)\right) dt = \int \mu(t)(t^3 + \sin(3t)) dt + c_1$

- Evaluate the integral on the lhs

$\mu(t) y = \int \mu(t)(t^3 + \sin(3t)) dt + c_1$

- Solve for y

$y = \frac{\int \mu(t)(t^3+\sin(3t))dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^t$

$y = \frac{\int (t^3+\sin(3t))e^t dt + c_1}{e^t}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{e^t \sin(3t)}{10} - \frac{3 e^t \cos(3t)}{10} + e^t t^3 - 3t^2 e^t + 6t e^t - 6 e^t + c_1}{e^t}$

- Simplify

$y = t^3 - 3t^2 + 6t + \frac{\sin(3t)}{10} - \frac{3\cos(3t)}{10} - 6 + c_1 e^{-t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(t),t)+y(t)=t^3+sin(3*t),y(t), singsol=all)
```

$$y(t) = t^3 - 3t^2 + 6t - 6 - \frac{3\cos(3t)}{10} + \frac{\sin(3t)}{10} + e^{-t}c_1$$

✓ Solution by Mathematica

Time used: 0.19 (sec). Leaf size: 42

```
DSolve[y'[t]+y[t]==t^3+Sin[3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t^3 - 3t^2 + 6t + \frac{1}{10}\sin(3t) - \frac{3}{10}\cos(3t) + c_1e^{-t} - 6$$

6.15 problem 23

Internal problem ID [13004]
Internal file name [`OUTPUT/11656_Tuesday_November_07_2023_11_54_10_PM_53582205/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = 2t - e^{4t}}$$

6.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= 2t - e^{4t} \end{aligned}$$

Hence the ode is

$$y' - 3y = 2t - e^{4t}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-3)dt} \\
&= \mathrm{e}^{-3t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(2t - \mathrm{e}^{4t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-3t} y\right) &= \left(\mathrm{e}^{-3t}\right)\left(2t - \mathrm{e}^{4t}\right) \\
\mathrm{d}\left(\mathrm{e}^{-3t} y\right) &= \left(\left(2t - \mathrm{e}^{4t}\right)\mathrm{e}^{-3t}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-3t} y &= \int \left(2t - \mathrm{e}^{4t}\right)\mathrm{e}^{-3t}\,\mathrm{d}t \\
\mathrm{e}^{-3t} y &= -\frac{2t\,\mathrm{e}^{-3t}}{3} - \frac{2\,\mathrm{e}^{-3t}}{9} - \mathrm{e}^{t} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-3t}$ results in

$$
y = \mathrm{e}^{3t}\left(-\frac{2t\,\mathrm{e}^{-3t}}{3} - \frac{2\,\mathrm{e}^{-3t}}{9} - \mathrm{e}^{t}\right) + c_1 \mathrm{e}^{3t}
$$

which simplifies to

$$
y = -\frac{2t}{3} - \frac{2}{9} - \mathrm{e}^{4t} + c_1 \mathrm{e}^{3t}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = -\frac{2t}{3} - \frac{2}{9} - \mathrm{e}^{4t} + c_1 \mathrm{e}^{3t} \tag{1}
$$

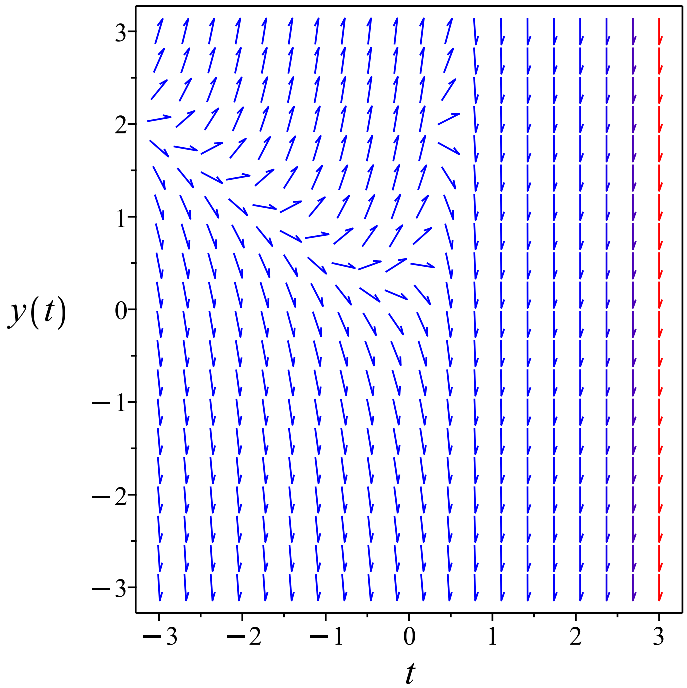


Figure 218: Slope field plot

Verification of solutions

$$y = -\frac{2t}{3} - \frac{2}{9} - e^{4t} + c_1 e^{3t}$$

Verified OK.

6.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= 3y + 2t - e^{4t} \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 218: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,y) &= 0 \\ \eta(t,y) &= e^{3t} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{3t}} dy \end{aligned}$$

Which results in

$$S = e^{-3t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = 3y + 2t - e^{4t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -3\,e^{-3t} y \\ S_y &= e^{-3t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \left(2t - e^{4t}\right) e^{-3t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \left(2R - e^{4R}\right) e^{-3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2\,\mathrm{e}^{-3R}R}{3} - \frac{2\,\mathrm{e}^{-3R}}{9} - \mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-3t}y = -\frac{2t\,\mathrm{e}^{-3t}}{3} - \frac{2\,\mathrm{e}^{-3t}}{9} - \mathrm{e}^{t} + c_1$$

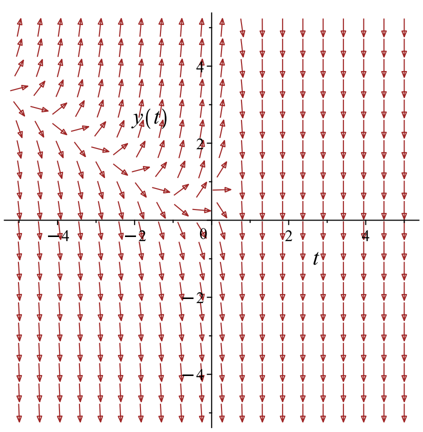
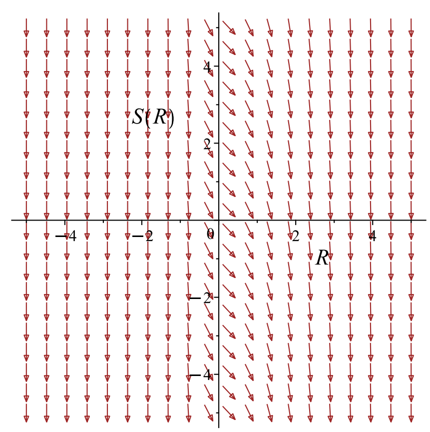
Which simplifies to

$$\mathrm{e}^{-3t}y = -\frac{2t\,\mathrm{e}^{-3t}}{3} - \frac{2\,\mathrm{e}^{-3t}}{9} - \mathrm{e}^{t} + c_1$$

Which gives

$$y = -\frac{(9\,\mathrm{e}^{4t} - 9c_1\mathrm{e}^{3t} + 6t + 2)\,\mathrm{e}^{3t}\mathrm{e}^{-3t}}{9}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 3y + 2t - \mathrm{e}^{4t}$ 	$R = t$ $S = \mathrm{e}^{-3t}y$	$\frac{dS}{dR} = \left(2R - \mathrm{e}^{4R}\right)\mathrm{e}^{-3R}$ 

Summary
The solution(s) found are the following

$$y = -\frac{(9\,\mathrm{e}^{4t} - 9c_1\mathrm{e}^{3t} + 6t + 2)\,\mathrm{e}^{3t}\mathrm{e}^{-3t}}{9} \tag{1}$$

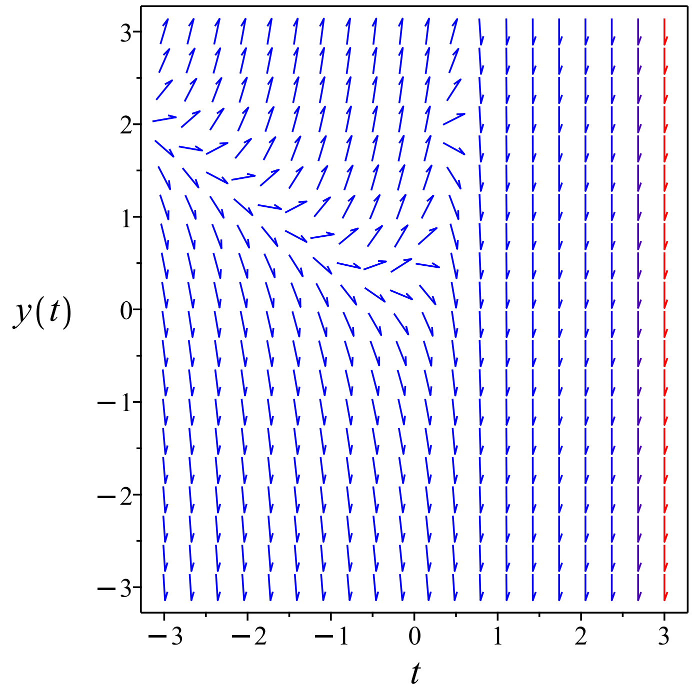


Figure 219: Slope field plot

Verification of solutions

$$y = -\frac{(9\,e^{4t} - 9c_1 e^{3t} + 6t + 2)\,e^{3t} e^{-3t}}{9}$$

Verified OK.

6.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\, dt + N(t, y)\, dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned} dy &= \left(3y + 2t - e^{4t}\right) dt \\ \left(-3y - 2t + e^{4t}\right) dt + dy &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, y) &= -3y - 2t + e^{4t} \\ N(t, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3y - 2t + e^{4t}\right) \\ &= -3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((-3) - (0)) \\ &= -3 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}t} \\ &= e^{\int -3\,\mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3t} \\ &= \mathrm{e}^{-3t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-3t}\left(-3y - 2t + \mathrm{e}^{4t}\right) \\ &= \left(-3y - 2t + \mathrm{e}^{4t}\right)\mathrm{e}^{-3t} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-3t}(1) \\ &= \mathrm{e}^{-3t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\left(-3y - 2t + \mathrm{e}^{4t}\right)\mathrm{e}^{-3t}\right) + \left(\mathrm{e}^{-3t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \left(-3y - 2t + \mathrm{e}^{4t}\right) \mathrm{e}^{-3t} \, \mathrm{d}t$$

$$\phi = \frac{\left(9\,\mathrm{e}^{4t} + 6t + 9y + 2\right) \mathrm{e}^{-3t}}{9} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-3t} = \mathrm{e}^{-3t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\left(9\,\mathrm{e}^{4t} + 6t + 9y + 2\right) \mathrm{e}^{-3t}}{9} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\left(9\,\mathrm{e}^{4t} + 6t + 9y + 2\right) \mathrm{e}^{-3t}}{9}$$

The solution becomes

$$y = -\frac{\left(9\,\mathrm{e}^{-3t}\mathrm{e}^{4t} + 6t\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-3t} - 9c_1\right) \mathrm{e}^{3t}}{9}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(9\,\mathrm{e}^{-3t}\mathrm{e}^{4t} + 6t\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-3t} - 9c_1\right)\mathrm{e}^{3t}}{9} \tag{1}$$

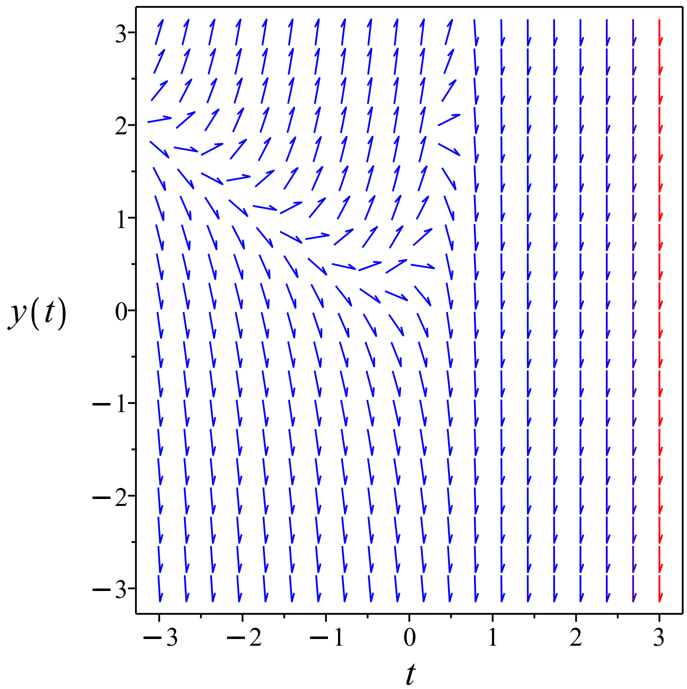


Figure 220: Slope field plot

Verification of solutions

$$y = -\frac{\left(9\,\mathrm{e}^{-3t}\mathrm{e}^{4t} + 6t\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-3t} - 9c_1\right)\mathrm{e}^{3t}}{9}$$

Verified OK.

6.15.4 Maple step by step solution

Let's solve

$y' - 3y = 2t - \mathrm{e}^{4t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = 3y + 2t - e^{4t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' - 3y = 2t - e^{4t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(y' - 3y\right) = \mu(t)\left(2t - e^{4t}\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$\mu(t)\left(y' - 3y\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

$\mu'(t) = -3\mu(t)$

- Solve to find the integrating factor

$\mu(t) = e^{-3t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\left(2t - e^{4t}\right) dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\, y = \int \mu(t)\left(2t - e^{4t}\right) dt + c_1$

- Solve for y

$y = \frac{\int \mu(t)\left(2t - e^{4t}\right) dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{-3t}$

$y = \frac{\int \left(2t - e^{4t}\right) e^{-3t} dt + c_1}{e^{-3t}}$

- Evaluate the integrals on the rhs

$y = \frac{-\frac{2t}{3\left(e^t\right)^3} - \frac{2}{9\left(e^t\right)^3} - e^t + c_1}{e^{-3t}}$

- Simplify

$y = -\frac{2t}{3} - \frac{2}{9} - e^{4t} + c_1 e^{3t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(t),t)-3*y(t)=2*t-exp(4*t),y(t), singsol=all)
```

$$y(t) = -\frac{2t}{3} - \frac{2}{9} - e^{4t} + c_1 e^{3t}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 30

```
DSolve[y'[t]-3*y[t]==2*t-Exp[4*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{2}{9}(3t+1) - e^{4t} + c_1 e^{3t}$$

6.16 problem 24

Internal problem ID [13005]
Internal file name [`OUTPUT/11657_Tuesday_November_07_2023_11_54_11_PM_33710285/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.8 page 121
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \cos(2t) + 3\sin(2t) + e^{-t}}$$

6.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \cos(2t) + 3\sin(2t) + e^{-t} \end{aligned}$$

Hence the ode is

$$y' + y = \cos(2t) + 3\sin(2t) + e^{-t}$$

The integrating factor μ is

$$\mu = e^{\int 1dt}$$
$$= e^t$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\cos(2t) + 3\sin(2t) + e^{-t}\right)$$
$$\frac{d}{dt}\left(e^t y\right) = \left(e^t\right)\left(\cos(2t) + 3\sin(2t) + e^{-t}\right)$$
$$d\left(e^t y\right) = \left(e^t \cos(2t) + 3\, e^t \sin(2t) + 1\right)\, dt$$

Integrating gives

$$e^t y = \int e^t \cos(2t) + 3\, e^t \sin(2t) + 1\, dt$$
$$e^t y = t - e^t \cos(2t) + e^t \sin(2t) + c_1$$

Dividing both sides by the integrating factor $\mu = e^t$ results in

$$y = e^{-t}\left(t - e^t \cos(2t) + e^t \sin(2t)\right) + c_1 e^{-t}$$

which simplifies to

$$y = (t + c_1)\, e^{-t} - \cos(2t) + \sin(2t)$$

Summary

The solution(s) found are the following

$$y = (t + c_1)\, e^{-t} - \cos(2t) + \sin(2t) \tag{1}$$

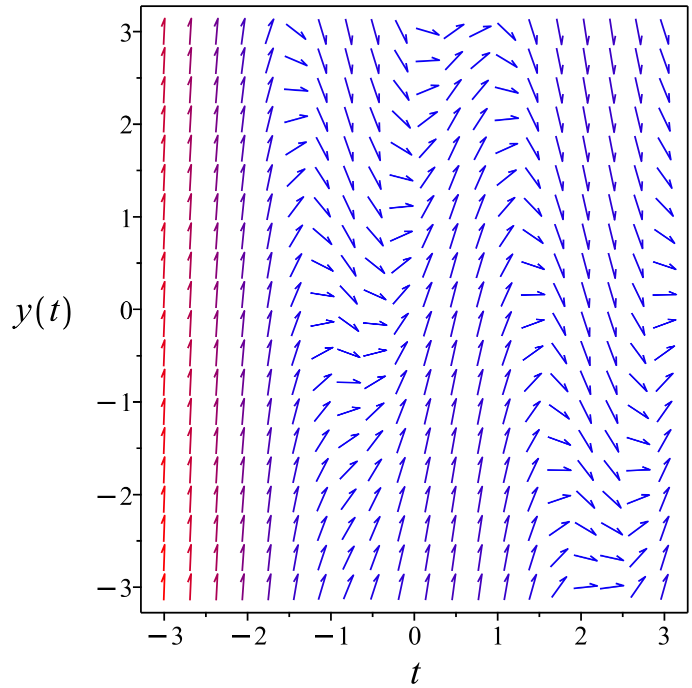


Figure 221: Slope field plot

Verification of solutions

$$y = (t + c_1)\,e^{-t} - \cos(2t) + \sin(2t)$$

Verified OK.

6.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + \cos(2t) + 3\sin(2t) + e^{-t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 221: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-t}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -y + \cos(2t) + 3\sin(2t) + \mathrm{e}^{-t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \mathrm{e}^{t} y \\ S_y &= \mathrm{e}^{t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{t} \cos(2t) + 3\,\mathrm{e}^{t} \sin(2t) + 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R} \cos(2R) + 3\,\mathrm{e}^{R} \sin(2R) + 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 - e^R(\cos(2R) - \sin(2R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^t y = t + c_1 - e^t(\cos(2t) - \sin(2t))$$

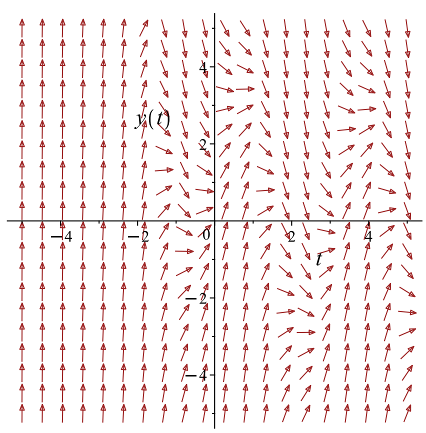
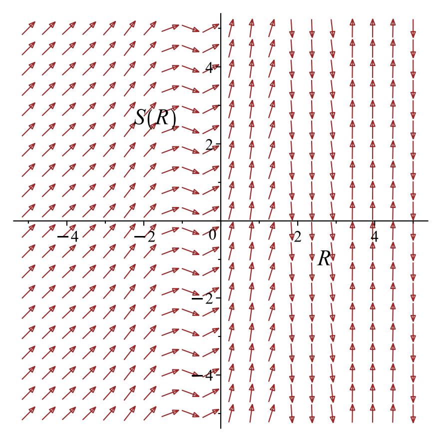
Which simplifies to

$$e^t y = t + c_1 - e^t(\cos(2t) - \sin(2t))$$

Which gives

$$y = -e^{-t}\left(e^t \cos(2t) - e^t \sin(2t) - c_1 - t\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -y + \cos(2t) + 3\sin(2t) + e^{-t}$  	$R = t$ $S = e^t y$	$\frac{dS}{dR} = e^R \cos(2R) + 3 e^R \sin(2R) + 1$ 

Summary

The solution(s) found are the following

$$y = -e^{-t}\left(e^t \cos(2t) - e^t \sin(2t) - c_1 - t\right) \tag{1}$$

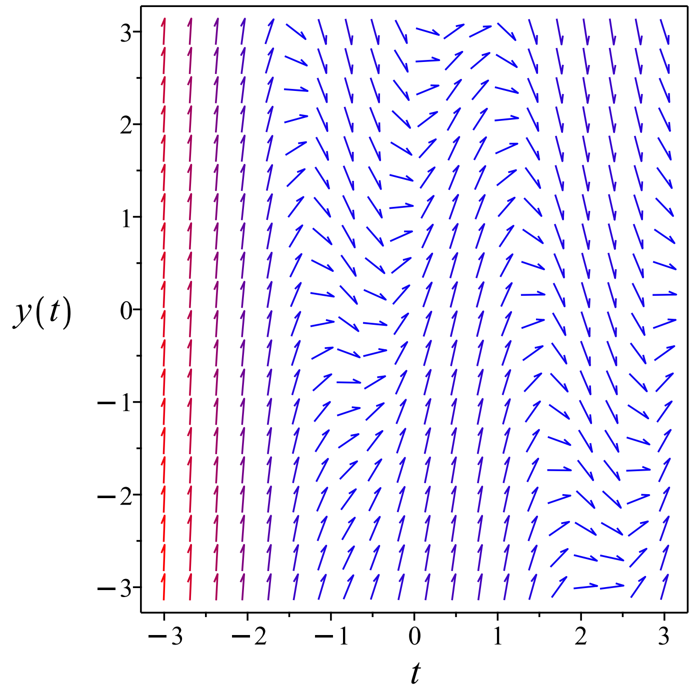


Figure 222: Slope field plot

Verification of solutions

$$y = -e^{-t}\left(e^t \cos(2t) - e^t \sin(2t) - c_1 - t\right)$$

Verified OK.

6.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-y + \cos(2t) + 3\sin(2t) + \mathrm{e}^{-t}\right)\mathrm{d}t$$
$$\left(y - \cos(2t) - 3\sin(2t) - \mathrm{e}^{-t}\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = y - \cos(2t) - 3\sin(2t) - \mathrm{e}^{-t}$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - \cos(2t) - 3\sin(2t) - \mathrm{e}^{-t}\right)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((1) - (0)) \\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 1 \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{t} \\ &= \mathrm{e}^{t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{t}\left(y - \cos(2t) - 3\sin(2t) - \mathrm{e}^{-t}\right) \\ &= -\mathrm{e}^{t}\cos(2t) - 3\,\mathrm{e}^{t}\sin(2t) + \mathrm{e}^{t}y - 1\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{t}(1) \\ &= \mathrm{e}^{t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(-\mathrm{e}^{t}\cos(2t) - 3\,\mathrm{e}^{t}\sin(2t) + \mathrm{e}^{t}y - 1\right) + \left(\mathrm{e}^{t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\mathrm{e}^{t} \cos(2t) - 3\,\mathrm{e}^{t} \sin(2t) + \mathrm{e}^{t} y - 1 \, \mathrm{d}t \\
\phi &= -t + \mathrm{e}^{t} y + \mathrm{e}^{t} \cos(2t) - \mathrm{e}^{t} \sin(2t) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{t} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{t}$. Therefore equation (4) becomes

$$
\mathrm{e}^{t} = \mathrm{e}^{t} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -t + \mathrm{e}^{t} y + \mathrm{e}^{t} \cos(2t) - \mathrm{e}^{t} \sin(2t) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -t + \mathrm{e}^{t} y + \mathrm{e}^{t} \cos(2t) - \mathrm{e}^{t} \sin(2t)
$$

The solution becomes

$$
y = -\mathrm{e}^{-t} \left(\mathrm{e}^{t} \cos(2t) - \mathrm{e}^{t} \sin(2t) - c_1 - t\right)
$$

Summary

The solution(s) found are the following

$$y = -e^{-t}\left(e^{t} \cos(2t) - e^{t} \sin(2t) - c_1 - t\right) \tag{1}$$

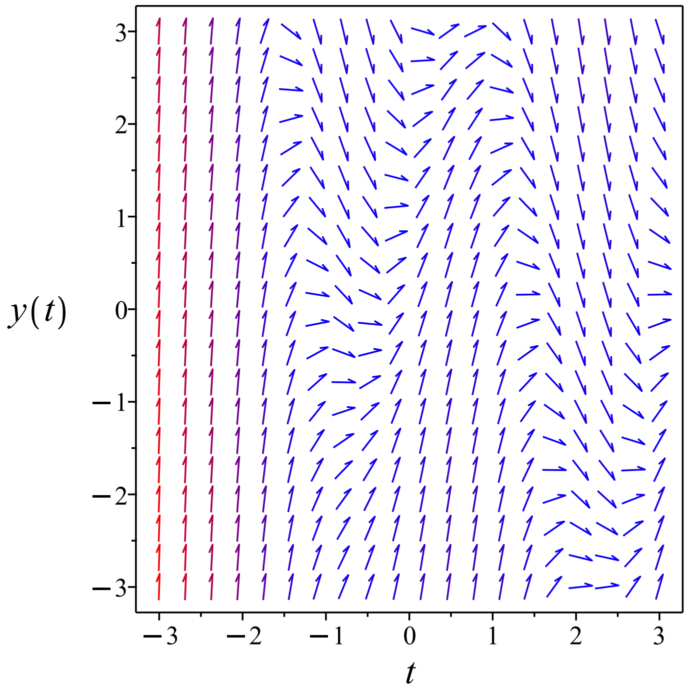


Figure 223: Slope field plot

Verification of solutions

$$y = -e^{-t}\left(e^{t} \cos(2t) - e^{t} \sin(2t) - c_1 - t\right)$$

Verified OK.

6.16.4 Maple step by step solution

Let's solve

$y' + y = \cos(2t) + 3\sin(2t) + e^{-t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + \cos(2t) + 3\sin(2t) + e^{-t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = \cos(2t) + 3\sin(2t) + \mathrm{e}^{-t}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(y' + y) = \mu(t)(\cos(2t) + 3\sin(2t) + \mathrm{e}^{-t})$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)(y' + y) = \mu'(t)\, y + \mu(t)\, y'$
- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)(\cos(2t) + 3\sin(2t) + \mathrm{e}^{-t})\, dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)(\cos(2t) + 3\sin(2t) + \mathrm{e}^{-t})\, dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t)(\cos(2t)+3\sin(2t)+\mathrm{e}^{-t})dt+c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{t}$

  $y = \frac{\int (\cos(2t)+3\sin(2t)+\mathrm{e}^{-t})\mathrm{e}^{t}dt+c_1}{\mathrm{e}^{t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{2(\cos(t)+2\sin(t))\mathrm{e}^{t}\cos(t)}{5} - \frac{\mathrm{e}^{t}}{5} + \frac{3\,\mathrm{e}^{t}(-2\cos(2t)+\sin(2t))}{5} + t + c_1}{\mathrm{e}^{t}}$
- Simplify

  $y = (t + c_1)\,\mathrm{e}^{-t} - \cos(2t) + \sin(2t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(t),t)+y(t)=cos(2*t)+3*sin(2*t)+exp(-t),y(t), singsol=all)
```

$$y(t) = (t + c_1)\,e^{-t} - \cos(2t) + \sin(2t)$$

✓ Solution by Mathematica

Time used: 0.239 (sec). Leaf size: 32

```
DSolve[y'[t]+y[t]==Cos[2*t]+3*Sin[2*t]+Exp[-t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}\left(t + e^t \sin(2t) - e^t \cos(2t) + c_1\right)$$

7 Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133

7.1 problem 1

Internal problem ID [13006]
Internal file name [`OUTPUT/11658_Tuesday_November_07_2023_11_54_12_PM_31871316/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{t} = 2}$$

7.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = 2$$

Hence the ode is

$$y' + \frac{y}{t} = 2$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{t}dt}$$
$$= t$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)(2)$$
$$\frac{d}{dt}(ty) = (t)(2)$$
$$d(ty) = (2t)\, dt$$

Integrating gives

$$ty = \int 2t\, dt$$
$$ty = t^2 + c_1$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$y = t + \frac{c_1}{t}$$

Summary

The solution(s) found are the following

$$y = t + \frac{c_1}{t} \tag{1}$$

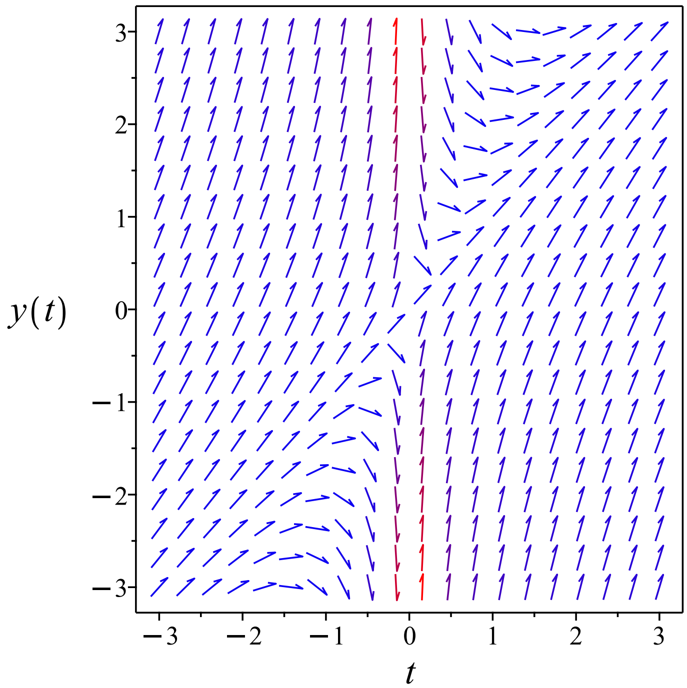


Figure 224: Slope field plot

<u>Verification of solutions</u>

$$y = t + \frac{c_1}{t}$$

Verified OK.

7.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + 2u(t) = 2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t, u) \\ &= f(t)g(u) \\ &= \frac{-2u + 2}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(u) = -2u + 2$. Integrating both sides gives

$$\frac{1}{-2u + 2}\,du = \frac{1}{t}\,dt$$

$$\int \frac{1}{-2u+2}\,du = \int \frac{1}{t}\,dt$$
$$-\frac{\ln(u-1)}{2} = \ln(t) + c_2$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{u-1}} = e^{\ln(t)+c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{u-1}} = c_3 t$$

Therefore the solution y is

$$\begin{aligned} y &= tu \\ &= \frac{(c_3^2 e^{2c_2} t^2 + 1)\, e^{-2c_2}}{t\, c_3^2} \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{(c_3^2 e^{2c_2} t^2 + 1)\, e^{-2c_2}}{t\, c_3^2} \tag{1}$$

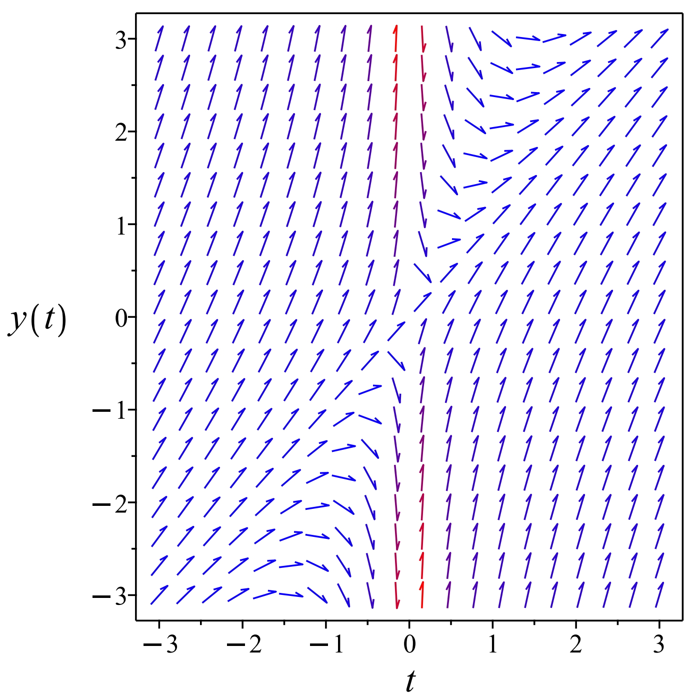


Figure 225: Slope field plot

Verification of solutions

$$y = \frac{(c_3^2 \mathrm{e}^{2c_2} t^2 + 1)\, \mathrm{e}^{-2c_2}}{t\, c_3^2}$$

Verified OK.

7.1.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{y}{t} + 2 \tag{1}$$

Which becomes

$$0 = (-t)\, dy + (2t - y)\, dt \tag{2}$$

But the RHS is complete differential because

$$(-t)\, dy + (2t - y)\, dt = d\big(t^2 - ty\big)$$

Hence (2) becomes

$$0 = d\big(t^2 - ty\big)$$

Integrating both sides gives gives these solutions

$$y = \frac{t^2 + c_1}{t} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{t^2 + c_1}{t} + c_1 \tag{1}$$

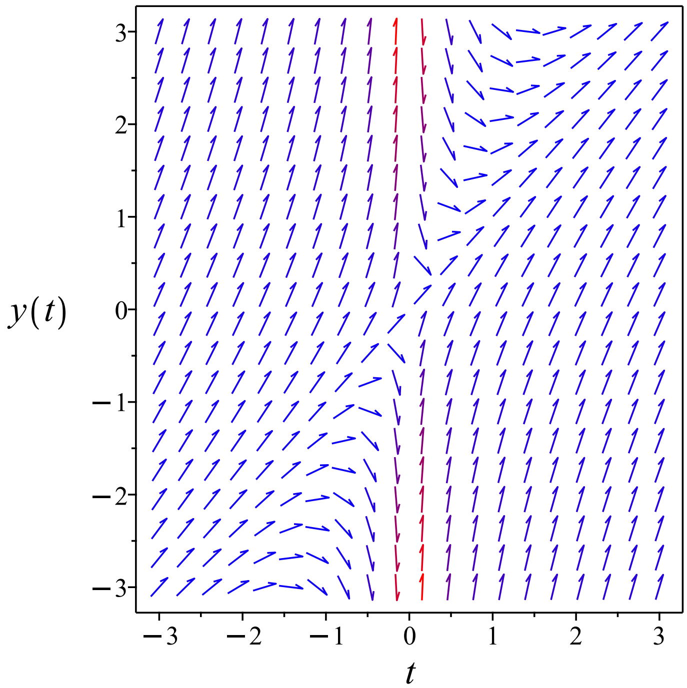


Figure 226: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{t^2 + c_1}{t} + c_1$$

Verified OK.

7.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-2t + y}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 224: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \frac{1}{t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t}} dy \end{aligned}$$

Which results in

$$S = ty$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{-2t + y}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= y \\ S_y &= t \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$ty = t^2 + c_1$$

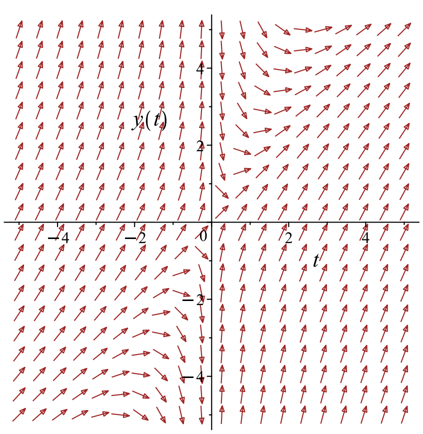
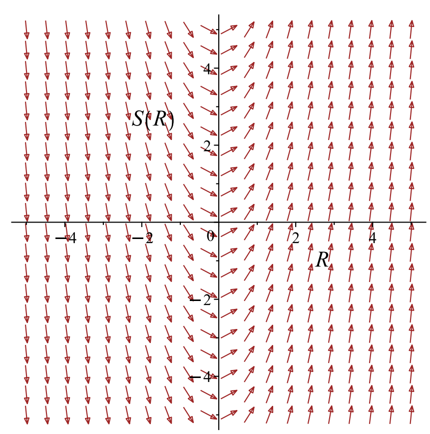
Which simplifies to

$$ty = t^2 + c_1$$

Which gives

$$y = \frac{t^2 + c_1}{t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{-2t+y}{t}$ 	$R = t$ $S = ty$	$\frac{dS}{dR} = 2R$ 

Summary

The solution(s) found are the following

$$y = \frac{t^2 + c_1}{t} \tag{1}$$

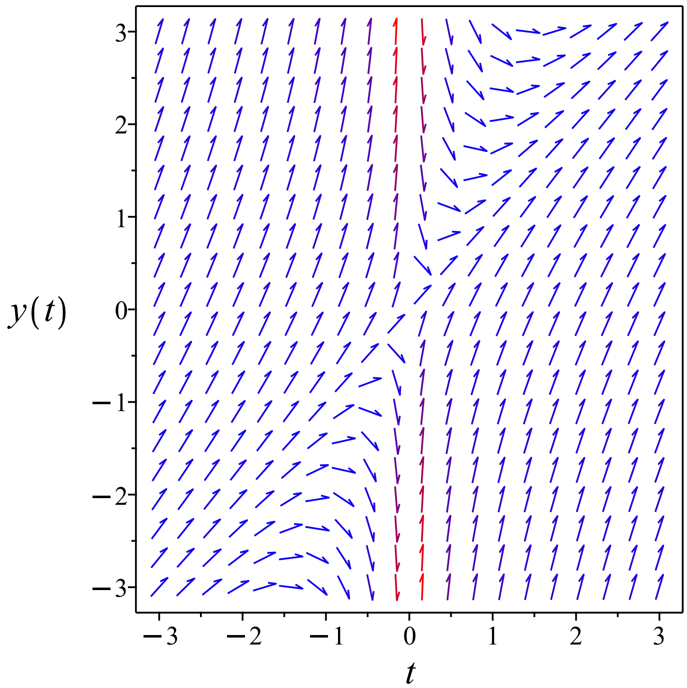


Figure 227: Slope field plot

Verification of solutions

$$y = \frac{t^2 + c_1}{t}$$

Verified OK.

7.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(t) \, \mathrm{d}y &= (2t - y) \, \mathrm{d}t \\
(-2t + y) \, \mathrm{d}t + (t) \, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -2t + y \\
N(t, y) &= t
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2t + y) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t=\int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t=\int -2t+y\,\mathrm{d}t$$
$$\phi=-t(t-y)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=t+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=t$. Therefore equation (4) becomes

$$t=t+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=-t(t-y)+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=-t(t-y)$$

The solution becomes

$$y = \frac{t^2 + c_1}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{t^2 + c_1}{t} \tag{1}$$

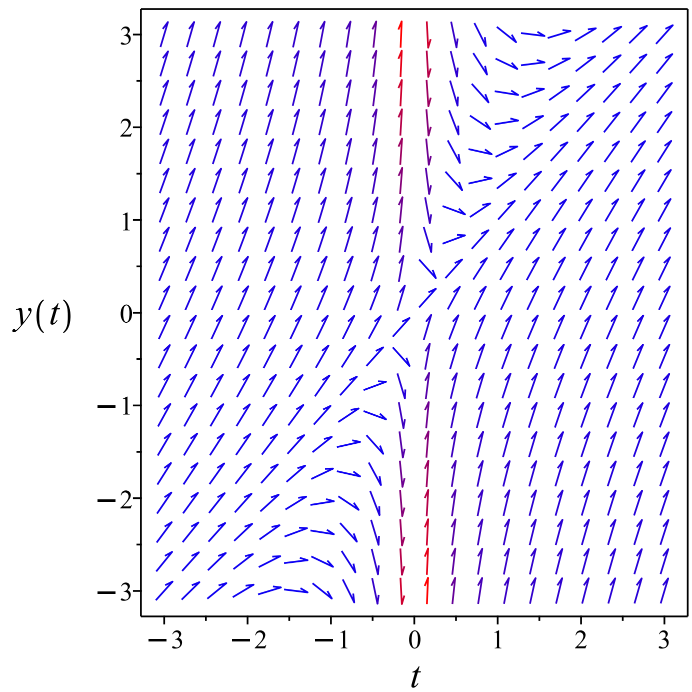


Figure 228: Slope field plot

Verification of solutions

$$y = \frac{t^2 + c_1}{t}$$

Verified OK.

7.1.6 Maple step by step solution

Let's solve

$y' + \frac{y}{t} = 2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{t} + 2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{t} = 2$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t) \left(y' + \frac{y}{t}\right) = 2\mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t) \left(y' + \frac{y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 2\mu(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 2\mu(t)\, dt + c_1$

- Solve for y

  $y = \frac{\int 2\mu(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t$

  $y = \frac{\int 2t dt + c_1}{t}$

- Evaluate the integrals on the rhs

  $y = \frac{t^2 + c_1}{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(t),t)=-y(t)/t+2,y(t), singsol=all)
```

$$y(t) = t + \frac{c_1}{t}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 13

```
DSolve[y'[t]==-y[t]/t+2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t + \frac{c_1}{t}$$

7.2 problem 2

Internal problem ID [13007]
Internal file name [`OUTPUT/11659_Wednesday_November_08_2023_03_28_11_AM_64086885/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{3y}{t} = t^5}$$

7.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{3}{t}$$
$$q(t) = t^5$$

Hence the ode is

$$y' - \frac{3y}{t} = t^5$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{t}dt}$$
$$= \frac{1}{t^3}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(t^5\right)$$
$$\frac{d}{dt}\left(\frac{y}{t^3}\right) = \left(\frac{1}{t^3}\right)\left(t^5\right)$$
$$d\left(\frac{y}{t^3}\right) = t^2\, dt$$

Integrating gives

$$\frac{y}{t^3} = \int t^2\, dt$$
$$\frac{y}{t^3} = \frac{t^3}{3} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^3}$ results in

$$y = \frac{1}{3}t^6 + t^3 c_1$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3}t^6 + t^3 c_1 \tag{1}$$

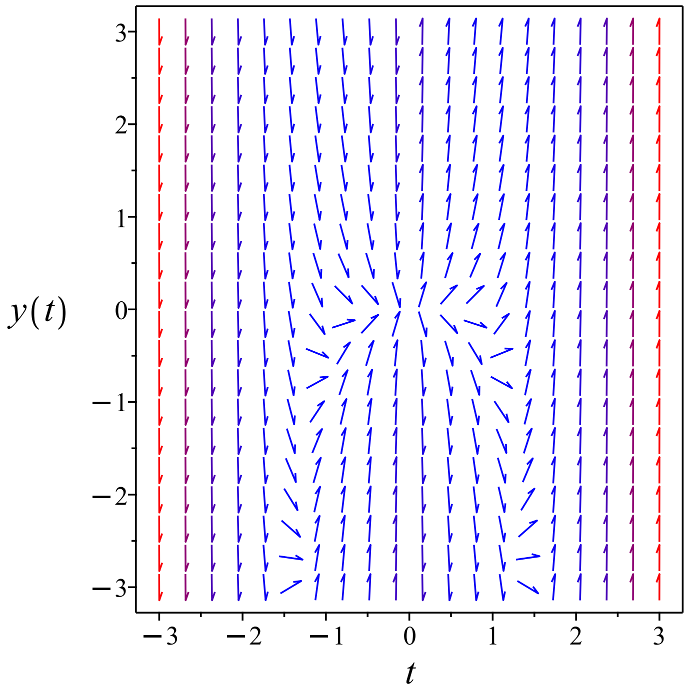


Figure 229: Slope field plot

Verification of solutions

$$y = \frac{1}{3}t^6 + t^3 c_1$$

Verified OK.

7.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{t^6 + 3y}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 227: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= t^3
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^3} dy \end{aligned}$$

Which results in

$$S = \frac{y}{t^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{t^6 + 3y}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{3y}{t^4} \\ S_y &= \frac{1}{t^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{t^3} = \frac{t^3}{3} + c_1$$

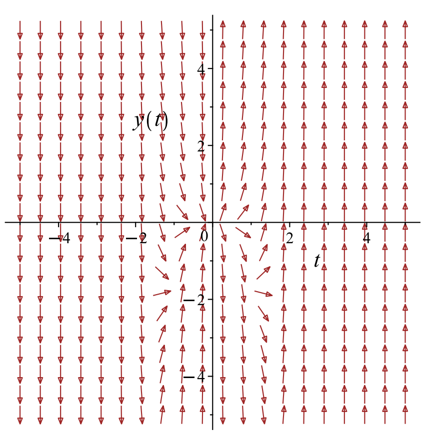
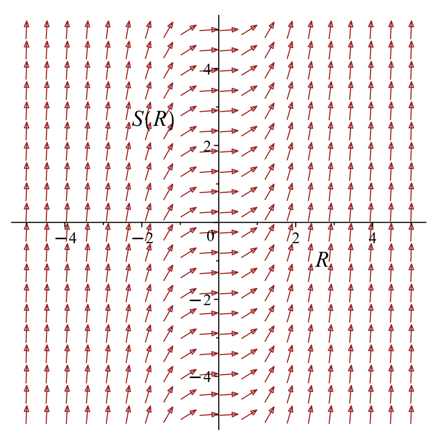
Which simplifies to

$$\frac{y}{t^3} = \frac{t^3}{3} + c_1$$

Which gives

$$y = \frac{t^3(t^3 + 3c_1)}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{t^6+3y}{t}$ 	$R = t$ $S = \frac{y}{t^3}$	$\frac{dS}{dR} = R^2$ 

Summary

The solution(s) found are the following

$$y = \frac{t^3(t^3 + 3c_1)}{3} \tag{1}$$

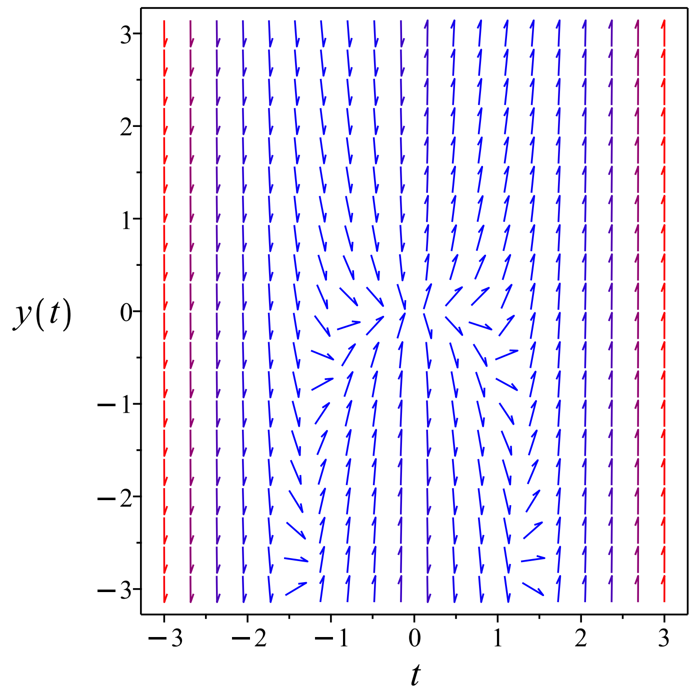


Figure 230: Slope field plot

Verification of solutions

$$y = \frac{t^3(t^3 + 3c_1)}{3}$$

Verified OK.

7.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\frac{3y}{t}+t^5\right)\mathrm{d}t$$

$$\left(-\frac{3y}{t}-t^5\right)\mathrm{d}t+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{3y}{t}-t^5\\ N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{3y}{t}-t^5\right)\\ &=-\frac{3}{t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(-\frac{3}{t}\right) - (0)\right)\\ &= -\frac{3}{t}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}t}\\ &= e^{\int -\frac{3}{t} \,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(t)}\\ &= \frac{1}{t^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{t^3}\left(-\frac{3y}{t} - t^5\right)\\ &= \frac{-t^6 - 3y}{t^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{t^3}(1)\\ &= \frac{1}{t^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\frac{-t^6 - 3y}{t^4}\right) + \left(\frac{1}{t^3}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \frac{-t^6 - 3y}{t^4}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{-t^6 + 3y}{3t^3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{t^3} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{t^3}$. Therefore equation (4) becomes

$$
\frac{1}{t^3} = \frac{1}{t^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-t^6 + 3y}{3t^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-t^6 + 3y}{3t^3}$$

The solution becomes

$$y = \frac{t^3(t^3 + 3c_1)}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{t^3(t^3 + 3c_1)}{3} \tag{1}$$

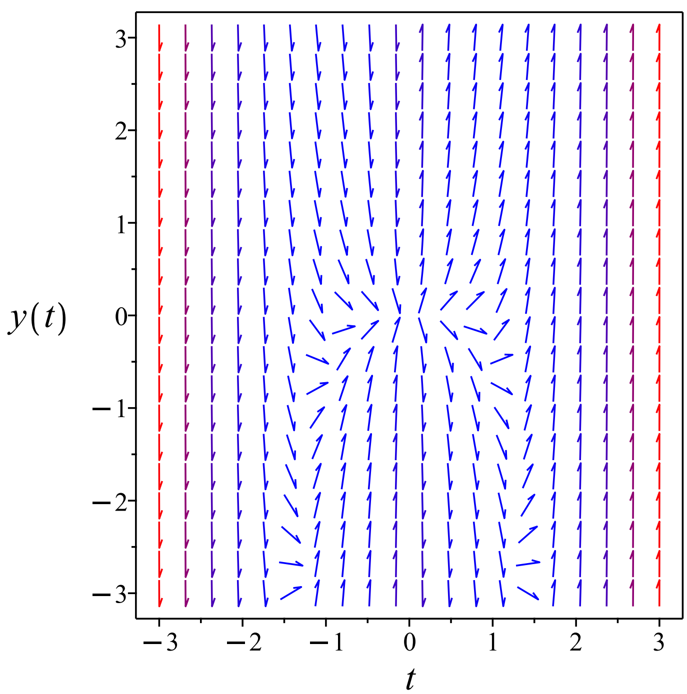


Figure 231: Slope field plot

Verification of solutions

$$y = \frac{t^3(t^3 + 3c_1)}{3}$$

Verified OK.

7.2.4 Maple step by step solution

Let's solve

$y' - \frac{3y}{t} = t^5$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{3y}{t} + t^5$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{3y}{t} = t^5$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{3y}{t}\right) = \mu(t)\, t^5$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - \frac{3y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{3\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{t^3}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\, t^5 dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\, t^5 dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t) t^5 dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{t^3}$

$y = t^3 \left(\int t^2 dt + c_1 \right)$

- Evaluate the integrals on the rhs

  $y = t^3 \left(\frac{t^3}{3} + c_1 \right)$

- Simplify

  $y = \frac{t^3 (t^3 + 3c_1)}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)=3/t*y(t)+t^5,y(t), singsol=all)
```

$$y(t) = \frac{(t^3 + 3c_1)\, t^3}{3}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 19

```
DSolve[y'[t]==3/t*y[t]+t^5,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{t^6}{3} + c_1 t^3$$

7.3 problem 3

Internal problem ID [13008]
Internal file name [`OUTPUT/11660_Wednesday_November_08_2023_03_28_14_AM_59055855/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{1+t} = t^2}$$

7.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{1+t}$$
$$q(t) = t^2$$

Hence the ode is

$$y' + \frac{y}{1+t} = t^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{1+t}dt} \\
&= 1+t
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(t^2\right) \\
\frac{d}{dt}((1+t)\,y) &= (1+t)\left(t^2\right) \\
d((1+t)\,y) &= \left(t^2(1+t)\right)\,dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(1+t)\,y &= \int t^2(1+t)\,dt \\
(1+t)\,y &= \frac{1}{4}t^4 + \frac{1}{3}t^3 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = 1+t$ results in

$$
y = \frac{\frac{1}{4}t^4 + \frac{1}{3}t^3}{1+t} + \frac{c_1}{1+t}
$$

which simplifies to

$$
y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t}
$$

Summary
The solution(s) found are the following

$$
y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t} \tag{1}
$$

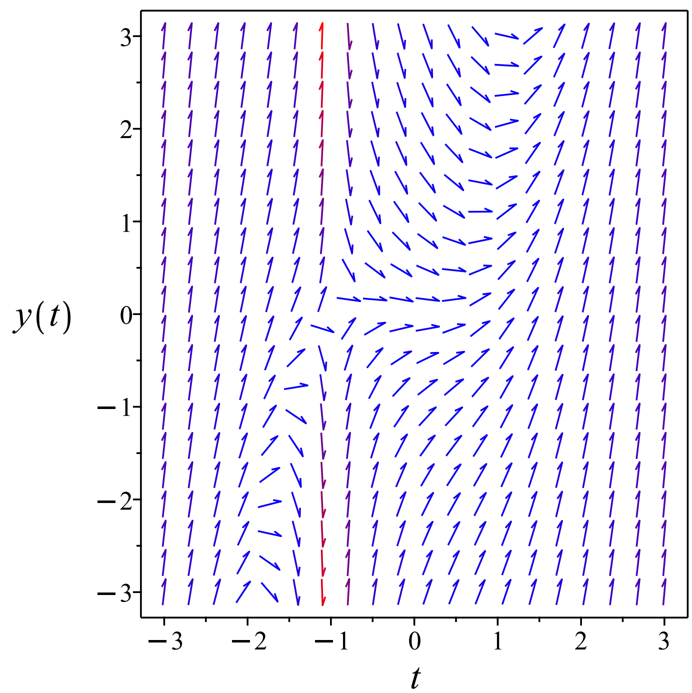


Figure 232: Slope field plot

Verification of solutions

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t}$$

Verified OK.

7.3.2 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{y}{1+t} + t^2 \tag{1}$$

Which becomes

$$0 = (-1-t)\,dy + \left(t^3 + t^2 - y\right) dt \tag{2}$$

But the RHS is complete differential because

$$(-1-t)\,dy + \left(t^3 + t^2 - y\right) dt = d\left(\frac{1}{4}t^4 + \frac{1}{3}t^3 - ty - y\right)$$

Hence (2) becomes

$$0 = d\left(\frac{1}{4}t^4 + \frac{1}{3}t^3 - ty - y\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t} + c_1$$

Summary

The solution(s) found are the following

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t} + c_1 \tag{1}$$

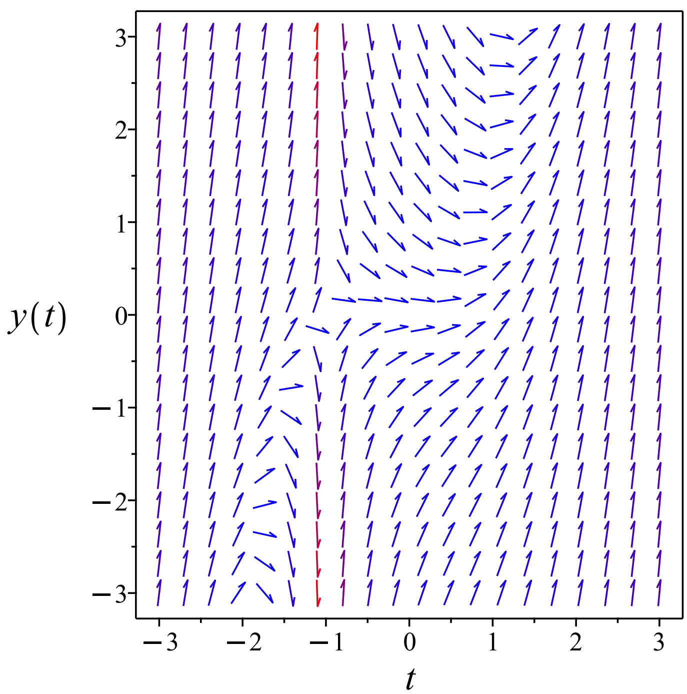


Figure 233: Slope field plot

Verification of solutions

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t} + c_1$$

Verified OK.

7.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-t^3 - t^2 + y}{1+t} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 230: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,y) &= 0 \\ \eta(t,y) &= \frac{1}{1+t} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{1+t}} dy \end{aligned}$$

Which results in

$$S = (1+t)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\frac{-t^3 - t^2 + y}{1+t}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= y \\
S_y &= 1+t
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = t^2(1+t) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R^2(R+1)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{1}{4}R^4 + \frac{1}{3}R^3 + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$
(1+t)\, y = \frac{1}{4}t^4 + \frac{1}{3}t^3 + c_1
$$

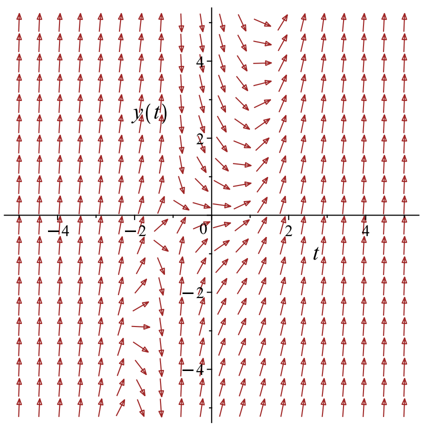
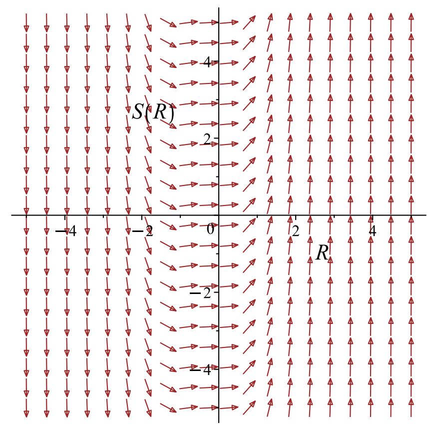
Which simplifies to

$$
(1+t)\, y = \frac{1}{4}t^4 + \frac{1}{3}t^3 + c_1
$$

Which gives

$$
y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{-t^3-t^2+y}{1+t}$ 	$R = t$ $S = (1+t)\, y$	$\frac{dS}{dR} = R^2(R+1)$ 

Summary
The solution(s) found are the following

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t} \tag{1}$$

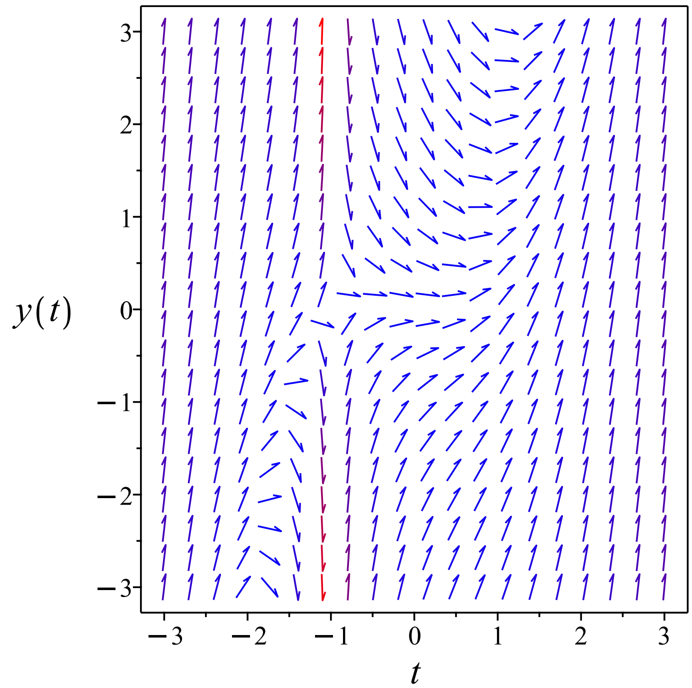


Figure 234: Slope field plot

Verification of solutions

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t}$$

Verified OK.

7.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(1+t)\,\mathrm{d}y &= \left(t^3+t^2-y\right)\mathrm{d}t\\ \left(-t^3-t^2+y\right)\mathrm{d}t + (1+t)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -t^3-t^2+y\\ N(t,y) &= 1+t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-t^3-t^2+y\right)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1+t)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t^3 - t^2 + y \, \mathrm{d}t$$

$$\phi = -\frac{1}{4}t^4 - \frac{1}{3}t^3 + ty + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1 + t$. Therefore equation (4) becomes

$$1 + t = t + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (1) \, \mathrm{d}y$$

$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{4}t^4 - \frac{1}{3}t^3 + ty + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{4}t^4 - \frac{1}{3}t^3 + ty + y$$

The solution becomes

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t}$$

Summary

The solution(s) found are the following

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t} \tag{1}$$

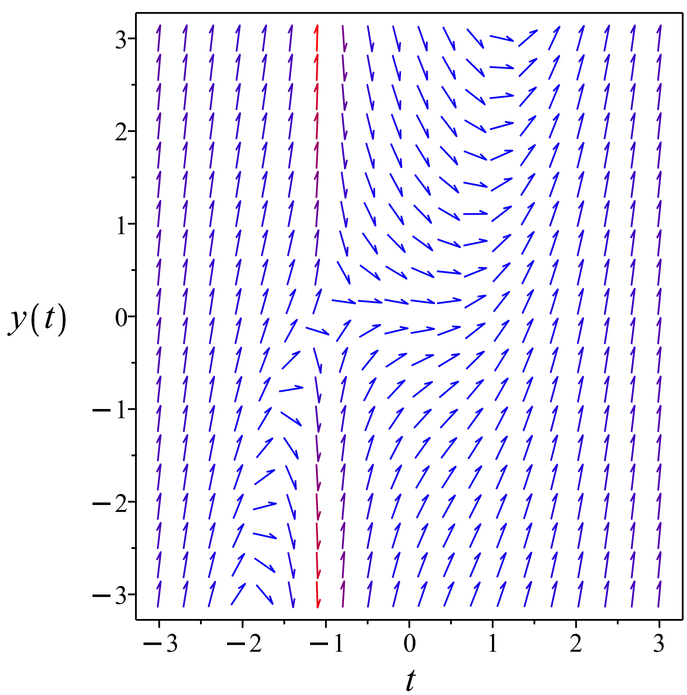


Figure 235: Slope field plot

Verification of solutions

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t}$$

Verified OK.

7.3.5 Maple step by step solution

Let's solve

$y' + \frac{y}{1+t} = t^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{1+t} + t^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{1+t} = t^2$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{y}{1+t}\right) = \mu(t)\, t^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{y}{1+t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{1+t}$

- Solve to find the integrating factor

  $\mu(t) = 1 + t$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\, t^2 dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\, t^2 dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t) t^2 dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = 1 + t$

  $y = \frac{\int t^2 (1+t) dt + c_1}{1+t}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{1}{4} t^4 + \frac{1}{3} t^3 + c_1}{1+t}$

- Simplify

$$y = \frac{3t^4 + 4t^3 + 12c_1}{12 + 12t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(t),t)=-y(t)/(1+t)+t^2,y(t), singsol=all)
```

$$y(t) = \frac{3t^4 + 4t^3 + 12c_1}{12t + 12}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 28

```
DSolve[y'[t]==-y[t]/(1+t)+t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{3t^4 + 4t^3 + 12c_1}{12t + 12}$$

7.4 problem 4

Internal problem ID [13009]
Internal file name [`OUTPUT/11661_Wednesday_November_08_2023_03_28_15_AM_27323432/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + 2ty = 4\,\mathrm{e}^{-t^2}}$$

7.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 2t \\ q(t) &= 4\,\mathrm{e}^{-t^2} \end{aligned}$$

Hence the ode is

$$y' + 2ty = 4\,\mathrm{e}^{-t^2}$$

The integrating factor μ is

$$\mu = e^{\int 2tdt}$$
$$= e^{t^2}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(4\,e^{-t^2}\right)$$
$$\frac{d}{dt}\left(e^{t^2} y\right) = \left(e^{t^2}\right)\left(4\,e^{-t^2}\right)$$
$$d\left(e^{t^2} y\right) = 4\,dt$$

Integrating gives

$$e^{t^2} y = \int 4\,dt$$
$$e^{t^2} y = 4t + c_1$$

Dividing both sides by the integrating factor $\mu = e^{t^2}$ results in

$$y = 4\,e^{-t^2} t + c_1 e^{-t^2}$$

which simplifies to

$$y = e^{-t^2}(4t + c_1)$$

Summary
The solution(s) found are the following

$$y = e^{-t^2}(4t + c_1) \tag{1}$$

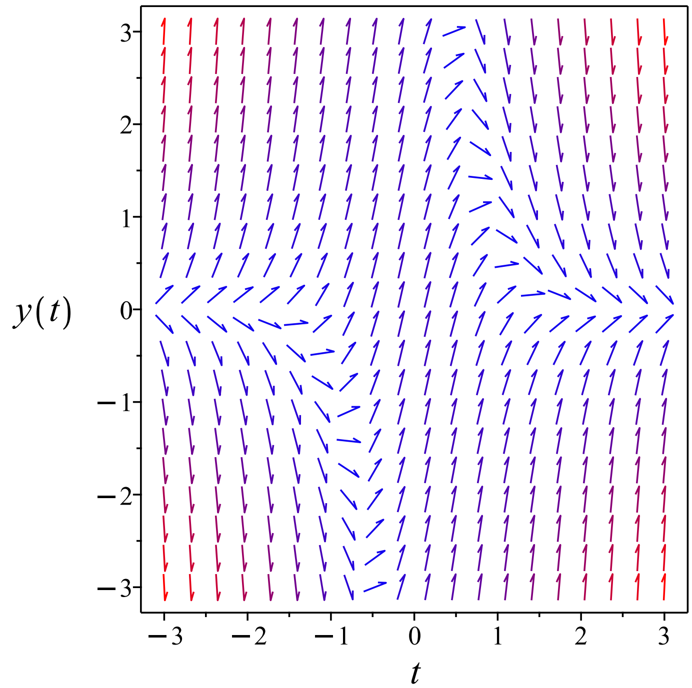


Figure 236: Slope field plot

Verification of solutions

$$y = e^{-t^2}(4t + c_1)$$

Verified OK.

7.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -2ty + 4\,e^{-t^2} \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 233: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, y) &= 0 \\ \eta(t, y) &= \mathrm{e}^{-t^2}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-t^2}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{t^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -2ty + 4\,\mathrm{e}^{-t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2t\,\mathrm{e}^{t^2} y \\ S_y &= \mathrm{e}^{t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 4R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$y\,\mathrm{e}^{t^2} = 4t + c_1$$

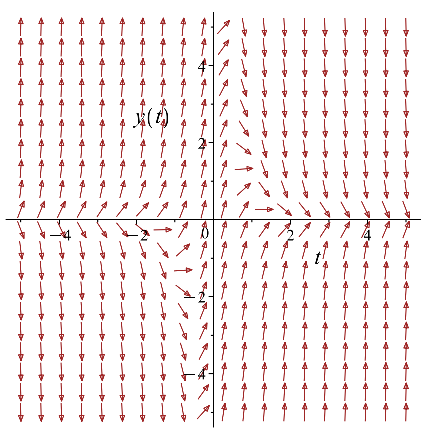
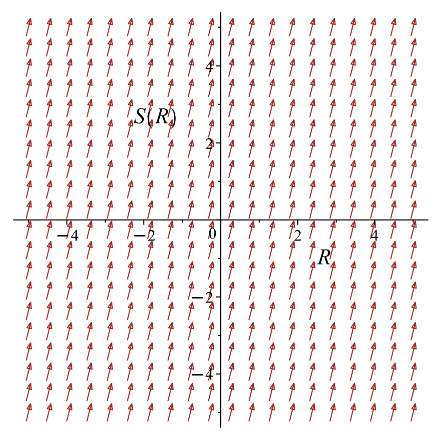
Which simplifies to

$$y\,\mathrm{e}^{t^2} = 4t + c_1$$

Which gives

$$y = \mathrm{e}^{-t^2}(4t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dt} = -2ty + 4\,\mathrm{e}^{-t^2}$

</td><td>$R = t$
$S = \mathrm{e}^{t^2} y$</td><td>$\frac{dS}{dR} = 4$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-t^2}(4t + c_1) \tag{1}$$

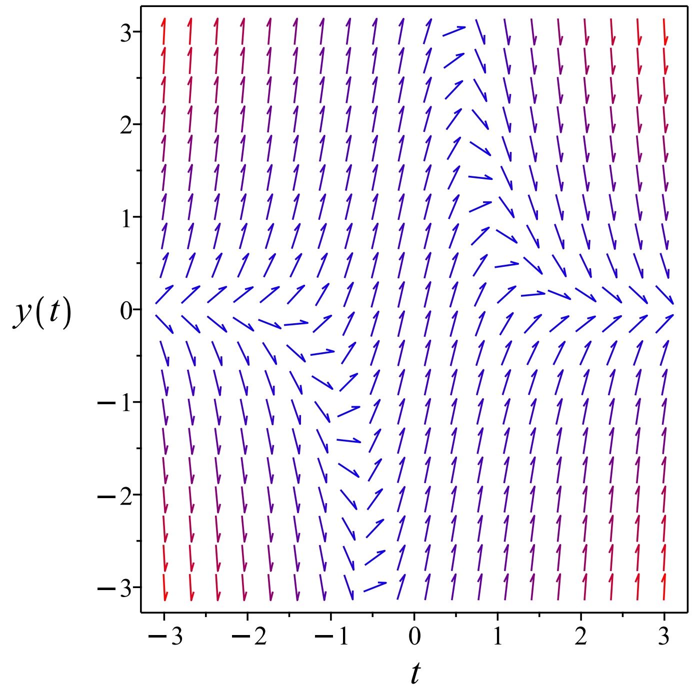


Figure 237: Slope field plot

Verification of solutions

$$y = e^{-t^2}(4t + c_1)$$

Verified OK.

7.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-2ty + 4\,\mathrm{e}^{-t^2}\right)\mathrm{d}t$$
$$\left(2ty - 4\,\mathrm{e}^{-t^2}\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = 2ty - 4\,\mathrm{e}^{-t^2}$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2ty - 4\,\mathrm{e}^{-t^2}\right)$$
$$= 2t$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((2t) - (0)) \\
&= 2t
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int 2t \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{t^2} \\
&= \mathrm{e}^{t^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{t^2}\left(2ty - 4\,\mathrm{e}^{-t^2}\right) \\
&= 2t\,\mathrm{e}^{t^2} y - 4
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{t^2}(1) \\
&= \mathrm{e}^{t^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(2t\,\mathrm{e}^{t^2} y - 4\right) + \left(\mathrm{e}^{t^2}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int 2t\,\mathrm{e}^{t^2} y - 4 \,\mathrm{d}t$$

$$\phi = -4t + \mathrm{e}^{t^2} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{t^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{t^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{t^2} = \mathrm{e}^{t^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -4t + \mathrm{e}^{t^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -4t + \mathrm{e}^{t^2} y$$

The solution becomes

$$y = e^{-t^2}(4t + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-t^2}(4t + c_1) \tag{1}$$

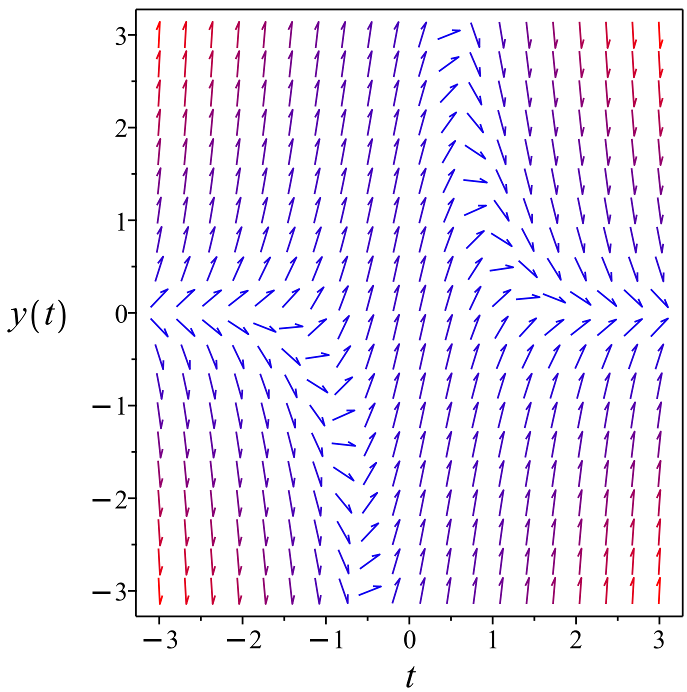


Figure 238: Slope field plot

Verification of solutions

$$y = e^{-t^2}(4t + c_1)$$

Verified OK.

7.4.4 Maple step by step solution

Let's solve

$y' + 2ty = 4\,\mathrm{e}^{-t^2}$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = -2ty + 4\,\mathrm{e}^{-t^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2ty = 4\,\mathrm{e}^{-t^2}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 2ty) = 4\mu(t)\,\mathrm{e}^{-t^2}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + 2ty) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)\,t$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t^2}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 4\mu(t)\,\mathrm{e}^{-t^2} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 4\mu(t)\,\mathrm{e}^{-t^2} dt + c_1$
- Solve for y

  $y = \frac{\int 4\mu(t)\mathrm{e}^{-t^2} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{t^2}$

  $y = \frac{\int 4\,\mathrm{e}^{-t^2}\mathrm{e}^{t^2} dt + c_1}{\mathrm{e}^{t^2}}$
- Evaluate the integrals on the rhs

  $y = \frac{4t + c_1}{\mathrm{e}^{t^2}}$
- Simplify

$$y = e^{-t^2}(4t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)=-2*t*y(t)+4*exp(-t^2),y(t), singsol=all)
```

$$y(t) = (4t + c_1)\,e^{-t^2}$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 19

```
DSolve[y'[t]==-2*t*y[t]+4*Exp[-t^2],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t^2}(4t + c_1)$$

7.5 problem 5

Internal problem ID [13010]
Internal file name [`OUTPUT/11662_Wednesday_November_08_2023_03_28_15_AM_27558722/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{2ty}{t^2+1} = 3}$$

7.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{2t}{t^2+1}$$
$$q(t) = 3$$

Hence the ode is

$$y' - \frac{2ty}{t^2+1} = 3$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2t}{t^2+1}dt}$$
$$= \frac{1}{t^2+1}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)(3)$$
$$\frac{d}{dt}\left(\frac{y}{t^2+1}\right) = \left(\frac{1}{t^2+1}\right)(3)$$
$$d\left(\frac{y}{t^2+1}\right) = \left(\frac{3}{t^2+1}\right) dt$$

Integrating gives

$$\frac{y}{t^2+1} = \int \frac{3}{t^2+1} dt$$
$$\frac{y}{t^2+1} = 3\arctan(t) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2+1}$ results in

$$y = 3(t^2+1)\arctan(t) + c_1(t^2+1)$$

which simplifies to

$$y = (t^2+1)(3\arctan(t) + c_1)$$

Summary

The solution(s) found are the following

$$y = (t^2+1)(3\arctan(t) + c_1) \tag{1}$$

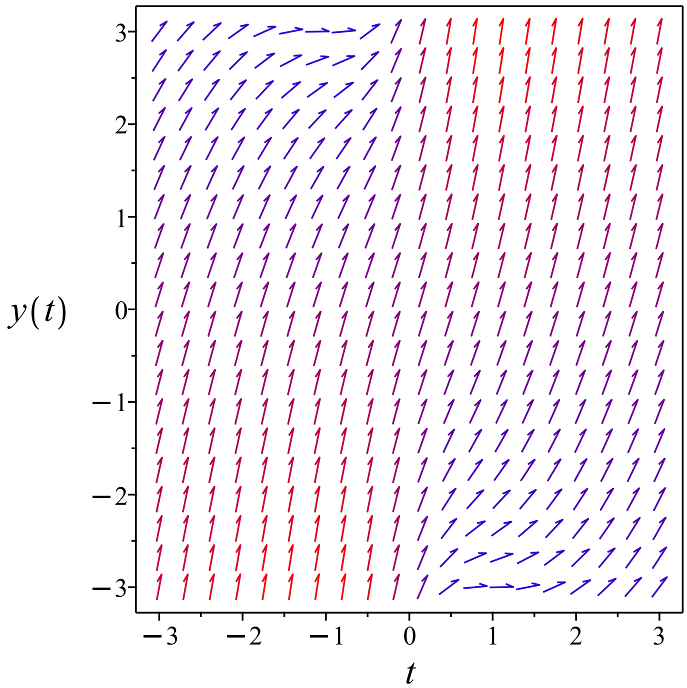


Figure 239: Slope field plot

Verification of solutions

$$y = (t^2 + 1)(3 \arctan(t) + c_1)$$

Verified OK.

7.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3t^2 + 2ty + 3}{t^2 + 1}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 236: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= t^2 + 1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^2+1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{t^2+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{3t^2 + 2ty + 3}{t^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{2yt}{(t^2+1)^2} \\ S_y &= \frac{1}{t^2+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3}{t^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 3\arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{t^2+1} = 3\arctan(t) + c_1$$

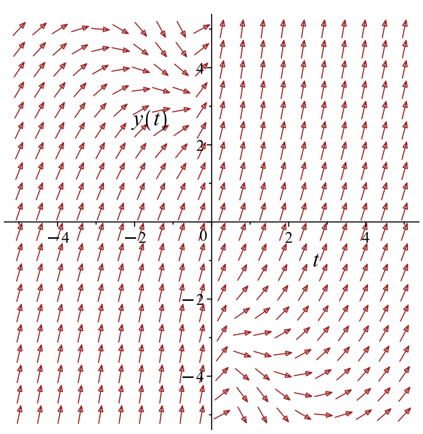
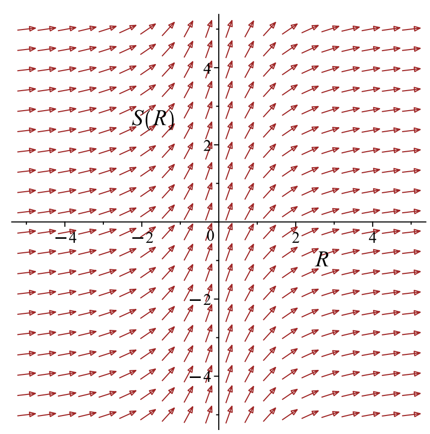
Which simplifies to

$$\frac{y}{t^2+1} = 3\arctan(t) + c_1$$

Which gives

$$y = (t^2+1)(3\arctan(t) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{3t^2+2ty+3}{t^2+1}$ 	$R = t$ $S = \frac{y}{t^2+1}$	$\frac{dS}{dR} = \frac{3}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = (t^2+1)(3\arctan(t) + c_1) \tag{1}$$

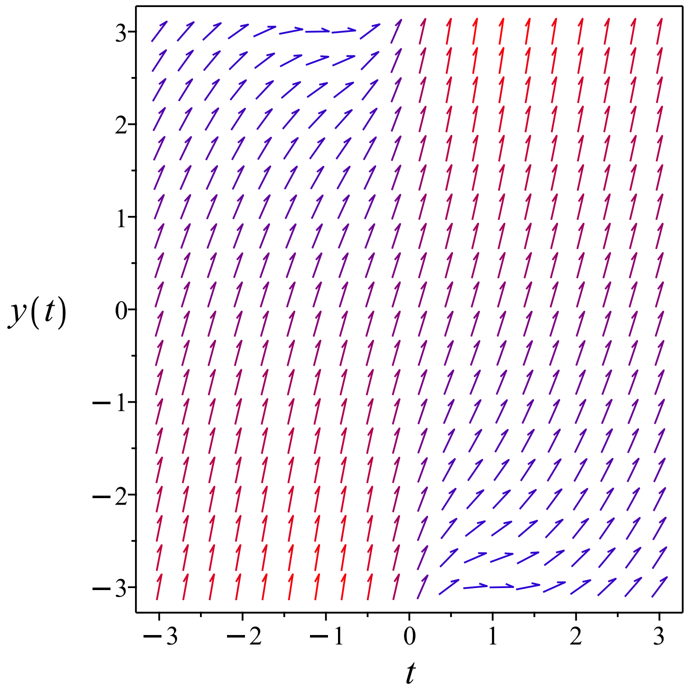


Figure 240: Slope field plot

Verification of solutions

$$y = (t^2 + 1)(3\arctan(t) + c_1)$$

Verified OK.

7.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{2ty}{t^2+1} + 3\right)\mathrm{d}t$$
$$\left(-\frac{2ty}{t^2+1} - 3\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,y) = -\frac{2ty}{t^2+1} - 3$$
$$N(t,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2ty}{t^2+1} - 3\right)\\ &= -\frac{2t}{t^2+1}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1\left(\left(-\frac{2t}{t^2+1}\right) - (0)\right)\\ &= -\frac{2t}{t^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}t}\\ &= e^{\int -\frac{2t}{t^2+1} \,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(t^2+1)}\\ &= \frac{1}{t^2+1}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{t^2+1}\left(-\frac{2ty}{t^2+1} - 3\right)\\ &= \frac{-3t^2 - 2ty - 3}{(t^2+1)^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{t^2+1}(1)\\ &= \frac{1}{t^2+1}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$
$$\left(\frac{-3t^2 - 2ty - 3}{(t^2+1)^2}\right) + \left(\frac{1}{t^2+1}\right)\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \frac{-3t^2 - 2ty - 3}{(t^2+1)^2}\,\mathrm{d}t$$
$$\phi = \frac{y}{t^2+1} - 3\arctan(t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{t^2+1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{t^2+1}$. Therefore equation (4) becomes

$$\frac{1}{t^2+1} = \frac{1}{t^2+1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{t^2+1} - 3\arctan(t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{t^2+1} - 3\arctan(t)$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{y}{t^2+1} - 3\arctan(t) = c_1 \tag{1}$$

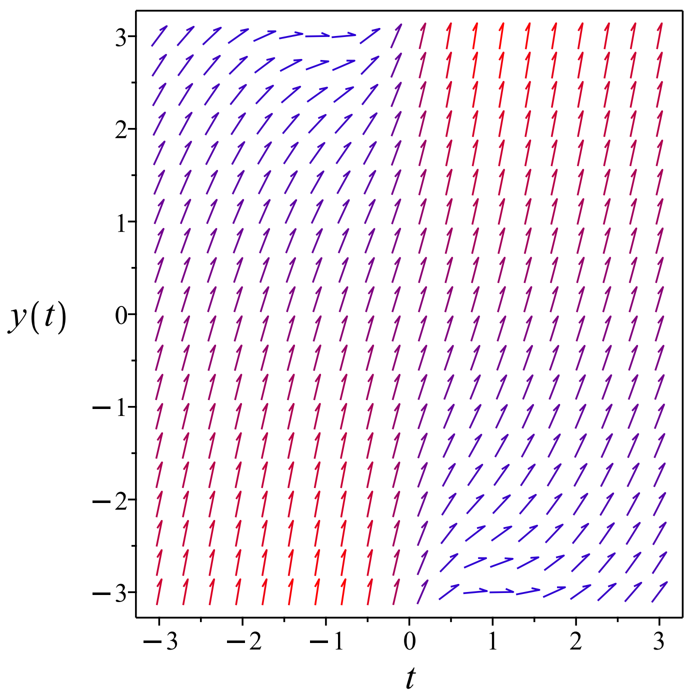


Figure 241: Slope field plot

<u>Verification of solutions</u>

$$\frac{y}{t^2+1} - 3\arctan(t) = c_1$$

Verified OK.

7.5.4 Maple step by step solution

Let's solve

$y' - \frac{2ty}{t^2+1} = 3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2ty}{t^2+1} + 3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2ty}{t^2+1} = 3$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{2ty}{t^2+1}\right) = 3\mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - \frac{2ty}{t^2+1}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{2\mu(t)t}{t^2+1}$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{t^2+1}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 3\mu(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 3\mu(t)\, dt + c_1$

- Solve for y

  $y = \frac{\int 3\mu(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{t^2+1}$

  $y = (t^2 + 1)\left(\int \frac{3}{t^2+1}dt + c_1\right)$

- Evaluate the integrals on the rhs

  $y = (t^2 + 1)\,(3\arctan(t) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)-2*t/(1+t^2)*y(t)=3,y(t), singsol=all)
```

$$y(t) = (3 \arctan(t) + c_1)(t^2 + 1)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 18

```
DSolve[y'[t]-2*t/(1+t^2)*y[t]==3,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to (t^2 + 1)(3 \arctan(t) + c_1)$$

7.6 problem 6

Internal problem ID [13011]
Internal file name [`OUTPUT/11663_Wednesday_November_08_2023_03_28_16_AM_40367264/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{2y}{t} = \mathrm{e}^{t} t^{3}}$$

7.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{2}{t}$$
$$q(t) = \mathrm{e}^{t} t^{3}$$

Hence the ode is

$$y' - \frac{2y}{t} = \mathrm{e}^{t} t^{3}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2}{t}dt} \\
&= \frac{1}{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(e^t t^3\right) \\
\frac{d}{dt}\left(\frac{y}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(e^t t^3\right) \\
d\left(\frac{y}{t^2}\right) &= \left(t\,e^t\right)\,dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{t^2} &= \int t\,e^t\,dt \\
\frac{y}{t^2} &= e^t(t-1) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$
y = t^2 e^t (t-1) + t^2 c_1
$$

which simplifies to

$$
y = t^2\left(e^t(t-1) + c_1\right)
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = t^2\left(e^t(t-1) + c_1\right) \tag{1}
$$

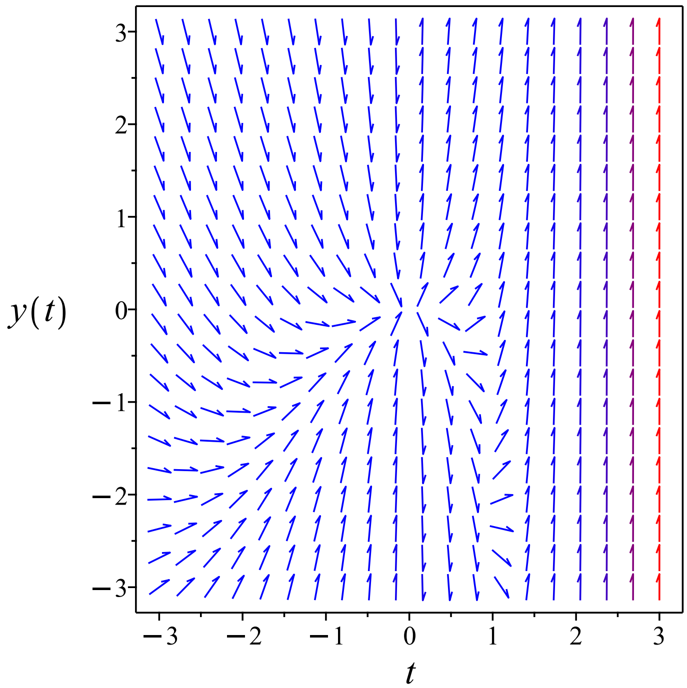


Figure 242: Slope field plot

Verification of solutions

$$y = t^2\left(e^t(t-1)+c_1\right)$$

Verified OK.

7.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{e^t t^4 + 2y}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 239: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= 0\\ \eta(t,y) &= t^2\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right)S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{t^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{e^t t^4 + 2y}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{2y}{t^3} \\ S_y &= \frac{1}{t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t\, e^t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\, e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = (R-1)\,\mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{t^2} = \mathrm{e}^{t}(t-1) + c_1$$

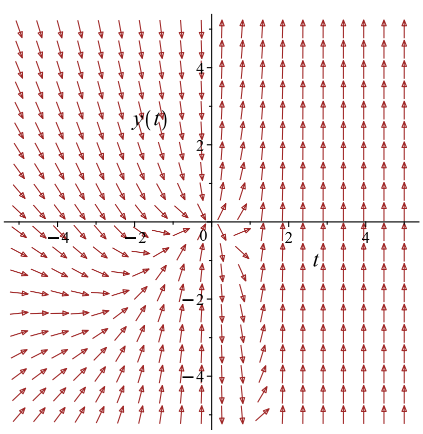
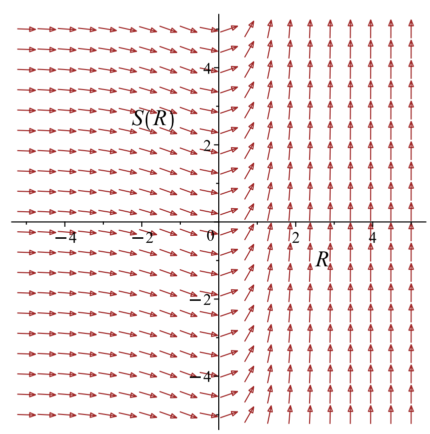
Which simplifies to

$$\frac{y}{t^2} = \mathrm{e}^{t}(t-1) + c_1$$

Which gives

$$y = t^2\left(t\,\mathrm{e}^{t} - \mathrm{e}^{t} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{\mathrm{e}^{t} t^4 + 2y}{t}$ 	$R = t$ $S = \frac{y}{t^2}$	$\frac{dS}{dR} = R\,\mathrm{e}^{R}$ 

Summary

The solution(s) found are the following

$$y = t^2\left(t\,\mathrm{e}^{t} - \mathrm{e}^{t} + c_1\right) \tag{1}$$

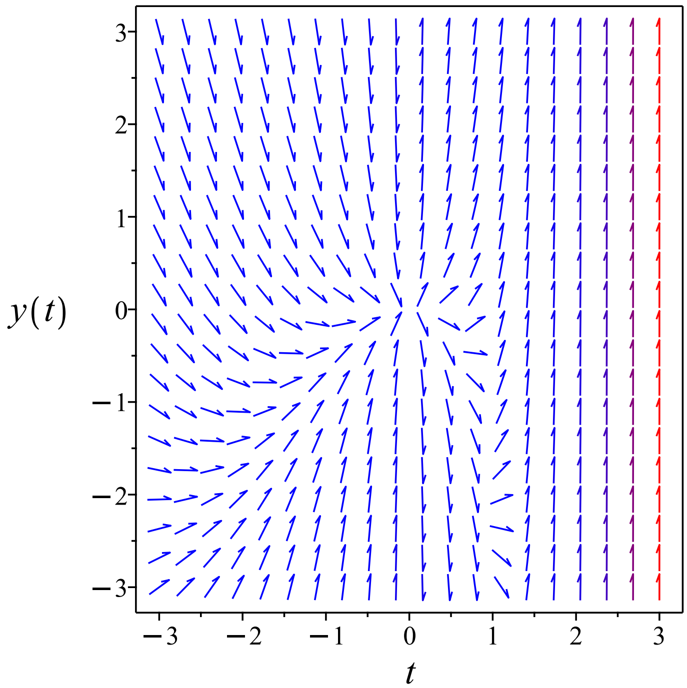


Figure 243: Slope field plot

Verification of solutions

$$y = t^2\left(t\,e^t - e^t + c_1\right)$$

Verified OK.

7.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{2y}{t} + \mathrm{e}^t t^3\right)\mathrm{d}t$$
$$\left(-\frac{2y}{t} - \mathrm{e}^t t^3\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -\frac{2y}{t} - \mathrm{e}^t t^3$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{2y}{t} - \mathrm{e}^t t^3\right)$$
$$= -\frac{2}{t}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(-\frac{2}{t}\right) - (0)\right) \\
&= -\frac{2}{t}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{2}{t} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{t^2}\left(-\frac{2y}{t} - \mathrm{e}^t t^3\right) \\
&= \frac{-\mathrm{e}^t t^4 - 2y}{t^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{t^2}(1) \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$
$$\left(\frac{-\mathrm{e}^{t}t^{4} - 2y}{t^{3}}\right) + \left(\frac{1}{t^{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \frac{-\mathrm{e}^{t}t^{4} - 2y}{t^{3}}\,\mathrm{d}t$$
$$\phi = \frac{(-t^{3} + t^{2})\,\mathrm{e}^{t} + y}{t^{2}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{t^{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{t^{2}}$. Therefore equation (4) becomes

$$\frac{1}{t^{2}} = \frac{1}{t^{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_{1}$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(-t^3 + t^2)\,e^t + y}{t^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(-t^3 + t^2)\,e^t + y}{t^2}$$

The solution becomes

$$y = t^2 (t\,e^t - e^t + c_1)$$

Summary

The solution(s) found are the following

$$y = t^2 (t\,e^t - e^t + c_1) \tag{1}$$

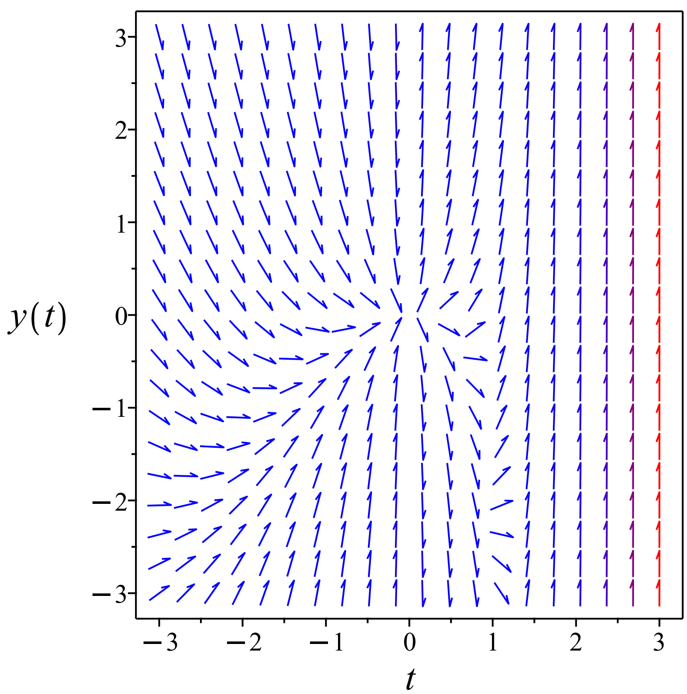


Figure 244: Slope field plot

Verification of solutions

$$y = t^2 \left(t\, e^t - e^t + c_1\right)$$

Verified OK.

7.6.4 Maple step by step solution

Let's solve

$y' - \frac{2y}{t} = e^t t^3$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = \frac{2y}{t} + e^t t^3$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{t} = e^t t^3$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{2y}{t}\right) = \mu(t)\, e^t t^3$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - \frac{2y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{2\mu(t)}{t}$
- Solve to find the integrating factor

  $\mu(t) = \frac{1}{t^2}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\, e^t t^3 dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\, e^t t^3 dt + c_1$
- Solve for y

  $y = \frac{\int \mu(t) e^t t^3 dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \frac{1}{t^2}$

  $y = t^2 \left(\int t\, e^t dt + c_1\right)$

- Evaluate the integrals on the rhs

  $y = t^2(e^t(t-1) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)-2/t*y(t)=t^3*exp(t),y(t), singsol=all)
```

$$y(t) = \left(e^t(t-1) + c_1\right) t^2$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 19

```
DSolve[y'[t]-2/t*y[t]==t^3*Exp[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t^2\left(e^t(t-1) + c_1\right)$$

7.7 problem 7

Internal problem ID [13012]
Internal file name [`OUTPUT/11664_Wednesday_November_08_2023_03_28_17_AM_27176372/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{1+t} = 2}$$

With initial conditions

$$[y(0) = 3]$$

7.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{1+t} \\ q(t) &= 2 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{1+t} = 2$$

The domain of $p(t) = \frac{1}{1+t}$ is

$$\{t < -1 \lor -1 < t\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.7.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{1+t}dt} \\ &= 1 + t \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)(2) \\ \frac{\mathrm{d}}{\mathrm{d}t}((1+t)\,y) &= (1+t)\,(2) \\ \mathrm{d}((1+t)\,y) &= (2t+2)\ \mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} (1+t)\,y &= \int 2t + 2\,\mathrm{d}t \\ (1+t)\,y &= t^2 + 2t + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = 1 + t$ results in

$$y = \frac{t^2 + 2t}{1 + t} + \frac{c_1}{1 + t}$$

which simplifies to

$$y = \frac{t^2 + c_1 + 2t}{1 + t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

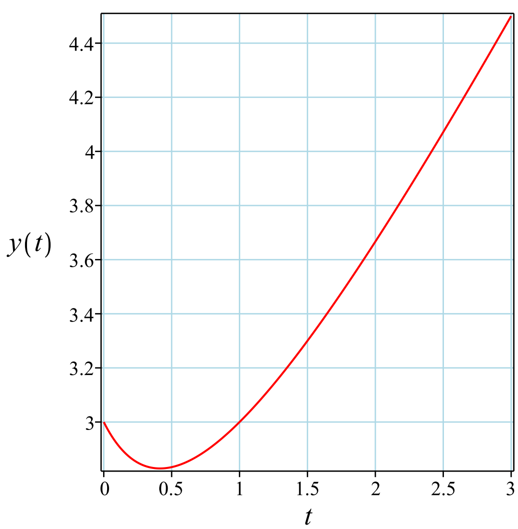
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

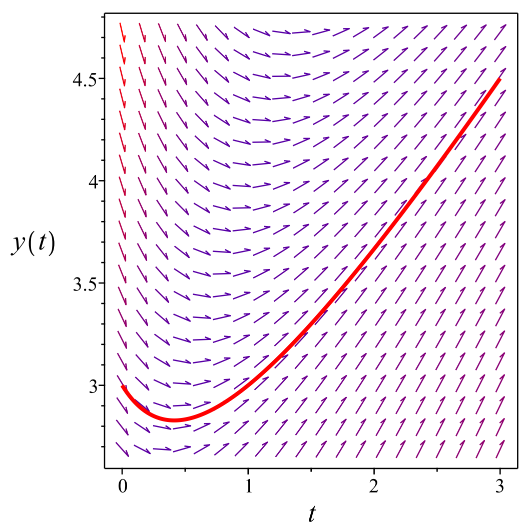
$$y = \frac{t^2 + 2t + 3}{1 + t}$$

Summary
The solution(s) found are the following

$$y = \frac{t^2 + 2t + 3}{1 + t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{t^2 + 2t + 3}{1 + t}$$

Verified OK.

7.7.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{y}{1 + t} + 2 \tag{1}$$

Which becomes

$$0 = (-1 - t)\, dy + (-y + 2t + 2)\, dt \tag{2}$$

But the RHS is complete differential because

$$(-1 - t)\, dy + (-y + 2t + 2)\, dt = d\left(t^2 - ty + 2t - y\right)$$

Hence (2) becomes

$$0 = d\left(t^2 - ty + 2t - y\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{t^2 + c_1 + 2t}{1 + t} + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 2c_1$$

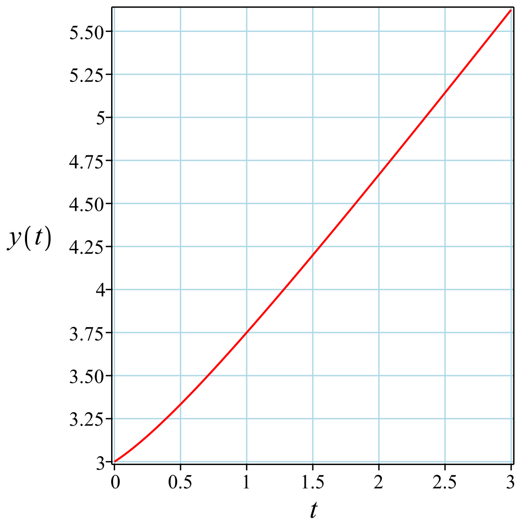
$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

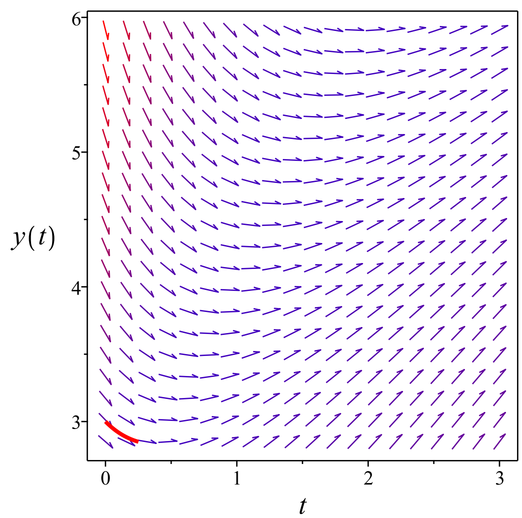
$$y = \frac{2t^2 + 7t + 6}{2t + 2}$$

Summary
The solution(s) found are the following

$$y = \frac{2t^2 + 7t + 6}{2t + 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2t^2 + 7t + 6}{2t + 2}$$

Verified OK.

7.7.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = t + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{-2X - 2x_0 - 2 + Y(X) + y_0}{1 + X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{-2X + Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{-2X + Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 2X - Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{du}{dX}X + u = 2 - u$$
$$\frac{du}{dX} = \frac{2 - 2u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{2 - 2u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 2u(X) - 2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= \frac{2 - 2u}{X} \end{aligned}$$

Where $f(X) = \frac{1}{X}$ and $g(u) = 2 - 2u$. Integrating both sides gives

$$\frac{1}{2 - 2u}\,du = \frac{1}{X}\,dX$$
$$\int \frac{1}{2 - 2u}\,du = \int \frac{1}{X}\,dX$$
$$-\frac{\ln(-1 + u)}{2} = \ln(X) + c_2$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-1+u}} = \mathrm{e}^{\ln(X)+c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{-1+u}} = c_3 X$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \frac{\left(c_3^2 \mathrm{e}^{2c_2} X^2 + 1\right) \mathrm{e}^{-2c_2}}{X\, c_3^2}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{\left(c_3^2 \mathrm{e}^{2c_2} X^2 + 1\right) \mathrm{e}^{-2c_2}}{X\, c_3^2}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= t + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y \\ X &= t - 1 \end{aligned}$$

Then the solution in y becomes

$$y = \frac{\left(c_3^2 \mathrm{e}^{2c_2} (1+t)^2 + 1\right) \mathrm{e}^{-2c_2}}{(1+t)\, c_3^2}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{\mathrm{e}^{-2c_2} \mathrm{e}^{2c_2} c_3^2 + \mathrm{e}^{-2c_2}}{c_3^2}$$

$$c_2 = -\frac{\ln\left(2c_3^2\right)}{2}$$

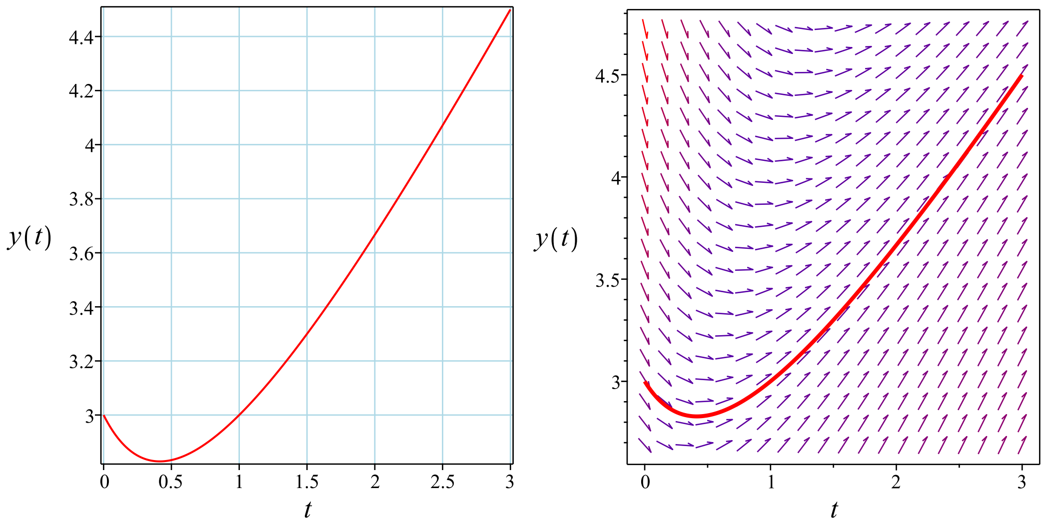
Substituting c_2 found above in the general solution gives

$$y = \frac{t^2 + 2t + 3}{1+t}$$

Summary

The solution(s) found are the following

$$y = \frac{t^2 + 2t + 3}{1 + t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{t^2 + 2t + 3}{1 + t}$$

Verified OK.

7.7.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y - 2t - 2}{1 + t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 242: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \frac{1}{1+t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{1+t}} dy \end{aligned}$$

Which results in

$$S = (1 + t) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -\frac{y - 2t - 2}{1 + t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= y \\ S_y &= 1 + t \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2t + 2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R + 2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + 2R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$(1 + t)\, y = t^2 + c_1 + 2t$$

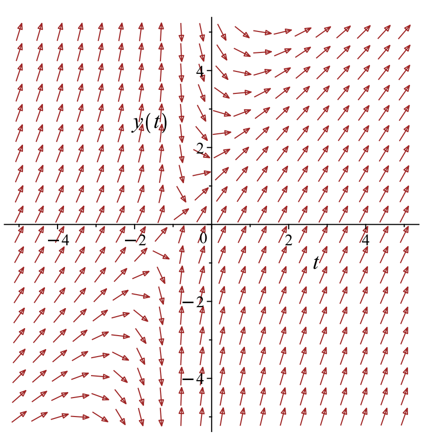
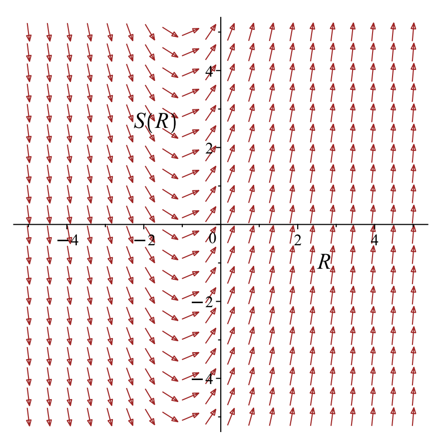
Which simplifies to

$$(1 + t)\, y = t^2 + c_1 + 2t$$

Which gives

$$y = \frac{t^2 + c_1 + 2t}{1 + t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{y-2t-2}{1+t}$ 	$R = t$ $S = (1 + t)\, y$	$\frac{dS}{dR} = 2R + 2$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

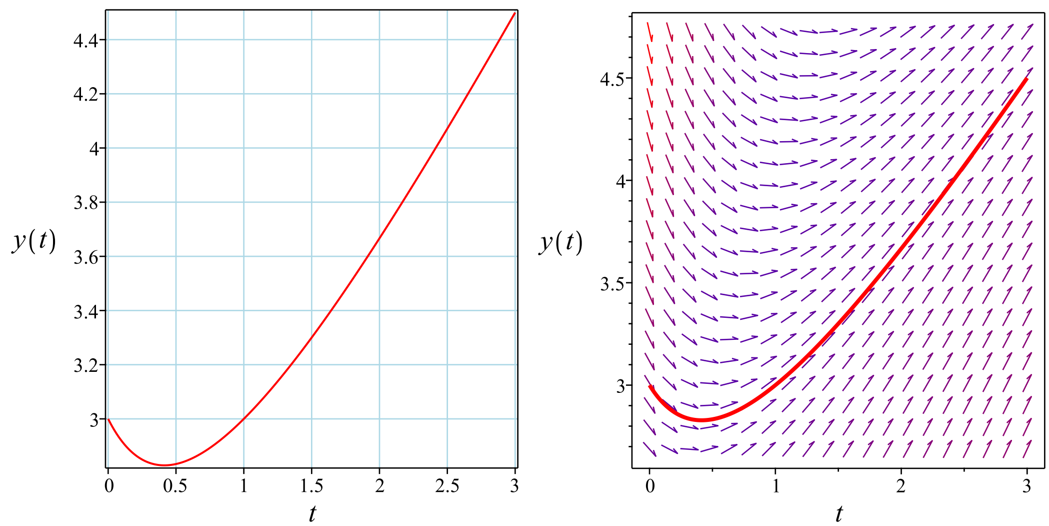
Substituting c_1 found above in the general solution gives

$$y = \frac{t^2 + 2t + 3}{1 + t}$$

Summary

The solution(s) found are the following

$$y = \frac{t^2 + 2t + 3}{1 + t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{t^2 + 2t + 3}{1 + t}$$

Verified OK.

7.7.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(1+t)\,\mathrm{d}y&=(-y+2t+2)\,\mathrm{d}t\\ (y-2t-2)\,\mathrm{d}t+(1+t)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=y-2t-2\\ N(t,y)&=1+t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - 2t - 2)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1 + t)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int M\, \mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int y - 2t - 2\, \mathrm{d}t\end{aligned}$$

$$\phi = -t(t - y + 2) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1 + t$. Therefore equation (4) becomes

$$1 + t = t + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -t(t - y + 2) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t(t - y + 2) + y$$

The solution becomes

$$y = \frac{t^2 + c_1 + 2t}{1 + t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

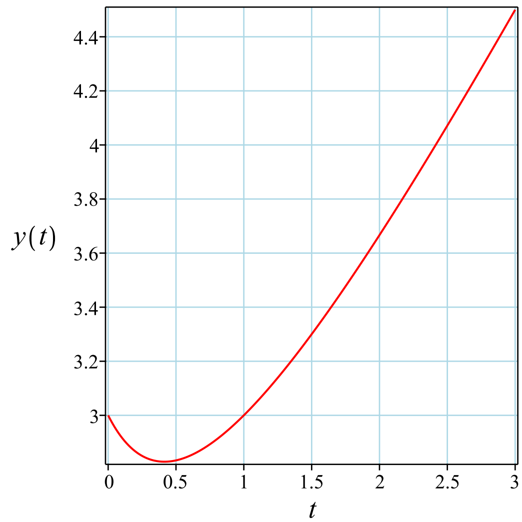
Substituting c_1 found above in the general solution gives

$$y = \frac{t^2 + 2t + 3}{1 + t}$$

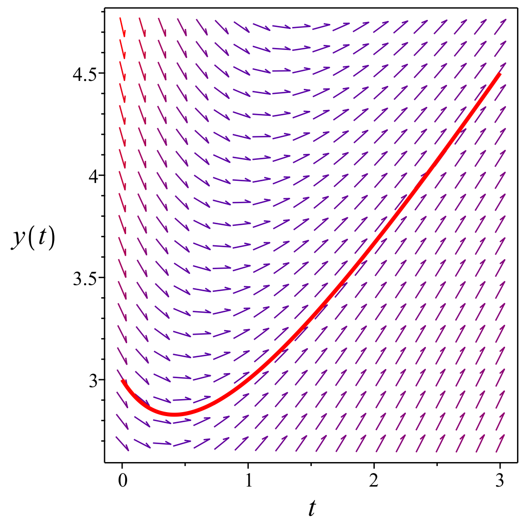
Summary

The solution(s) found are the following

$$y = \frac{t^2 + 2t + 3}{1 + t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{t^2 + 2t + 3}{1 + t}$$

Verified OK.

7.7.7 Maple step by step solution

Let's solve

$\left[y' + \frac{y}{1+t} = 2, y(0) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{1+t} + 2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{1+t} = 2$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{y}{1+t}\right) = 2\mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$\mu(t)\left(y' + \frac{y}{1+t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{1+t}$

- Solve to find the integrating factor

  $\mu(t) = 1 + t$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 2\mu(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 2\mu(t)\, dt + c_1$

- Solve for y

  $y = \frac{\int 2\mu(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = 1 + t$

  $y = \frac{\int (2t+2)dt + c_1}{1+t}$

- Evaluate the integrals on the rhs

  $y = \frac{t^2 + c_1 + 2t}{1+t}$

- Use initial condition $y(0) = 3$

  $3 = c_1$

- Solve for c_1

  $c_1 = 3$

- Substitute $c_1 = 3$ into general solution and simplify

  $y = \frac{t^2 + 2t + 3}{1+t}$

- Solution to the IVP

  $y = \frac{t^2 + 2t + 3}{1+t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([diff(y(t),t)=-y(t)/(1+t)+2,y(0) = 3],y(t), singsol=all)
```

$$y(t) = \frac{t^2 + 2t + 3}{t + 1}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 19

```
DSolve[{y'[t]==-y[t]/(1+t)+2,{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{t^2 + 2t + 3}{t + 1}$$

7.8 problem 8

Internal problem ID [13013]
Internal file name [`OUTPUT/11665_Wednesday_November_08_2023_03_28_18_AM_42393074/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{1+t} = 4t^2 + 4t}$$

With initial conditions

$$[y(1) = 10]$$

7.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{1}{1+t}$$
$$q(t) = 4(1+t)\,t$$

Hence the ode is

$$y' - \frac{y}{1+t} = 4(1+t)\,t$$

The domain of $p(t) = -\frac{1}{1+t}$ is

$$\{t < -1 \lor -1 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = 4(1+t)\,t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

7.8.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{1+t}dt}\\ &= \frac{1}{1+t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\,(4(1+t)\,t)\\ \frac{d}{dt}\left(\frac{y}{1+t}\right) &= \left(\frac{1}{1+t}\right)(4(1+t)\,t)\\ d\left(\frac{y}{1+t}\right) &= (4t)\;dt\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{1+t} &= \int 4t\,dt\\ \frac{y}{1+t} &= 2t^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{1+t}$ results in

$$y = 2t^2(1+t) + c_1(1+t)$$

which simplifies to

$$y = 2\left(t^2 + \frac{c_1}{2}\right)(1+t)$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 10$ in the above solution gives an equation to solve for the constant of integration.

$$10 = 2c_1 + 4$$

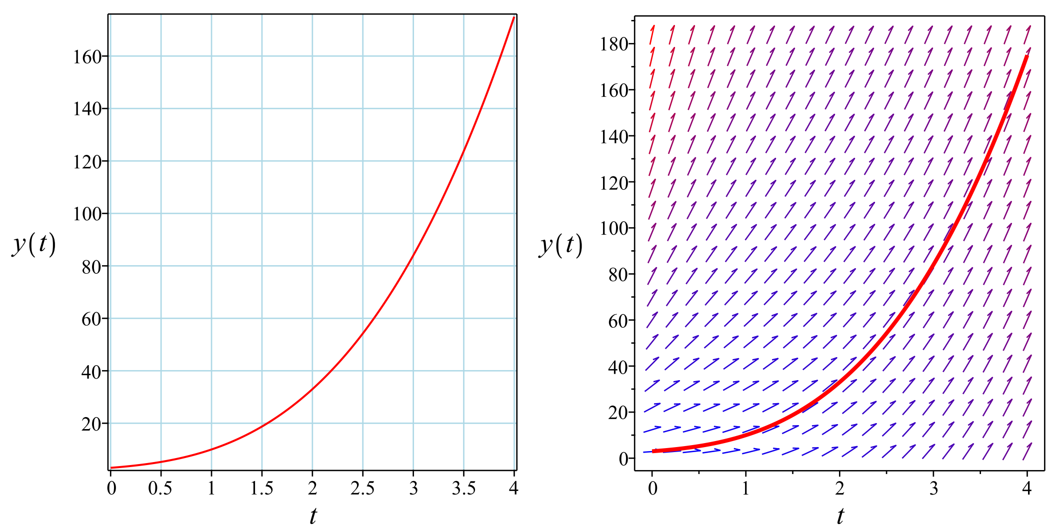
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$y = (2t^2 + 3)(1+t)$$

Summary
The solution(s) found are the following

$$y = (2t^2 + 3)(1+t) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (2t^2 + 3)(1+t)$$

Verified OK.

7.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{4t^3+8t^2+4t+y}{1+t}\\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t+\omega(\eta_y-\xi_t)-\omega^2\xi_y-\omega_t\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 245: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y'=f(x)y(x)+g(x)$	0	$e^{\int f dx}$
separable ode	$y'=f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y'=f(x)$	0	1
quadrature ode	$y'=g(y)$	1	0
homogeneous ODEs of Class A	$y'=f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y'=(a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y'=g(x)\,e^{h(x)+by}+f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y'=f(x)\,y+g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y'=f_1(x)\,y+f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= 0\\ \eta(t,y) &= 1+t\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{1+t} dy\end{aligned}$$

Which results in

$$S = \frac{y}{1+t}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{4t^3 + 8t^2 + 4t + y}{1+t}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= -\frac{y}{(1+t)^2} \\
S_y &= \frac{1}{1+t}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 4t \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 4R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = 2R^2 + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$
\frac{y}{1+t} = 2t^2 + c_1
$$

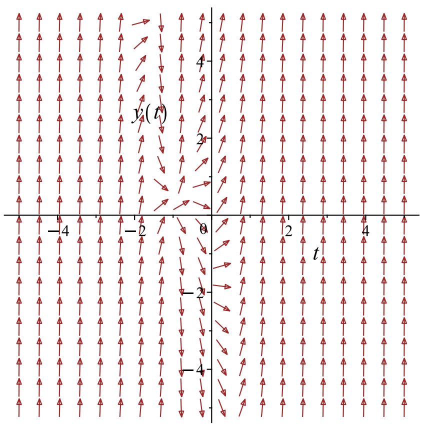
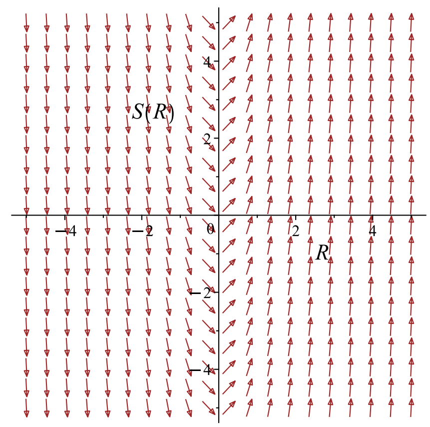
Which simplifies to

$$
\frac{y}{1+t} = 2t^2 + c_1
$$

Which gives

$$
y = \left(2t^2 + c_1\right)(1+t)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{4t^3+8t^2+4t+y}{1+t}$ 	$R = t$ $S = \frac{y}{1+t}$	$\frac{dS}{dR} = 4R$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 10$ in the above solution gives an equation to solve for the constant of integration.

$$10 = 2c_1 + 4$$

$$c_1 = 3$$

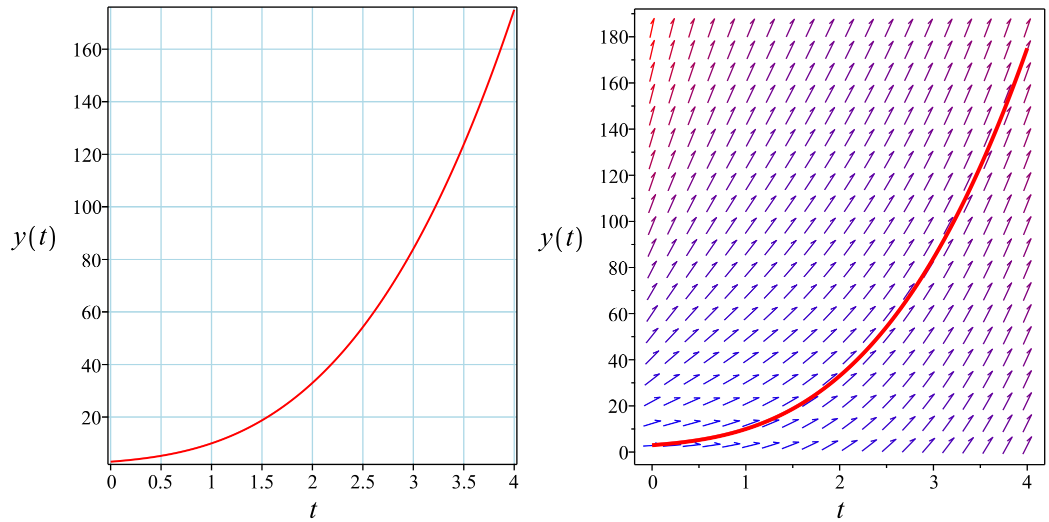
Substituting c_1 found above in the general solution gives

$$y = (2t^2 + 3)(1 + t)$$

Summary

The solution(s) found are the following

$$y = (2t^2 + 3)(1 + t) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = (2t^2 + 3)(1 + t)$$

Verified OK.

7.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{y}{1+t} + 4t^2 + 4t\right)\mathrm{d}t$$

$$\left(-\frac{y}{1+t} - 4t^2 - 4t\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, y) &= -\frac{y}{1+t} - 4t^2 - 4t \\ N(t, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{1+t} - 4t^2 - 4t\right) \\ &= -\frac{1}{1+t} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1\left(\left(-\frac{1}{1+t}\right) - (0)\right) \\ &= -\frac{1}{1+t} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int -\frac{1}{1+t} \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(1+t)} \\ &= \frac{1}{1+t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{1+t}\left(-\frac{y}{1+t} - 4t^2 - 4t\right) \\ &= \frac{-4t^3 - 8t^2 - 4t - y}{(1+t)^2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{1+t}(1) \\ &= \frac{1}{1+t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\frac{-4t^3 - 8t^2 - 4t - y}{(1+t)^2}\right) + \left(\frac{1}{1+t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \frac{-4t^3 - 8t^2 - 4t - y}{(1+t)^2} \, \mathrm{d}t$$

$$\phi = -2t^2 + \frac{y}{1+t} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{1+t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1+t}$. Therefore equation (4) becomes

$$\frac{1}{1+t} = \frac{1}{1+t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -2t^2 + \frac{y}{1+t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2t^2 + \frac{y}{1+t}$$

The solution becomes

$$y = (2t^2 + c_1)(1 + t)$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 10$ in the above solution gives an equation to solve for the constant of integration.

$$10 = 2c_1 + 4$$

$$c_1 = 3$$

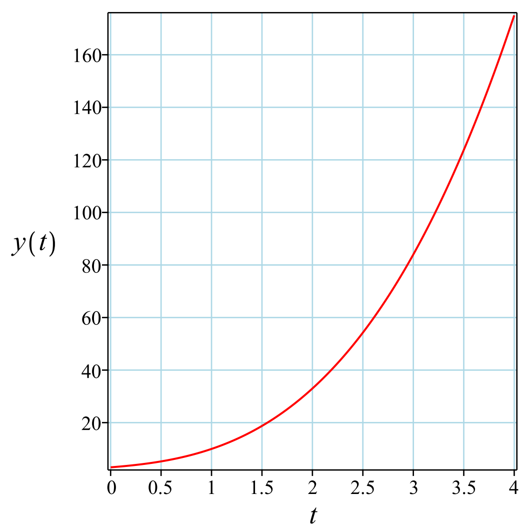
Substituting c_1 found above in the general solution gives

$$y = (2t^2 + 3)(1 + t)$$

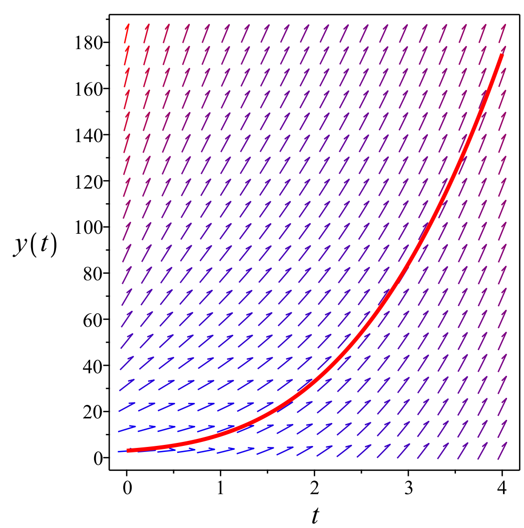
Summary

The solution(s) found are the following

$$y = (2t^2 + 3)(1 + t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (2t^2 + 3)(1 + t)$$

Verified OK.

7.8.5 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{1+t} = 4t^2 + 4t, y(1) = 10\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{1+t} + 4t^2 + 4t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{1+t} = 4t^2 + 4t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{y}{1+t}\right) = \mu(t)\left(4t^2 + 4t\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - \frac{y}{1+t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{\mu(t)}{1+t}$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{1+t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\left(4t^2 + 4t\right) dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\left(4t^2 + 4t\right) dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t)(4t^2+4t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{1+t}$

  $y = (1+t)\left(\int \frac{4t^2+4t}{1+t} dt + c_1\right)$

- Evaluate the integrals on the rhs

  $y = (2t^2 + c_1)(1+t)$

- Use initial condition $y(1) = 10$

$$10 = 2c_1 + 4$$

- Solve for c_1

$$c_1 = 3$$

- Substitute $c_1 = 3$ into general solution and simplify

$$y = 2t^3 + 2t^2 + 3t + 3$$

- Solution to the IVP

$$y = 2t^3 + 2t^2 + 3t + 3$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(t),t)=y(t)/(1+t)+4*t^2+4*t,y(1) = 10],y(t), singsol=all)
```

$$y(t) = 2t^3 + 2t^2 + 3t + 3$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 20

```
DSolve[{y'[t]==y[t]/(1+t)+4*t^2+4*t,{y[1]==10}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2t^3 + 2t^2 + 3t + 3$$

7.9 problem 9

Internal problem ID [13014]
Internal file name [`OUTPUT/11666_Wednesday_November_08_2023_03_28_18_AM_91064150/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{t} = 2}$$

With initial conditions

$$[y(1) = 3]$$

7.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= 2 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{t} = 2$$

The domain of $p(t) = \frac{1}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

7.9.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{t} dt} \\ &= t \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu y) &= (\mu)(2) \\ \frac{d}{dt}(ty) &= (t)(2) \\ d(ty) &= (2t)\, dt \end{aligned}$$

Integrating gives

$$\begin{aligned} ty &= \int 2t \, dt \\ ty &= t^2 + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$y = t + \frac{c_1}{t}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 1 + c_1$$

$$c_1 = 2$$

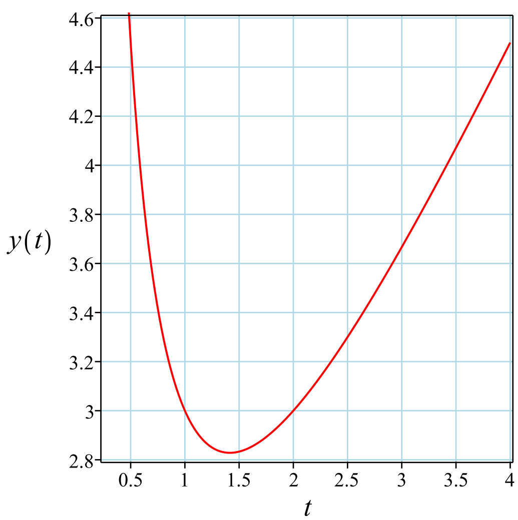
Substituting c_1 found above in the general solution gives

$$y = \frac{t^2 + 2}{t}$$

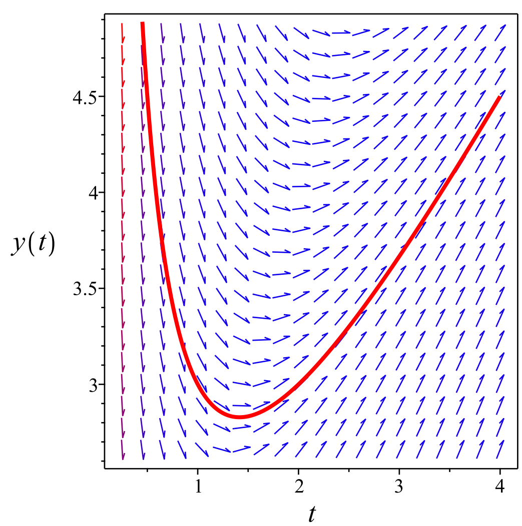
Summary

The solution(s) found are the following

$$y = \frac{t^2 + 2}{t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{t^2 + 2}{t}$$

Verified OK.

7.9.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + 2u(t) = 2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(t,u)\\ &= f(t)g(u)\\ &= \frac{-2u+2}{t}\end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(u) = -2u + 2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-2u+2}\,du &= \frac{1}{t}\,dt\\ \int \frac{1}{-2u+2}\,du &= \int \frac{1}{t}\,dt\\ -\frac{\ln(u-1)}{2} &= \ln(t) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{u-1}} = \mathrm{e}^{\ln(t)+c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{u-1}} = c_3 t$$

Therefore the solution y is

$$\begin{aligned}y &= tu\\ &= \frac{\left(c_3^2\mathrm{e}^{2c_2}t^2 + 1\right)\mathrm{e}^{-2c_2}}{t\,c_3^2}\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{\mathrm{e}^{-2c_2}\mathrm{e}^{2c_2}c_3^2 + \mathrm{e}^{-2c_2}}{c_3^2}$$

$$c_2 = -\frac{\ln(2c_3^2)}{2}$$

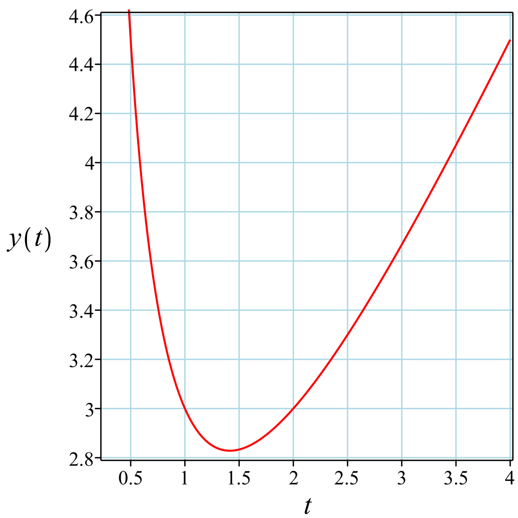
Substituting c_2 found above in the general solution gives

$$y = \frac{t^2 + 2}{t}$$

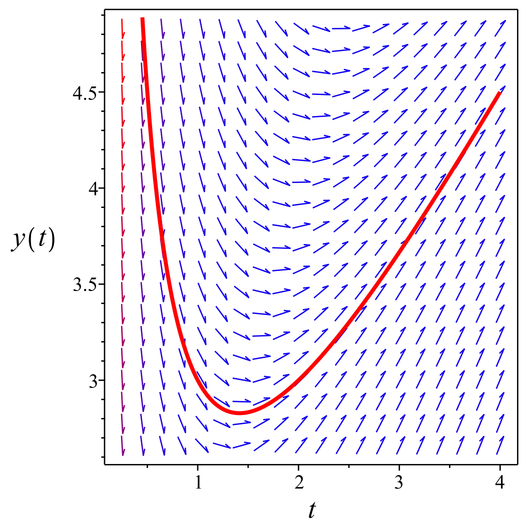
Summary

The solution(s) found are the following

$$y = \frac{t^2 + 2}{t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{t^2 + 2}{t}$$

Verified OK.

7.9.4 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{y}{t} + 2 \tag{1}$$

Which becomes

$$0 = (-t)\, dy + (2t - y)\, dt \tag{2}$$

But the RHS is complete differential because

$$(-t)\, dy + (2t - y)\, dt = d\left(t^2 - ty\right)$$

Hence (2) becomes

$$0 = d\left(t^2 - ty\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{t^2 + c_1}{t} + c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 2c_1 + 1$$

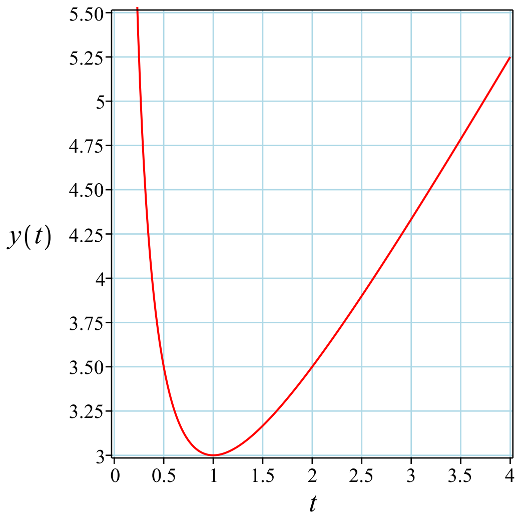
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

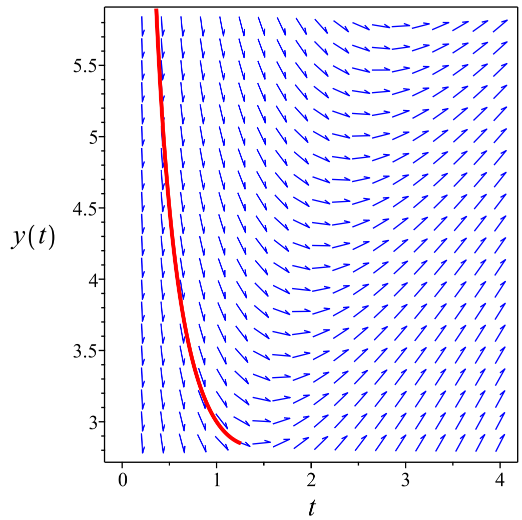
$$y = \frac{t^2 + t + 1}{t}$$

Summary
The solution(s) found are the following

$$y = \frac{t^2 + t + 1}{t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{t^2 + t + 1}{t}$$

Verified OK.

7.9.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-2t + y}{t} \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 248: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \frac{1}{t}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{t}} dy \end{aligned}$$

Which results in

$$S = ty$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\frac{-2t + y}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= y \\ S_y &= t \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$ty = t^2 + c_1$$

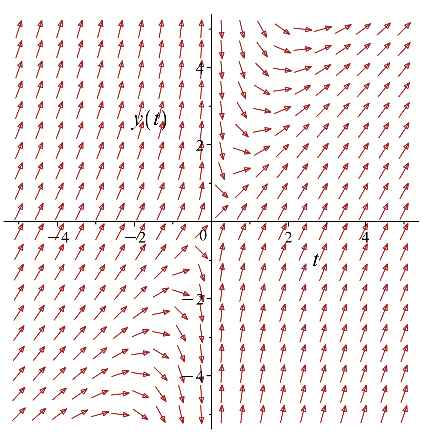
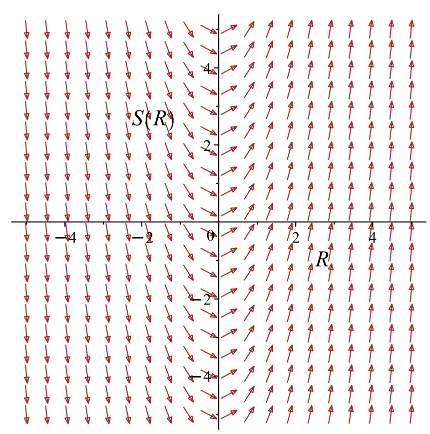
Which simplifies to

$$ty = t^2 + c_1$$

Which gives

$$y = \frac{t^2 + c_1}{t}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{-2t+y}{t}$ 	$R = t$ $S = ty$	$\frac{dS}{dR} = 2R$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 1 + c_1$$

$$c_1 = 2$$

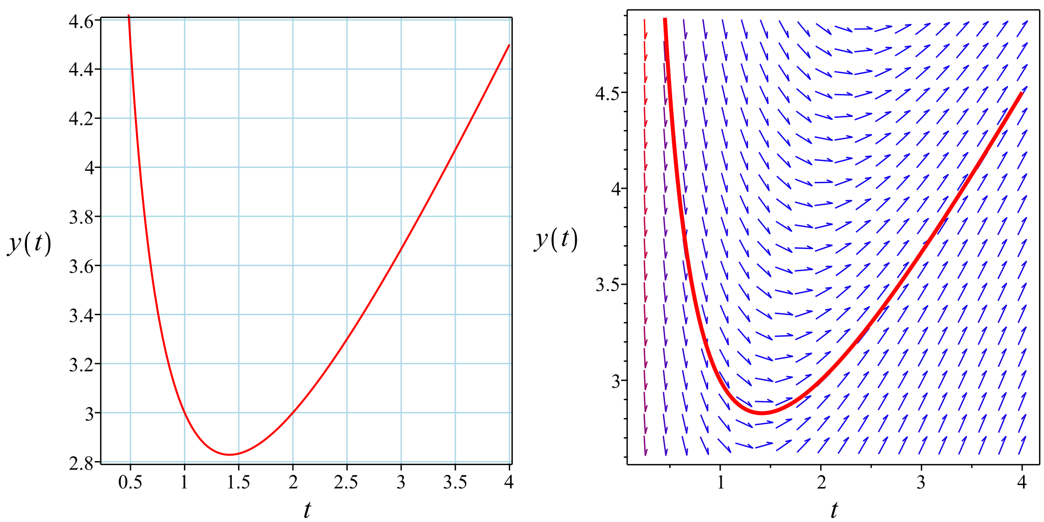
Substituting c_1 found above in the general solution gives

$$y = \frac{t^2 + 2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{t^2 + 2}{t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{t^2 + 2}{t}$$

Verified OK.

7.9.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(t)\,\mathrm{d}y &= (2t-y)\,\mathrm{d}t\\ (-2t+y)\,\mathrm{d}t + (t)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -2t+y\\ N(t,y) &= t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2t + y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -2t + y \,\mathrm{d}t \\ \phi &= -t(t - y) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = t + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = t$. Therefore equation (4) becomes

$$t = t + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -t(t - y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t(t - y)$$

The solution becomes

$$y = \frac{t^2 + c_1}{t}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 1 + c_1$$

$$c_1 = 2$$

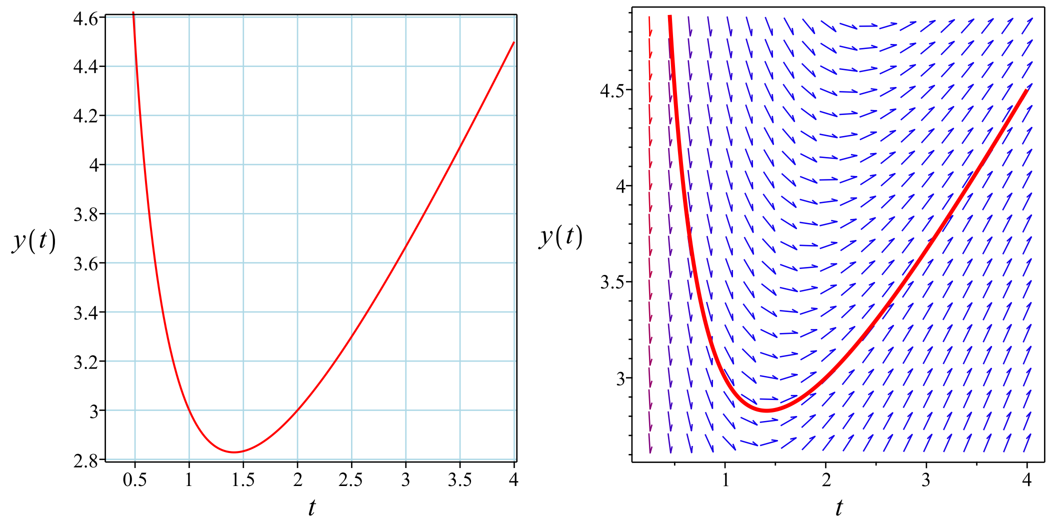
Substituting c_1 found above in the general solution gives

$$y = \frac{t^2 + 2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{t^2 + 2}{t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{t^2+2}{t}$$

Verified OK.

7.9.7 Maple step by step solution

Let's solve

$\left[y' + \frac{y}{t} = 2, y(1) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{t} + 2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{t} = 2$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{y}{t}\right) = 2\mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = t$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 2\mu(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 2\mu(t)\, dt + c_1$

- Solve for y

  $y = \frac{\int 2\mu(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = t$

  $y = \frac{\int 2tdt + c_1}{t}$

- Evaluate the integrals on the rhs

  $y = \frac{t^2 + c_1}{t}$

- Use initial condition $y(1) = 3$

  $3 = 1 + c_1$

- Solve for c_1

  $c_1 = 2$

- Substitute $c_1 = 2$ into general solution and simplify

  $y = \frac{t^2 + 2}{t}$

- Solution to the IVP

  $y = \frac{t^2 + 2}{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([diff(y(t),t)=-y(t)/t+2,y(1) = 3],y(t), singsol=all)
```

$$y(t) = t + \frac{2}{t}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 12

```
DSolve[{y'[t]==-y[t]/t+2,{y[1]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t + \frac{2}{t}$$

7.10 problem 10

Internal problem ID [13015]
Internal file name [`OUTPUT/11667_Wednesday_November_08_2023_03_28_19_AM_11615872/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + 2ty = 4\,\mathrm{e}^{-t^2}}$$

With initial conditions

$$[y(0) = 3]$$

7.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 2t$$
$$q(t) = 4\,\mathrm{e}^{-t^2}$$

Hence the ode is

$$y' + 2ty = 4\,\mathrm{e}^{-t^2}$$

The domain of $p(t) = 2t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4\,\mathrm{e}^{-t^2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.10.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 2t dt}\\ &= \mathrm{e}^{t^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(4\,\mathrm{e}^{-t^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{t^2} y\right) &= \left(\mathrm{e}^{t^2}\right)\left(4\,\mathrm{e}^{-t^2}\right)\\ \mathrm{d}\left(\mathrm{e}^{t^2} y\right) &= 4\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{t^2} y &= \int 4\,\mathrm{d}t\\ \mathrm{e}^{t^2} y &= 4t + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{t^2}$ results in

$$y = 4\,\mathrm{e}^{-t^2} t + c_1 \mathrm{e}^{-t^2}$$

which simplifies to

$$y = \mathrm{e}^{-t^2}(4t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

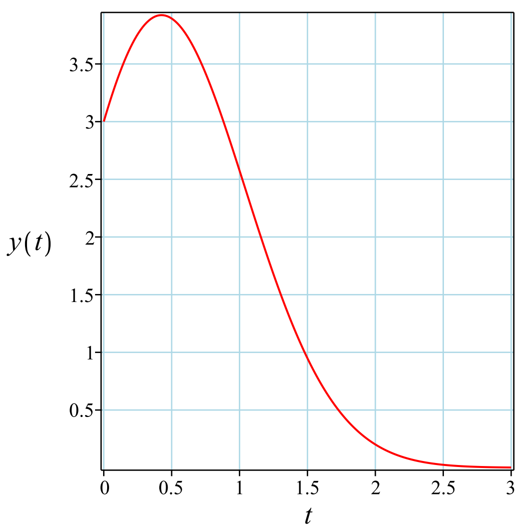
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

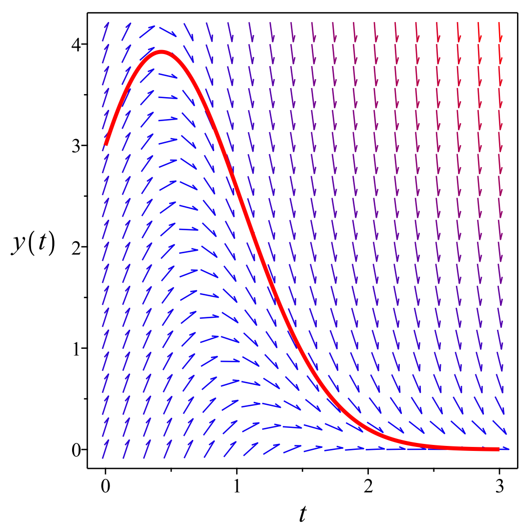
$$y = \mathrm{e}^{-t^2}(3 + 4t)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{-t^2}(3 + 4t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{-t^2}(3 + 4t)$$

Verified OK.

7.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -2ty + 4\,\mathrm{e}^{-t^2} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 251: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t,y) &= 0\\ \eta(t,y) &= \mathrm{e}^{-t^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right)S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\mathrm{e}^{-t^2}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{t^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -2ty + 4\,\mathrm{e}^{-t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1\\ R_y &= 0\\ S_t &= 2t\,\mathrm{e}^{t^2} y\\ S_y &= \mathrm{e}^{t^2}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 4R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$y\,\mathrm{e}^{t^2} = 4t + c_1$$

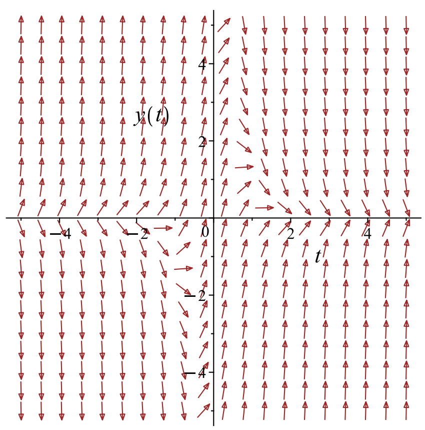
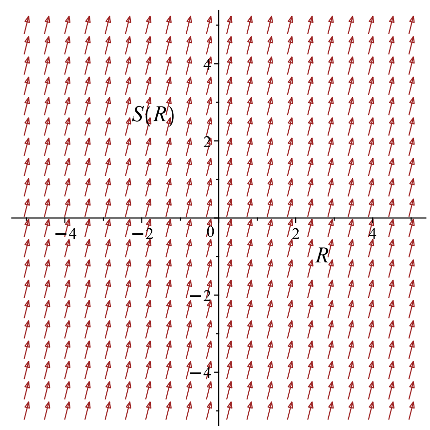
Which simplifies to

$$y\,\mathrm{e}^{t^2} = 4t + c_1$$

Which gives

$$y = \mathrm{e}^{-t^2}(4t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -2ty + 4\,\mathrm{e}^{-t^2}$ 	$R = t$ $S = \mathrm{e}^{t^2} y$	$\frac{dS}{dR} = 4$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

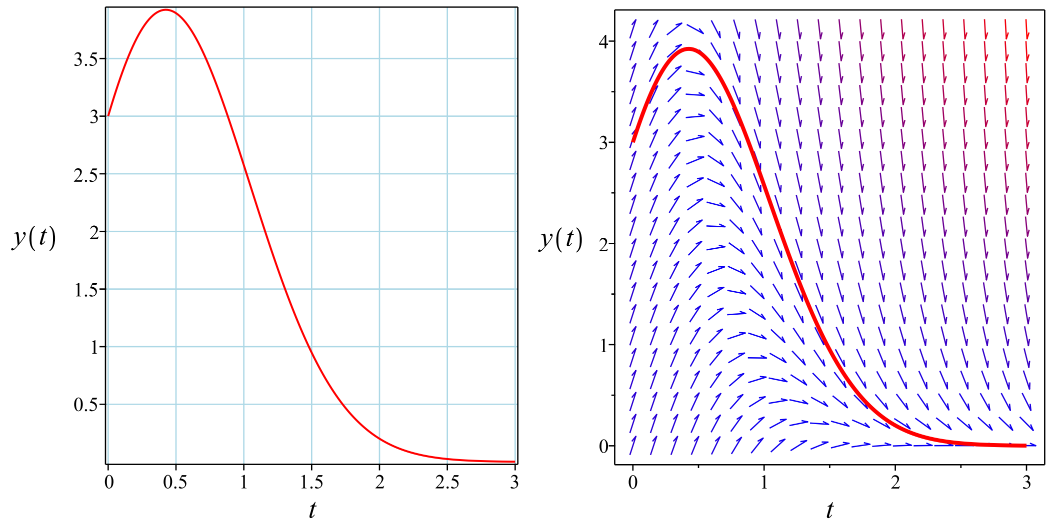
Substituting c_1 found above in the general solution gives

$$y = 4\,\mathrm{e}^{-t^2} t + 3\,\mathrm{e}^{-t^2}$$

<u>Summary</u>

The solution(s) found are the following

$$y = 4\,\mathrm{e}^{-t^2} t + 3\,\mathrm{e}^{-t^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 4\,\mathrm{e}^{-t^2}t + 3\,\mathrm{e}^{-t^2}$$

Verified OK.

7.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-2ty + 4\,\mathrm{e}^{-t^2}\right)\mathrm{d}t\\ \left(2ty - 4\,\mathrm{e}^{-t^2}\right)\mathrm{d}t + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= 2ty - 4\,\mathrm{e}^{-t^2}\\ N(t,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2ty - 4\,\mathrm{e}^{-t^2}\right)\\ &= 2t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((2t) - (0))\\ &= 2t\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int 2t\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{t^2}\\ &= \mathrm{e}^{t^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{t^2}\left(2ty - 4\,\mathrm{e}^{-t^2}\right)\\ &= 2t\,\mathrm{e}^{t^2}y - 4\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{t^2}(1)\\ &= \mathrm{e}^{t^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \left(2t\,\mathrm{e}^{t^2}y - 4\right) + \left(\mathrm{e}^{t^2}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int 2t\,\mathrm{e}^{t^2}y - 4\,\mathrm{d}t\end{aligned}$$

$$\phi = -4t + \mathrm{e}^{t^2}y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{t^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{t^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{t^2} = \mathrm{e}^{t^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -4t + \mathrm{e}^{t^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -4t + \mathrm{e}^{t^2} y$$

The solution becomes

$$y = \mathrm{e}^{-t^2}(4t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

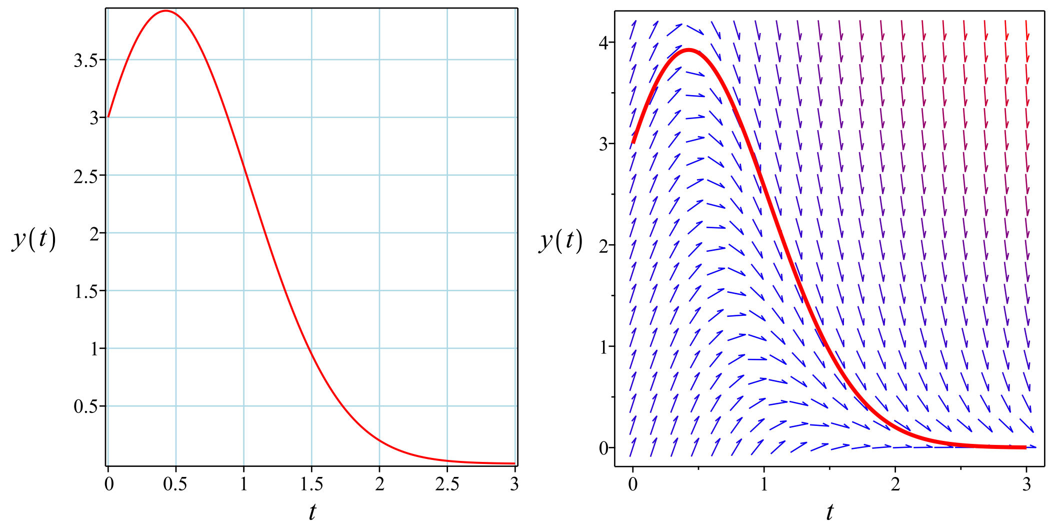
Substituting c_1 found above in the general solution gives

$$y = 4\,\mathrm{e}^{-t^2} t + 3\,\mathrm{e}^{-t^2}$$

Summary

The solution(s) found are the following

$$y = 4\,e^{-t^2}t + 3\,e^{-t^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 4\,e^{-t^2}t + 3\,e^{-t^2}$$

Verified OK.

7.10.5 Maple step by step solution

Let's solve

$\left[y' + 2ty = 4\,e^{-t^2}, y(0) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2ty + 4\,e^{-t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2ty = 4\,e^{-t^2}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y'+2ty\right)=4\mu(t)\,\mathrm{e}^{-t^2}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y'+2ty\right)=\mu'(t)\,y+\mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t)=2\mu(t)\,t$
- Solve to find the integrating factor

  $\mu(t)=\mathrm{e}^{t^2}$
- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int 4\mu(t)\,\mathrm{e}^{-t^2}dt+c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y=\int 4\mu(t)\,\mathrm{e}^{-t^2}dt+c_1$
- Solve for y

  $y=\frac{\int 4\mu(t)\mathrm{e}^{-t^2}dt+c_1}{\mu(t)}$
- Substitute $\mu(t)=\mathrm{e}^{t^2}$

  $y=\frac{\int 4\,\mathrm{e}^{-t^2}\mathrm{e}^{t^2}dt+c_1}{\mathrm{e}^{t^2}}$
- Evaluate the integrals on the rhs

  $y=\frac{4t+c_1}{\mathrm{e}^{t^2}}$
- Simplify

  $y=\mathrm{e}^{-t^2}(4t+c_1)$
- Use initial condition $y(0)=3$

  $3=c_1$
- Solve for c_1

  $c_1=3$
- Substitute $c_1=3$ into general solution and simplify

  $y=\mathrm{e}^{-t^2}(3+4t)$
- Solution to the IVP

  $y=\mathrm{e}^{-t^2}(3+4t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(t),t)=-2*t*y(t)+4*exp(-t^2),y(0) = 3],y(t), singsol=all)
```

$$y(t) = (4t + 3)\,e^{-t^2}$$

✓ Solution by Mathematica
Time used: 0.09 (sec). Leaf size: 18

```
DSolve[{y'[t]==-2*t*y[t]+4*Exp[-t^2],{y[0]==3}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t^2}(4t + 3)$$

7.11 problem 11

Internal problem ID [13016]
Internal file name [`OUTPUT/11668_Wednesday_November_08_2023_03_28_20_AM_39873401/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{2y}{t} = 2t^2}$$

With initial conditions

$$[y(-2) = 4]$$

7.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{2}{t}$$
$$q(t) = 2t^2$$

Hence the ode is

$$y' - \frac{2y}{t} = 2t^2$$

The domain of $p(t) = -\frac{2}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = -2$ is inside this domain. The domain of $q(t) = 2t^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = -2$ is also inside this domain. Hence solution exists and is unique.

7.11.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{2}{t}dt} \\
&= \frac{1}{t^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(2t^2\right) \\
\frac{d}{dt}\left(\frac{y}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(2t^2\right) \\
d\left(\frac{y}{t^2}\right) &= 2\,dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{t^2} &= \int 2\,dt \\
\frac{y}{t^2} &= 2t + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$y = c_1 t^2 + 2t^3$$

which simplifies to

$$y = t^2(2t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = -2$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = -16 + 4c_1$$

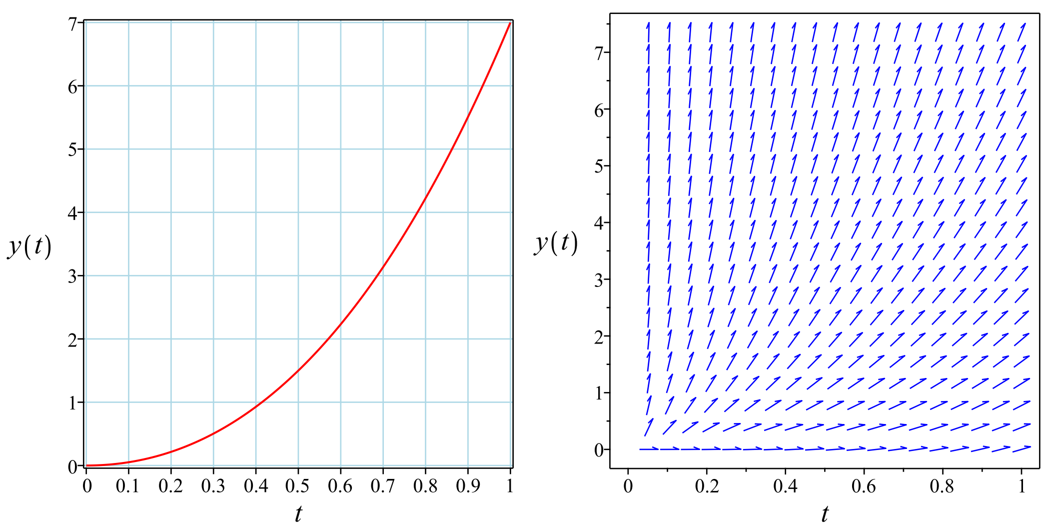
$$c_1 = 5$$

Substituting c_1 found above in the general solution gives

$$y = t^2(2t + 5)$$

Summary
The solution(s) found are the following

$$y = t^2(2t + 5) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = t^2(2t + 5)$$

Verified OK.

7.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2t^3 + 2y}{t} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 254: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,y) = 0$$
$$\eta(t,y) = t^2 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{t^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{2t^3 + 2y}{t}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= -\frac{2y}{t^3} \\
S_y &= \frac{1}{t^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = 2R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$
\frac{y}{t^2} = 2t + c_1
$$

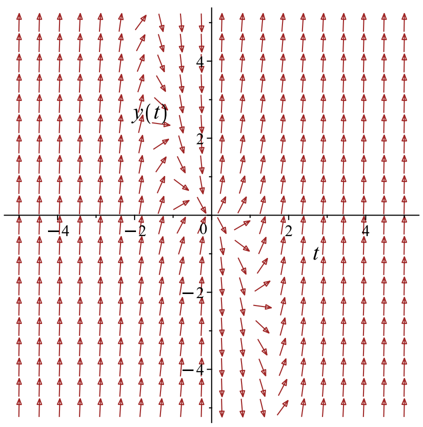
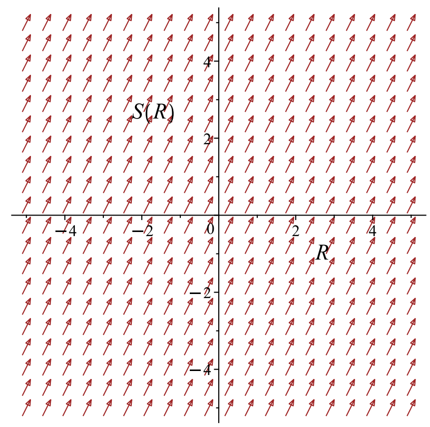
Which simplifies to

$$
\frac{y}{t^2} = 2t + c_1
$$

Which gives

$$
y = t^2(2t + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{2t^3+2y}{t}$ 	$R = t$ $S = \frac{y}{t^2}$	$\frac{dS}{dR} = 2$ 

Initial conditions are used to solve for c_1. Substituting $t = -2$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = -16 + 4c_1$$

$$c_1 = 5$$

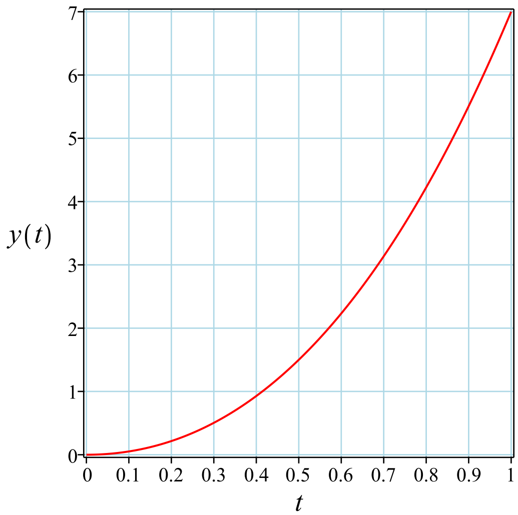
Substituting c_1 found above in the general solution gives

$$y = t^2(2t + 5)$$

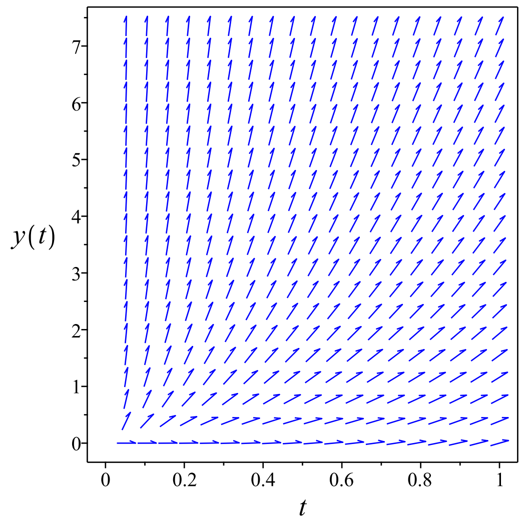
Summary

The solution(s) found are the following

$$y = t^2(2t + 5) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = t^2(2t + 5)$$

Verified OK.

7.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{2y}{t} + 2t^2\right) \mathrm{d}t$$

$$\left(-\frac{2y}{t} - 2t^2\right) \mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, y) &= -\frac{2y}{t} - 2t^2 \\ N(t, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2y}{t} - 2t^2\right) \\ &= -\frac{2}{t} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(-\frac{2}{t}\right) - (0)\right) \\
&= -\frac{2}{t}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{2}{t} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{t^2}\left(-\frac{2y}{t} - 2t^2\right) \\
&= \frac{-2t^3 - 2y}{t^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{t^2}(1) \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\frac{-2t^3 - 2y}{t^3}\right) + \left(\frac{1}{t^2}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \frac{-2t^3 - 2y}{t^3} \, \mathrm{d}t$$

$$\phi = -2t + \frac{y}{t^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{t^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{t^2}$. Therefore equation (4) becomes

$$\frac{1}{t^2} = \frac{1}{t^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -2t + \frac{y}{t^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2t + \frac{y}{t^2}$$

The solution becomes

$$y = t^2(2t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = -2$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = -16 + 4c_1$$

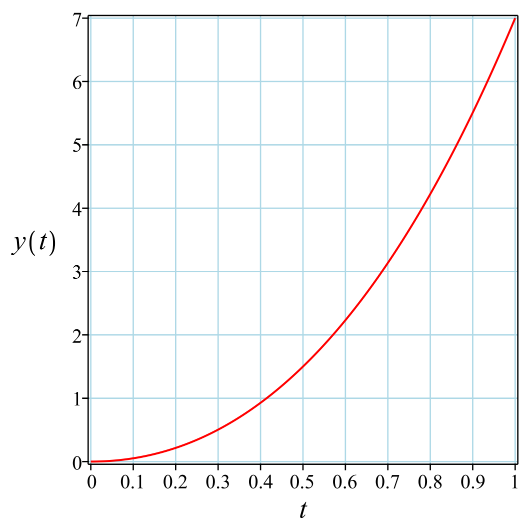
$$c_1 = 5$$

Substituting c_1 found above in the general solution gives

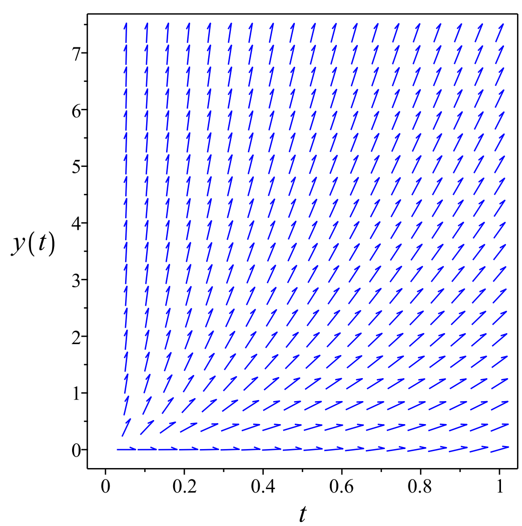
$$y = t^2(2t + 5)$$

Summary
The solution(s) found are the following

$$y = t^2(2t + 5) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = t^2(2t + 5)$$

Verified OK.

7.11.5 Maple step by step solution

Let's solve

$\left[y' - \frac{2y}{t} = 2t^2, y(-2) = 4\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{t} + 2t^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{t} = 2t^2$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{2y}{t}\right) = 2\mu(t)\, t^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - \frac{2y}{t}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{2\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{t^2}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 2\mu(t)\, t^2 dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 2\mu(t)\, t^2 dt + c_1$

- Solve for y

  $y = \frac{\int 2\mu(t) t^2 dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{t^2}$

  $y = t^2\left(\int 2dt + c_1\right)$

- Evaluate the integrals on the rhs

  $y = t^2(2t + c_1)$

- Use initial condition $y(-2) = 4$

  $4 = -16 + 4c_1$

- Solve for c_1

  $c_1 = 5$
- Substitute $c_1 = 5$ into general solution and simplify

  $y = 2t^3 + 5t^2$
- Solution to the IVP

  $y = 2t^3 + 5t^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(y(t),t)-2*y(t)/t=2*t^2,y(-2) = 4],y(t), singsol=all)
```

$$y(t) = 2t^3 + 5t^2$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 14

```
DSolve[{y'[t]-2*y[t]/t==2*t^2,{y[-2]==4}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t^2(2t + 5)$$

7.12 problem 12

Internal problem ID [13017]
Internal file name [`OUTPUT/11669_Wednesday_November_08_2023_03_28_20_AM_47236562/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{3y}{t} = 2\,\mathrm{e}^{2t}t^3}$$

With initial conditions

$$[y(1) = 0]$$

7.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{3}{t}$$
$$q(t) = 2\,\mathrm{e}^{2t}t^3$$

Hence the ode is

$$y' - \frac{3y}{t} = 2\,\mathrm{e}^{2t} t^3$$

The domain of $p(t) = -\frac{3}{t}$ is

$$\{t < 0 \lor 0 < t\}$$

And the point $t_0 = 1$ is inside this domain. The domain of $q(t) = 2\,\mathrm{e}^{2t} t^3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is also inside this domain. Hence solution exists and is unique.

7.12.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{3}{t}dt} \\
&= \frac{1}{t^3}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(2\,\mathrm{e}^{2t} t^3\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{t^3}\right) &= \left(\frac{1}{t^3}\right)\left(2\,\mathrm{e}^{2t} t^3\right) \\
\mathrm{d}\left(\frac{y}{t^3}\right) &= \left(2\,\mathrm{e}^{2t}\right)\,\mathrm{d}t
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{t^3} &= \int 2\,\mathrm{e}^{2t}\,\mathrm{d}t \\
\frac{y}{t^3} &= \mathrm{e}^{2t} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^3}$ results in

$$y = \mathrm{e}^{2t} t^3 + t^3 c_1$$

which simplifies to

$$y = t^3\left(\mathrm{e}^{2t} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \mathrm{e}^{2} + c_1$$

$$c_1 = -\mathrm{e}^{2}$$

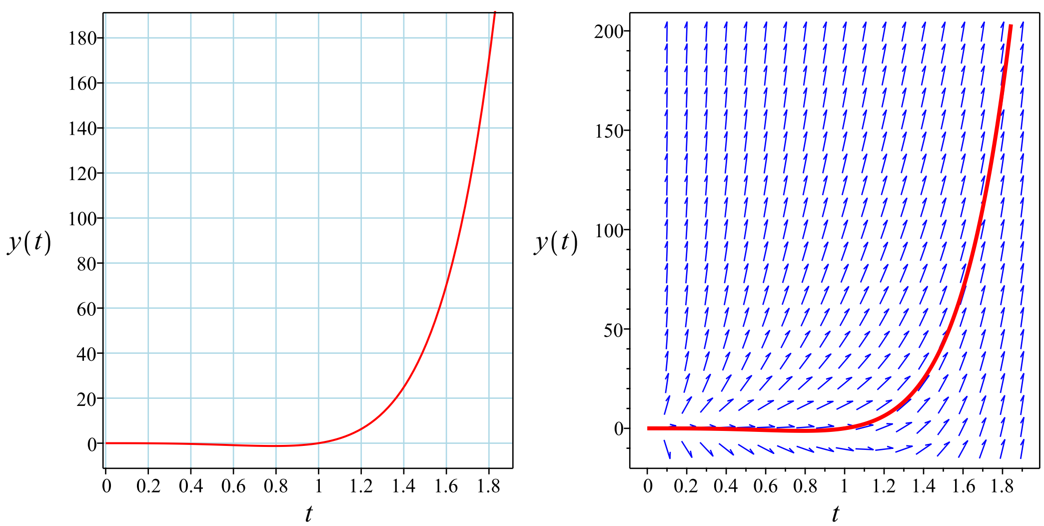
Substituting c_1 found above in the general solution gives

$$y = t^{3}\left(\mathrm{e}^{2t} - \mathrm{e}^{2}\right)$$

Summary

The solution(s) found are the following

$$y = t^{3}\left(\mathrm{e}^{2t} - \mathrm{e}^{2}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = t^{3}\left(\mathrm{e}^{2t} - \mathrm{e}^{2}\right)$$

Verified OK.

7.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2t^4 e^{2t} + 3y}{t} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 257: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,y) = 0$$
$$\eta(t,y) = t^3 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^3} dy \end{aligned}$$

Which results in

$$S = \frac{y}{t^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{2t^4 e^{2t} + 3y}{t}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= -\frac{3y}{t^4} \\
S_y &= \frac{1}{t^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\,\mathrm{e}^{2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\,\mathrm{e}^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{2R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{t^3} = \mathrm{e}^{2t} + c_1$$

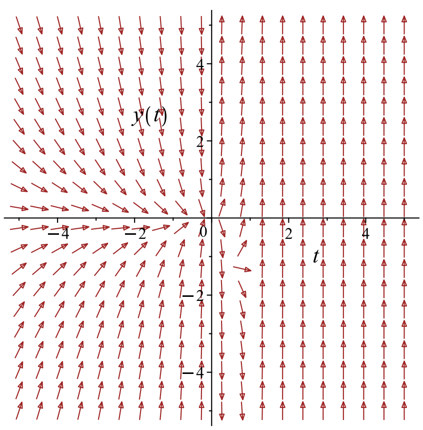
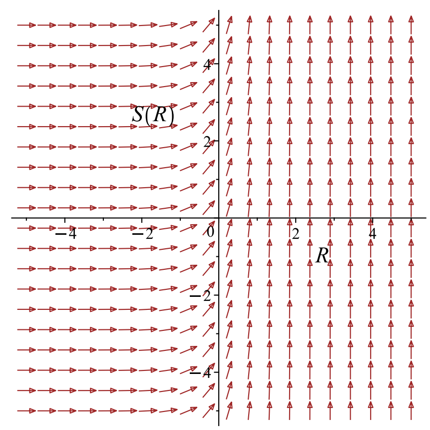
Which simplifies to

$$\frac{y}{t^3} = \mathrm{e}^{2t} + c_1$$

Which gives

$$y = t^3\left(\mathrm{e}^{2t} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{2t^4 e^{2t}+3y}{t}$ 	$R = t$ $S = \frac{y}{t^3}$	$\frac{dS}{dR} = 2\,e^{2R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e^2 + c_1$$

$$c_1 = -e^2$$

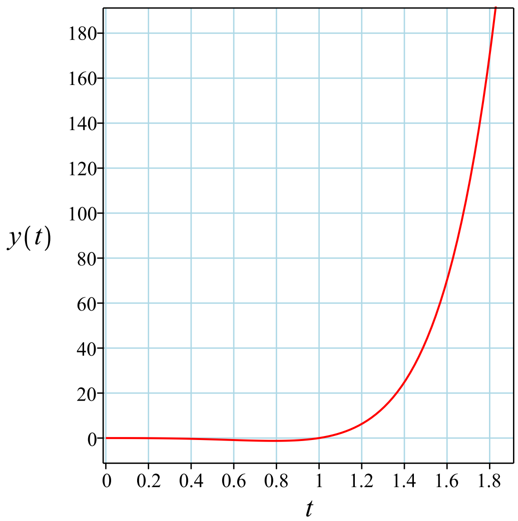
Substituting c_1 found above in the general solution gives

$$y = t^3 \left(e^{2t} - e^2\right)$$

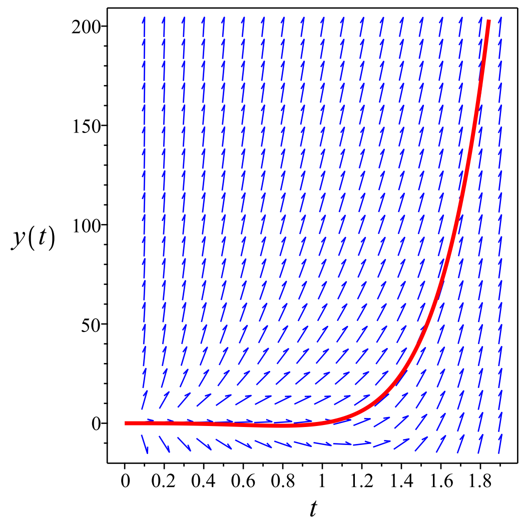
Summary

The solution(s) found are the following

$$y = t^3 \left(e^{2t} - e^2\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = t^3 \left(e^{2t} - e^2\right)$$

Verified OK.

7.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\frac{3y}{t}+2\,\mathrm{e}^{2t}t^3\right)\mathrm{d}t$$

$$\left(-\frac{3y}{t}-2\,\mathrm{e}^{2t}t^3\right)\mathrm{d}t+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{3y}{t}-2\,\mathrm{e}^{2t}t^3\\N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{3y}{t}-2\,\mathrm{e}^{2t}t^3\right)\\&=-\frac{3}{t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t}&=\frac{\partial}{\partial t}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1\left(\left(-\frac{3}{t}\right) - (0)\right) \\ &= -\frac{3}{t} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int -\frac{3}{t} \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3\ln(t)} \\ &= \frac{1}{t^3} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{t^3}\left(-\frac{3y}{t} - 2\,\mathrm{e}^{2t} t^3\right) \\ &= \frac{-2t^4 \mathrm{e}^{2t} - 3y}{t^4} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{t^3}(1) \\ &= \frac{1}{t^3} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(\frac{-2t^4 \mathrm{e}^{2t} - 3y}{t^4}\right) + \left(\frac{1}{t^3}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \frac{-2t^4 \mathrm{e}^{2t} - 3y}{t^4} \, \mathrm{d}t$$

$$\phi = \frac{-\mathrm{e}^{2t} t^3 + y}{t^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{t^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{t^3}$. Therefore equation (4) becomes

$$\frac{1}{t^3} = \frac{1}{t^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-\mathrm{e}^{2t} t^3 + y}{t^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-\mathrm{e}^{2t} t^3 + y}{t^3}$$

The solution becomes

$$y = t^3(e^{2t} + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e^2 + c_1$$

$$c_1 = -e^2$$

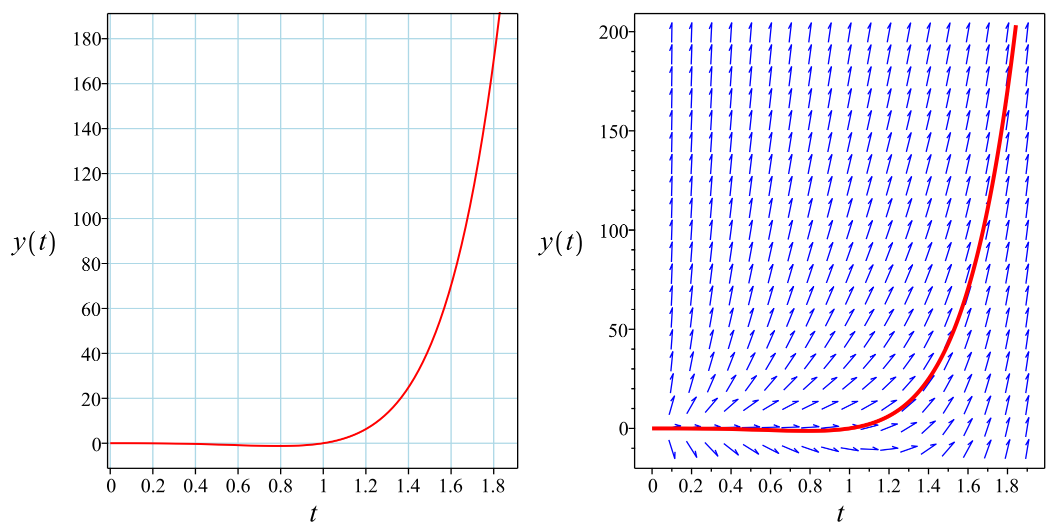
Substituting c_1 found above in the general solution gives

$$y = t^3(e^{2t} - e^2)$$

Summary

The solution(s) found are the following

$$y = t^3(e^{2t} - e^2) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = t^3(e^{2t} - e^2)$$

Verified OK.

7.12.5 Maple step by step solution

Let's solve

$\left[y' - \frac{3y}{t} = 2\,\mathrm{e}^{2t}t^3, y(1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{3y}{t} + 2\,\mathrm{e}^{2t}t^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{3y}{t} = 2\,\mathrm{e}^{2t}t^3$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{3y}{t}\right) = 2\mu(t)\,\mathrm{e}^{2t}t^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\left(y' - \frac{3y}{t}\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{3\mu(t)}{t}$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{t^3}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 2\mu(t)\,\mathrm{e}^{2t}t^3 dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 2\mu(t)\,\mathrm{e}^{2t}t^3 dt + c_1$

- Solve for y

  $y = \frac{\int 2\mu(t)\mathrm{e}^{2t}t^3 dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{t^3}$

  $y = t^3\left(\int 2\,\mathrm{e}^{2t}dt + c_1\right)$

- Evaluate the integrals on the rhs

  $y = t^3(\mathrm{e}^{2t} + c_1)$

- Use initial condition $y(1) = 0$

  $0 = \mathrm{e}^2 + c_1$

- Solve for c_1

  $c_1 = -e^2$

- Substitute $c_1 = -e^2$ into general solution and simplify

  $y = t^3(e^{2t} - e^2)$

- Solution to the IVP

  $y = t^3(e^{2t} - e^2)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(t),t)-3/t*y(t)=2*t^3*exp(2*t),y(1) = 0],y(t), singsol=all)
```

$$y(t) = -\left(-e^{2t} + e^2\right) t^3$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 20

```
DSolve[{y'[t]-3/t*y[t]==2*t^3*Exp[2*t],{y[1]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \left(e^{2t} - e^2\right) t^3$$

7.13 problem 13

Internal problem ID [13018]
Internal file name [`OUTPUT/11670_Wednesday_November_08_2023_03_28_21_AM_66253612/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \sin(t)\, y = 4}$$

7.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\sin(t) \\ q(t) &= 4 \end{aligned}$$

Hence the ode is

$$y' - \sin(t)\, y = 4$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\sin(t) dt} \\
&= \mathrm{e}^{\cos(t)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)(4) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\cos(t)} y\right) &= \left(\mathrm{e}^{\cos(t)}\right)(4) \\
\mathrm{d}\left(\mathrm{e}^{\cos(t)} y\right) &= \left(4\,\mathrm{e}^{\cos(t)}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\cos(t)} y &= \int 4\,\mathrm{e}^{\cos(t)}\,\mathrm{d}t \\
\mathrm{e}^{\cos(t)} y &= \int 4\,\mathrm{e}^{\cos(t)} dt + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\cos(t)}$ results in

$$
y = \mathrm{e}^{-\cos(t)} \left(\int 4\,\mathrm{e}^{\cos(t)} dt\right) + c_1 \mathrm{e}^{-\cos(t)}
$$

which simplifies to

$$
y = \mathrm{e}^{-\cos(t)} \left(4\left(\int \mathrm{e}^{\cos(t)} dt\right) + c_1\right)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-\cos(t)} \left(4\left(\int \mathrm{e}^{\cos(t)} dt\right) + c_1\right) \tag{1}
$$

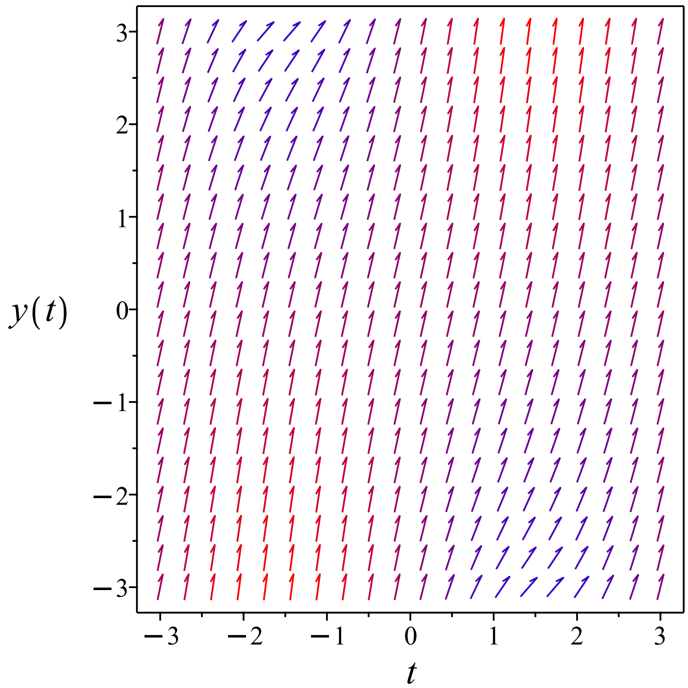


Figure 267: Slope field plot

Verification of solutions

$$y = e^{-\cos(t)} \left(4\left(\int e^{\cos(t)}dt\right) + c_1\right)$$

Verified OK.

7.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \sin(t) y + 4$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 260: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-\cos(t)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\cos(t)}} dy \end{aligned}$$

Which results in

$$S = e^{\cos(t)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \sin(t) y + 4$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\sin(t) e^{\cos(t)} y \\ S_y &= e^{\cos(t)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4 e^{\cos(t)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4 e^{\cos(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int 4\,\mathrm{e}^{\cos(R)} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{\cos(t)} y = \int 4\,\mathrm{e}^{\cos(t)} dt + c_1$$

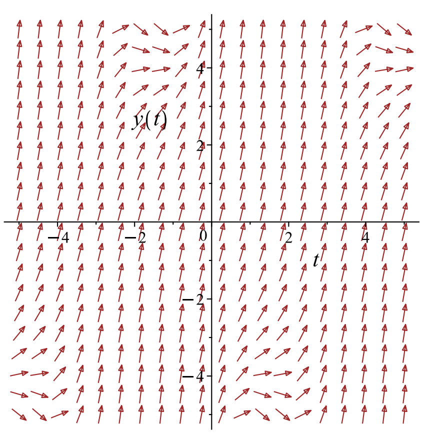
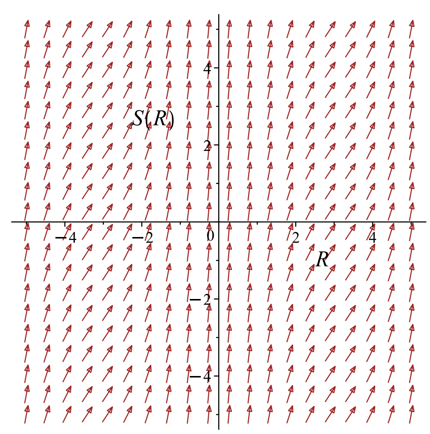
Which simplifies to

$$\mathrm{e}^{\cos(t)} y = \int 4\,\mathrm{e}^{\cos(t)} dt + c_1$$

Which gives

$$y = \left(\int 4\,\mathrm{e}^{\cos(t)} dt + c_1\right) \mathrm{e}^{-\cos(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \sin(t)\,y + 4$ 	$R = t$ $S = \mathrm{e}^{\cos(t)} y$	$\frac{dS}{dR} = 4\,\mathrm{e}^{\cos(R)}$ 

Summary

The solution(s) found are the following

$$y = \left(\int 4\,\mathrm{e}^{\cos(t)} dt + c_1\right) \mathrm{e}^{-\cos(t)} \tag{1}$$

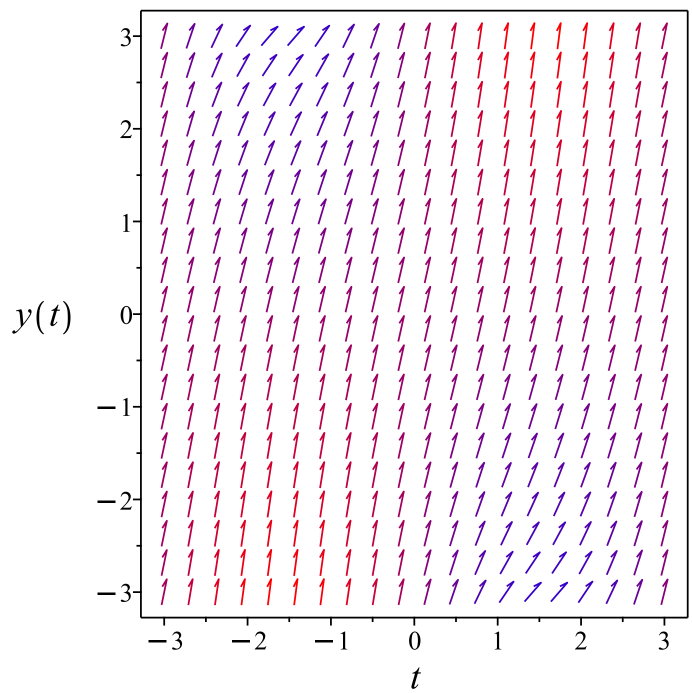


Figure 268: Slope field plot

Verification of solutions

$$y = \left(\int 4\,\mathrm{e}^{\cos(t)}dt + c_1\right)\mathrm{e}^{-\cos(t)}$$

Verified OK.

7.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= (\sin(t)\, y + 4)\,\mathrm{d}t \\
(-\sin(t)\, y - 4)\,\mathrm{d}t + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -\sin(t)\, y - 4 \\
N(t, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(t)\, y - 4) \\
&= -\sin(t)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((-\sin(t)) - (0)) \\
&= -\sin(t)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\sin(t) \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\cos(t)} \\
&= \mathrm{e}^{\cos(t)}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\cos(t)}(-\sin(t)\, y - 4) \\
&= -\mathrm{e}^{\cos(t)}(\sin(t)\, y + 4)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\cos(t)}(1) \\
&= \mathrm{e}^{\cos(t)}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{\cos(t)}(\sin(t)\, y + 4)\right) + \left(\mathrm{e}^{\cos(t)}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int \overline{M} \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \mathrm{d}t = \int -\mathrm{e}^{\cos(t)}(\sin(t) y + 4) \mathrm{d}t$$
$$\phi = \int^{t} -\mathrm{e}^{\cos(\_a)}(\sin(\_a) y + 4) d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\cos(t)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\cos(t)}$. Therefore equation (4) becomes

$$\mathrm{e}^{\cos(t)} = \mathrm{e}^{\cos(t)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{t} -\mathrm{e}^{\cos(\_a)}(\sin(\_a) y + 4) d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{t} -\mathrm{e}^{\cos(\_a)}(\sin(\_a) y + 4) d\_a$$

Summary

The solution(s) found are the following

$$\int^{t} -\mathrm{e}^{\cos(\_a)}(\sin(\_a) y + 4) d\_a = c_1 \tag{1}$$

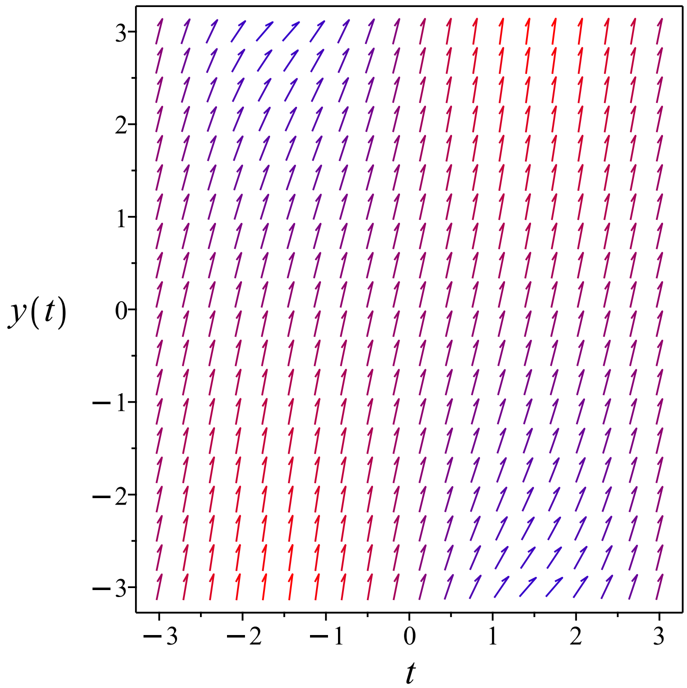


Figure 269: Slope field plot

Verification of solutions

$$\int^{t} -\mathrm{e}^{\cos(\_a)} (\sin(\_a)\, y + 4)\, d\_a = c_1$$

Verified OK.

7.13.4 Maple step by step solution

Let's solve

$y' - \sin(t)\, y = 4$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \sin(t)\, y + 4$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \sin(t)\, y = 4$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(y' - \sin(t)\, y\right) = 4\mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - \sin(t)\, y\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)\sin(t)$

- Solve to find the integrating factor

  $\mu(t) = e^{\cos(t)}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 4\mu(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 4\mu(t)\, dt + c_1$

- Solve for y

  $y = \frac{\int 4\mu(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{\cos(t)}$

  $y = \frac{\int 4\, e^{\cos(t)} dt + c_1}{e^{\cos(t)}}$

- Simplify

  $y = \left(4\left(\int e^{\cos(t)} dt\right) + c_1\right) e^{-\cos(t)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(t),t)=sin(t)*y(t)+4,y(t), singsol=all)
```

$$y(t) = \left(4\left(\int e^{\cos(t)} dt\right) + c_1\right) e^{-\cos(t)}$$

✓ Solution by Mathematica

Time used: 0.486 (sec). Leaf size: 29

```
DSolve[y'[t]==Sin[t]*y[t]+4,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-\cos(t)} \left(\int_1^t 4e^{\cos(K[1])} dK[1] + c_1 \right)$$

7.14 problem 14

Internal problem ID [13019]
Internal file name [`OUTPUT/11671_Wednesday_November_08_2023_03_28_22_AM_17103159/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - t^2 y = 4}$$

7.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -t^2$$
$$q(t) = 4$$

Hence the ode is

$$y' - t^2 y = 4$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -t^2 dt} \\
&= e^{-\frac{t^3}{3}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)(4) \\
\frac{d}{dt}\left(e^{-\frac{t^3}{3}} y\right) &= \left(e^{-\frac{t^3}{3}}\right)(4) \\
d\left(e^{-\frac{t^3}{3}} y\right) &= \left(4\,e^{-\frac{t^3}{3}}\right) dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-\frac{t^3}{3}} y &= \int 4\,e^{-\frac{t^3}{3}} dt \\
e^{-\frac{t^3}{3}} y &= \frac{4\,3^{\frac{1}{3}}\left(\frac{3t3^{\frac{5}{6}} e^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{t^3}{3}\right)}{4(t^3)^{\frac{1}{6}}} + \frac{3\,3^{\frac{5}{6}} e^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{7}{6},\frac{2}{3},\frac{t^3}{3}\right)}{t^2(t^3)^{\frac{1}{6}}}\right)}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{t^3}{3}}$ results in

$$
y = \frac{4\,e^{\frac{t^3}{3}} 3^{\frac{1}{3}}\left(\frac{3t3^{\frac{5}{6}} e^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{t^3}{3}\right)}{4(t^3)^{\frac{1}{6}}} + \frac{3\,3^{\frac{5}{6}} e^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{7}{6},\frac{2}{3},\frac{t^3}{3}\right)}{t^2(t^3)^{\frac{1}{6}}}\right)}{3} + c_1 e^{\frac{t^3}{3}}
$$

which simplifies to

$$
y = \frac{3\,3^{\frac{1}{6}} \operatorname{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{t^3}{3}\right) t\,e^{\frac{t^3}{6}}}{(t^3)^{\frac{1}{6}}} + c_1 e^{\frac{t^3}{3}} + 4t
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{3\,3^{\frac{1}{6}} \operatorname{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{t^3}{3}\right) t\,e^{\frac{t^3}{6}}}{(t^3)^{\frac{1}{6}}} + c_1 e^{\frac{t^3}{3}} + 4t \tag{1}
$$

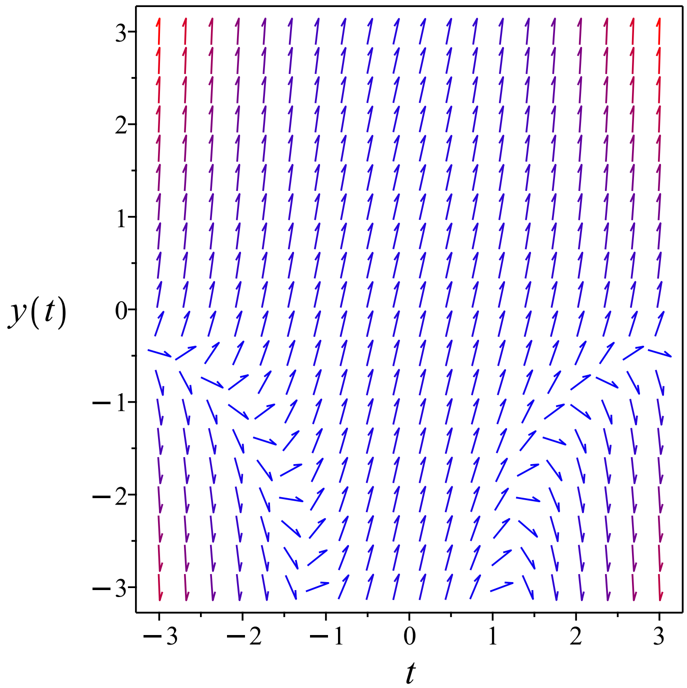


Figure 270: Slope field plot

Verification of solutions

$$y = \frac{3\,3^{\frac{1}{6}}\,\text{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t\,\mathrm{e}^{\frac{t^3}{6}}}{\left(t^3\right)^{\frac{1}{6}}} + c_1 \mathrm{e}^{\frac{t^3}{3}} + 4t$$

Verified OK.

7.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y\,t^2 + 4$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 263: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{\frac{t^3}{3}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{t^3}{3}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{t^3}{3}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = y\,t^2 + 4$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -t^2 e^{-\frac{t^3}{3}} y \\ S_y &= e^{-\frac{t^3}{3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4\, e^{-\frac{t^3}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4\, e^{-\frac{R^3}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R\,\mathrm{e}^{-\frac{R^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{R^3}{3}\right) 243^{\frac{5}{6}}}{27\left(R^3\right)^{\frac{1}{6}}} + \frac{4\,\mathrm{e}^{-\frac{R^3}{6}} \operatorname{WhittakerM}\left(\frac{7}{6}, \frac{2}{3}, \frac{R^3}{3}\right) 243^{\frac{5}{6}}}{27R^2\left(R^3\right)^{\frac{1}{6}}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-\frac{t^3}{3}} y = \frac{t\,\mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) 243^{\frac{5}{6}}}{27\left(t^3\right)^{\frac{1}{6}}} + \frac{4\,\mathrm{e}^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{7}{6}, \frac{2}{3}, \frac{t^3}{3}\right) 243^{\frac{5}{6}}}{27t^2\left(t^3\right)^{\frac{1}{6}}} + c_1$$

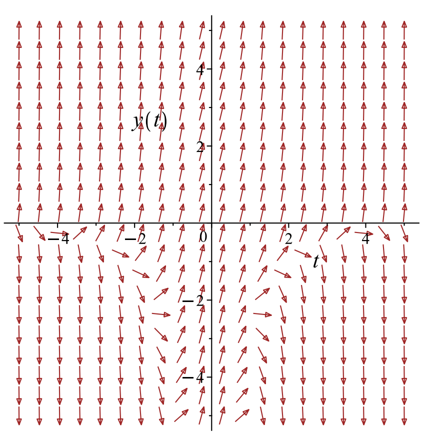
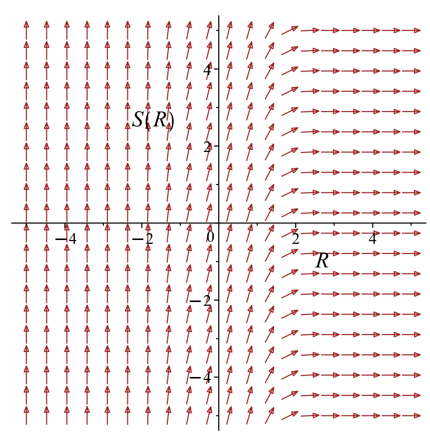
Which simplifies to

$$\left(-3\,3^{\frac{1}{6}}\sqrt{t}\,\operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right)\mathrm{e}^{\frac{t^3}{6}} - c_1\mathrm{e}^{\frac{t^3}{3}} - 4t + y\right)\mathrm{e}^{-\frac{t^3}{3}} = 0$$

Which gives

$$y = 3\,3^{\frac{1}{6}}\sqrt{t}\,\operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right)\mathrm{e}^{\frac{t^3}{6}} + c_1\mathrm{e}^{\frac{t^3}{3}} + 4t$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y\,t^2 + 4$ 	$R = t$ $S = \mathrm{e}^{-\frac{t^3}{3}} y$	$\frac{dS}{dR} = 4\,\mathrm{e}^{-\frac{R^3}{3}}$ 

Summary

The solution(s) found are the following

$$y = 3\,3^{\frac{1}{6}} \sqrt{t}\, \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^{3}}{3}\right) \mathrm{e}^{\frac{t^{3}}{6}} + c_{1} \mathrm{e}^{\frac{t^{3}}{3}} + 4t \tag{1}$$

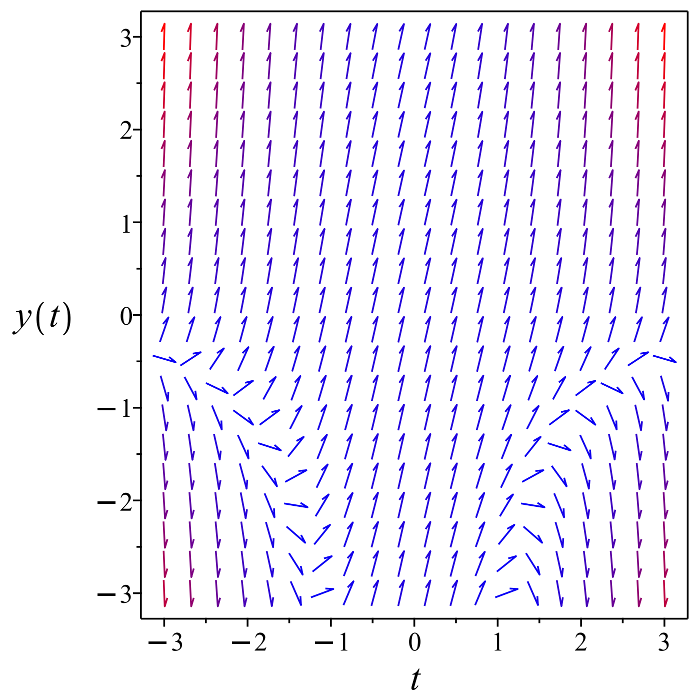


Figure 271: Slope field plot

Verification of solutions

$$y = 3\,3^{\frac{1}{6}} \sqrt{t}\, \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^{3}}{3}\right) \mathrm{e}^{\frac{t^{3}}{6}} + c_{1} \mathrm{e}^{\frac{t^{3}}{3}} + 4t$$

Verified OK.

7.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(y\,t^2+4\right)\mathrm{d}t\\ \left(-y\,t^2-4\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -y\,t^2-4\\ N(t,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y\,t^2 - 4\right) \\ &= -t^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1\left(\left(-t^2\right) - (0)\right) \\ &= -t^2\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int -t^2 \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{t^3}{3}} \\ &= \mathrm{e}^{-\frac{t^3}{3}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-\frac{t^3}{3}}\left(-y\,t^2 - 4\right) \\ &= -\mathrm{e}^{-\frac{t^3}{3}}\left(y\,t^2 + 4\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-\frac{t^3}{3}}(1) \\ &= \mathrm{e}^{-\frac{t^3}{3}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$
$$\left(-\mathrm{e}^{-\frac{t^3}{3}}\left(y\,t^2+4\right)\right) + \left(\mathrm{e}^{-\frac{t^3}{3}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int\overline{M}\,\mathrm{d}t$$
$$\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int -\mathrm{e}^{-\frac{t^3}{3}}\left(y\,t^2+4\right)\mathrm{d}t$$
$$\phi = \frac{-3\,3^{\frac{1}{6}}\mathrm{e}^{-\frac{t^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{6},\frac{2}{3},\frac{t^3}{3}\right)t - 4\,\mathrm{e}^{-\frac{t^3}{3}}\left(t^3\right)^{\frac{1}{6}}t + \mathrm{e}^{-\frac{t^3}{3}}y\left(t^3\right)^{\frac{1}{6}} - \left(t^3\right)^{\frac{1}{6}}y}{\left(t^3\right)^{\frac{1}{6}}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{\mathrm{e}^{-\frac{t^3}{3}}\left(t^3\right)^{\frac{1}{6}} - \left(t^3\right)^{\frac{1}{6}}}{\left(t^3\right)^{\frac{1}{6}}} + f'(y) \tag{4}$$
$$= -1 + \mathrm{e}^{-\frac{t^3}{3}} + f'(y)$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{-\frac{t^3}{3}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{t^3}{3}} = -1 + \mathrm{e}^{-\frac{t^3}{3}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-3\,3^{\frac{1}{6}}\mathrm{e}^{-\frac{t^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t - 4\,\mathrm{e}^{-\frac{t^3}{3}}\left(t^3\right)^{\frac{1}{6}} t + \mathrm{e}^{-\frac{t^3}{3}} y\left(t^3\right)^{\frac{1}{6}} - \left(t^3\right)^{\frac{1}{6}} y}{\left(t^3\right)^{\frac{1}{6}}} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-3\,3^{\frac{1}{6}}\mathrm{e}^{-\frac{t^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t - 4\,\mathrm{e}^{-\frac{t^3}{3}}\left(t^3\right)^{\frac{1}{6}} t + \mathrm{e}^{-\frac{t^3}{3}} y\left(t^3\right)^{\frac{1}{6}} - \left(t^3\right)^{\frac{1}{6}} y}{\left(t^3\right)^{\frac{1}{6}}} + y$$

The solution becomes

$$y = \frac{\mathrm{e}^{\frac{t^3}{3}}\left(3\,3^{\frac{1}{6}}\mathrm{e}^{-\frac{t^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t + 4\,\mathrm{e}^{-\frac{t^3}{3}}\left(t^3\right)^{\frac{1}{6}} t + c_1\left(t^3\right)^{\frac{1}{6}}\right)}{\left(t^3\right)^{\frac{1}{6}}}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{\frac{t^3}{3}}\left(3\,3^{\frac{1}{6}}\mathrm{e}^{-\frac{t^3}{6}}\,\mathrm{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t + 4\,\mathrm{e}^{-\frac{t^3}{3}}\left(t^3\right)^{\frac{1}{6}} t + c_1\left(t^3\right)^{\frac{1}{6}}\right)}{\left(t^3\right)^{\frac{1}{6}}} \tag{1}$$

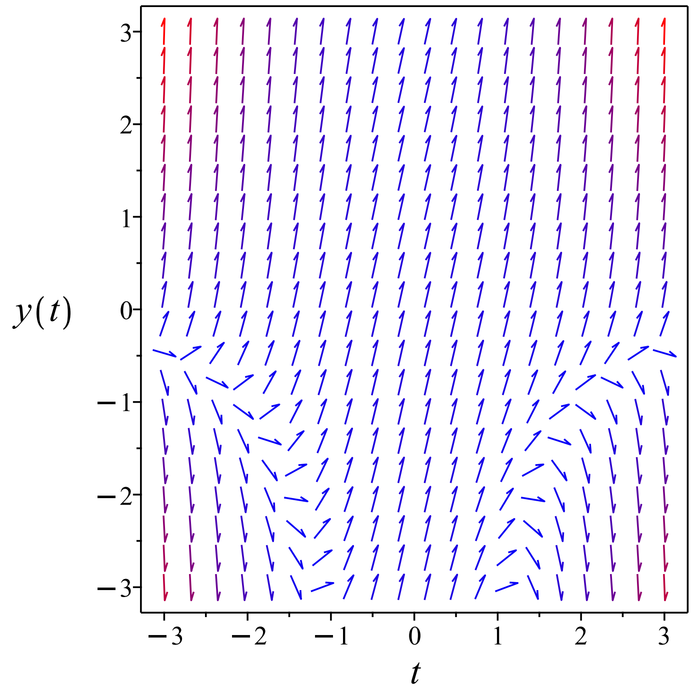


Figure 272: Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{t^3}{3}}\left(3\,3^{\frac{1}{6}} e^{-\frac{t^3}{6}} \operatorname{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t + 4\,e^{-\frac{t^3}{3}} (t^3)^{\frac{1}{6}} t + c_1 (t^3)^{\frac{1}{6}}\right)}{(t^3)^{\frac{1}{6}}}$$

Verified OK.

7.14.4 Maple step by step solution

Let's solve

$y' - t^2 y = 4$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = t^2 y + 4$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - t^2 y = 4$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $$\mu(t)\left(y' - t^2 y\right) = 4\mu(t)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $$\mu(t)\left(y' - t^2 y\right) = \mu'(t)\, y + \mu(t)\, y'$$

- Isolate $\mu'(t)$

  $$\mu'(t) = -\mu(t)\, t^2$$

- Solve to find the integrating factor

  $$\mu(t) = \mathrm{e}^{-\frac{t^3}{3}}$$

- Integrate both sides with respect to t

  $$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 4\mu(t)\, dt + c_1$$

- Evaluate the integral on the lhs

  $$\mu(t)\, y = \int 4\mu(t)\, dt + c_1$$

- Solve for y

  $$y = \frac{\int 4\mu(t) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{-\frac{t^3}{3}}$

  $$y = \frac{\int 4\,\mathrm{e}^{-\frac{t^3}{3}} dt + c_1}{\mathrm{e}^{-\frac{t^3}{3}}}$$

- Evaluate the integrals on the rhs

  $$y = \frac{\frac{4\, 3^{\frac{1}{3}} \left(\frac{3 t 3^{\frac{5}{6}} \mathrm{e}^{-\frac{t^3}{6}}\, \mathit{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right)}{4\left(t^3\right)^{\frac{1}{6}}} + \frac{3\, 3^{\frac{5}{6}} \mathrm{e}^{-\frac{t^3}{6}}\, \mathit{WhittakerM}\left(\frac{7}{6}, \frac{2}{3}, \frac{t^3}{3}\right)}{t^2 \left(t^3\right)^{\frac{1}{6}}}\right)}{3} + c_1}{\mathrm{e}^{-\frac{t^3}{3}}}$$

- Simplify

  $$y = \frac{3\, 3^{\frac{1}{6}}\, \mathit{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) t\, \mathrm{e}^{\frac{t^3}{6}}}{\left(t^3\right)^{\frac{1}{6}}} + c_1 \mathrm{e}^{\frac{t^3}{3}} + 4t$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(t),t)=t^2*y(t)+4,y(t), singsol=all)
```

$$y(t) = \frac{3\,3^{\frac{1}{6}} t \,\mathrm{WhittakerM}\left(\frac{1}{6}, \frac{2}{3}, \frac{t^3}{3}\right) \mathrm{e}^{\frac{t^3}{6}}}{(t^3)^{\frac{1}{6}}} + c_1 \mathrm{e}^{\frac{t^3}{3}} + 4t$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 49

```
DSolve[y'[t]==t^2*y[t]+4,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{3} e^{\frac{t^3}{3}} \left(-\frac{4\sqrt[3]{3} t \Gamma\left(\frac{1}{3}, \frac{t^3}{3}\right)}{\sqrt[3]{t^3}} + 3c_1\right)$$

7.15 problem 15

Internal problem ID [13020]
Internal file name [`OUTPUT/11672_Wednesday_November_08_2023_03_28_23_AM_61629781/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{y}{t^2} = 4\cos(t)}$$

7.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{1}{t^2}$$
$$q(t) = 4\cos(t)$$

Hence the ode is

$$y' - \frac{y}{t^2} = 4\cos(t)$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{t^2} dt}$$
$$= e^{\frac{1}{t}}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)(4\cos(t))$$
$$\frac{d}{dt}\left(e^{\frac{1}{t}} y\right) = \left(e^{\frac{1}{t}}\right)(4\cos(t))$$
$$d\left(e^{\frac{1}{t}} y\right) = \left(4\cos(t) e^{\frac{1}{t}}\right) dt$$

Integrating gives

$$e^{\frac{1}{t}} y = \int 4\cos(t) e^{\frac{1}{t}} dt$$
$$e^{\frac{1}{t}} y = \int 4\cos(t) e^{\frac{1}{t}} dt + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{1}{t}}$ results in

$$y = e^{-\frac{1}{t}}\left(\int 4\cos(t) e^{\frac{1}{t}} dt\right) + c_1 e^{-\frac{1}{t}}$$

which simplifies to

$$y = e^{-\frac{1}{t}}\left(4\left(\int \cos(t) e^{\frac{1}{t}} dt\right) + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{1}{t}}\left(4\left(\int \cos(t) e^{\frac{1}{t}} dt\right) + c_1\right) \tag{1}$$

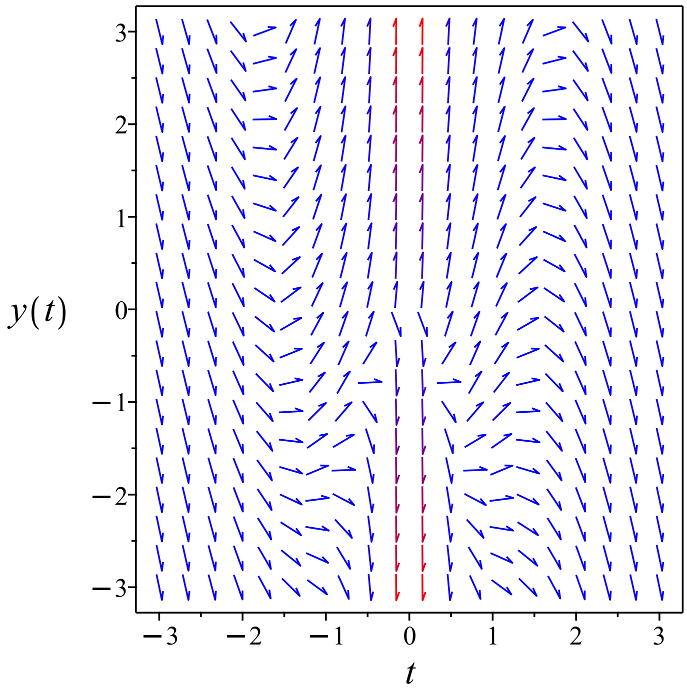


Figure 273: Slope field plot

Verification of solutions

$$y = e^{-\frac{1}{t}}\left(4\left(\int \cos(t)\, e^{\frac{1}{t}} dt\right) + c_1\right)$$

Verified OK.

7.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + 4\cos(t)\, t^2}{t^2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 266: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, y) &= 0\\ \eta(t, y) &= \mathrm{e}^{-\frac{1}{t}}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{1}{t}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{1}{t}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{y + 4\cos(t)\, t^2}{t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{\mathrm{e}^{\frac{1}{t}} y}{t^2} \\ S_y &= \mathrm{e}^{\frac{1}{t}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4\cos(t)\, \mathrm{e}^{\frac{1}{t}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4\cos(R)\, \mathrm{e}^{\frac{1}{R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \int 4\cos(R)\, e^{\frac{1}{R}} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{\frac{1}{t}} y = \int 4\cos(t)\, e^{\frac{1}{t}} dt + c_1$$

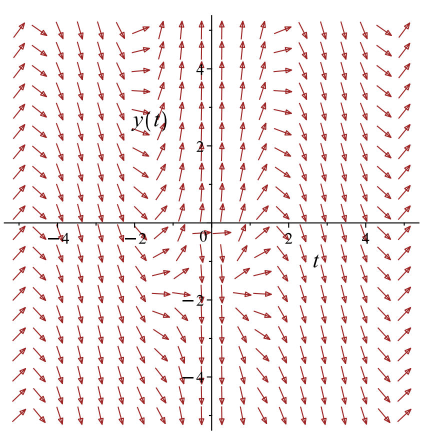
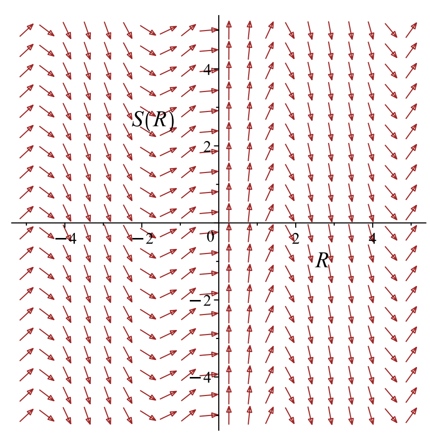
Which simplifies to

$$e^{\frac{1}{t}} y = \int 4\cos(t)\, e^{\frac{1}{t}} dt + c_1$$

Which gives

$$y = \left(\int 4\cos(t)\, e^{\frac{1}{t}} dt + c_1\right) e^{-\frac{1}{t}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{y+4\cos(t)t^2}{t^2}$ 	$R = t$ $S = e^{\frac{1}{t}} y$	$\frac{dS}{dR} = 4\cos(R)\, e^{\frac{1}{R}}$ 

Summary

The solution(s) found are the following

$$y = \left(\int 4\cos(t)\, \mathrm{e}^{\frac{1}{t}} dt + c_1\right) \mathrm{e}^{-\frac{1}{t}} \tag{1}$$

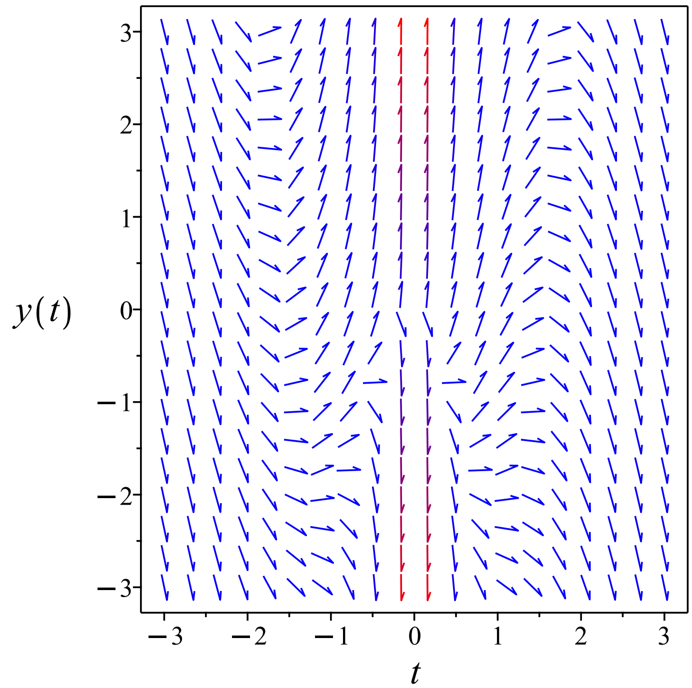


Figure 274: Slope field plot

Verification of solutions

$$y = \left(\int 4\cos(t)\, \mathrm{e}^{\frac{1}{t}} dt + c_1\right) \mathrm{e}^{-\frac{1}{t}}$$

Verified OK.

7.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(\frac{y}{t^2} + 4\cos(t)\right)\mathrm{d}t \\ \left(-\frac{y}{t^2} - 4\cos(t)\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -\frac{y}{t^2} - 4\cos(t) \\ N(t, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{t^2} - 4\cos(t)\right) \\ &= -\frac{1}{t^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1\left(\left(-\frac{1}{t^2}\right) - (0)\right) \\ &= -\frac{1}{t^2}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t} \\ &= e^{\int -\frac{1}{t^2}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{1}{t}} \\ &= \mathrm{e}^{\frac{1}{t}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{\frac{1}{t}}\left(-\frac{y}{t^2} - 4\cos(t)\right) \\ &= \frac{\left(-y - 4\cos(t)\,t^2\right)\mathrm{e}^{\frac{1}{t}}}{t^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{1}{t}}(1) \\
&= \mathrm{e}^{\frac{1}{t}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\frac{\left(-y - 4\cos\left(t\right)t^2\right)\mathrm{e}^{\frac{1}{t}}}{t^2}\right) + \left(\mathrm{e}^{\frac{1}{t}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \frac{\left(-y - 4\cos\left(t\right)t^2\right)\mathrm{e}^{\frac{1}{t}}}{t^2}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \int^t \frac{\left(-y - 4\cos\left(\_a\right)\_a^2\right)\mathrm{e}^{\frac{1}{\_a}}}{\_a^2} d\_a + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{1}{t}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{1}{t}}$. Therefore equation (4) becomes

$$
\mathrm{e}^{\frac{1}{t}} = \mathrm{e}^{\frac{1}{t}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^t \frac{\left(-y - 4\cos\left(\_a\right)\_a^2\right) e^{\frac{1}{\_a}}}{\_a^2} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^t \frac{\left(-y - 4\cos\left(\_a\right)\_a^2\right) e^{\frac{1}{\_a}}}{\_a^2} d\_a$$

Summary

The solution(s) found are the following

$$\int^t \frac{\left(-y - 4\cos\left(\_a\right)\_a^2\right) e^{\frac{1}{\_a}}}{\_a^2} d\_a = c_1 \tag{1}$$

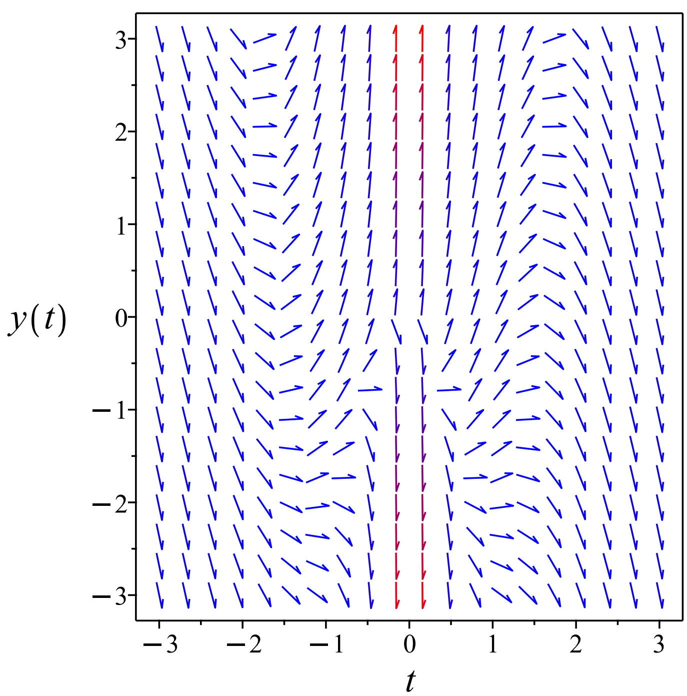


Figure 275: Slope field plot

Verification of solutions

$$\int^{t} \frac{\left(-y-4 \cos \left(\_a\right) \_a^{2}\right) \mathrm{e}^{-\frac{1}{\_a}}}{\_a^{2}} d\_a=c_{1}$$

Verified OK.

7.15.4 Maple step by step solution

Let's solve

$y' - \frac{y}{t^2} = 4\cos(t)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{t^2} + 4\cos(t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{t^2} = 4\cos(t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{y}{t^2}\right) = 4\mu(t)\cos(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - \frac{y}{t^2}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{\mu(t)}{t^2}$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{\frac{1}{t}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 4\mu(t)\cos(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 4\mu(t)\cos(t)\, dt + c_1$

- Solve for y

  $y = \frac{\int 4\mu(t)\cos(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{\frac{1}{t}}$

$$y = \frac{\int 4\cos(t)\mathrm{e}^{\frac{1}{t}}dt + c_1}{\mathrm{e}^{\frac{1}{t}}}$$

- Simplify

$$y = \left(4\left(\int \cos(t)\,\mathrm{e}^{\frac{1}{t}}dt\right) + c_1\right)\mathrm{e}^{-\frac{1}{t}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(t),t)=y(t)/t^2+4*cos(t),y(t), singsol=all)
```

$$y(t) = \left(4\left(\int \cos(t)\,\mathrm{e}^{\frac{1}{t}}dt\right) + c_1\right)\mathrm{e}^{-\frac{1}{t}}$$

✓ Solution by Mathematica

Time used: 3.836 (sec). Leaf size: 34

```
DSolve[y'[t]==y[t]/t^2+4*Cos[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow e^{-1/t}\left(\int_1^t 4e^{\frac{1}{K[1]}}\cos(K[1])dK[1] + c_1\right)$$

7.16 problem 16

Internal problem ID [13021]
Internal file name [`OUTPUT/11673_Wednesday_November_08_2023_03_28_24_AM_40855923/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 4\cos\left(t^2\right)}$$

7.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 4\cos\left(t^2\right) \end{aligned}$$

Hence the ode is

$$y' - y = 4\cos\left(t^2\right)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-1)dt} \\
&= e^{-t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(4\cos\left(t^2\right)\right) \\
\frac{d}{dt}\left(e^{-t}y\right) &= \left(e^{-t}\right)\left(4\cos\left(t^2\right)\right) \\
d\left(e^{-t}y\right) &= \left(4\cos\left(t^2\right)e^{-t}\right)\, dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-t}y &= \int 4\cos\left(t^2\right)e^{-t}\, dt \\
e^{-t}y &= \frac{\sqrt{\pi}\, e^{\frac{i}{4}} \operatorname{erf}\left(\sqrt{-i}\, t + \frac{1}{2\sqrt{-i}}\right)}{\sqrt{-i}} - \sqrt{\pi}\, e^{-\frac{i}{4}} (-1)^{\frac{3}{4}} \operatorname{erf}\left((-1)^{\frac{1}{4}}\, t - \frac{(-1)^{\frac{3}{4}}}{2}\right) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-t}$ results in

$$
y = e^{t}\left(\frac{\sqrt{\pi}\, e^{\frac{i}{4}} \operatorname{erf}\left(\sqrt{-i}\, t + \frac{1}{2\sqrt{-i}}\right)}{\sqrt{-i}} - \sqrt{\pi}\, e^{-\frac{i}{4}} (-1)^{\frac{3}{4}} \operatorname{erf}\left((-1)^{\frac{1}{4}}\, t - \frac{(-1)^{\frac{3}{4}}}{2}\right)\right) + c_1 e^{t}
$$

which simplifies to

$$
y = \left(\frac{1}{4} - \frac{i}{4}\right)\left(2\, e^{-\frac{i}{4}} \operatorname{erf}\left(\frac{(1 - i + (2 + 2i)\, t)\sqrt{2}}{4}\right)\sqrt{\pi} + 2i\sqrt{\pi}\, e^{\frac{i}{4}} \operatorname{erf}\left(\left(\frac{1}{4} - \frac{i}{4}\right)\sqrt{2}\,(2t + i)\right) + (1 + i)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = \left(\frac{1}{4} - \frac{i}{4}\right)\left(2\, e^{-\frac{i}{4}} \operatorname{erf}\left(\frac{(1 - i + (2 + 2i)\, t)\sqrt{2}}{4}\right)\sqrt{\pi}\right. \\
\left. + 2i\sqrt{\pi}\, e^{\frac{i}{4}} \operatorname{erf}\left(\left(\frac{1}{4} - \frac{i}{4}\right)\sqrt{2}\,(2t + i)\right) + (1 + i)\, c_1\sqrt{2}\right) e^{t}\sqrt{2}
\end{aligned} \tag{1}
$$

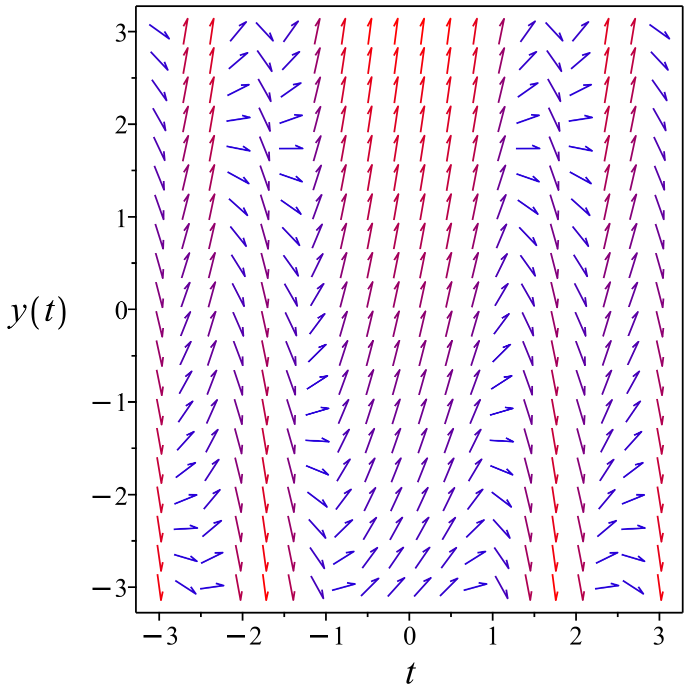


Figure 276: Slope field plot

Verification of solutions

$$y = \left(\frac{1}{4}-\frac{i}{4}\right)\left(2\,\mathrm{e}^{-\frac{i}{4}}\operatorname{erf}\left(\frac{(1-i+(2+2i)\,t)\sqrt{2}}{4}\right)\sqrt{\pi} + 2i\sqrt{\pi}\,\mathrm{e}^{\frac{i}{4}}\operatorname{erf}\left(\left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{2}\,(2t+i)\right) + (1+i)\,c_1\sqrt{2}\right)\mathrm{e}^{t}\sqrt{2}$$

Verified OK.

7.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + 4\cos\left(t^2\right)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 269: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{t}} dy \end{aligned}$$

Which results in

$$S = e^{-t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = y + 4 \cos\left(t^2\right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -e^{-t} y \\ S_y &= e^{-t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4 \cos\left(t^2\right) e^{-t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4 \cos\left(R^2\right) e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\sqrt{\pi}\, \mathrm{e}^{\frac{i}{4}} \operatorname{erf}\left(\sqrt{-i}\, R + \frac{1}{2\sqrt{-i}}\right)}{\sqrt{-i}} - \sqrt{\pi}\, \mathrm{e}^{-\frac{i}{4}} (-1)^{\frac{3}{4}} \operatorname{erf}\left((-1)^{\frac{1}{4}} R - \frac{(-1)^{\frac{3}{4}}}{2}\right) + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-t} y = \frac{\sqrt{\pi}\, \mathrm{e}^{\frac{i}{4}} \operatorname{erf}\left(\sqrt{-i}\, t + \frac{1}{2\sqrt{-i}}\right)}{\sqrt{-i}} - \sqrt{\pi}\, \mathrm{e}^{-\frac{i}{4}} (-1)^{\frac{3}{4}} \operatorname{erf}\left((-1)^{\frac{1}{4}} t - \frac{(-1)^{\frac{3}{4}}}{2}\right) + c_1$$

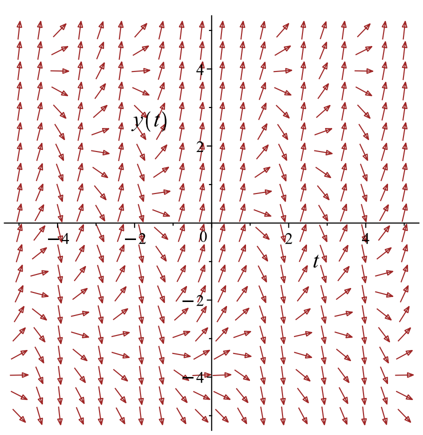
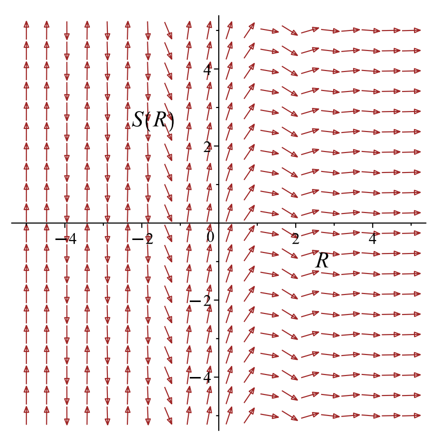
Which simplifies to

$$\mathrm{e}^{-t} y = \frac{\sqrt{\pi}\, \mathrm{e}^{\frac{i}{4}} \operatorname{erf}\left(\sqrt{-i}\, t + \frac{1}{2\sqrt{-i}}\right)}{\sqrt{-i}} - \sqrt{\pi}\, \mathrm{e}^{-\frac{i}{4}} (-1)^{\frac{3}{4}} \operatorname{erf}\left((-1)^{\frac{1}{4}} t - \frac{(-1)^{\frac{3}{4}}}{2}\right) + c_1$$

Which gives

$$y = -\frac{\left(i\sqrt{\pi}\, \mathrm{e}^{-\frac{i}{4}} \operatorname{erf}\left((-1)^{\frac{1}{4}} t - \frac{(-1)^{\frac{3}{4}}}{2}\right) + \sqrt{\pi}\, \mathrm{e}^{\frac{i}{4}} \operatorname{erf}\left(\frac{2it-1}{2\sqrt{-i}}\right) - c_1\sqrt{-i}\right) \mathrm{e}^{t}}{\sqrt{-i}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y + 4\cos(t^2)$ 	$R = t$ $S = \mathrm{e}^{-t} y$	$\frac{dS}{dR} = 4\cos(R^2)\, \mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-\frac{i}{4}}\,\mathrm{erf}\left((-1)^{\frac{1}{4}}\,t-\frac{(-1)^{\frac{3}{4}}}{2}\right)+\sqrt{\pi}\,\mathrm{e}^{\frac{i}{4}}\,\mathrm{erf}\left(\frac{2it-1}{2\sqrt{-i}}\right)-c_1\sqrt{-i}\right)\mathrm{e}^{t}}{\sqrt{-i}} \tag{1}$$

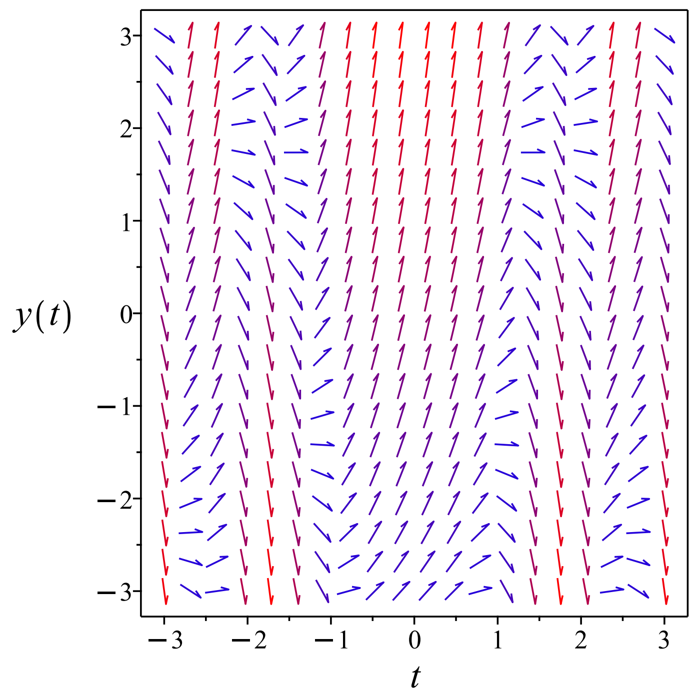


Figure 277: Slope field plot

Verification of solutions

$$y = -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-\frac{i}{4}}\,\mathrm{erf}\left((-1)^{\frac{1}{4}}\,t-\frac{(-1)^{\frac{3}{4}}}{2}\right)+\sqrt{\pi}\,\mathrm{e}^{\frac{i}{4}}\,\mathrm{erf}\left(\frac{2it-1}{2\sqrt{-i}}\right)-c_1\sqrt{-i}\right)\mathrm{e}^{t}}{\sqrt{-i}}$$

Verified OK.

7.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(y+4\cos\left(t^2\right)\right)\mathrm{d}t\\ \left(-y-4\cos\left(t^2\right)\right)\mathrm{d}t+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-y-4\cos\left(t^2\right)\\ N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - 4\cos\left(t^2\right)\right)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((-1) - (0))\\ &= -1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -1\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-t}\\ &= \mathrm{e}^{-t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-t}\left(-y - 4\cos\left(t^2\right)\right)\\ &= -\mathrm{e}^{-t}\left(y + 4\cos\left(t^2\right)\right)\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-t}(1) \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{-t}\left(y + 4\cos\left(t^2\right)\right)\right) + \left(\mathrm{e}^{-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\mathrm{e}^{-t}\left(y + 4\cos\left(t^2\right)\right)\mathrm{d}t \\
\phi &= \int^{t} -\mathrm{e}^{-\_a}\left(y + 4\cos\left(\_a^2\right)\right) d\_a + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-t} = \mathrm{e}^{-t} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^t -\mathrm{e}^{-\_a}\left(y + 4\cos\left(\_a^2\right)\right) d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^t -\mathrm{e}^{-\_a}\left(y + 4\cos\left(\_a^2\right)\right) d\_a$$

Summary

The solution(s) found are the following

$$\int^t -\mathrm{e}^{-\_a}\left(y + 4\cos\left(\_a^2\right)\right) d\_a = c_1 \tag{1}$$

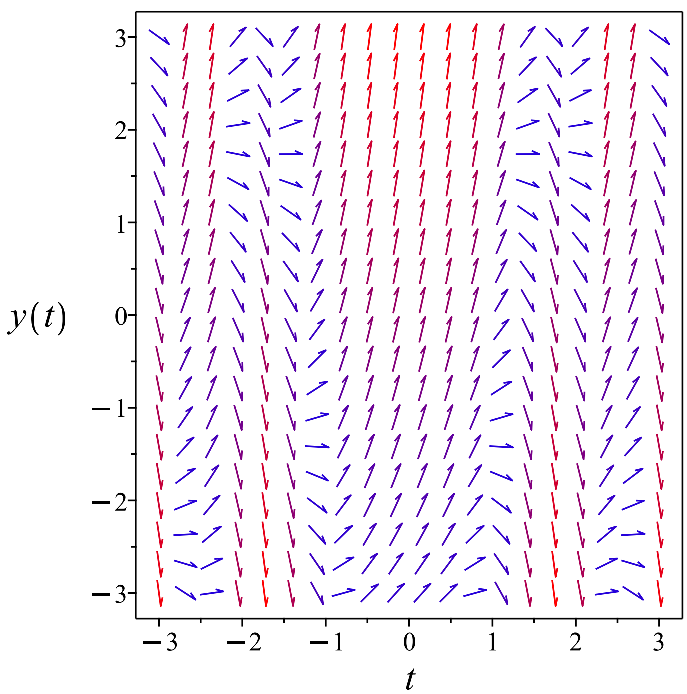


Figure 278: Slope field plot

Verification of solutions

$$\int^{t} -e^{-\_a} \left(y + 4\cos\left(\_a^2\right)\right) d\_a = c_1$$

Verified OK.

7.16.4 Maple step by step solution

Let's solve

$y' - y = 4\cos(t^2)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + 4\cos(t^2)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 4\cos(t^2)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - y) = 4\mu(t)\cos(t^2)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = e^{-t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 4\mu(t)\cos(t^2)\,dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 4\mu(t)\cos(t^2)\,dt + c_1$

- Solve for y

  $y = \frac{\int 4\mu(t)\cos(t^2)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{-t}$

  $y = \frac{\int 4\cos(t^2)e^{-t}dt + c_1}{e^{-t}}$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{I}{4}}\,\mathrm{erf}\left(\sqrt{-I}\,t+\frac{1}{2\sqrt{-I}}\right)}{\sqrt{-I}} - \sqrt{\pi}\,\mathrm{e}^{-\frac{I}{4}}\,(-1)^{\frac{3}{4}}\,\mathrm{erf}\left((-1)^{\frac{1}{4}}\,t - \frac{(-1)^{\frac{3}{4}}}{2}\right) + c_1}{\mathrm{e}^{-t}}$$

- Simplify

$$y = \left(\frac{1}{4} - \frac{I}{4}\right)\left(2\,\mathrm{e}^{-\frac{I}{4}}\mathrm{erf}\left(\frac{(1-I+(2+2\,I)t)\sqrt{2}}{4}\right)\sqrt{\pi} + 2\,I\sqrt{\pi}\,\mathrm{e}^{\frac{I}{4}}\mathrm{erf}\left(\left(\frac{1}{4} - \frac{I}{4}\right)\sqrt{2}\,(2t+I)\right) + (1+I)\,c_1\sqrt{2}\right)\mathrm{e}^{t}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 68

```
dsolve(diff(y(t),t)=y(t)+4*cos(t^2),y(t), singsol=all)
```

$$y(t) = \left(\frac{1}{4} - \frac{i}{4}\right)\sqrt{2}\,\mathrm{e}^{t}\left(2\,\mathrm{e}^{-\frac{i}{4}}\,\mathrm{erf}\left(\frac{(1-i+(2+2i)\,t)\sqrt{2}}{4}\right)\sqrt{\pi} + 2i\sqrt{\pi}\,\mathrm{e}^{\frac{i}{4}}\,\mathrm{erf}\left(\left(\frac{1}{4} - \frac{i}{4}\right)\sqrt{2}\,(2t+i)\right) + (1+i)\sqrt{2}\,c_1\right)$$

✓ Solution by Mathematica

Time used: 0.137 (sec). Leaf size: 77

```
DSolve[y'[t]==y[t]+4*Cos[t^2],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{t}\left(c_1 - \sqrt[4]{-1}e^{-\frac{i}{4}}\sqrt{\pi}\left(\mathrm{erfi}\left(\frac{1}{2}(-1)^{3/4}(2t-i)\right) + ie^{\frac{i}{2}}\mathrm{erfi}\left(\frac{1}{2}\sqrt[4]{-1}(2t+i)\right)\right)\right)$$

7.17 problem 17

Internal problem ID [13022]
Internal file name [`OUTPUT/11674_Wednesday_November_08_2023_03_28_25_AM_28842069/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + e^{-t^2} y = \cos(t)}$$

7.17.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = e^{-t^2}$$
$$q(t) = \cos(t)$$

Hence the ode is

$$y' + e^{-t^2} y = \cos(t)$$

The integrating factor μ is

$$\mu = e^{\int e^{-t^2} dt}$$
$$= e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)(\cos(t))$$
$$\frac{d}{dt}\left(e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} y\right) = \left(e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}}\right)(\cos(t))$$
$$d\left(e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} y\right) = \left(\cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}}\right) dt$$

Integrating gives

$$e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} y = \int \cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}}\, dt$$
$$e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} y = \int \cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} dt + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}}$ results in

$$y = e^{-\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} \left(\int \cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} dt\right) + c_1 e^{-\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}}$$

which simplifies to

$$y = e^{-\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} \left(\int \cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} dt + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} \left(\int \cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} dt + c_1\right) \tag{1}$$

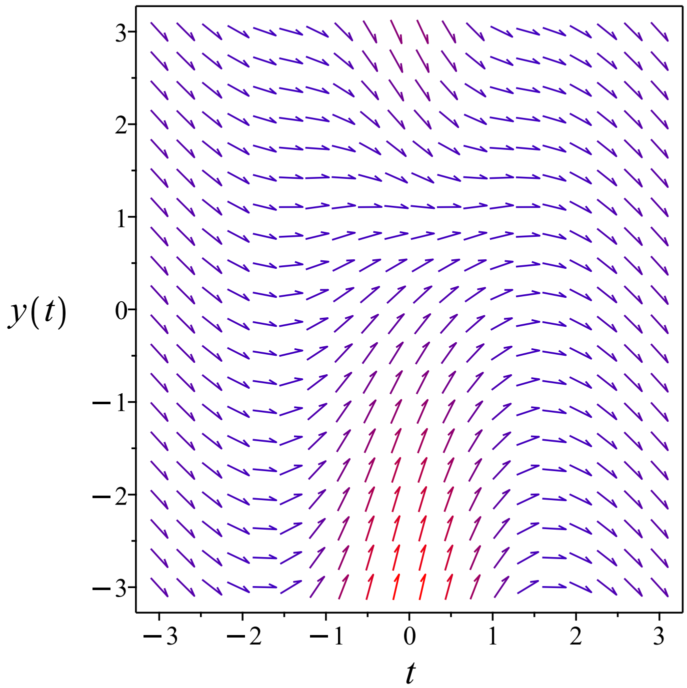


Figure 279: Slope field plot

Verification of solutions

$$y = e^{-\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}} \left(\int \cos(t)\, e^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}} dt + c_1\right)$$

Verified OK.

7.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \left(-y + \cos(t)\, e^{t^2}\right) e^{-t^2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 272: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \left(-y + \cos(t)\, \mathrm{e}^{t^2}\right) \mathrm{e}^{-t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1 \\ R_y &= 0 \\ S_t &= y\, \mathrm{e}^{-t^2 + \frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} \\ S_y &= \mathrm{e}^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(t)\, \mathrm{e}^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)\, \mathrm{e}^{\frac{\sqrt{\pi}\, \mathrm{erf}(R)}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \cos(R)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(R)}{2}} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} y = \int \cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} dt + c_1$$

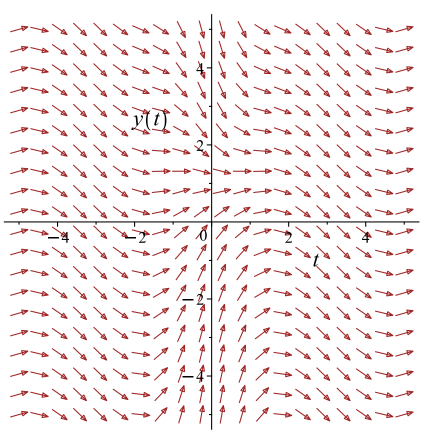
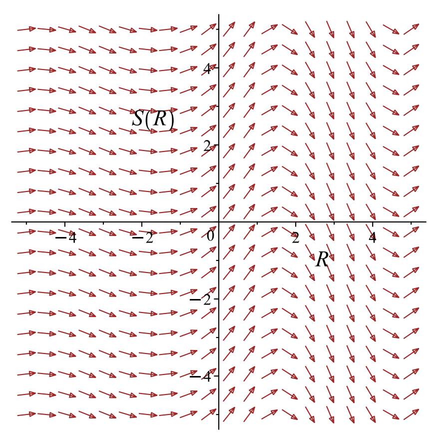
Which simplifies to

$$e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} y = \int \cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} dt + c_1$$

Which gives

$$y = \left(\int \cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} dt + c_1\right) e^{-\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dt} = \left(-y + \cos(t)\, e^{t^2}\right) e^{-t^2}$
</td><td>$R = t$
$S = e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} y$</td><td>$\frac{dS}{dR} = \cos(R)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(R)}{2}}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \left(\int \cos(t)\, e^{\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} dt + c_1\right) e^{-\frac{\sqrt{\pi}\, \operatorname{erf}(t)}{2}} \tag{1}$$

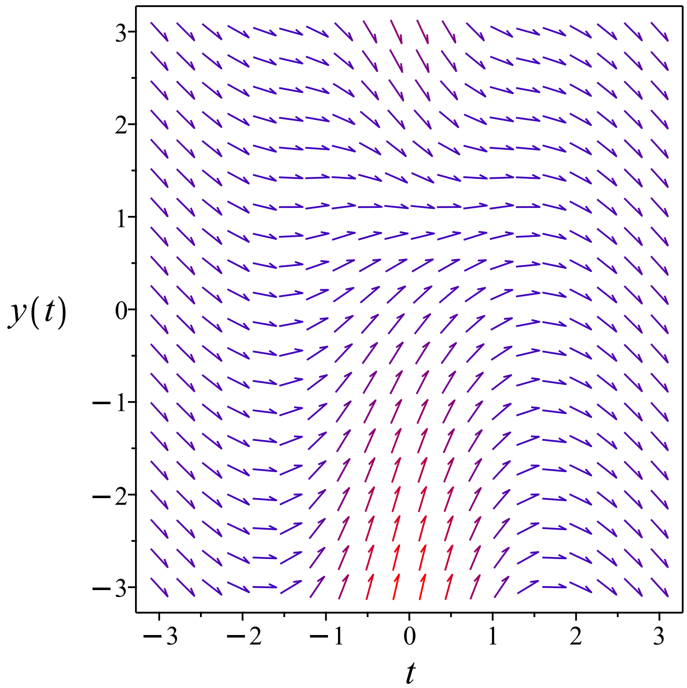


Figure 280: Slope field plot

Verification of solutions

$$y = \left(\int \cos(t)\, e^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} dt + c_1\right) e^{-\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}$$

Verified OK.

7.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-y\,\mathrm{e}^{-t^2} + \cos(t)\right)\mathrm{d}t\\ \left(-\cos(t) + y\,\mathrm{e}^{-t^2}\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -\cos(t) + y\,\mathrm{e}^{-t^2}\\ N(t, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\cos(t) + y\,\mathrm{e}^{-t^2}\right)\\ &= \mathrm{e}^{-t^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(\mathrm{e}^{-t^2}\right) - (0)\right) \\
&= \mathrm{e}^{-t^2}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int \mathrm{e}^{-t^2} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} \\
&= \mathrm{e}^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}\left(-\cos(t) + y\,\mathrm{e}^{-t^2}\right) \\
&= \mathrm{e}^{-t^2 + \frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}\left(-\cos(t)\,\mathrm{e}^{t^2} + y\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}(1) \\
&= \mathrm{e}^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(\mathrm{e}^{-t^2 + \frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}\left(-\cos(t)\,\mathrm{e}^{t^2} + y\right)\right) + \left(\mathrm{e}^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \mathrm{e}^{-t^2 + \frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}} \left(-\cos(t)\,\mathrm{e}^{t^2} + y\right) \mathrm{d}t$$

$$\phi = \int^t \mathrm{e}^{-\_a^2 + \frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} \left(-\cos(\_a)\,\mathrm{e}^{\_a^2} + y\right) d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \int^t \mathrm{e}^{-\_a^2 + \frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} d\_a + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}}$. Therefore equation (4) becomes

$$\mathrm{e}^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}} = \int^t \mathrm{e}^{-\_a^2 + \frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} d\_a + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\left(\int^t \mathrm{e}^{-\_a^2 + \frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} d\_a\right) + \mathrm{e}^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(-\left(\int^t \mathrm{e}^{-\_a^2 + \frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} d\_a\right) + \mathrm{e}^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}}\right) \mathrm{d}y$$

$$f(y) = \left(-\left(\int^t \mathrm{e}^{-\_a^2 + \frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} d\_a\right) + \mathrm{e}^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}}\right) y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{t} e^{-\_a^2+\frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} \left(-\cos(\_a)\, e^{-\_a^2} + y\right) d\_a + \left(-\left(\int^{t} e^{-\_a^2+\frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} d\_a\right) + e^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}}\right) y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{t} e^{-\_a^2+\frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} \left(-\cos(\_a)\, e^{-\_a^2} + y\right) d\_a + \left(-\left(\int^{t} e^{-\_a^2+\frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} d\_a\right) + e^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}}\right) y$$

Summary

The solution(s) found are the following

$$\int^{t} e^{-\_a^2+\frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} \left(-\cos(\_a)\, e^{-\_a^2} + y\right) d\_a + \left(-\left(\int^{t} e^{-\_a^2+\frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}} d\_a\right) + e^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}}\right) y = c_1 \tag{1}$$

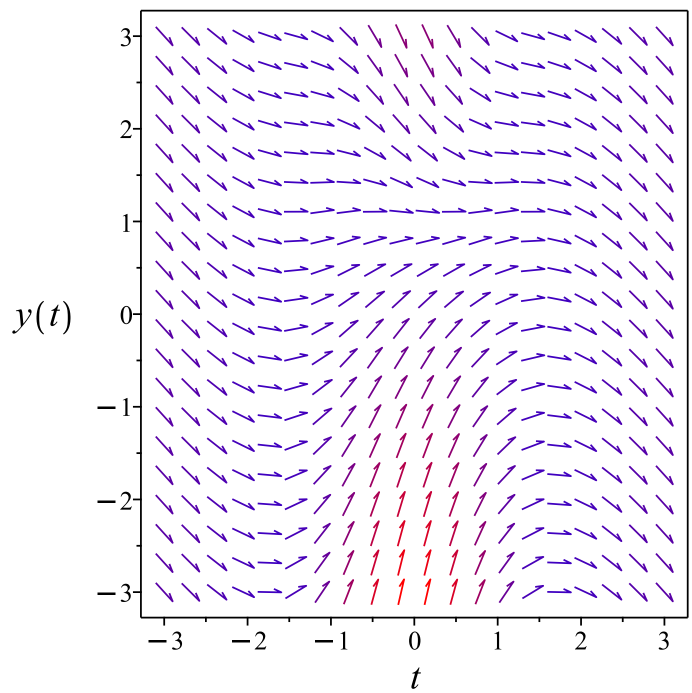


Figure 281: Slope field plot

Verification of solutions

$$\int^{t} e^{-\_a^2+\frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}}\left(-\cos(\_a)\,e^{-\_a^2}+y\right)d\_a + \left(-\left(\int^{t} e^{-\_a^2+\frac{\sqrt{\pi}\,\mathrm{erf}(\_a)}{2}}d\_a\right)+e^{\frac{\sqrt{\pi}\,\mathrm{erf}(t)}{2}}\right)y = c_1$$

Verified OK.

7.17.4 Maple step by step solution

Let's solve

$y' + \frac{y}{e^{t^2}} = \cos(t)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{e^{t^2}} + \cos(t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{y}{e^{t^2}} = \cos(t)$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t) \left(y' + \frac{y}{e^{t^2}}\right) = \mu(t) \cos(t)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$$\mu(t) \left(y' + \frac{y}{e^{t^2}}\right) = \mu'(t)\, y + \mu(t)\, y'$$

- Isolate $\mu'(t)$

$$\mu'(t) = \frac{\mu(t)}{e^{t^2}}$$

- Solve to find the integrating factor

$$\mu(t) = e^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} e^{-t^2} e^{t^2}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t) \cos(t)\, dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\, y = \int \mu(t) \cos(t)\, dt + c_1$$

- Solve for y

$$y = \frac{\int \mu(t) \cos(t) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = e^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} e^{-t^2} e^{t^2}$

$$y = \frac{\int e^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} e^{-t^2} e^{t^2} \cos(t) dt + c_1}{e^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} e^{-t^2} e^{t^2}}$$

- Simplify

$$y = \left(\int \cos(t)\, e^{\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}} dt + c_1\right) e^{-\frac{\sqrt{\pi}\, \mathrm{erf}(t)}{2}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(t),t)=-y(t)/exp(t^2)+cos(t),y(t), singsol=all)
```

$$y(t) = \left(\int \cos\left(t\right) \mathrm{e}^{\frac{\sqrt{\pi}\,\operatorname{erf}\left(t\right)}{2}}dt + c_1\right) \mathrm{e}^{-\frac{\sqrt{\pi}\,\operatorname{erf}\left(t\right)}{2}}$$

✓ Solution by Mathematica

Time used: 1.093 (sec). Leaf size: 47

```
DSolve[y'[t]==-y[t]/Exp[t^2]+Cos[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-\frac{1}{2}\sqrt{\pi}\text{erf}(t)}\left(\int_1^t e^{\frac{1}{2}\sqrt{\pi}\text{erf}(K[1])}\cos(K[1])dK[1] + c_1\right)$$

7.18 problem 18

Internal problem ID [13023]
Internal file name [`OUTPUT/11675_Wednesday_November_08_2023_03_28_27_AM_88496519/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{y}{\sqrt{t^3-3}} = t}$$

7.18.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{1}{\sqrt{t^3-3}}$$
$$q(t) = t$$

Hence the ode is

$$y' - \frac{y}{\sqrt{t^3-3}} = t$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{\sqrt{t^3-3}}dt}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\int -\frac{1}{\sqrt{t^3-3}}dt} y\right) &= \left(e^{\int -\frac{1}{\sqrt{t^3-3}}dt}\right)(t) \\
\mathrm{d}\left(e^{\int -\frac{1}{\sqrt{t^3-3}}dt} y\right) &= \left(t\, e^{-\left(\int \frac{1}{\sqrt{t^3-3}}dt\right)}\right) \mathrm{d}t
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{\int -\frac{1}{\sqrt{t^3-3}}dt} y &= \int t\, e^{-\left(\int \frac{1}{\sqrt{t^3-3}}dt\right)} \mathrm{d}t \\
e^{\int -\frac{1}{\sqrt{t^3-3}}dt} y &= \int t\, e^{-\left(\int \frac{1}{\sqrt{t^3-3}}dt\right)} dt + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{1}{\sqrt{t^3-3}}dt}$ results in

$$y = e^{\int \frac{1}{\sqrt{t^3-3}}dt}\left(\int t\, e^{-\left(\int \frac{1}{\sqrt{t^3-3}}dt\right)} dt\right) + c_1 e^{\int \frac{1}{\sqrt{t^3-3}}dt}$$

which simplifies to

$$y = e^{\int \frac{1}{\sqrt{t^3-3}}dt}\left(\int t\, e^{-\left(\int \frac{1}{\sqrt{t^3-3}}dt\right)} dt + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{\int \frac{1}{\sqrt{t^3-3}}dt}\left(\int t\, e^{-\left(\int \frac{1}{\sqrt{t^3-3}}dt\right)} dt + c_1\right) \tag{1}$$

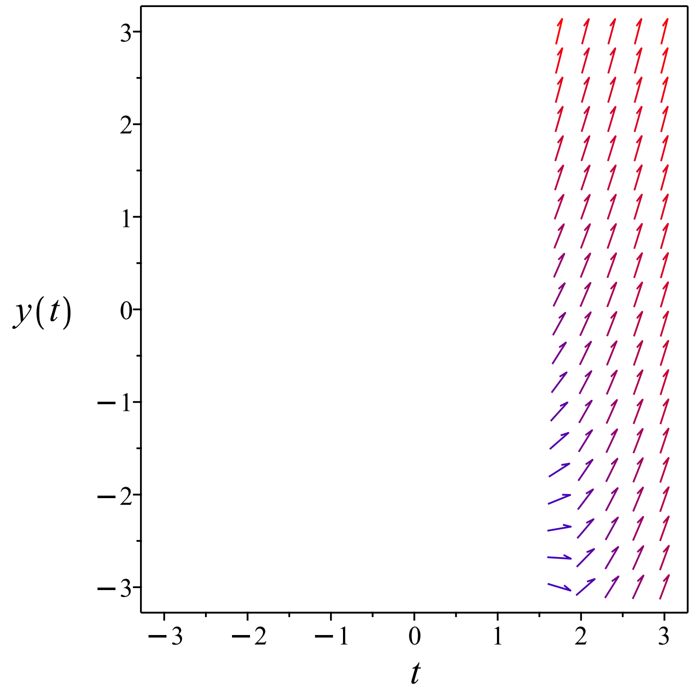


Figure 282: Slope field plot

Verification of solutions

$$y = e^{\int \frac{1}{\sqrt{t^3-3}}dt} \left(\int t\, e^{-\left(\int \frac{1}{\sqrt{t^3-3}}dt\right)}dt + c_1\right)$$

Verified OK.

7.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{t\sqrt{t^3-3} + y}{\sqrt{t^3-3}}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 275: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,y) = 0$$

$$\eta(t,y) = \mathrm{e}^{\frac{2i3^{\frac{5}{6}}\sqrt{-i\left(t+\frac{3^{\frac{1}{3}}}{2}+\frac{i3^{\frac{5}{6}}}{2}\right)3^{\frac{1}{6}}}\sqrt{\frac{t-3^{\frac{1}{3}}}{-\frac{3\,3^{\frac{1}{3}}}{2}-\frac{i3^{\frac{5}{6}}}{2}}}\sqrt{i\left(t-\frac{i3^{\frac{5}{6}}}{2}+\frac{3^{\frac{1}{3}}}{2}\right)3^{\frac{1}{6}}}\ \mathrm{EllipticF}\left(\frac{\sqrt{3}\sqrt{-i\left(t+\frac{3^{\frac{1}{3}}}{2}+\frac{i3^{\frac{5}{6}}}{2}\right)3^{\frac{1}{6}}}}{3}, \sqrt{-\frac{i3^{\frac{5}{6}}}{-\frac{3\,3^{\frac{1}{3}}}{2}-\frac{i3^{\frac{5}{6}}}{2}}}\right)}{3\sqrt{t^3-3}}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t}+\eta \frac{\partial}{\partial y}\right) S(t, y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{2 i 3^{\frac{5}{6}} \sqrt{-i\left(t+\frac{3 3^{\frac{1}{3}}}{2}+\frac{i 3^{\frac{5}{6}}}{2}\right) 3^{\frac{1}{6}}}\, \sqrt{\frac{t-3^{\frac{1}{3}}}{-\frac{3 3^{\frac{1}{3}}}{2}-\frac{i 3^{\frac{5}{6}}}{2}}}\, \sqrt{i\left(t-\frac{i 3^{\frac{5}{6}}}{2}+\frac{3 3^{\frac{1}{3}}}{2}\right) 3^{\frac{1}{6}}}\, \operatorname{EllipticF}\left(\frac{\sqrt{3}\, \sqrt{-i\left(t+\frac{3 3^{\frac{1}{3}}}{2}+\frac{i 3^{\frac{5}{6}}}{2}\right) 3^{\frac{1}{6}}}}{3}, \sqrt{-\frac{i 3^{\frac{5}{6}}}{-\frac{3 3^{\frac{1}{3}}}{2}-\frac{i 3^{\frac{5}{6}}}{2}}}\right)}{3 \sqrt{t^{3}-3}}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{2 i \operatorname{EllipticF}\left(\frac{\sqrt{6}\, \sqrt{-i\left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 t\right) 3^{\frac{1}{6}}}}{6}, \sqrt{-\frac{i 3^{\frac{5}{6}}}{-\frac{3 3^{\frac{1}{3}}}{2}-\frac{i 3^{\frac{5}{6}}}{2}}}\right) \sqrt{\frac{t-3^{\frac{1}{3}}}{-\frac{3 3^{\frac{1}{3}}}{2}-\frac{i 3^{\frac{5}{6}}}{2}}}\, \sqrt{i\left(t-\frac{i 3^{\frac{5}{6}}}{2}+\frac{3 3^{\frac{1}{3}}}{2}\right) 3^{\frac{1}{6}}}\, \sqrt{-i\left(t+\frac{3 3^{\frac{1}{3}}}{2}+\frac{i 3^{\frac{5}{6}}}{2}\right) 3^{\frac{1}{6}}}\, 3^{\frac{5}{6}}}{3 \sqrt{t^{3}-3}}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = \frac{t\sqrt{t^3-3} + y}{\sqrt{t^3-3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{12\, e^{-\frac{2\sqrt{3^{\frac{1}{3}}-t}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\, \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6\, 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 t\right)}{\sqrt{i 3^{\frac{5}{6}}+3\, 3^{\frac{1}{3}}}\, \sqrt{t^{3}-3}\, \sqrt{2\, 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}} \left(-3^{\frac{5}{6}} t+i 3^{\frac{2}{3}}+i 3^{\frac{1}{3}} t-2 i t^{2}+3\, 3^{\frac{1}{6}}\right) 3}{\sqrt{18-6 i \sqrt{3}+6 i 3^{\frac{1}{6}} t-6\, 3^{\frac{2}{3}} t}\, \sqrt{18+6 i \sqrt{3}+12 i 3^{\frac{1}{6}} t}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\, \sqrt{t^{3}-3}\, \sqrt{i 3^{\frac{5}{6}}+3\, 3^{\frac{1}{3}}}\, \sqrt{3^{\frac{1}{3}}}} \\ S_y &= e^{-\frac{2\sqrt{3^{\frac{1}{3}}-t}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\, \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6\, 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 t\right)}{\sqrt{i 3^{\frac{5}{6}}+3\, 3^{\frac{1}{3}}}\, \sqrt{t^{3}-3}\, \sqrt{2\, 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{24\left(\frac{\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{18+6i\sqrt{3}+12i3^{\frac{1}{6}}t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\left(t\sqrt{t^{3}-3}+y\right)\sqrt{18-6i\sqrt{3}+6i3^{\frac{1}{6}}t-6\,3^{\frac{2}{3}}t}}{24} + \left(i3^{\frac{7}{12}}t^{2} - \frac{i3^{\frac{11}{12}}t}{2} - \frac{3i3}{2}\right.\right.}{\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{t^{3}-3}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{18-6i\sqrt{3}+6i3^{\frac{1}{6}}t-}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-12iS(R)\,3^{\frac{11}{12}}R + 24iS(R)\,3^{\frac{7}{12}}R^{2} + 36S(R)\,3^{\frac{5}{12}}R + R\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2iR}\,\sqrt{18-6i\sqrt{3}+6i3^{\frac{1}{6}}R-6}}{}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \left(\int R\,\mathrm{e}^{-\frac{2i\sqrt{3^{\frac{1}{3}}-R}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2iR}\,\operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12iR}}{6},\,\frac{i}{2}+\frac{\sqrt{3}}{2}\right)3^{\frac{5}{6}}+2\sqrt{3^{\frac{1}{3}}-R}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2iR}\,\operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}}}{6}\right.}{}}\right. \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$y\,\mathrm{e}^{\frac{\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{-3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{2}\,\operatorname{EllipticF}\left(\frac{i3^{\frac{7}{12}}\sqrt{2}\,\sqrt{-3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}}{6},\,\frac{i}{2}+\frac{\sqrt{3}}{2}\right)}{\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{t^{3}-3}}} = \left(\int t\,\mathrm{e}^{-\frac{2i\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3}}{}\right.}{}}\right.$$

Which simplifies to

$$y\,\mathrm{e}^{\frac{\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{-3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{2}\,\mathrm{EllipticF}\left(\frac{i3^{\frac{7}{12}}\sqrt{2}\,\sqrt{-3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)}{\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{t^3-3}}} = \left(\int t\,\mathrm{e}^{-\frac{2i\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3}}{}\right.}{}}\right.$$

Which gives

$$y=\mathrm{e}^{\int \frac{-12i3^{\frac{11}{12}}t+24i3^{\frac{7}{12}}t^2+36\,3^{\frac{5}{12}}t+\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{18+6i\sqrt{3}+12i3^{\frac{1}{6}}t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{18-6i\sqrt{3}+6i3^{\frac{1}{6}}t-6\,3^{\frac{2}{3}}t}-36\,3^{\frac{3}{4}}-36i3^{\frac{1}{4}}}{\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{t^3-3}\,\sqrt{18-6i\sqrt{3}+6i3^{\frac{1}{6}}t-6\,3^{\frac{2}{3}}t}\,\sqrt{18+6i\sqrt{3}+12i3^{\frac{1}{6}}t}}dt}\left(\int t\,\mathrm{e}^{-\frac{2i\sqrt{}}{}}\right.$$

Summary

The solution(s) found are the following

$$y \tag{1}$$
$$=\mathrm{e}^{\int \frac{-12i3^{\frac{11}{12}}t+24i3^{\frac{7}{12}}t^2+36\,3^{\frac{5}{12}}t+\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{18+6i\sqrt{3}+12i3^{\frac{1}{6}}t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{18-6i\sqrt{3}+6i3^{\frac{1}{6}}t-6\,3^{\frac{2}{3}}t}-36\,3^{\frac{3}{4}}-36i3^{\frac{1}{4}}}{\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{t^3-3}\,\sqrt{18-6i\sqrt{3}+6i3^{\frac{1}{6}}t-6\,3^{\frac{2}{3}}t}\,\sqrt{18+6i\sqrt{3}+12i3^{\frac{1}{6}}t}}dt}\left(\int t\,\mathrm{e}^{-\frac{2i\sqrt{3^{\frac{1}{3}}}}{}}\right.$$
$$\left.+\,c_1\right)\mathrm{e}^{-\frac{\mathrm{EllipticF}\left(\frac{i\sqrt{-2\,3^{\frac{5}{6}}+2i3^{\frac{1}{3}}+4it}\,3^{\frac{7}{12}}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{-3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{2\,3^{\frac{1}{3}}-2t}}{\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{t^3-3}}}$$

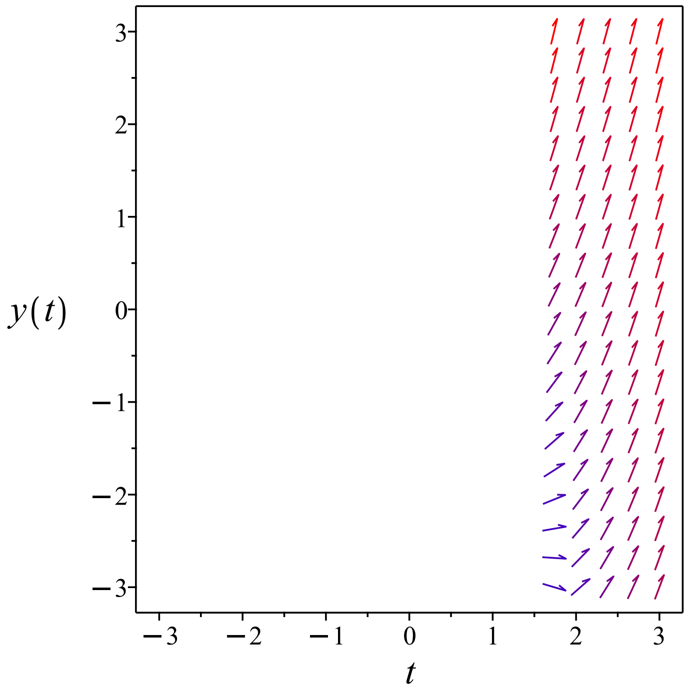


Figure 283: Slope field plot

<u>Verification of solutions</u>

$$y = e^{\int \frac{-12i3^{\frac{11}{12}} t+24i3^{\frac{7}{12}} t^2+36\,3^{\frac{5}{12}} t+\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{18+6i\sqrt{3}+12i3^{\frac{1}{6}} t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{18-6i\sqrt{3}+6i3^{\frac{1}{6}} t-6\,3^{\frac{2}{3}} t}-36\,3^{\frac{3}{4}}-36i3^{\frac{1}{4}}}{\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{3^{\frac{1}{3}}-t}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{t^3-3}\,\sqrt{18-6i\sqrt{3}+6i3^{\frac{1}{6}} t-6\,3^{\frac{2}{3}} t}\,\sqrt{18+6i\sqrt{3}+12i3^{\frac{1}{6}} t}}dt} \left(\int t\, e^{-\frac{2i\sqrt{3^{\frac{1}{3}}}}{}}\right.$$

$$\left. + c_1\right) e^{-\frac{\operatorname{EllipticF}\left(\frac{i\sqrt{-2\,3^{\frac{5}{6}}+2i3^{\frac{1}{3}}+4it}\,3^{\frac{7}{12}}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{-3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\sqrt{2\,3^{\frac{1}{3}}-2t}}{\sqrt{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}\,\sqrt{t^3-3}}}$$

Verified OK.

7.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\frac{y}{\sqrt{t^3-3}}+t\right)\mathrm{d}t$$

$$\left(-\frac{y}{\sqrt{t^3-3}}-t\right)\mathrm{d}t+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,y) &= -\frac{y}{\sqrt{t^3-3}} - t\\
N(t,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{\sqrt{t^3-3}} - t\right)\\
&= -\frac{1}{\sqrt{t^3-3}}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\
&= 1\left(\left(-\frac{1}{\sqrt{t^3-3}}\right) - (0)\right)\\
&= -\frac{1}{\sqrt{t^3-3}}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t}\\
&= e^{\int -\frac{1}{\sqrt{t^3-3}} \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{2 i 3^{\frac{5}{6}} \sqrt{-i\left(t+\frac{3 3^{\frac{1}{3}}}{2}+\frac{i 3^{\frac{5}{6}}}{2}\right) 3^{\frac{1}{6}}}\, \sqrt{\frac{t-3^{\frac{1}{3}}}{-\frac{3 3^{\frac{1}{3}}}{2}-\frac{i 3^{\frac{5}{6}}}{2}}}\, \sqrt{i\left(t-\frac{i 3^{\frac{5}{6}}}{2}+\frac{3 3^{\frac{1}{3}}}{2}\right) 3^{\frac{1}{6}}}\ \operatorname{EllipticF}\left(\frac{\sqrt{3}\, \sqrt{-i\left(t+\frac{3 3^{\frac{1}{3}}}{2}+\frac{i 3^{\frac{5}{6}}}{2}\right) 3^{\frac{1}{6}}}}{3}, \sqrt{-\frac{i 3^{\frac{5}{6}}}{-\frac{3 3^{\frac{1}{3}}}{2}-\frac{i 3^{\frac{5}{6}}}{2}}}\right)}{3 \sqrt{t^{3}-3}}} \\
&= \mathrm{e}^{-\frac{2 \sqrt{\frac{3^{\frac{1}{3}}-t}{i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}}}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\ \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 t\right)}{\sqrt{t^{3}-3}\, \sqrt{2 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{2 \sqrt{\frac{3^{\frac{1}{3}}-t}{i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}}}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\ \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 t\right)}{\sqrt{t^{3}-3}\, \sqrt{2 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}}\left(-\frac{y}{\sqrt{t^{3}-3}}-t\right) \\
&= \frac{\left(-t \sqrt{t^{3}-3}-y\right) \mathrm{e}^{-\frac{2 \sqrt{\frac{3^{\frac{1}{3}}-t}{i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}}}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\ \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 t\right)}{\sqrt{t^{3}-3}\, \sqrt{2 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}}}{\sqrt{t^{3}-3}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{2 \sqrt{\frac{3^{\frac{1}{3}}-t}{i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}}}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\ \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 t\right)}{\sqrt{t^{3}-3}\, \sqrt{2 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}}(1) \\
&= \mathrm{e}^{-\frac{2 \sqrt{\frac{3^{\frac{1}{3}}-t}{i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}}}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\ \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 t\right)}{\sqrt{t^{3}-3}\, \sqrt{2 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\left(\frac{\left(-t\sqrt{t^3-3}-y\right) \mathrm{e}^{-\frac{2\sqrt{\frac{3 3^{\frac{1}{3}}-t}{i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}+2 t\right)}{\sqrt{t^3-3}\sqrt{2 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}}}{\sqrt{t^3-3}}\right)+\left(\mathrm{e}^{-\frac{2\sqrt{\frac{3 3^{\frac{1}{3}}-t}{i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}\cdots}}{}}\right.$$

The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial\phi}{\partial t}=\overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t=\int\overline{M}\,\mathrm{d}t$$

$$\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t$$

$$=\int\frac{\left(-t\sqrt{t^3-3}-y\right) \mathrm{e}^{-\frac{2\sqrt{\frac{3 3^{\frac{1}{3}}-t}{i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}+2 t\right)}{\sqrt{t^3-3}\sqrt{2 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}}}{\sqrt{t^3-3}}\,\mathrm{d}t$$

$$\phi=\int^{t}\frac{\left(-\_a\sqrt{\_a^3-3}-y\right) \mathrm{e}^{-\frac{2\sqrt{\frac{3 3^{\frac{1}{3}}-\_a}{i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i\_a}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i 3^{\frac{5}{6}}+3 3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^3-3}\sqrt{2 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i\_a}}}}{\sqrt{\_a^3-3}}d\_ + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\left(\int^{t} \frac{\mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^{3}-3}\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^{3}-3}}d\_a\right) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-t}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12it}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2t\right)}{\sqrt{t^{3}-3}\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4it}}}$.
Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-t}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12it}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2t\right)}{\sqrt{t^{3}-3}\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4it}}} = -\left(\int^{t} \frac{\mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^{3}-3}\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^{3}-3}}d\_a\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \int^{t} \frac{\mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^{3}-3}\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^{3}-3}}d\_a + \mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-t}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12it}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2t\right)}{\sqrt{t^{3}-3}\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4it}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\int^{t} \frac{\mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^{3}-3}}d\_a + \mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-t}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12it}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2t\right)}{\sqrt{t^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4it}}}\right)\mathrm{d}y$$

$$f(y) = \left(\int^{t} \frac{\mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^{3}-3}}d\_a + \mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-t}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12it}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2t\right)}{\sqrt{t^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4it}}}\right) y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\begin{aligned}
\phi \\
&= \int^{t} \frac{\left(-\_a\sqrt{\_a^{3}-3}-y\right) \mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^{3}-3}}d\_a \\
&+ \left(\int^{t} \frac{\mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^{3}-3}}d\_a \right. \\
&\left. + \mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-t}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12it}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2t\right)}{\sqrt{t^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4it}}}\right) y + c_1
\end{aligned}
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^t \frac{\left(-\_a\sqrt{\_a^3-3}-y\right) e^{-\frac{2\sqrt{\frac{3\,3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^3-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^3-3}}d\_a + \left(\int^t \frac{e^{-\frac{2\sqrt{\frac{3\,3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^3-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^3-3}}d\_a + e^{-\frac{2\sqrt{\frac{3\,3^{\frac{1}{3}}-t}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\,\mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12it}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2t\right)}{\sqrt{t^3-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4it}}}\right) y$$

Summary

The solution(s) found are the following

$$
\int^{t} \frac{\left(-\_a\sqrt{\_a^{3}-3}-y\right) \mathrm{e}^{-\frac{2 \sqrt{\frac{3^{\frac{1}{3}}-\_a}{i 3^{\frac{5}{6}}+3\, 3^{\frac{1}{3}}}}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i \_a}\, \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6\, 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i \_a}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right) \left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 \_a\right)}{\sqrt{\_a^{3}-3}\, \sqrt{2\, 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i \_a}}}}{\sqrt{\_a^{3}-3}} d\_a
+ \left(\int^{t} \frac{\mathrm{e}^{-\frac{2 \sqrt{\frac{3^{\frac{1}{3}}-\_a}{i 3^{\frac{5}{6}}+3\, 3^{\frac{1}{3}}}}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i \_a}\, \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6\, 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i \_a}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right) \left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 \_a\right)}{\sqrt{\_a^{3}-3}\, \sqrt{2\, 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i \_a}}}}{\sqrt{\_a^{3}-3}} d\_a
+ \mathrm{e}^{-\frac{2 \sqrt{\frac{3^{\frac{1}{3}}-t}{i 3^{\frac{5}{6}}+3\, 3^{\frac{1}{3}}}}\, \sqrt{3^{\frac{5}{6}}+i 3^{\frac{1}{3}}+2 i t}\, \operatorname{EllipticF}\left(\frac{3^{\frac{1}{12}} \sqrt{6\, 3^{\frac{5}{6}}-6 i 3^{\frac{1}{3}}-12 i t}}{6}, \frac{i}{2}+\frac{\sqrt{3}}{2}\right) \left(i 3^{\frac{5}{6}}+3^{\frac{1}{3}}+2 t\right)}{\sqrt{t^{3}-3}\, \sqrt{2\, 3^{\frac{5}{6}}-2 i 3^{\frac{1}{3}}-4 i t}}}\right) y = c_{1} \tag{1}
$$

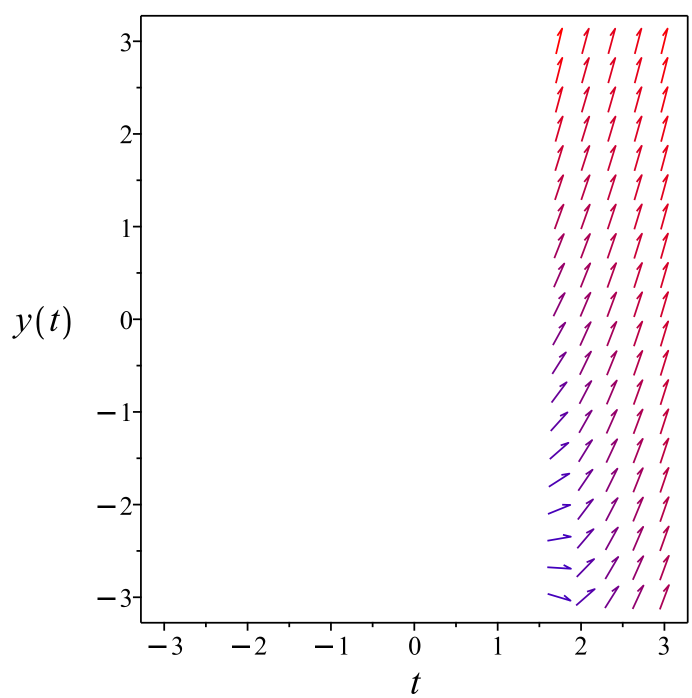


Figure 284: Slope field plot

Verification of solutions

$$\int^{t} \frac{\left(-\_a\sqrt{\_a^{3}-3}-y\right) e^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\ \mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^{3}-3}} d\_a + \left(\int^{t} \frac{e^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-\_a}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2i\_a}\ \mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12i\_a}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2\_a\right)}{\sqrt{\_a^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4i\_a}}}}{\sqrt{\_a^{3}-3}} d\_a + e^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-t}{i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+i3^{\frac{1}{3}}+2it}\ \mathrm{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6i3^{\frac{1}{3}}-12it}}{6},\frac{i}{2}+\frac{\sqrt{3}}{2}\right)\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}+2t\right)}{\sqrt{t^{3}-3}\,\sqrt{2\,3^{\frac{5}{6}}-2i3^{\frac{1}{3}}-4it}}}\right) y = c_1$$

Verified OK.

7.18.4 Maple step by step solution

Let's solve

$y' - \frac{y}{\sqrt{t^3-3}} = t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{\sqrt{t^3-3}} + t$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{\sqrt{t^3-3}} = t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - \frac{y}{\sqrt{t^3-3}}\right) = \mu(t)\, t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$$\mu(t)\left(y'-\frac{y}{\sqrt{t^3-3}}\right)=\mu'(t)\,y+\mu(t)\,y'$$

- Isolate $\mu'(t)$

$$\mu'(t)=-\frac{\mu(t)}{\sqrt{t^3-3}}$$

- Solve to find the integrating factor

$$\mu(t)=\mathrm{e}^{\int -\frac{1}{\sqrt{t^3-3}}dt}$$

- Integrate both sides with respect to t

$$\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int \mu(t)\,tdt+c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\,y=\int \mu(t)\,tdt+c_1$$

- Solve for y

$$y=\frac{\int \mu(t)tdt+c_1}{\mu(t)}$$

- Substitute $\mu(t)=\mathrm{e}^{\int -\frac{1}{\sqrt{t^3-3}}dt}$

$$y=\frac{\int t\,\mathrm{e}^{\int -\frac{1}{\sqrt{t^3-3}}dt}dt+c_1}{\mathrm{e}^{\int -\frac{1}{\sqrt{t^3-3}}dt}}$$

- Evaluate the integrals on the rhs

$$y=\frac{\int t\,\mathrm{e}^{\frac{-\frac{2\mathrm{I}}{3}3^{\frac{5}{6}}\sqrt{-\mathrm{I}\left(t+\frac{3^{\frac{1}{3}}}{2}+\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}\right)3^{\frac{1}{6}}}\,\sqrt{\frac{t-3^{\frac{1}{3}}}{-\frac{3\,3^{\frac{1}{3}}}{2}-\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}}}\,\sqrt{\mathrm{I}\left(t-\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}+\frac{3^{\frac{1}{3}}}{2}\right)3^{\frac{1}{6}}}\,\mathit{EllipticF}\left(\frac{\sqrt{3}\,\sqrt{-\mathrm{I}\left(t+\frac{3^{\frac{1}{3}}}{2}+\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}\right)3^{\frac{1}{6}}}}{3},\sqrt{\frac{-\mathrm{I}\,3^{\frac{5}{6}}}{-\frac{3\,3^{\frac{1}{3}}}{2}-\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}}}\right)}{\sqrt{t^3-3}}}dt+c}{\mathrm{e}^{\frac{-\frac{2\mathrm{I}}{3}3^{\frac{5}{6}}\sqrt{-\mathrm{I}\left(t+\frac{3^{\frac{1}{3}}}{2}+\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}\right)3^{\frac{1}{6}}}\,\sqrt{\frac{t-3^{\frac{1}{3}}}{-\frac{3\,3^{\frac{1}{3}}}{2}-\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}}}\,\sqrt{\mathrm{I}\left(t-\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}+\frac{3^{\frac{1}{3}}}{2}\right)3^{\frac{1}{6}}}\,\mathit{EllipticF}\left(\frac{\sqrt{3}\,\sqrt{-\mathrm{I}\left(t+\frac{3^{\frac{1}{3}}}{2}+\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}\right)3^{\frac{1}{6}}}}{3},\sqrt{\frac{-\mathrm{I}\,3^{\frac{5}{6}}}{-\frac{3\,3^{\frac{1}{3}}}{2}-\frac{\mathrm{I}\,3^{\frac{5}{6}}}{2}}}\right)}{\sqrt{t^3-3}}}}$$

- Simplify

$$y=\left(\int \mathrm{e}^{-\frac{2\sqrt{\frac{3^{\frac{1}{3}}-t}{\mathrm{I}\,3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+\mathrm{I}\,3^{\frac{1}{3}}+2\,\mathrm{I}t}\,\mathit{EllipticF}\left(\frac{3^{\frac{1}{12}}\sqrt{6\,3^{\frac{5}{6}}-6\,\mathrm{I}\,3^{\frac{1}{3}}-12\,\mathrm{I}t}}{6},\frac{\mathrm{I}}{2}+\frac{\sqrt{3}}{2}\right)\left(\mathrm{I}\,3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}+2t\right)}{\sqrt{t^3-3}\,\sqrt{2\,3^{\frac{5}{6}}-2\,\mathrm{I}\,3^{\frac{1}{3}}-4\,\mathrm{I}t}}}tdt+c_1\right)\mathrm{e}^{\frac{2\sqrt{\frac{3^{\frac{1}{3}}-t}{\mathrm{I}\,3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}}}\,\sqrt{3^{\frac{5}{6}}+\cdots}}{\cdots}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(t),t)=y(t)/sqrt(t^3-3)+t,y(t), singsol=all)
```

$$y(t) = \left(\int t\,\mathrm{e}^{-\left(\int \frac{1}{\sqrt{t^3-3}}dt\right)}dt + c_1\right)\mathrm{e}^{\int \frac{1}{\sqrt{t^3-3}}dt}$$

✓ Solution by Mathematica

Time used: 20.591 (sec). Leaf size: 110

```
DSolve[y'[t]==y[t]/Sqrt[t^3-3]+t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{\frac{t\sqrt{1-\frac{t^3}{3}}\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{2},\frac{4}{3},\frac{t^3}{3}\right)}{\sqrt{t^3-3}}}\left(\int_1^t \exp\left(-\frac{\text{Hypergeometric2F1}\left(\frac{1}{3},\frac{1}{2},\frac{4}{3},\frac{K[1]^3}{3}\right)K[1]\sqrt{1-\frac{K[1]^3}{3}}}{\sqrt{K[1]^3-3}}\right)K[1]d\right. \left. + c_1\right)$$

7.19 problem 19

Internal problem ID [13024]
Internal file name [`OUTPUT/11676_Wednesday_November_08_2023_03_28_38_AM_47368872/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - aty = 4\,\mathrm{e}^{-t^2}}$$

7.19.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -ta$$
$$q(t) = 4\,\mathrm{e}^{-t^2}$$

Hence the ode is

$$y' - aty = 4\,\mathrm{e}^{-t^2}$$

The integrating factor μ is

$$\mu = e^{\int -t a dt}$$
$$= e^{-\frac{t^2 a}{2}}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(4\,e^{-t^2}\right)$$
$$\frac{d}{dt}\left(e^{-\frac{t^2 a}{2}} y\right) = \left(e^{-\frac{t^2 a}{2}}\right)\left(4\,e^{-t^2}\right)$$
$$d\left(e^{-\frac{t^2 a}{2}} y\right) = \left(4\,e^{-\frac{t^2(a+2)}{2}}\right) dt$$

Integrating gives

$$e^{-\frac{t^2 a}{2}} y = \int 4\,e^{-\frac{t^2(a+2)}{2}}\, dt$$
$$e^{-\frac{t^2 a}{2}} y = \frac{4\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right)}{\sqrt{2a+4}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{t^2 a}{2}}$ results in

$$y = \frac{4\,e^{\frac{t^2 a}{2}}\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right)}{\sqrt{2a+4}} + c_1 e^{\frac{t^2 a}{2}}$$

Summary

The solution(s) found are the following

$$y = \frac{4\,e^{\frac{t^2 a}{2}}\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right)}{\sqrt{2a+4}} + c_1 e^{\frac{t^2 a}{2}} \tag{1}$$

Verification of solutions

$$y = \frac{4\,e^{\frac{t^2 a}{2}}\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right)}{\sqrt{2a+4}} + c_1 e^{\frac{t^2 a}{2}}$$

Verified OK.

7.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= aty + 4\,\mathrm{e}^{-t^2} \\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 278: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= \mathrm{e}^{\frac{t^2 a}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{\frac{t^2 a}{2}}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{-\frac{t^2 a}{2}} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = aty + 4\,\mathrm{e}^{-t^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= -ta\,\mathrm{e}^{-\frac{t^2 a}{2}} y \\
S_y &= \mathrm{e}^{-\frac{t^2 a}{2}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4\,\mathrm{e}^{-\frac{t^2(a+2)}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4\,\mathrm{e}^{-\frac{R^2(a+2)}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{4\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2a+4}\,R}{2}\right)}{\sqrt{2a+4}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-\frac{t^2 a}{2}} y = \frac{4\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right)}{\sqrt{2a+4}} + c_1$$

Which simplifies to

$$\mathrm{e}^{-\frac{t^2 a}{2}} y = \frac{4\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right)}{\sqrt{2a+4}} + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{\frac{t^2 a}{2}}\left(4\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right) + \sqrt{2a+4}\,c_1\right)}{\sqrt{2a+4}}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{\frac{t^2 a}{2}}\left(4\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right) + \sqrt{2a+4}\,c_1\right)}{\sqrt{2a+4}} \tag{1}$$

Verification of solutions

$$y = \frac{\mathrm{e}^{\frac{t^2 a}{2}}\left(4\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right) + \sqrt{2a+4}\,c_1\right)}{\sqrt{2a+4}}$$

Verified OK.

7.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(aty+4\,\mathrm{e}^{-t^2}\right)\mathrm{d}t\\ \left(-aty-4\,\mathrm{e}^{-t^2}\right)\mathrm{d}t+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-aty-4\,\mathrm{e}^{-t^2}\\ N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-aty - 4\,\mathrm{e}^{-t^2}\right)\\ &= -ta\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((-ta) - (0))\\ &= -ta\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -ta\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{t^2 a}{2}}\\ &= \mathrm{e}^{-\frac{t^2 a}{2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-\frac{t^2 a}{2}}\left(-aty - 4\,\mathrm{e}^{-t^2}\right)\\ &= -\mathrm{e}^{-\frac{t^2 a}{2}}\left(aty + 4\,\mathrm{e}^{-t^2}\right)\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{t^2 a}{2}}(1) \\
&= \mathrm{e}^{-\frac{t^2 a}{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{-\frac{t^2 a}{2}}\left(aty + 4\,\mathrm{e}^{-t^2}\right)\right) + \left(\mathrm{e}^{-\frac{t^2 a}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\mathrm{e}^{-\frac{t^2 a}{2}}\left(aty + 4\,\mathrm{e}^{-t^2}\right)\mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{\mathrm{e}^{-\frac{t^2 a}{2}} y\sqrt{2a+4} - 4\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right)}{\sqrt{2a+4}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{t^2 a}{2}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{t^2 a}{2}}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-\frac{t^2 a}{2}} = \mathrm{e}^{-\frac{t^2 a}{2}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{e^{-\frac{t^2 a}{2}} y\sqrt{2a+4} - 4\sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2a+4}\, t}{2}\right)}{\sqrt{2a+4}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{e^{-\frac{t^2 a}{2}} y\sqrt{2a+4} - 4\sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2a+4}\, t}{2}\right)}{\sqrt{2a+4}}$$

The solution becomes

$$y = \frac{e^{\frac{t^2 a}{2}} \left(4\sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2a+4}\, t}{2}\right) + \sqrt{2a+4}\, c_1\right)}{\sqrt{2a+4}}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{\frac{t^2 a}{2}} \left(4\sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2a+4}\, t}{2}\right) + \sqrt{2a+4}\, c_1\right)}{\sqrt{2a+4}} \tag{1}$$

Verification of solutions

$$y = \frac{e^{\frac{t^2 a}{2}} \left(4\sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2a+4}\, t}{2}\right) + \sqrt{2a+4}\, c_1\right)}{\sqrt{2a+4}}$$

Verified OK.

7.19.4 Maple step by step solution

Let's solve

$y' - aty = 4\,\mathrm{e}^{-t^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = aty + 4\,\mathrm{e}^{-t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - aty = 4\,\mathrm{e}^{-t^2}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - aty) = 4\mu(t)\,\mathrm{e}^{-t^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - aty) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)\,ta$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-\frac{t^2a}{2}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right)dt = \int 4\mu(t)\,\mathrm{e}^{-t^2}dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 4\mu(t)\,\mathrm{e}^{-t^2}dt + c_1$

- Solve for y

  $y = \frac{\int 4\mu(t)\mathrm{e}^{-t^2}dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-\frac{t^2a}{2}}$

  $y = \frac{\int 4\,\mathrm{e}^{-t^2}\mathrm{e}^{-\frac{t^2a}{2}}dt + c_1}{\mathrm{e}^{-\frac{t^2a}{2}}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{4\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{2a+4}\,t}{2}\right)}{\sqrt{2a+4}} + c_1}{\mathrm{e}^{-\frac{t^2a}{2}}}$

- Simplify

$$y = \frac{e^{\frac{t^2 a}{2}} \left(4\sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2a+4}\, t}{2}\right) + \sqrt{2a+4}\, c_1\right)}{\sqrt{2a+4}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(t),t)=a*t*y(t)+4*exp(-t^2),y(t), singsol=all)
```

$$y(t) = \frac{\left(c_1\sqrt{2a+4} + 4\sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2a+4}\, t}{2}\right)\right) e^{\frac{a\,t^2}{2}}}{\sqrt{2a+4}}$$

✓ Solution by Mathematica

Time used: 0.213 (sec). Leaf size: 58

```
DSolve[y'[t]==a*t*y[t]+4*Exp[-t^2],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{\frac{at^2}{2}} \left(2\sqrt{2\pi}\mathrm{erf}\left(\frac{\sqrt{a+2}t}{\sqrt{2}}\right) + \sqrt{a+2}c_1\right)}{\sqrt{a+2}}$$

7.20 problem 20

Internal problem ID [13025]
Internal file name [`OUTPUT/11677_Wednesday_November_08_2023_03_28_39_AM_78626353/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - t^r y = 4}$$

7.20.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -t^r$$
$$q(t) = 4$$

Hence the ode is

$$y' - t^r y = 4$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -t^r dt} \\
&= e^{-\frac{t^{1+r}}{1+r}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)(4) \\
\frac{d}{dt}\left(e^{-\frac{t^{1+r}}{1+r}} y\right) &= \left(e^{-\frac{t^{1+r}}{1+r}}\right)(4) \\
d\left(e^{-\frac{t^{1+r}}{1+r}} y\right) &= \left(4\, e^{-\frac{t^{1+r}}{1+r}}\right) dt
\end{aligned}
$$

Integrating gives

$$
e^{-\frac{t^{1+r}}{1+r}} y = \int 4\, e^{-\frac{t^{1+r}}{1+r}}\, dt
$$

$$
e^{-\frac{t^{1+r}}{1+r}} y = \frac{4\left(\frac{1}{1+r}\right)^{-\frac{1}{1+r}} \left(\frac{(1+r)^2 t^{\frac{1}{1+r}+\frac{r}{1+r}-1-r} \left(\frac{1}{1+r}\right)^{\frac{1}{1+r}} \left(\frac{t^{1+r} r^2}{1+r}+\frac{2t^{1+r} r}{1+r}+r^2+\frac{t^{1+r}}{1+r}+3r+2\right) \left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}} e^{-\frac{t^{1+r}}{2(1+r)}} \text{WhittakerM}}{(r+2)(2r+3)}\right.}{}
$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{t^{1+r}}{1+r}}$ results in

$$
y = \frac{4\, e^{\frac{t^{1+r}}{1+r}} \left(\frac{1}{1+r}\right)^{-\frac{1}{1+r}} \left(\frac{(1+r)^2 t^{\frac{1}{1+r}+\frac{r}{1+r}-1-r} \left(\frac{1}{1+r}\right)^{\frac{1}{1+r}} \left(\frac{t^{1+r} r^2}{1+r}+\frac{2t^{1+r} r}{1+r}+r^2+\frac{t^{1+r}}{1+r}+3r+2\right) \left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}} e^{-\frac{t^{1+r}}{2(1+r)}} \text{WhittakerM}\left(\right.}{(r+2)(2r+3)}\right.}{}
$$

which simplifies to

$$
y = \frac{4\, e^{\frac{t\, t^r}{2+2r}} \left(t^{-r} \left(\frac{t\, t^r}{1+r}\right)^{\frac{-r-2}{2+2r}} (1+r)(r+2)^2 \text{WhittakerM}\left(\frac{r+2}{2+2r}, \frac{2r+3}{2+2r}, \frac{t\, t^r}{1+r}\right) + \left((r+2)\, t^{-r} + t\right) \left(\frac{t\, t^r}{1+r}\right)^{\frac{-r-2}{2+2r}} (1+\right.}{2r^2+7r+6}
$$

Summary

The solution(s) found are the following

$$
y = \frac{4\, e^{\frac{t\, t^r}{2+2r}} \left(t^{-r} \left(\frac{t\, t^r}{1+r}\right)^{\frac{-r-2}{2+2r}} (1+r)(r+2)^2 \text{WhittakerM}\left(\frac{r+2}{2+2r}, \frac{2r+3}{2+2r}, \frac{t\, t^r}{1+r}\right) + \left((r+2)\, t^{-r} + t\right) \left(\frac{t\, t^r}{1+r}\right)^{\frac{-r-2}{2+2r}} (1+r\right.}{2r^2+7r+6} \tag{1}
$$

Verification of solutions

$$y = \frac{4\,\mathrm{e}^{\frac{t\,t^{r}}{2+2r}}\left(t^{-r}\left(\frac{t\,t^{r}}{1+r}\right)^{\frac{-r-2}{2+2r}}(1+r)\,(r+2)^{2}\,\mathrm{WhittakerM}\left(\frac{r+2}{2+2r},\frac{2r+3}{2+2r},\frac{t\,t^{r}}{1+r}\right)+\left((r+2)\,t^{-r}+t\right)\left(\frac{t\,t^{r}}{1+r}\right)^{\frac{-r-2}{2+2r}}(1+r}{2r^{2}+7r+6}$$

Verified OK.

7.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = t^{r}y+4$$
$$y' = \omega(t,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t+\omega(\eta_y-\xi_t)-\omega^2\xi_y-\omega_t\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 281: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{\frac{t^{1+r}}{1+r}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{t^{1+r}}{1+r}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{t^{1+r}}{1+r}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = t^r y + 4$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -y\, t^r e^{-\frac{t^{1+r}}{1+r}} \\ S_y &= e^{-\frac{t^{1+r}}{1+r}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4\, e^{-\frac{t^{1+r}}{1+r}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4\, e^{-\frac{R^{1+r}}{1+r}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{4\left(\frac{1}{1+r}\right)^{-\frac{1}{1+r}} \left(\frac{(1+r)^2 R^{\frac{1}{1+r}+\frac{r}{1+r}-1-r} \left(\frac{1}{1+r}\right)^{\frac{1}{1+r}} \left(\frac{R^{1+r} r^2}{1+r} + \frac{2R^{1+r} r}{1+r} + r^2 + \frac{R^{1+r}}{1+r} + 3r + 2\right) \left(\frac{R^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}} e^{-\frac{R^{1+r}}{2(1+r)}} \text{WhittakerM}}{(r+2)(2r+3)}}{} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-\frac{t^{1+r}}{1+r}} y = \frac{4\left(\frac{1}{1+r}\right)^{-\frac{1}{1+r}} \left(\frac{(1+r)^2 t^{\frac{1}{1+r}+\frac{r}{1+r}-1-r} \left(\frac{1}{1+r}\right)^{\frac{1}{1+r}} \left(\frac{t^{1+r} r^2}{1+r} + \frac{2t^{1+r} r}{1+r} + r^2 + \frac{t^{1+r}}{1+r} + 3r + 2\right) \left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}} e^{-\frac{t^{1+r}}{2(1+r)}} \text{WhittakerM}}{(r+2)(2r+3)}}{}$$

Which simplifies to

$$\frac{-4 e^{-\frac{t^{1+r}}{2+2r}} t^{-\frac{3r}{2}-1} (1+r)^{\frac{3r+4}{2+2r}} (r+2)^2 \text{WhittakerM}\left(\frac{r+2}{2+2r}, \frac{2r+3}{2+2r}, \frac{t^{1+r}}{1+r}\right) - 4(1+r)^{\frac{3r+4}{2+2r}} e^{-\frac{t^{1+r}}{2+2r}} (1+r) \left((r+2) t}{2r^2 + 7r + 6}$$

Which gives

$$y = \frac{\left(4r^2 t^{-\frac{3r}{2}-1} (1+r)^{\frac{3r+4}{2+2r}} e^{-\frac{t^{1+r}}{2(1+r)}} \text{WhittakerM}\left(-\frac{r}{2(1+r)}, \frac{2r+3}{2+2r}, \frac{t^{1+r}}{1+r}\right) + 4r^2 t^{-\frac{3r}{2}-1} (1+r)^{\frac{3r+4}{2+2r}} e^{-\frac{t^{1+r}}{2(1+r)}} \text{Whit}}{}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(4r^2 t^{-\frac{3r}{2}-1} (1+r)^{\frac{3r+4}{2+2r}} e^{-\frac{t^{1+r}}{2(1+r)}} \text{WhittakerM}\left(-\frac{r}{2(1+r)}, \frac{2r+3}{2+2r}, \frac{t^{1+r}}{1+r}\right) + 4r^2 t^{-\frac{3r}{2}-1} (1+r)^{\frac{3r+4}{2+2r}} e^{-\frac{t^{1+r}}{2(1+r)}} \text{Whitta}}{} \tag{1}$$

Verification of solutions

$$y = \frac{\left(4r^2 t^{-\frac{3r}{2}-1} (1+r)^{\frac{3r+4}{2+2r}} e^{-\frac{t^{1+r}}{2(1+r)}} \text{WhittakerM}\left(-\frac{r}{2(1+r)}, \frac{2r+3}{2+2r}, \frac{t^{1+r}}{1+r}\right) + 4r^2 t^{-\frac{3r}{2}-1} (1+r)^{\frac{3r+4}{2+2r}} e^{-\frac{t^{1+r}}{2(1+r)}} \text{Whitta}}{}$$

Verified OK.

7.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(t^r y+4)\,\mathrm{d}t\\ (-t^r y-4)\,\mathrm{d}t+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-t^r y-4\\ N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-t^r y - 4)\\ &= -t^r\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((-t^r) - (0))\\ &= -t^r\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -t^r\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{t^{1+r}}{1+r}}\\ &= \mathrm{e}^{-\frac{t^{1+r}}{1+r}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-\frac{t^{1+r}}{1+r}}(-t^r y - 4)\\ &= -\mathrm{e}^{-\frac{t^{1+r}}{1+r}}(t^r y + 4)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-\frac{t^{1+r}}{1+r}}(1) \\ &= \mathrm{e}^{-\frac{t^{1+r}}{1+r}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(-\mathrm{e}^{-\frac{t^{1+r}}{1+r}}(t^r y + 4)\right) + \left(\mathrm{e}^{-\frac{t^{1+r}}{1+r}}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\mathrm{e}^{-\frac{t^{1+r}}{1+r}}(t^r y + 4)\,\mathrm{d}t\end{aligned}$$

$$\phi = \frac{-4t^{-r}\mathrm{e}^{-\frac{t^{1+r}}{2+2r}}\left(\frac{t^{1+r}}{1+r}\right)^{\frac{-r-2}{2+2r}}(1+r)(r+2)^2\,\mathrm{WhittakerM}\left(\frac{r+2}{2+2r}, \frac{2r+3}{2+2r}, \frac{t^{1+r}}{1+r}\right) - 4((r+2)t^{-r} + t)\left(\frac{t^{1+r}}{1+r}\right.}{2r^2 + 7r +} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{2\left(r + \frac{3}{2}\right)\left(\mathrm{e}^{-\frac{t^{1+r}}{1+r}} - 1\right)(r+2)}{2r^2 + 7r + 6} + f'(y) \qquad (4) \\ &= \mathrm{e}^{-\frac{t^{1+r}}{1+r}} - 1 + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{-\frac{t^{1+r}}{1+r}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{t^{1+r}}{1+r}} = \mathrm{e}^{-\frac{t^{1+r}}{1+r}} - 1 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-4t^{-r}\mathrm{e}^{-\frac{t^{1+r}}{2+2r}}\left(\frac{t^{1+r}}{1+r}\right)^{\frac{-r-2}{2+2r}}(1+r)(r+2)^2 \operatorname{WhittakerM}\left(\frac{r+2}{2+2r}, \frac{2r+3}{2+2r}, \frac{t^{1+r}}{1+r}\right) - 4((r+2)t^{-r}+t)\left(\frac{t^{1+r}}{1+r}\right)^{\frac{-r-2}{2+2r}}\mathrm{e}}{2r^2+7r+6} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-4t^{-r}\mathrm{e}^{-\frac{t^{1+r}}{2+2r}}\left(\frac{t^{1+r}}{1+r}\right)^{\frac{-r-2}{2+2r}}(1+r)(r+2)^2 \operatorname{WhittakerM}\left(\frac{r+2}{2+2r}, \frac{2r+3}{2+2r}, \frac{t^{1+r}}{1+r}\right) - 4((r+2)t^{-r}+t)\left(\frac{t^{1+r}}{1+r}\right)^{\frac{-r-2}{2+2r}}\mathrm{e}}{2r^2+7r+6} + y$$

The solution becomes

$$y = \frac{\left(4r^3t^{-r}\mathrm{e}^{-\frac{t^{1+r}}{2(1+r)}}\left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}} \operatorname{WhittakerM}\left(-\frac{r}{2(1+r)}, \frac{2r+3}{2+2r}, \frac{t^{1+r}}{1+r}\right) + 4r^3t^{-r}\mathrm{e}^{-\frac{t^{1+r}}{2(1+r)}}\left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}} \operatorname{Whittaker}\right.}{}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(4r^3t^{-r}\mathrm{e}^{-\frac{t^{1+r}}{2(1+r)}}\left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}}\mathrm{WhittakerM}\left(-\frac{r}{2(1+r)},\frac{2r+3}{2+2r},\frac{t^{1+r}}{1+r}\right)+4r^3t^{-r}\mathrm{e}^{-\frac{t^{1+r}}{2(1+r)}}\left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}}\mathrm{Whittaker}}{} \tag{1}$$

Verification of solutions

$$y = \frac{\left(4r^3t^{-r}\mathrm{e}^{-\frac{t^{1+r}}{2(1+r)}}\left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}}\mathrm{WhittakerM}\left(-\frac{r}{2(1+r)},\frac{2r+3}{2+2r},\frac{t^{1+r}}{1+r}\right)+4r^3t^{-r}\mathrm{e}^{-\frac{t^{1+r}}{2(1+r)}}\left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}}\mathrm{Whittaker}}{}$$

Verified OK.

7.20.4 Maple step by step solution

Let's solve

$y' - t^r y = 4$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = t^r y + 4$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - t^r y = 4$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - t^r y\right) = 4\mu(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - t^r y\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)\, t^r$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-\frac{t\,t^r}{1+r}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 4\mu(t)\, dt + c_1$

- Evaluate the integral on the lhs

  $$\mu(t)\, y = \int 4\mu(t)\, dt + c_1$$

- Solve for y

  $$y = \frac{\int 4\mu(t)dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{-\frac{t\,t^r}{1+r}}$

  $$y = \frac{\int 4\,\mathrm{e}^{-\frac{t\,t^r}{1+r}} dt + c_1}{\mathrm{e}^{-\frac{t\,t^r}{1+r}}}$$

- Evaluate the integrals on the rhs

  $$y = \frac{4\left(\frac{1}{1+r}\right)^{-\frac{1}{1+r}} \left(\frac{(1+r)^2 t^{\frac{1}{1+r}+\frac{r}{1+r}-1-r} \left(\frac{1}{1+r}\right)^{\frac{1}{1+r}} \left(\frac{t^{1+r} r^2}{1+r} + \frac{2t^{1+r} r}{1+r} + r^2 + \frac{t^{1+r}}{1+r} + 3r + 2\right) \left(\frac{t^{1+r}}{1+r}\right)^{-\frac{r+2}{2(1+r)}} \mathrm{e}^{-\frac{t^{1+r}}{2(1+r)}}\, \mathit{WhittakerM}\left(\frac{1}{1+r} - \right.}{(r+2)(2r+3)}\right.}{}$$

- Simplify

  $$y = \frac{4\,\mathrm{e}^{\frac{t\,t^r}{2+2r}} \left(t^{-r} \left(\frac{t\,t^r}{1+r}\right)^{\frac{-r-2}{2+2r}} (1+r)(r+2)^2\, \mathit{WhittakerM}\left(\frac{r+2}{2+2r}, \frac{2r+3}{2+2r}, \frac{t\,t^r}{1+r}\right) + ((r+2)t^{-r}+t) \left(\frac{t\,t^r}{1+r}\right)^{\frac{-r-2}{2+2r}} (1+r)^2\, \mathit{WhittakerM}\left(-\frac{r}{2+2r}, \right.\right.}{2r^2+7r+6}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 202

```
dsolve(diff(y(t),t)=t^r*y(t)+4,y(t), singsol=all)
```

$$y(t) = \frac{4\,\mathrm{e}^{\frac{t^r t}{2r+2}} \left(t^{-r} \left(\frac{t\,t^r}{r+1}\right)^{\frac{-r-2}{2r+2}} (r+1)\,(r+2)^2 \,\mathrm{WhittakerM}\left(\frac{r+2}{2r+2}, \frac{2r+3}{2r+2}, \frac{t\,t^r}{r+1}\right) + (r+1)^2 \left((r+2)\,t^{-r} + t\right) \left(\frac{t\,t^r}{r+1}\right)^{\frac{-r-}{2r+}} \right.}{2r^2 + 7r + 6}$$

✓ Solution by Mathematica

Time used: 0.12 (sec). Leaf size: 66

```
DSolve[y'[t]==t^r*y[t]+4,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{\frac{t^{r+1}}{r+1}} \left(-\frac{4t \left(\frac{t^{r+1}}{r+1} \right)^{-\frac{1}{r+1}} \Gamma\left(\frac{1}{r+1}, \frac{t^{r+1}}{r+1} \right)}{r+1} + c_1 \right)$$

7.21 problem 21

Internal problem ID [13026]
Internal file name [`OUTPUT/11678_Wednesday_November_08_2023_03_28_41_AM_84814416/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$v' + \frac{2v}{5} = 3\cos(2t)$$

7.21.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$v' + p(t)v = q(t)$$

Where here

$$p(t) = \frac{2}{5}$$
$$q(t) = 3\cos(2t)$$

Hence the ode is

$$v' + \frac{2v}{5} = 3\cos(2t)$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{5}dt}$$
$$= e^{\frac{2t}{5}}$$

The ode becomes

$$\frac{d}{dt}(\mu v) = (\mu)(3\cos(2t))$$
$$\frac{d}{dt}\left(e^{\frac{2t}{5}} v\right) = \left(e^{\frac{2t}{5}}\right)(3\cos(2t))$$
$$d\left(e^{\frac{2t}{5}} v\right) = \left(3\cos(2t)\, e^{\frac{2t}{5}}\right) dt$$

Integrating gives

$$e^{\frac{2t}{5}} v = \int 3\cos(2t)\, e^{\frac{2t}{5}}\, dt$$
$$e^{\frac{2t}{5}} v = \frac{15\cos(2t)\, e^{\frac{2t}{5}}}{52} + \frac{75\sin(2t)\, e^{\frac{2t}{5}}}{52} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{2t}{5}}$ results in

$$v = e^{-\frac{2t}{5}}\left(\frac{15\cos(2t)\, e^{\frac{2t}{5}}}{52} + \frac{75\sin(2t)\, e^{\frac{2t}{5}}}{52}\right) + c_1 e^{-\frac{2t}{5}}$$

which simplifies to

$$v = \frac{75\sin(2t)}{52} + \frac{15\cos(2t)}{52} + c_1 e^{-\frac{2t}{5}}$$

Summary

The solution(s) found are the following

$$v = \frac{75\sin(2t)}{52} + \frac{15\cos(2t)}{52} + c_1 e^{-\frac{2t}{5}} \tag{1}$$

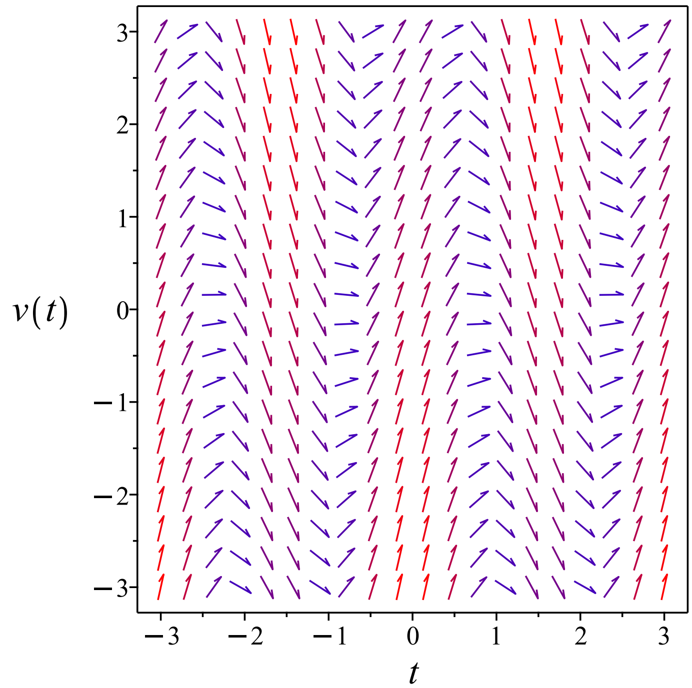


Figure 285: Slope field plot

Verification of solutions

$$v = \frac{75 \sin (2t)}{52} + \frac{15 \cos (2t)}{52} + c_1 e^{-\frac{2t}{5}}$$

Verified OK.

7.21.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$v' = -\frac{2v}{5} + 3 \cos (2t)$$
$$v' = \omega(t, v)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_v - \xi_t) - \omega^2 \xi_v - \omega_t \xi - \omega_v \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 284: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,v) &= 0 \\ \eta(t,v) &= e^{-\frac{2t}{5}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,v) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dv}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial v}\right) S(t,v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{2t}{5}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{2t}{5}} v$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, v) S_v}{R_t + \omega(t, v) R_v} \tag{2}$$

Where in the above R_t, R_v, S_t, S_v are all partial derivatives and $\omega(t, v)$ is the right hand side of the original ode given by

$$\omega(t, v) = -\frac{2v}{5} + 3\cos(2t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_v &= 0 \\ S_t &= \frac{2 e^{\frac{2t}{5}} v}{5} \\ S_v &= e^{\frac{2t}{5}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3\cos(2t) e^{\frac{2t}{5}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3\cos(2R) e^{\frac{2R}{5}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 + \frac{15\,\mathrm{e}^{\frac{2R}{5}} \left(\cos\left(2R\right) + 5\sin\left(2R\right)\right)}{52} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, v coordinates. This results in

$$\mathrm{e}^{\frac{2t}{5}} v = \frac{15\,\mathrm{e}^{\frac{2t}{5}} \left(\cos\left(2t\right) + 5\sin\left(2t\right)\right)}{52} + c_1$$

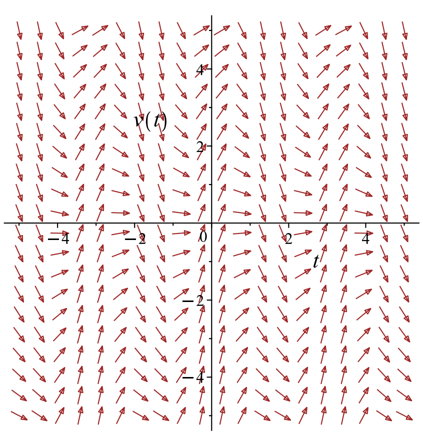
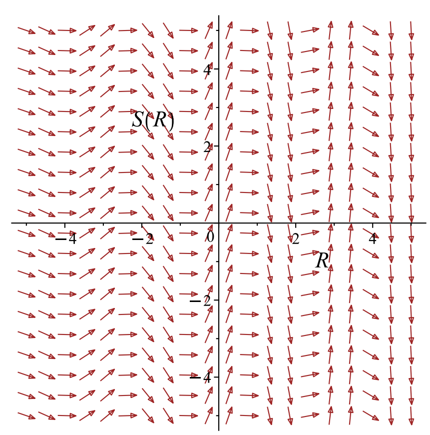
Which simplifies to

$$\mathrm{e}^{\frac{2t}{5}} v = \frac{15\,\mathrm{e}^{\frac{2t}{5}} \left(\cos\left(2t\right) + 5\sin\left(2t\right)\right)}{52} + c_1$$

Which gives

$$v = \frac{\mathrm{e}^{-\frac{2t}{5}} \left(15\cos\left(2t\right)\mathrm{e}^{\frac{2t}{5}} + 75\sin\left(2t\right)\mathrm{e}^{\frac{2t}{5}} + 52c_1\right)}{52}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in t, v coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dv}{dt} = -\frac{2v}{5} + 3\cos\left(2t\right)$
</td><td>$R = t$
$S = \mathrm{e}^{\frac{2t}{5}} v$</td><td>$\frac{dS}{dR} = 3\cos\left(2R\right)\mathrm{e}^{\frac{2R}{5}}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$v = \frac{e^{-\frac{2t}{5}} \left(15 \cos(2t) e^{\frac{2t}{5}} + 75 \sin(2t) e^{\frac{2t}{5}} + 52 c_1\right)}{52} \tag{1}$$

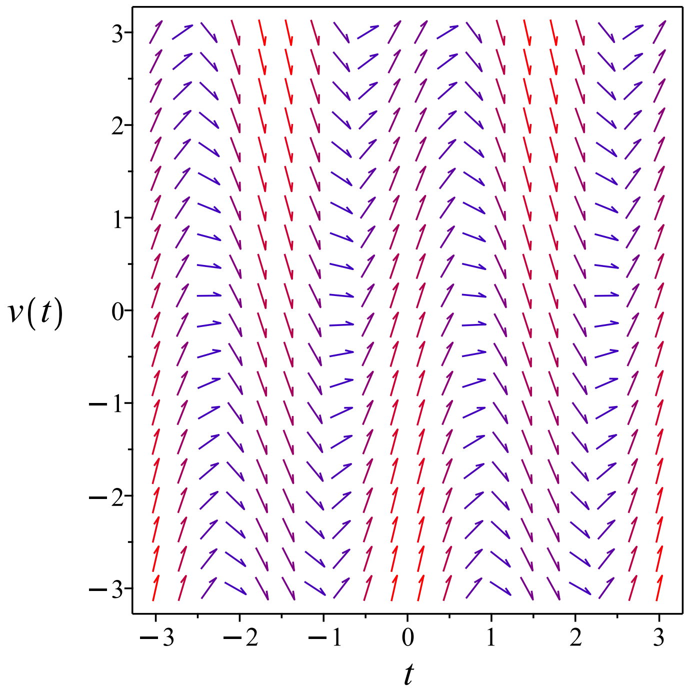


Figure 286: Slope field plot

Verification of solutions

$$v = \frac{e^{-\frac{2t}{5}} \left(15 \cos(2t) e^{\frac{2t}{5}} + 75 \sin(2t) e^{\frac{2t}{5}} + 52 c_1\right)}{52}$$

Verified OK.

7.21.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,v)\,\mathrm{d}t+N(t,v)\,\mathrm{d}v=0 \tag{1A}$$

Therefore

$$\mathrm{d}v=\left(-\frac{2v}{5}+3\cos(2t)\right)\mathrm{d}t$$

$$\left(\frac{2v}{5}-3\cos(2t)\right)\mathrm{d}t+\mathrm{d}v=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,v)&=\frac{2v}{5}-3\cos(2t)\\ N(t,v)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}\left(\frac{2v}{5} - 3\cos(2t)\right) \\ &= \frac{2}{5}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial v} - \frac{\partial N}{\partial t}\right) \\ &= 1\left(\left(\frac{2}{5}\right) - (0)\right) \\ &= \frac{2}{5}\end{aligned}$$

Since A does not depend on v, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \frac{2}{5} \, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{2t}{5}} \\ &= \mathrm{e}^{\frac{2t}{5}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{\frac{2t}{5}}\left(\frac{2v}{5} - 3\cos(2t)\right) \\ &= -\frac{(-2v + 15\cos(2t))\,\mathrm{e}^{\frac{2t}{5}}}{5}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{2t}{5}}(1) \\
&= \mathrm{e}^{\frac{2t}{5}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}v}{\mathrm{d}t} &= 0 \\
\left(-\frac{(-2v + 15\cos(2t))\,\mathrm{e}^{\frac{2t}{5}}}{5}\right) + \left(\mathrm{e}^{\frac{2t}{5}}\right)\frac{\mathrm{d}v}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, v)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial v} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{(-2v + 15\cos(2t))\,\mathrm{e}^{\frac{2t}{5}}}{5}\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{\mathrm{e}^{\frac{2t}{5}}\left(-52v + 15\cos(2t) + 75\sin(2t)\right)}{52} + f(v) \tag{3}
$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both t and v. Taking derivative of equation (3) w.r.t v gives

$$
\frac{\partial \phi}{\partial v} = \mathrm{e}^{\frac{2t}{5}} + f'(v) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = \mathrm{e}^{\frac{2t}{5}}$. Therefore equation (4) becomes

$$
\mathrm{e}^{\frac{2t}{5}} = \mathrm{e}^{\frac{2t}{5}} + f'(v) \tag{5}
$$

Solving equation (5) for $f'(v)$ gives

$$f'(v) = 0$$

Therefore

$$f(v) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(v)$ into equation (3) gives ϕ

$$\phi = -\frac{e^{\frac{2t}{5}}(-52v + 15\cos(2t) + 75\sin(2t))}{52} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{e^{\frac{2t}{5}}(-52v + 15\cos(2t) + 75\sin(2t))}{52}$$

The solution becomes

$$v = \frac{e^{-\frac{2t}{5}}\left(15\cos(2t)\,e^{\frac{2t}{5}} + 75\sin(2t)\,e^{\frac{2t}{5}} + 52c_1\right)}{52}$$

Summary

The solution(s) found are the following

$$v = \frac{e^{-\frac{2t}{5}}\left(15\cos(2t)\,e^{\frac{2t}{5}} + 75\sin(2t)\,e^{\frac{2t}{5}} + 52c_1\right)}{52} \tag{1}$$

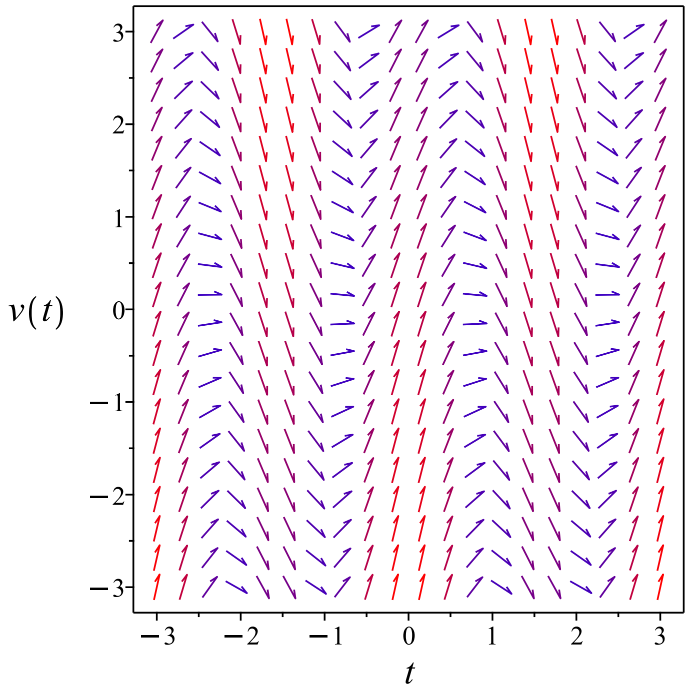


Figure 287: Slope field plot

Verification of solutions

$$v = \frac{e^{-\frac{2t}{5}} \left(15 \cos (2t) e^{\frac{2t}{5}} + 75 \sin (2t) e^{\frac{2t}{5}} + 52 c_1\right)}{52}$$

Verified OK.

7.21.4 Maple step by step solution

Let's solve

$v' + \frac{2v}{5} = 3 \cos (2t)$

- Highest derivative means the order of the ODE is 1

  v'

- Isolate the derivative

  $v' = -\frac{2v}{5} + 3 \cos (2t)$

- Group terms with v on the lhs of the ODE and the rest on the rhs of the ODE

  $v' + \frac{2v}{5} = 3 \cos (2t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t) \left(v' + \frac{2v}{5}\right) = 3\mu(t) \cos(2t)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, v)$

$$\mu(t) \left(v' + \frac{2v}{5}\right) = \mu'(t)\, v + \mu(t)\, v'$$

- Isolate $\mu'(t)$

$$\mu'(t) = \frac{2\mu(t)}{5}$$

- Solve to find the integrating factor

$$\mu(t) = e^{\frac{2t}{5}}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\, v)\right) dt = \int 3\mu(t) \cos(2t)\, dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\, v = \int 3\mu(t) \cos(2t)\, dt + c_1$$

- Solve for v

$$v = \frac{\int 3\mu(t) \cos(2t) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = e^{\frac{2t}{5}}$

$$v = \frac{\int 3\cos(2t) e^{\frac{2t}{5}} dt + c_1}{e^{\frac{2t}{5}}}$$

- Evaluate the integrals on the rhs

$$v = \frac{\frac{15\cos(2t) e^{\frac{2t}{5}}}{52} + \frac{75\sin(2t) e^{\frac{2t}{5}}}{52} + c_1}{e^{\frac{2t}{5}}}$$

- Simplify

$$v = \frac{75\sin(2t)}{52} + \frac{15\cos(2t)}{52} + c_1 e^{-\frac{2t}{5}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(v(t),t)+4/10*v(t)=3*cos(2*t),v(t), singsol=all)
```

$$v(t) = \frac{15\cos(2t)}{52} + \frac{75\sin(2t)}{52} + e^{-\frac{2t}{5}} c_1$$

✓ Solution by Mathematica

Time used: 0.152 (sec). Leaf size: 31

```
DSolve[v'[t]+4/10*v[t]==3*Cos[2*t],v[t],t,IncludeSingularSolutions -> True]
```

$$v(t) \to \frac{15}{52}(5\sin(2t) + \cos(2t)) + c_1 e^{-2t/5}$$

7.22 problem 22 (f)

Internal problem ID [13027]
Internal file name [`OUTPUT/11679_Wednesday_November_08_2023_03_28_41_AM_7293898/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 22 (f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + 2ty = 4\,\mathrm{e}^{-t^2}}$$

7.22.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 2t \\ q(t) &= 4\,\mathrm{e}^{-t^2} \end{aligned}$$

Hence the ode is

$$y' + 2ty = 4\,\mathrm{e}^{-t^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2t dt} \\
&= e^{t^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(4\,e^{-t^2}\right) \\
\frac{d}{dt}\left(e^{t^2} y\right) &= \left(e^{t^2}\right)\left(4\,e^{-t^2}\right) \\
d\left(e^{t^2} y\right) &= 4\,dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{t^2} y &= \int 4\,dt \\
e^{t^2} y &= 4t + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{t^2}$ results in

$$
y = 4\,e^{-t^2} t + c_1 e^{-t^2}
$$

which simplifies to

$$
y = e^{-t^2}(4t + c_1)
$$

Summary

The solution(s) found are the following

$$
y = e^{-t^2}(4t + c_1) \tag{1}
$$

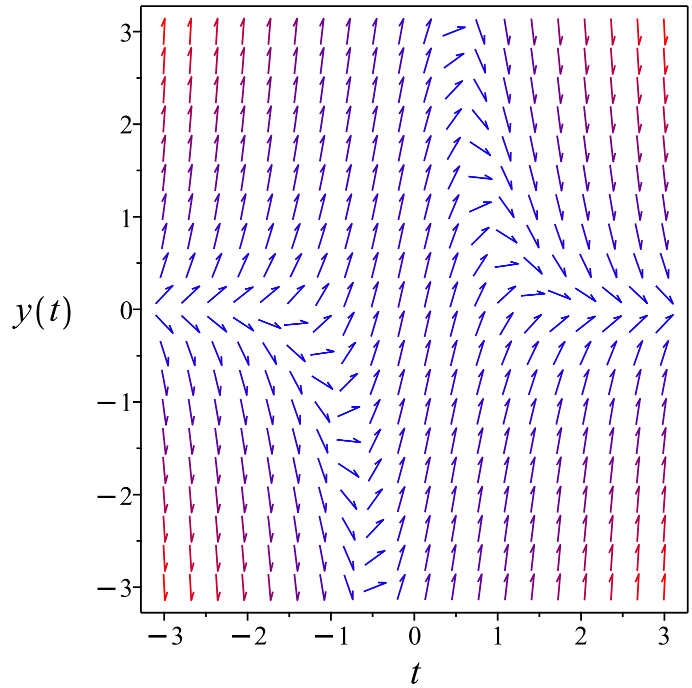


Figure 288: Slope field plot

Verification of solutions

$$y = e^{-t^2}(4t + c_1)$$

Verified OK.

7.22.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2ty + 4\,e^{-t^2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 287: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-t^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-t^2}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{t^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -2ty + 4\,\mathrm{e}^{-t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2t\,\mathrm{e}^{t^2} y \\ S_y &= \mathrm{e}^{t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 4R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$y\,\mathrm{e}^{t^2} = 4t + c_1$$

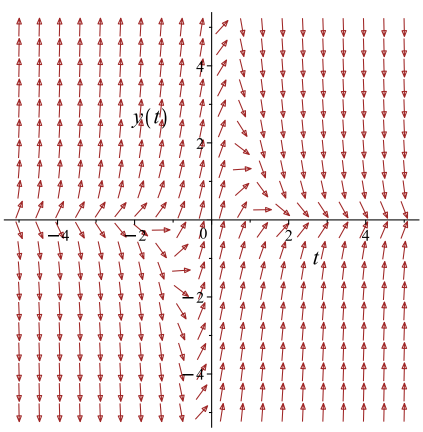
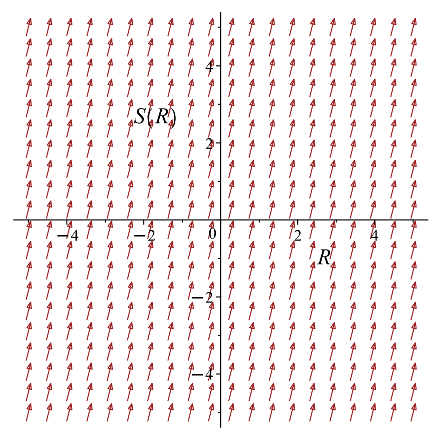
Which simplifies to

$$y\,\mathrm{e}^{t^2} = 4t + c_1$$

Which gives

$$y = \mathrm{e}^{-t^2}(4t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -2ty + 4\,\mathrm{e}^{-t^2}$ 	$R = t$ $S = \mathrm{e}^{t^2} y$	$\frac{dS}{dR} = 4$ 

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-t^2}(4t + c_1) \tag{1}$$

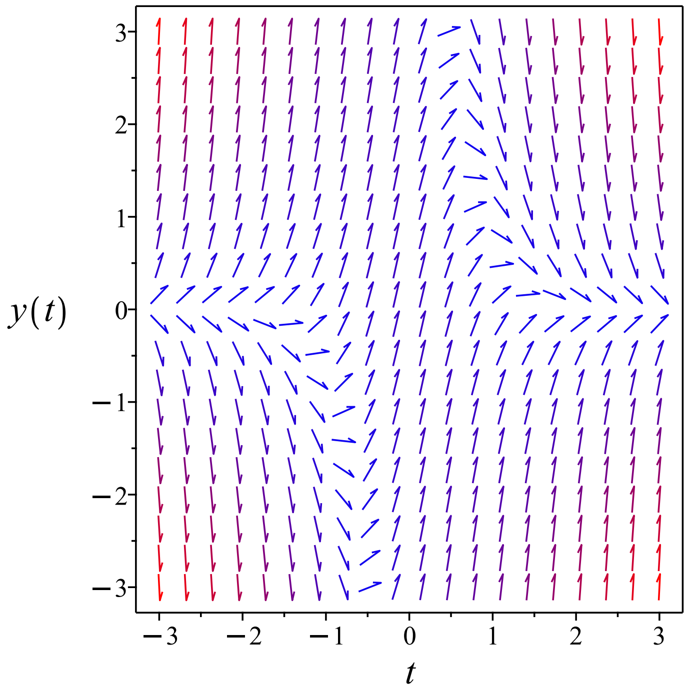


Figure 289: Slope field plot

Verification of solutions

$$y = e^{-t^2}(4t + c_1)$$

Verified OK.

7.22.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-2ty + 4\,\mathrm{e}^{-t^2}\right)\mathrm{d}t$$

$$\left(2ty - 4\,\mathrm{e}^{-t^2}\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= 2ty - 4\,\mathrm{e}^{-t^2} \\ N(t, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2ty - 4\,\mathrm{e}^{-t^2}\right) \\ &= 2t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((2t) - (0)) \\
&= 2t
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}t} \\
&= e^{\int 2t \,\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{t^2} \\
&= \mathrm{e}^{t^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{t^2}\left(2ty - 4\,\mathrm{e}^{-t^2}\right) \\
&= 2t\,\mathrm{e}^{t^2} y - 4
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{t^2}(1) \\
&= \mathrm{e}^{t^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(2t\,\mathrm{e}^{t^2} y - 4\right) + \left(\mathrm{e}^{t^2}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int 2t\,\mathrm{e}^{t^2} y - 4\,\mathrm{d}t$$
$$\phi = -4t + \mathrm{e}^{t^2} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \mathrm{e}^{t^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{t^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{t^2} = \mathrm{e}^{t^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -4t + \mathrm{e}^{t^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -4t + \mathrm{e}^{t^2} y$$

The solution becomes

$$y = e^{-t^2}(4t + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-t^2}(4t + c_1) \tag{1}$$

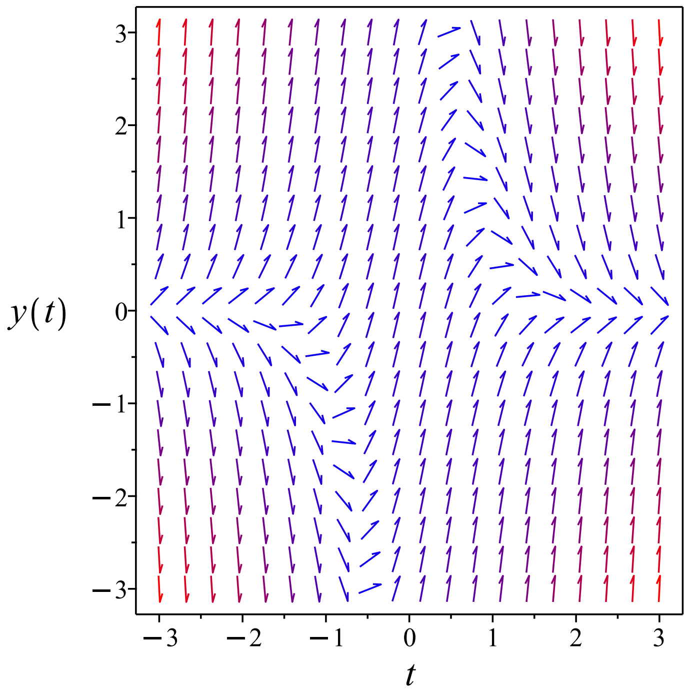


Figure 290: Slope field plot

Verification of solutions

$$y = e^{-t^2}(4t + c_1)$$

Verified OK.

7.22.4 Maple step by step solution

Let's solve

$y' + 2ty = 4\,\mathrm{e}^{-t^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2ty + 4\,\mathrm{e}^{-t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2ty = 4\,\mathrm{e}^{-t^2}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 2ty) = 4\mu(t)\,\mathrm{e}^{-t^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' + 2ty) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)\,t$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t^2}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 4\mu(t)\,\mathrm{e}^{-t^2} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 4\mu(t)\,\mathrm{e}^{-t^2} dt + c_1$

- Solve for y

  $y = \frac{\int 4\mu(t)\mathrm{e}^{-t^2} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{t^2}$

  $y = \frac{\int 4\,\mathrm{e}^{-t^2}\mathrm{e}^{t^2} dt + c_1}{\mathrm{e}^{t^2}}$

- Evaluate the integrals on the rhs

  $y = \frac{4t + c_1}{\mathrm{e}^{t^2}}$

- Simplify

$$y = e^{-t^2}(4t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)=-2*t*y(t)+4*exp(-t^2),y(t), singsol=all)
```

$$y(t) = (4t + c_1)\, e^{-t^2}$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 19

```
DSolve[y'[t]==-2*t*y[t]+4*Exp[-t^2],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t^2}(4t + c_1)$$

7.23 problem 23

Internal problem ID [13028]
Internal file name [`OUTPUT/11680_Wednesday_November_08_2023_03_28_42_AM_33364285/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Exercises section 1.9 page 133
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = 3\,\mathrm{e}^{-2t}}$$

7.23.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 2$$
$$q(t) = 3\,\mathrm{e}^{-2t}$$

Hence the ode is

$$y' + 2y = 3\,\mathrm{e}^{-2t}$$

The integrating factor μ is

$$\mu = e^{\int 2dt}$$
$$= e^{2t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(3\,e^{-2t}\right)$$
$$\frac{d}{dt}\left(e^{2t}y\right) = \left(e^{2t}\right)\left(3\,e^{-2t}\right)$$
$$d\left(e^{2t}y\right) = 3\,dt$$

Integrating gives

$$e^{2t}y = \int 3\,dt$$
$$e^{2t}y = 3t + c_1$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$y = 3t\,e^{-2t} + c_1 e^{-2t}$$

which simplifies to

$$y = e^{-2t}(3t + c_1)$$

Summary
The solution(s) found are the following

$$y = e^{-2t}(3t + c_1) \tag{1}$$

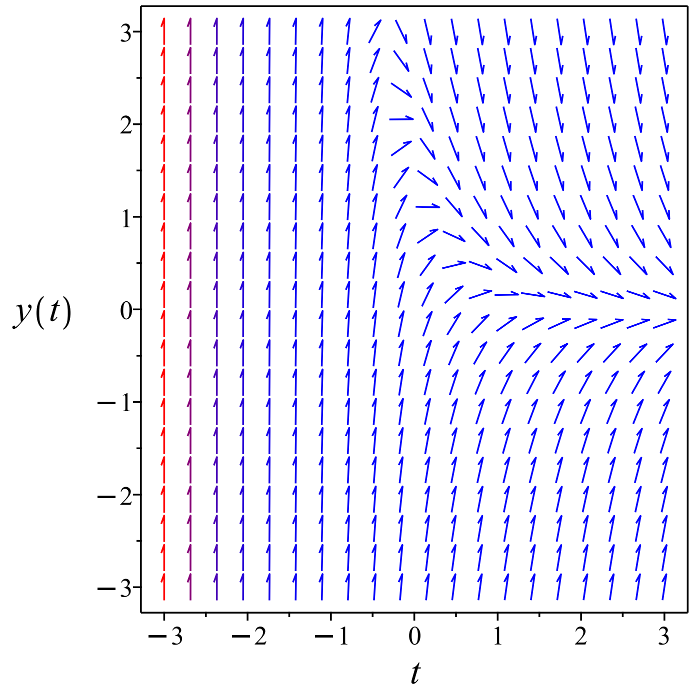


Figure 291: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-2t}(3t + c_1)$$

Verified OK.

7.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2y + 3\,\mathrm{e}^{-2t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 290: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= 0 \\ \eta(t, y) &= \mathrm{e}^{-2t} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2t}} dy \end{aligned}$$

Which results in

$$S = e^{2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -2y + 3 e^{-2t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 2 e^{2t} y \\ S_y &= e^{2t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{2t} y = 3t + c_1$$

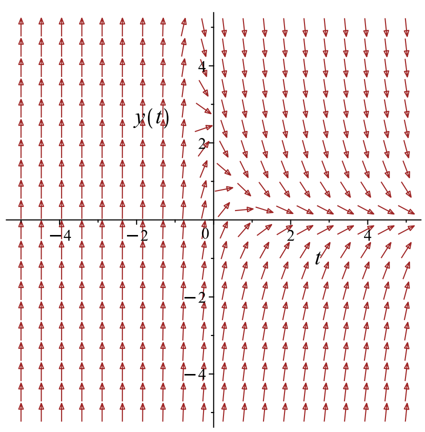
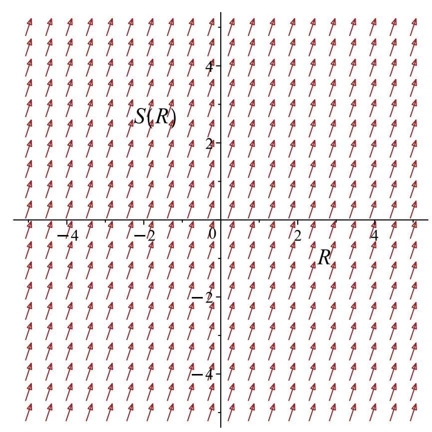
Which simplifies to

$$e^{2t} y = 3t + c_1$$

Which gives

$$y = e^{-2t}(3t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -2y + 3\,e^{-2t}$ 	$R = t$ $S = e^{2t} y$	$\frac{dS}{dR} = 3$ 

Summary
The solution(s) found are the following

$$y = e^{-2t}(3t + c_1) \tag{1}$$

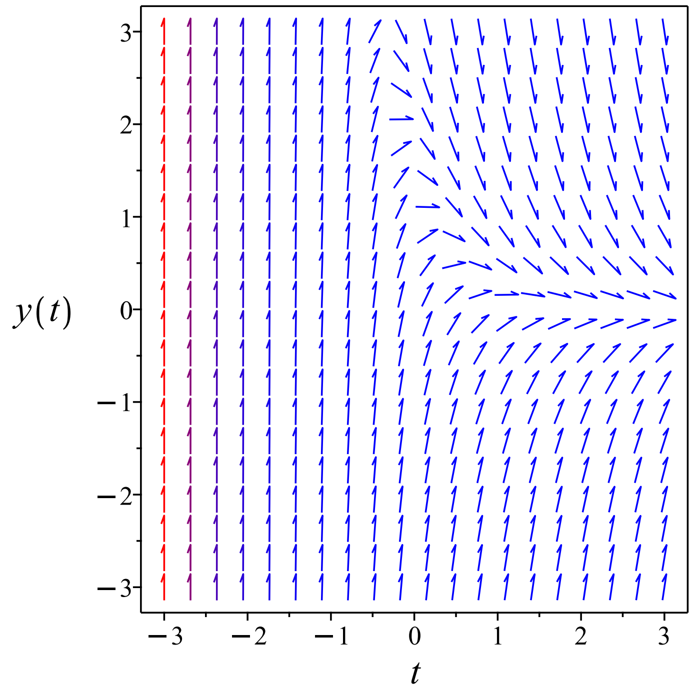


Figure 292: Slope field plot

Verification of solutions

$$y = e^{-2t}(3t + c_1)$$

Verified OK.

7.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\, \mathrm{d}t + N(t, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-2y + 3\,\mathrm{e}^{-2t}\right) \mathrm{d}t \\ \left(2y - 3\,\mathrm{e}^{-2t}\right) \mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= 2y - 3\,\mathrm{e}^{-2t} \\ N(t, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - 3\,\mathrm{e}^{-2t}\right) \\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((2) - (0)) \\ &= 2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 2 \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2t} \\ &= \mathrm{e}^{2t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{2t}\left(2y - 3\,\mathrm{e}^{-2t}\right) \\ &= 2\,\mathrm{e}^{2t} y - 3 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{2t}(1) \\ &= \mathrm{e}^{2t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(2\,\mathrm{e}^{2t} y - 3\right) + \left(\mathrm{e}^{2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int 2\,\mathrm{e}^{2t} y - 3 \, \mathrm{d}t$$

$$\phi = -3t + \mathrm{e}^{2t} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{2t} = \mathrm{e}^{2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -3t + \mathrm{e}^{2t} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3t + \mathrm{e}^{2t} y$$

The solution becomes

$$y = \mathrm{e}^{-2t}(3t + c_1)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2t}(3t + c_1) \tag{1}$$

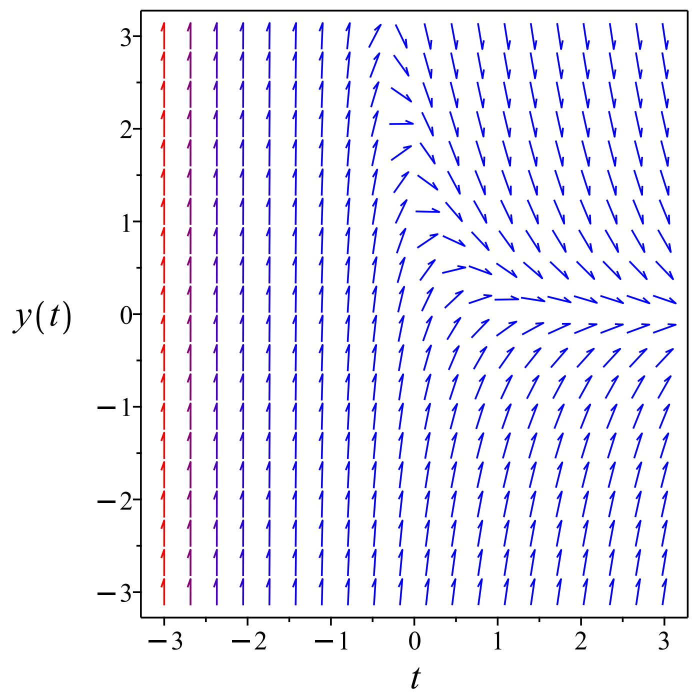


Figure 293: Slope field plot

Verification of solutions

$$y = e^{-2t}(3t + c_1)$$

Verified OK.

7.23.4 Maple step by step solution

Let's solve

$y' + 2y = 3\,e^{-2t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y + 3\,e^{-2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = 3\,e^{-2t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' + 2y) = 3\mu(t)\,e^{-2t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\,(y' + 2y) = \mu'(t)\, y + \mu(t)\, y'$
- Isolate $\mu'(t)$

  $\mu'(t) = 2\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{2t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int 3\mu(t)\, \mathrm{e}^{-2t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int 3\mu(t)\, \mathrm{e}^{-2t} dt + c_1$
- Solve for y

  $y = \frac{\int 3\mu(t)\mathrm{e}^{-2t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{2t}$

  $y = \frac{\int 3\, \mathrm{e}^{-2t} \mathrm{e}^{2t} dt + c_1}{\mathrm{e}^{2t}}$
- Evaluate the integrals on the rhs

  $y = \frac{3t + c_1}{\mathrm{e}^{2t}}$
- Simplify

  $y = \mathrm{e}^{-2t}(3t + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(t),t)+2*y(t)=3*exp(-2*t),y(t), singsol=all)
```

$$y(t) = (c_1 + 3t)\,\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 17

```
DSolve[y'[t]+2*y[t]==3*Exp[-2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-2t}(3t + c_1)$$

8 Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136

8.1 problem 2

Internal problem ID [13029]
Internal file name [`OUTPUT/11681_Wednesday_November_08_2023_03_28_42_AM_83223225/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 3y = 0}$$

8.1.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{3y}dy = \int dt$$
$$\frac{\ln(y)}{3} = t + c_1$$

Raising both side to exponential gives

$$y^{\frac{1}{3}} = e^{t+c_1}$$

Which simplifies to

$$y^{\frac{1}{3}} = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = c_2^3 e^{3t} \tag{1}$$

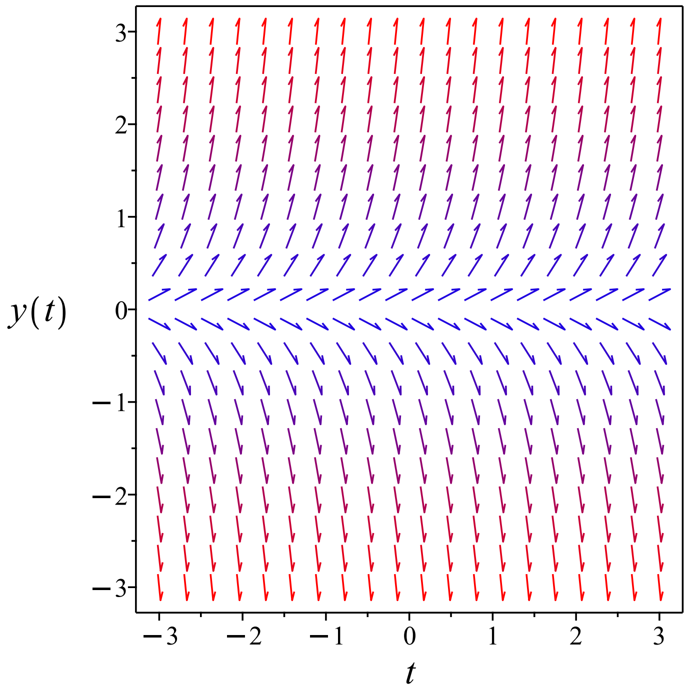


Figure 294: Slope field plot

Verification of solutions

$$y = c_2^3 e^{3t}$$

Verified OK.

8.1.2 Maple step by step solution

Let's solve

$y' - 3y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 3$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int 3dt + c_1$

- Evaluate integral

$$\ln(y) = 3t + c_1$$

- Solve for y

$$y = e^{3t+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(t),t)=3*y(t),y(t), singsol=all)
```

$$y(t) = c_1 e^{3t}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 18

```
DSolve[y'[t]==3*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^{3t}$$
$$y(t) \to 0$$

8.2 problem 3

Internal problem ID [13030]
Internal file name [`OUTPUT/11682_Wednesday_November_08_2023_03_28_43_AM_26569994/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = t^2(t^2+1)}$$

8.2.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int t^2(t^2+1)\, \mathrm{d}t \\ &= \frac{1}{5}t^5 + \frac{1}{3}t^3 + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{5}t^5 + \frac{1}{3}t^3 + c_1 \tag{1}$$

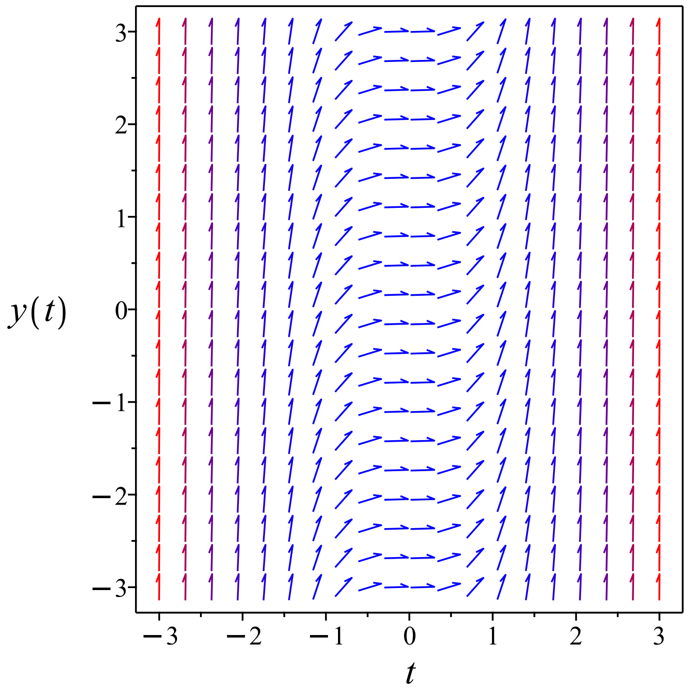


Figure 295: Slope field plot

Verification of solutions

$$y = \frac{1}{5}t^5 + \frac{1}{3}t^3 + c_1$$

Verified OK.

8.2.2 Maple step by step solution

Let's solve

$y' = t^2(t^2 + 1)$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to t

  $\int y' dt = \int t^2(t^2 + 1)\, dt + c_1$

- Evaluate integral

  $y = \frac{1}{5}t^5 + \frac{1}{3}t^3 + c_1$

- Solve for y

$$y = \frac{1}{5}t^5 + \frac{1}{3}t^3 + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)=t^2*(t^2+1),y(t), singsol=all)
```

$$y(t) = \frac{1}{5}t^5 + \frac{1}{3}t^3 + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 22

```
DSolve[y'[t]==t^2*(t^2+1),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \frac{t^5}{5} + \frac{t^3}{3} + c_1$$

8.3 problem 4

Internal problem ID [13031]
Internal file name [`OUTPUT/11683_Wednesday_November_08_2023_03_28_43_AM_97575103/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + \sin(y)^5 = 0}$$

8.3.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{\sin(y)^5} dy = \int dt$$
$$\int^y -\frac{1}{\sin(\_a)^5} d\_a = t + c_1$$

Summary
The solution(s) found are the following

$$\int^y -\frac{1}{\sin(\_a)^5} d\_a = t + c_1 \tag{1}$$

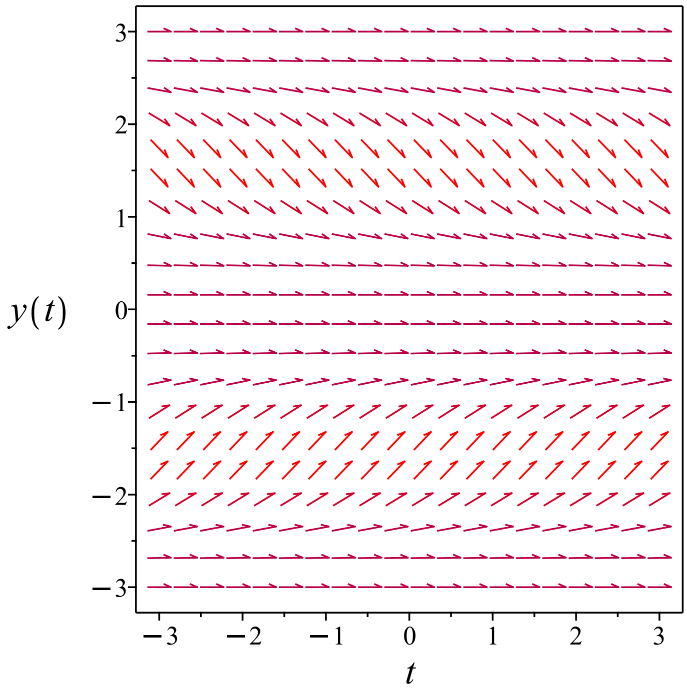


Figure 296: Slope field plot

Verification of solutions

$$\int^{y} -\frac{1}{\sin(\_a)^5} d\_a = t + c_1$$

Verified OK.

8.3.2 Maple step by step solution

Let's solve

$y' + \sin(y)^5 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sin(y)^5} = -1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\sin(y)^5} dt = \int (-1)\, dt + c_1$

- Evaluate integral

$$\left(-\frac{\csc(y)^3}{4}-\frac{3\csc(y)}{8}\right)\cot(y)+\frac{3\ln(\csc(y)-\cot(y))}{8}=-t+c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 190

```
dsolve(diff(y(t),t)=-sin(y(t))^5,y(t), singsol=all)
```

$$y(t) = \arctan\left(\frac{2\,e^{\mathrm{RootOf}\left(e^{8\_Z}+8\,e^{6\_Z}+64c_1e^{4\_Z}+24\_Ze^{4\_Z}+64t\,e^{4\_Z}-8\,e^{2\_Z}-1\right)}}{e^{2\,\mathrm{RootOf}\left(e^{8\_Z}+8\,e^{6\_Z}+64c_1e^{4\_Z}+24\_Ze^{4\_Z}+64t\,e^{4\_Z}-8\,e^{2\_Z}-1\right)}+1}, \frac{-e^{2\,\mathrm{RootOf}\left(e^{8\_Z}+8\,e^{6\_Z}+64c_1e^{4\_Z}+24\_Ze^{4\_Z}\right.}}{e^{2\,\mathrm{RootOf}\left(e^{8\_Z}+8\,e^{6\_Z}+64c_1e^{4\_Z}+24\_Ze^{4\_Z}+\right.}}\right.$$

✓ Solution by Mathematica

Time used: 1.165 (sec). Leaf size: 101

```
DSolve[y'[t]==-Sin[y[t]]^5,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to \text{InverseFunction}\left[\frac{1}{16}\left(-\frac{1}{64}\csc^4\left(\frac{\#1}{2}\right)-\frac{3}{32}\csc^2\left(\frac{\#1}{2}\right)+\frac{1}{64}\sec^4\left(\frac{\#1}{2}\right)+\frac{3}{32}\sec^2\left(\frac{\#1}{2}\right)+\frac{3}{8}\log\left(\sin\left(\frac{\#1}{2}\right)\right)-\frac{3}{8}\log\left(\cos\left(\frac{\#1}{2}\right)\right)\right)\&\right]\left[-\frac{t}{16}+c_1\right]$$

$$y(t)\to 0$$

8.4 problem 5

Internal problem ID [13032]
Internal file name [`OUTPUT/11684_Wednesday_November_08_2023_03_28_44_AM_66323007/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{(t^2-4)\,(y+1)\,e^{y}}{(t-1)\,(3-y)} = 0}$$

8.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= -\frac{(t^2-4)\,(y+1)\,e^{y}}{(t-1)\,(y-3)} \end{aligned}$$

Where $f(t) = -\frac{t^2-4}{t-1}$ and $g(y) = \frac{(y+1)e^y}{y-3}$. Integrating both sides gives

$$\frac{1}{\frac{(y+1)e^y}{y-3}}\,dy = -\frac{t^2-4}{t-1}\,dt$$

$$\int \frac{1}{\frac{(y+1)e^{y}}{y-3}}dy = \int -\frac{t^2-4}{t-1}dt$$

$$-e^{-y} + 4\,e\,\text{expIntegral}_1\,(y+1) = -\frac{t^2}{2} - t + 3\ln(t-1) + c_1$$

Which results in

$$y = -\operatorname{RootOf}\left(-8\,e\,\text{expIntegral}_1\,(1-\_Z) - t^2 + 2\,e^{-\_Z} + 6\ln(t-1) + 2c_1 - 2t\right)$$

Summary

The solution(s) found are the following

$$y = -\operatorname{RootOf}\left(-8\,e\,\text{expIntegral}_1\,(1-\_Z) - t^2 + 2\,e^{-\_Z} + 6\ln(t-1) + 2c_1 - 2t\right) \quad (1)$$

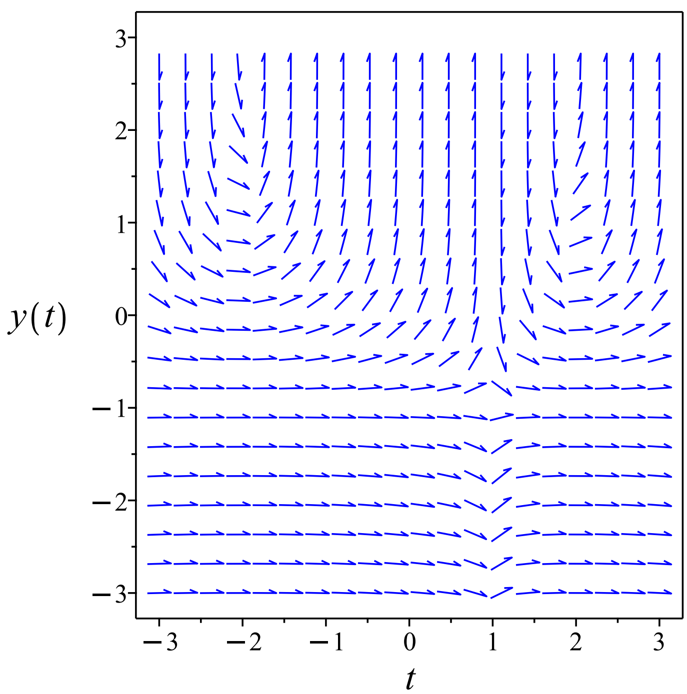


Figure 297: Slope field plot

Verification of solutions

$$y = -\operatorname{RootOf}\left(-8\,e\,\text{expIntegral}_1\,(1-\_Z) - t^2 + 2\,e^{-\_Z} + 6\ln(t-1) + 2c_1 - 2t\right)$$

Verified OK.

8.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{(t^2-4)(y+1)\,\mathrm{e}^{y}}{(t-1)(y-3)} \\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 296: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= -\frac{t-1}{t^{2}-4} \\
\eta(t,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{-\frac{t-1}{t^{2}-4}} dt
\end{aligned}
$$

Which results in

$$
S = -\frac{t^{2}}{2} - t + 3\ln\left(t-1\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = -\frac{\left(t^{2}-4\right)\left(y+1\right)\mathrm{e}^{y}}{\left(t-1\right)\left(y-3\right)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= -t - 1 + \frac{3}{t-1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\mathrm{e}^{-y}(y-3)}{y+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\mathrm{e}^{-R}(R-3)}{R+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + 4\,\mathrm{e}\ \mathrm{expIntegral}_1\left(R+1\right) + c_1 \tag{4}$$

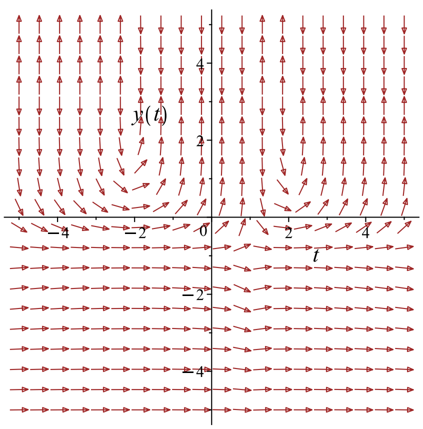
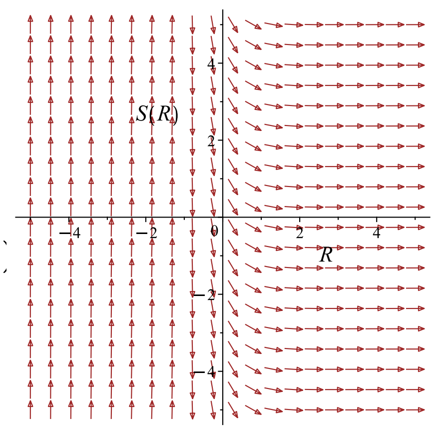
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{t^2}{2} - t + 3\ln\left(t-1\right) = -\mathrm{e}^{-y} + 4\,\mathrm{e}\ \mathrm{expIntegral}_1\left(y+1\right) + c_1$$

Which simplifies to

$$-\frac{t^2}{2} - t + 3\ln\left(t-1\right) = -\mathrm{e}^{-y} + 4\,\mathrm{e}\ \mathrm{expIntegral}_1\left(y+1\right) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{(t^2-4)(y+1)e^y}{(t-1)(y-3)}$ 	$R = y$ $S = -\frac{t^2}{2} - t + 3\ln(t-1)$	$\frac{dS}{dR} = \frac{e^{-R}(R-3)}{R+1}$ 

Summary
The solution(s) found are the following

$$-\frac{t^2}{2} - t + 3\ln(t-1) = -e^{-y} + 4\,e\,\text{expIntegral}_1(y+1) + c_1 \tag{1}$$

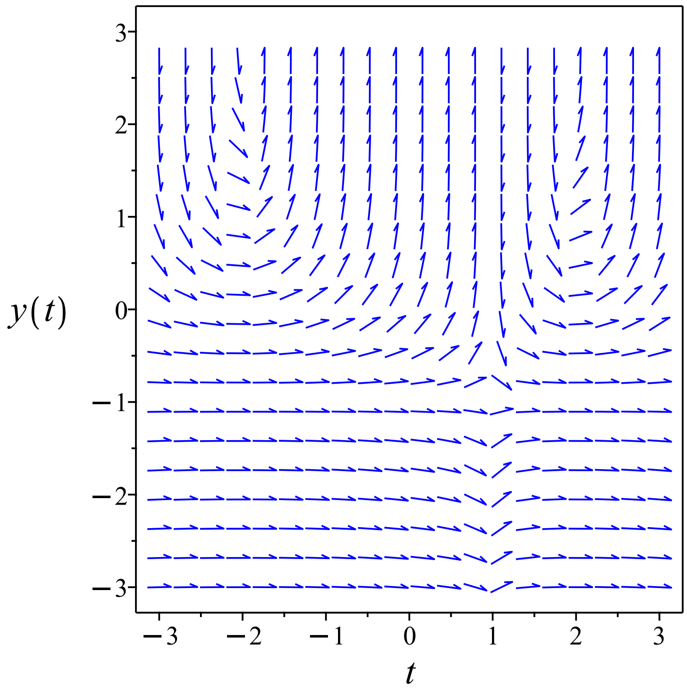


Figure 298: Slope field plot

Verification of solutions

$$-\frac{t^2}{2} - t + 3\ln(t-1) = -\mathrm{e}^{-y} + 4\,\mathrm{e}\ \mathrm{expIntegral}_1(y+1) + c_1$$

Verified OK.

8.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{\mathrm{e}^{-y}(y-3)}{y+1}\right) \mathrm{d}y &= \left(\frac{t^2-4}{t-1}\right) \mathrm{d}t \\
\left(-\frac{t^2-4}{t-1}\right) \mathrm{d}t + \left(-\frac{\mathrm{e}^{-y}(y-3)}{y+1}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -\frac{t^2-4}{t-1} \\
N(t, y) &= -\frac{\mathrm{e}^{-y}(y-3)}{y+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{t^2-4}{t-1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{\mathrm{e}^{-y}(y-3)}{y+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int M\, \mathrm{d}t
$$

$$
\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t = \int -\frac{t^2-4}{t-1}\, \mathrm{d}t
$$

$$
\phi = -\frac{t^2}{2} - t + 3\ln(t-1) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\mathrm{e}^{-y}(y-3)}{y+1}$. Therefore equation (4) becomes

$$
-\frac{\mathrm{e}^{-y}(y-3)}{y+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{\mathrm{e}^{-y}(y-3)}{y+1}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{\mathrm{e}^{-y}(y-3)}{y+1}\right)\mathrm{d}y$$

$$f(y) = \mathrm{e}^{-y} - 4\,\mathrm{e}\ \mathrm{expIntegral}_1\left(y+1\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - t + 3\ln\left(t-1\right) + \mathrm{e}^{-y} - 4\,\mathrm{e}\ \mathrm{expIntegral}_1\left(y+1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - t + 3\ln\left(t-1\right) + \mathrm{e}^{-y} - 4\,\mathrm{e}\ \mathrm{expIntegral}_1\left(y+1\right)$$

Summary

The solution(s) found are the following

$$-4\,\mathrm{e}\ \mathrm{expIntegral}_1\left(y+1\right) - \frac{t^2}{2} + 3\ln\left(t-1\right) + \mathrm{e}^{-y} - t = c_1 \tag{1}$$

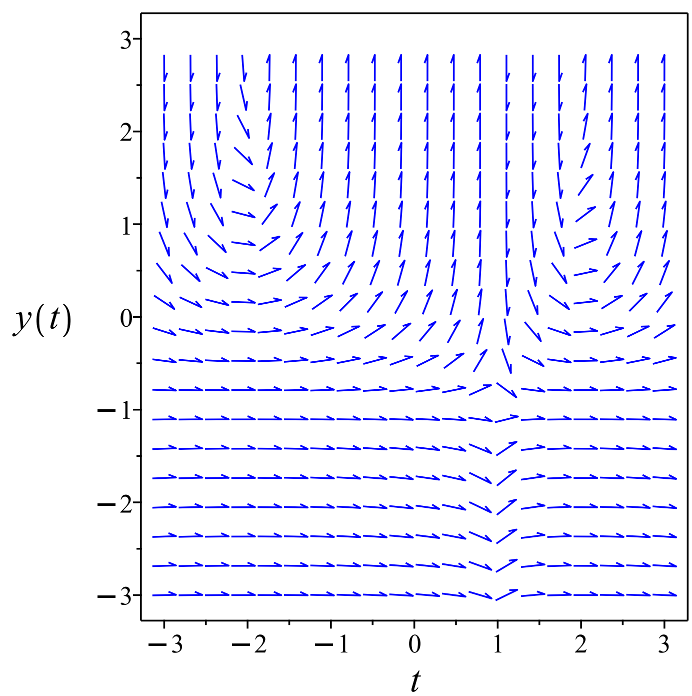


Figure 299: Slope field plot

Verification of solutions

$$-4\,\mathrm{e}\,\mathrm{expIntegral}_1\left(y+1\right)-\frac{t^2}{2}+3\ln\left(t-1\right)+\mathrm{e}^{-y}-t=c_1$$

Verified OK.

8.4.4 Maple step by step solution

Let's solve

$y'-\frac{(t^2-4)(y+1)\mathrm{e}^y}{(t-1)(3-y)}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(3-y)}{(y+1)\mathrm{e}^y}=\frac{t^2-4}{t-1}$

- Integrate both sides with respect to t

  $\int\frac{y'(3-y)}{(y+1)\mathrm{e}^y}dt=\int\frac{t^2-4}{t-1}dt+c_1$

- Evaluate integral

  $\mathrm{e}^{-y}-4\,\mathrm{e}\,\mathrm{Ei}_1(y+1)=\frac{t^2}{2}+t-3\ln\left(t-1\right)+c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 38

```
dsolve(diff(y(t),t)=(  (t^2-4)*(1+y(t))*exp(y(t)))/(   (t-1)*(3-y(t))),y(t), singsol=all)
```

$$y(t)=-\mathrm{RootOf}\left(8\,\mathrm{e}\,\mathrm{expIntegral}_1\left(1-\_Z\right)+t^2-2\,\mathrm{e}^{-\_Z}-6\ln\left(t-1\right)+2c_1+2t\right)$$

✓ Solution by Mathematica

Time used: 1.486 (sec). Leaf size: 53

```
DSolve[y'[t]==(  (t^2-4)*(1+y[t])*Exp[y[t]])/(   (t-1)*(3-y[t])),y[t],t,IncludeSingularSolut
```

$$y(t) \to \text{InverseFunction}\left[-4e \,\text{ExpIntegralEi}(-\#1-1) - e^{-\#1}\&\right]\left[-\frac{t^2}{2} - t + 3\log(t-1) + \frac{3}{2} + c_1\right]$$

$$y(t) \to -1$$

8.5 problem 6

Internal problem ID [13033]
Internal file name [`OUTPUT/11685_Wednesday_November_08_2023_03_28_45_AM_58753678/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sin(y)^2 = 0}$$

8.5.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{\sin(y)^2} dy &= t + c_1 \\ -\cot(y) &= t + c_1 \end{aligned}$$

Solving for y gives these solutions

$$y_1 = \pi - \operatorname{arccot}(t + c_1)$$

Summary
The solution(s) found are the following

$$y = \pi - \operatorname{arccot}(t + c_1) \tag{1}$$

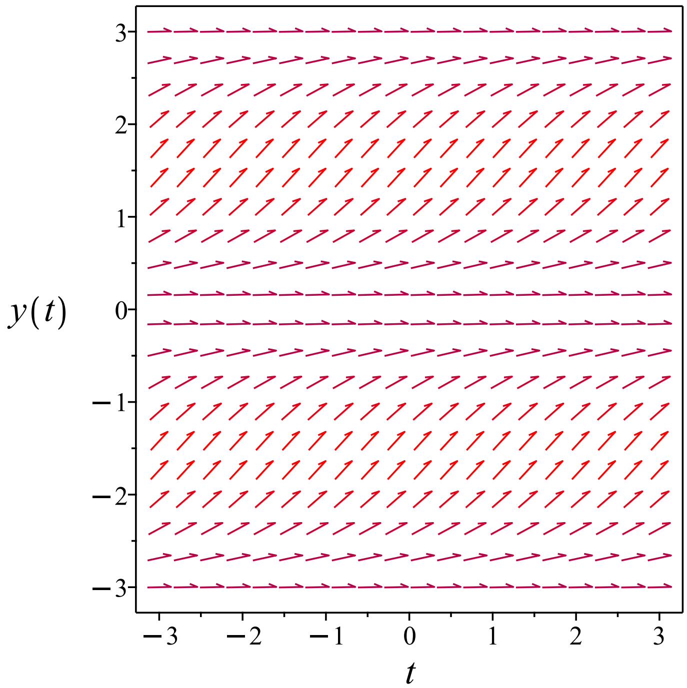


Figure 300: Slope field plot

Verification of solutions

$$y = \pi - \operatorname{arccot}(t + c_1)$$

Verified OK.

8.5.2 Maple step by step solution

Let's solve

$y' - \sin(y)^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sin(y)^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{\sin(y)^2} dt = \int 1 dt + c_1$

- Evaluate integral

$$-\cot(y) = t + c_1$$

- Solve for y

$$y = \pi - \operatorname{arccot}(t + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)=sin(y(t))^2,y(t), singsol=all)
```

$$y(t) = \frac{\pi}{2} + \arctan(t + c_1)$$

✓ Solution by Mathematica

Time used: 0.319 (sec). Leaf size: 19

```
DSolve[y'[t]==Sin[y[t]]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\cot^{-1}(t - 2c_1)$$
$$y(t) \to 0$$

8.6 problem 17

Internal problem ID [13034]
Internal file name [`OUTPUT/11686_Wednesday_November_08_2023_03_28_45_AM_49820772/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`x=_G(y,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - (y - 3)(\sin(y)\sin(t) + \cos(t) + 1) = 0}$$

With initial conditions

$$[y(0) = 4]$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve([diff(y(t),t)= (y(t)-3)*( sin(y(t))*sin(t)+cos(t)+1),y(0) = 4],y(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y'[t]==(y[t]-3)*( Sin[y[t]]*Sin[t]+Cos[t]+1),{y[0]==4}},y[t],t,IncludeSingularSoluti
```

Not solved

8.7 problem 20

Internal problem ID [13035]
Internal file name [`OUTPUT/11687_Wednesday_November_08_2023_03_28_48_AM_75053001/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = e^{-t}}$$

8.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = e^{-t}$$

Hence the ode is

$$y' - y = e^{-t}$$

The integrating factor μ is

$$\mu = e^{\int (-1)dt}$$
$$= e^{-t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(e^{-t}\right)$$
$$\frac{d}{dt}\left(e^{-t} y\right) = \left(e^{-t}\right)\left(e^{-t}\right)$$
$$d\left(e^{-t} y\right) = e^{-2t}\, dt$$

Integrating gives

$$e^{-t} y = \int e^{-2t}\, dt$$
$$e^{-t} y = -\frac{e^{-2t}}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-t}$ results in

$$y = -\frac{e^{t} e^{-2t}}{2} + c_1 e^{t}$$

which simplifies to

$$y = -\frac{e^{-t}}{2} + c_1 e^{t}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-t}}{2} + c_1 e^{t} \tag{1}$$

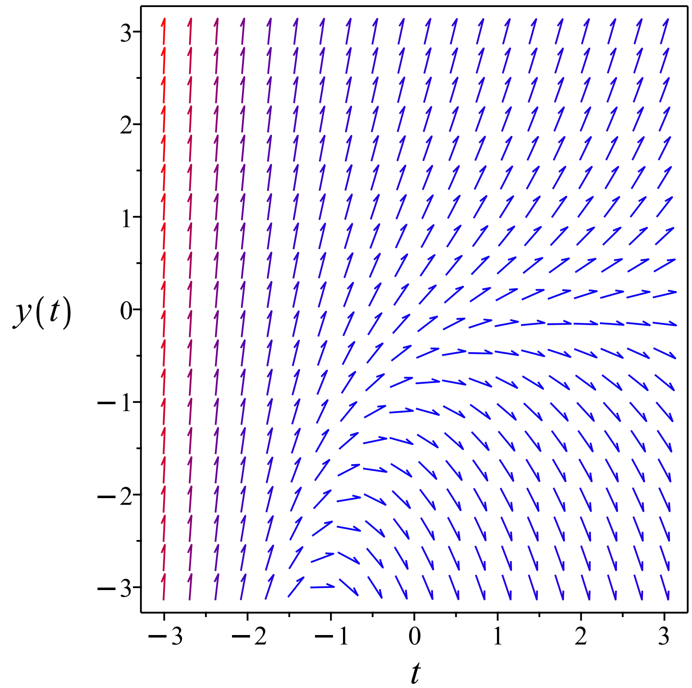


Figure 301: Slope field plot

Verification of solutions

$$y = -\frac{e^{-t}}{2} + c_1 e^{t}$$

Verified OK.

8.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + e^{-t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 300: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^t
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^t} dy \end{aligned}$$

Which results in

$$S = e^{-t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = y + e^{-t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -e^{-t} y \\ S_y &= e^{-t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{e^{-2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-t} y = -\frac{e^{-2t}}{2} + c_1$$

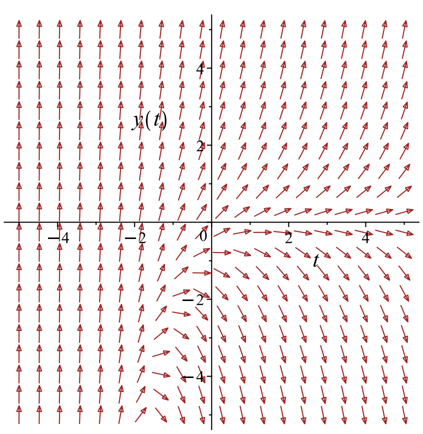
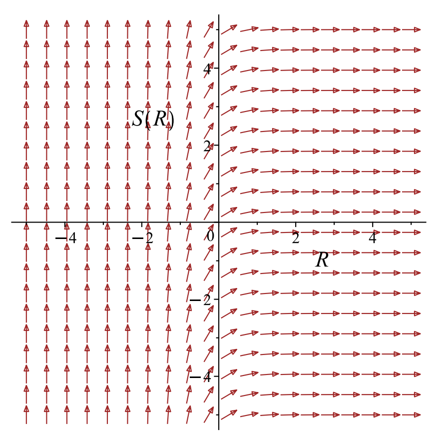
Which simplifies to

$$e^{-t} y = -\frac{e^{-2t}}{2} + c_1$$

Which gives

$$y = -\frac{(e^{-2t} - 2c_1) e^t}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y + e^{-t}$ 	$R = t$ $S = e^{-t} y$	$\frac{dS}{dR} = e^{-2R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{(e^{-2t} - 2c_1) e^t}{2} \tag{1}$$

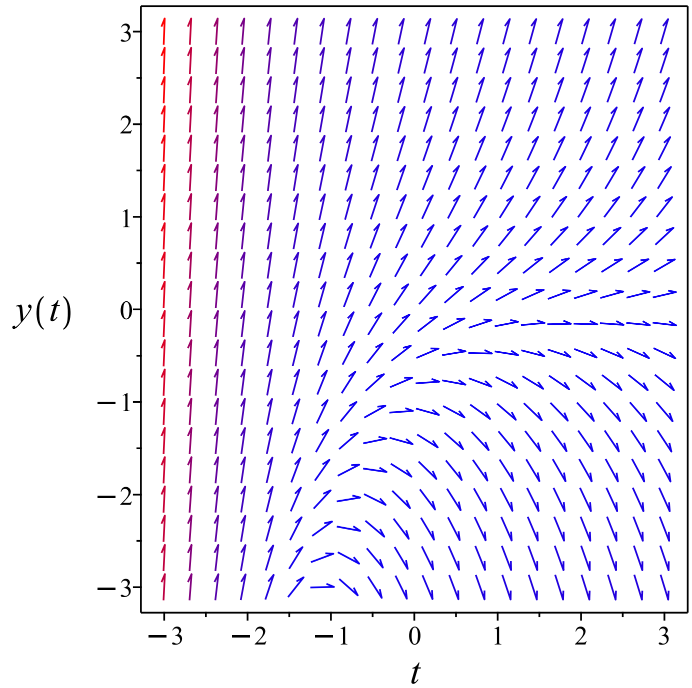


Figure 302: Slope field plot

Verification of solutions

$$y = -\frac{(e^{-2t} - 2c_1) e^t}{2}$$

Verified OK.

8.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(y + \mathrm{e}^{-t}\right)\mathrm{d}t \\
\left(-y - \mathrm{e}^{-t}\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -y - \mathrm{e}^{-t} \\
N(t, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - \mathrm{e}^{-t}\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((-1) - (0))\\ &= -1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -1\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-t}\\ &= \mathrm{e}^{-t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-t}(-y - \mathrm{e}^{-t})\\ &= -\mathrm{e}^{-t}(y + \mathrm{e}^{-t})\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-t}(1)\\ &= \mathrm{e}^{-t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ \left(-\mathrm{e}^{-t}(y + \mathrm{e}^{-t})\right) + \left(\mathrm{e}^{-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -\mathrm{e}^{-t}\left(y + \mathrm{e}^{-t}\right)\mathrm{d}t$$

$$\phi = \mathrm{e}^{-t} y + \frac{\mathrm{e}^{-2t}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-t} = \mathrm{e}^{-t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-t} y + \frac{\mathrm{e}^{-2t}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-t} y + \frac{\mathrm{e}^{-2t}}{2}$$

The solution becomes

$$y = -\frac{\left(\mathrm{e}^{-2t} - 2c_1\right)\mathrm{e}^{t}}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{(e^{-2t} - 2c_1)\, e^{t}}{2} \tag{1}$$

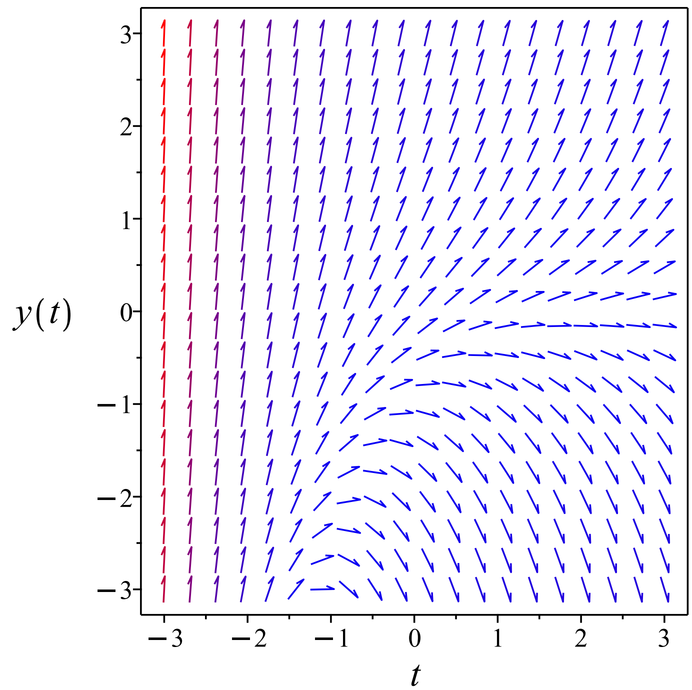


Figure 303: Slope field plot

Verification of solutions

$$y = -\frac{(e^{-2t} - 2c_1)\, e^{t}}{2}$$

Verified OK.

8.7.4 Maple step by step solution

Let's solve

$y' - y = e^{-t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = y + e^{-t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' - y = e^{-t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)(y' - y) = \mu(t) e^{-t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) y)$

$\mu(t)(y' - y) = \mu'(t) y + \mu(t) y'$

- Isolate $\mu'(t)$

$\mu'(t) = -\mu(t)$

- Solve to find the integrating factor

$\mu(t) = e^{-t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t) y)\right) dt = \int \mu(t) e^{-t} dt + c_1$

- Evaluate the integral on the lhs

$\mu(t) y = \int \mu(t) e^{-t} dt + c_1$

- Solve for y

$y = \frac{\int \mu(t) e^{-t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{-t}$

$y = \frac{\int (e^{-t})^2 dt + c_1}{e^{-t}}$

- Evaluate the integrals on the rhs

$y = \frac{-\frac{(e^{-t})^2}{2} + c_1}{e^{-t}}$

- Simplify

$y = -\frac{e^{-t}}{2} + c_1 e^{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(t),t)= y(t)+exp(-t),y(t), singsol=all)
```

$$y(t) = -\frac{e^{-t}}{2} + c_1 e^t$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 21

```
DSolve[y'[t]==y[t]+Exp[-t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{e^{-t}}{2} + c_1 e^t$$

8.8 problem 21

Internal problem ID [13036]
Internal file name [`OUTPUT/11688_Wednesday_November_08_2023_03_28_49_AM_31209479/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + 2y = 3}$$

8.8.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-2y+3} dy = \int dt$$
$$-\frac{\ln(-2y+3)}{2} = t + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2y+3}} = e^{t+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{-2y+3}} = c_2 e^t$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-2t}}{2c_2^2} + \frac{3}{2} \tag{1}$$

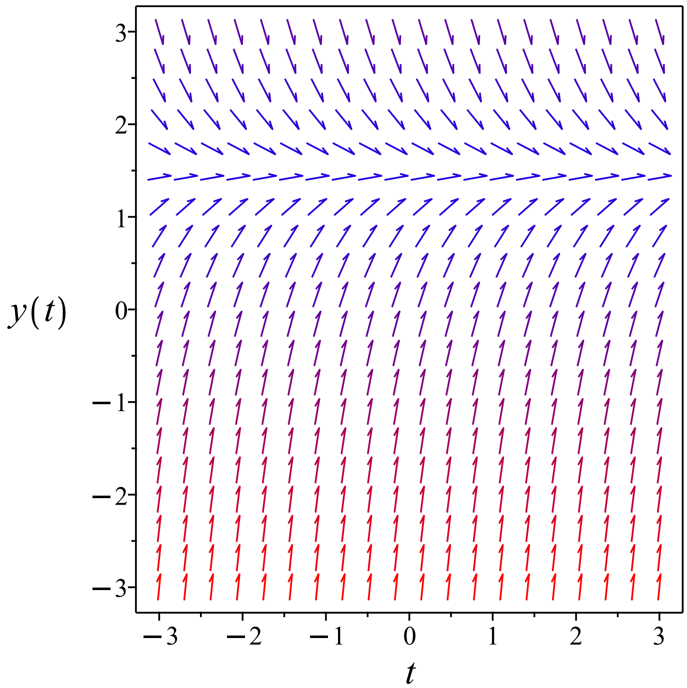


Figure 304: Slope field plot

Verification of solutions

$$y = -\frac{e^{-2t}}{2c_2^2} + \frac{3}{2}$$

Verified OK.

8.8.2 Maple step by step solution

Let's solve

$y' + 2y = 3$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'}{3-2y} = 1$$

- Integrate both sides with respect to t

$$\int \frac{y'}{3-2y} dt = \int 1 dt + c_1$$

- Evaluate integral

$$-\frac{\ln(3-2y)}{2} = t + c_1$$

- Solve for y

$$y = -\frac{\mathrm{e}^{-2t-2c_1}}{2} + \frac{3}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)= 3-2*y(t),y(t), singsol=all)
```

$$y(t) = \frac{3}{2} + \mathrm{e}^{-2t} c_1$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 24

```
DSolve[y'[t]==3-2*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{3}{2} + c_1 e^{-2t}$$
$$y(t) \to \frac{3}{2}$$

8.9 problem 22

Internal problem ID [13037]
Internal file name [`OUTPUT/11689_Wednesday_November_08_2023_03_28_49_AM_35410921/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - ty = 0}$$

8.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= ty \end{aligned}$$

Where $f(t) = t$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = t\,dt$$
$$\int \frac{1}{y}\,dy = \int t\,dt$$
$$\ln(y) = \frac{t^2}{2} + c_1$$
$$y = e^{\frac{t^2}{2}+c_1}$$
$$= c_1 e^{\frac{t^2}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{t^2}{2}} \tag{1}$$

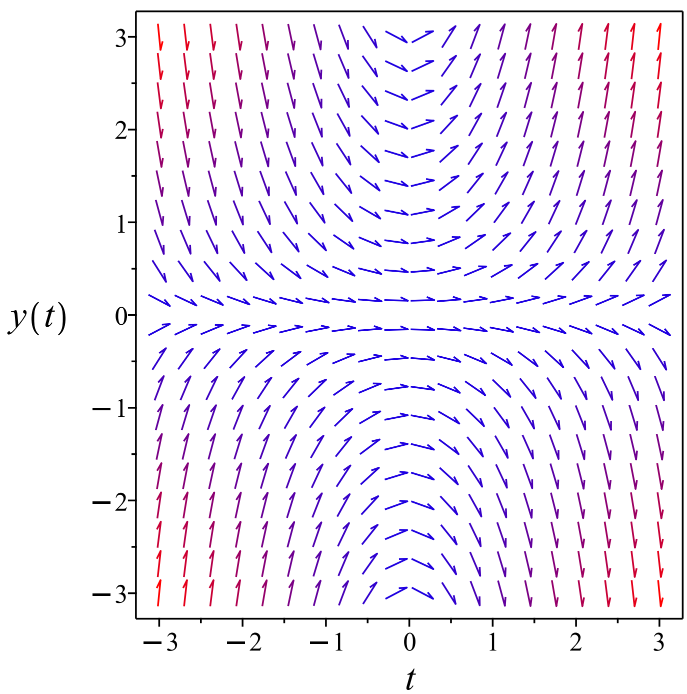


Figure 305: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{t^2}{2}}$$

Verified OK.

8.9.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -t$$
$$q(t) = 0$$

Hence the ode is

$$y' - ty = 0$$

The integrating factor μ is

$$\mu = e^{\int -t dt}$$
$$= e^{-\frac{t^2}{2}}$$

The ode becomes

$$\frac{d}{dt}\mu y = 0$$
$$\frac{d}{dt}\left(e^{-\frac{t^2}{2}}y\right) = 0$$

Integrating gives

$$e^{-\frac{t^2}{2}}y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{t^2}{2}}$ results in

$$y = c_1 e^{\frac{t^2}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\frac{t^2}{2}} \tag{1}$$

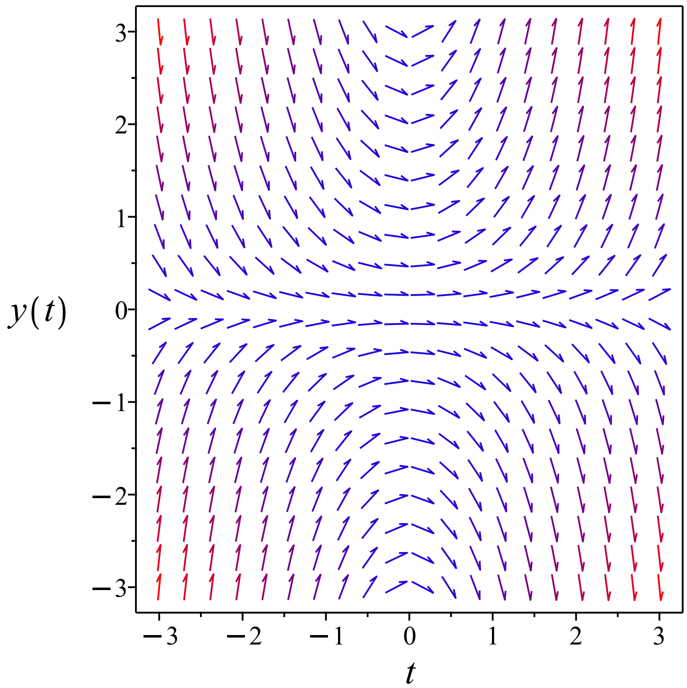


Figure 306: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{t^2}{2}}$$

Verified OK.

8.9.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) - t^2 u(t) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= \frac{u(t^2-1)}{t} \end{aligned}$$

Where $f(t) = \frac{t^2-1}{t}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{t^2-1}{t}\,dt \\
\int \frac{1}{u}\,du &= \int \frac{t^2-1}{t}\,dt \\
\ln(u) &= \frac{t^2}{2} - \ln(t) + c_2 \\
u &= \mathrm{e}^{\frac{t^2}{2} - \ln(t) + c_2} \\
&= c_2 \mathrm{e}^{\frac{t^2}{2} - \ln(t)}
\end{aligned}
$$

Which simplifies to

$$
u(t) = \frac{c_2 \mathrm{e}^{\frac{t^2}{2}}}{t}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= tu \\
&= \mathrm{e}^{\frac{t^2}{2}} c_2
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \mathrm{e}^{\frac{t^2}{2}} c_2 \tag{1}
$$

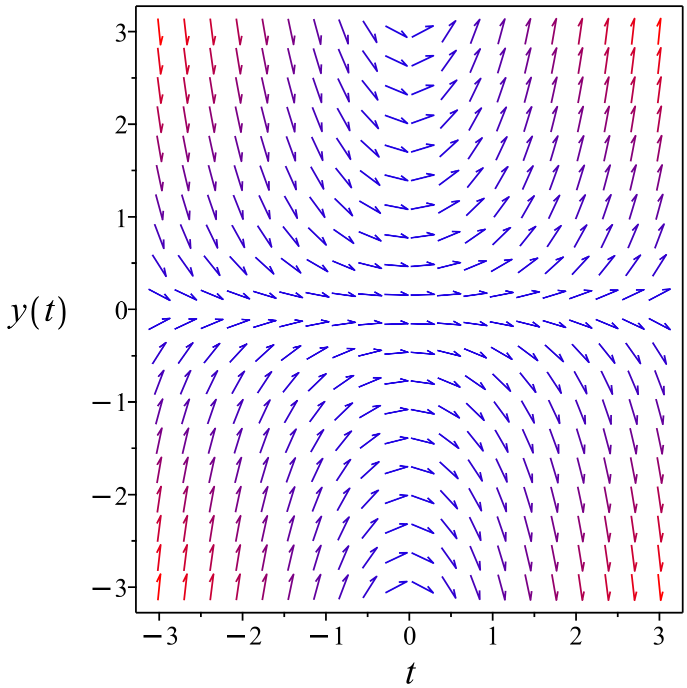


Figure 307: Slope field plot

Verification of solutions

$$y = e^{\frac{t^2}{2}} c_2$$

Verified OK.

8.9.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = ty$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 304: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, y) &= 0 \\ \eta(t, y) &= \mathrm{e}^{\frac{t^2}{2}}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{t^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{t^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = ty$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -t\, e^{-\frac{t^2}{2}} y \\ S_y &= e^{-\frac{t^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-\frac{t^2}{2}} y = c_1$$

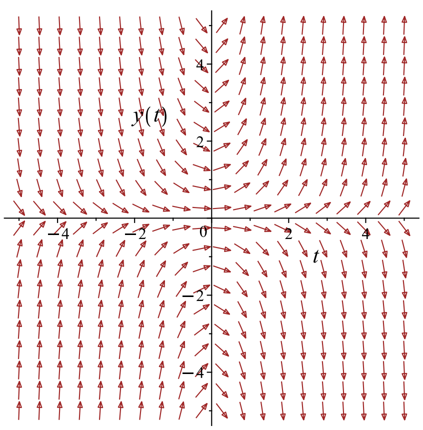
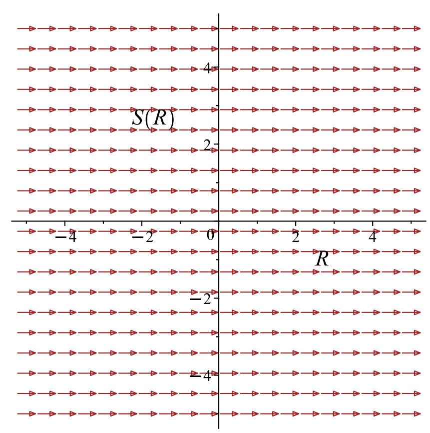
Which simplifies to

$$e^{-\frac{t^2}{2}} y = c_1$$

Which gives

$$y = c_1 e^{\frac{t^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = ty$ 	$R = t$ $S = e^{-\frac{t^2}{2}} y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{t^2}{2}} \tag{1}$$

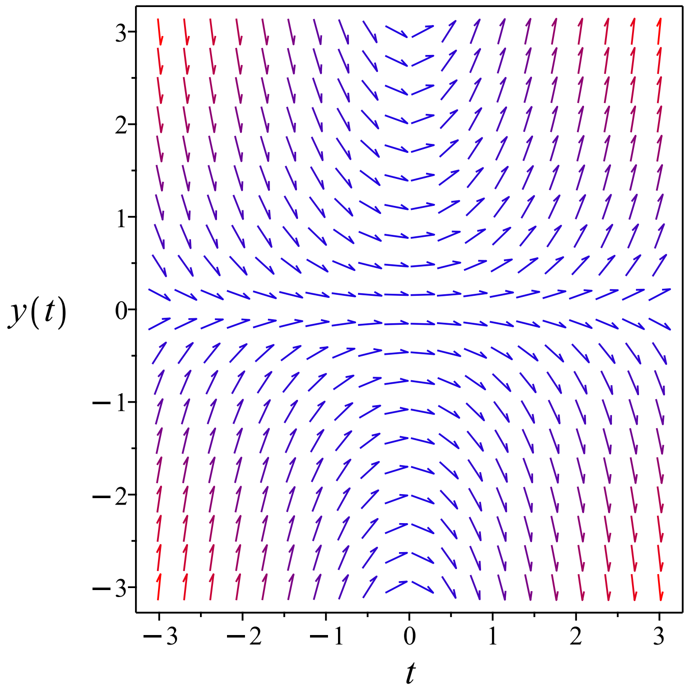


Figure 308: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{t^2}{2}}$$

Verified OK.

8.9.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right)\mathrm{d}y = (t)\,\mathrm{d}t$$
$$(-t)\,\mathrm{d}t + \left(\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,y) = -t$$
$$N(t,y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-t)$$
$$= 0$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -t \, \mathrm{d}t$$
$$\phi = -\frac{t^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} + \ln(y)$$

The solution becomes

$$y = e^{\frac{t^2}{2}+c_1}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{t^2}{2}+c_1} \tag{1}$$

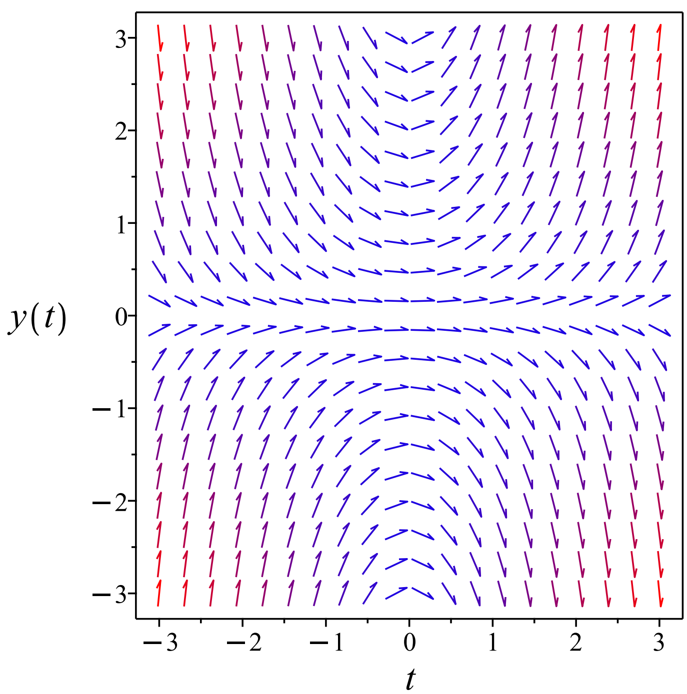


Figure 309: Slope field plot

Verification of solutions

$$y = e^{\frac{t^2}{2}+c_1}$$

Verified OK.

8.9.6 Maple step by step solution

Let's solve

$y' - ty = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = t$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int t dt + c_1$

- Evaluate integral

  $\ln(y) = \frac{t^2}{2} + c_1$

- Solve for y

  $y = e^{\frac{t^2}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(t),t)= t*y(t),y(t), singsol=all)
```

$$y(t) = e^{\frac{t^2}{2}} c_1$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 22

```
DSolve[y'[t]==t*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^{\frac{t^2}{2}}$$
$$y(t) \to 0$$

8.10 problem 23

Internal problem ID [13038]
Internal file name [`OUTPUT/11690_Wednesday_November_08_2023_03_28_50_AM_6445994/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = e^{7t}}$$

8.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= e^{7t} \end{aligned}$$

Hence the ode is

$$y' - 3y = e^{7t}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-3)dt} \\
&= \mathrm{e}^{-3t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\mathrm{e}^{7t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-3t} y\right) &= \left(\mathrm{e}^{-3t}\right)\left(\mathrm{e}^{7t}\right) \\
\mathrm{d}\left(\mathrm{e}^{-3t} y\right) &= \mathrm{e}^{4t}\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-3t} y &= \int \mathrm{e}^{4t}\,\mathrm{d}t \\
\mathrm{e}^{-3t} y &= \frac{\mathrm{e}^{4t}}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-3t}$ results in

$$
y = \frac{\mathrm{e}^{3t}\mathrm{e}^{4t}}{4} + c_1 \mathrm{e}^{3t}
$$

which simplifies to

$$
y = \frac{\mathrm{e}^{7t}}{4} + c_1 \mathrm{e}^{3t}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\mathrm{e}^{7t}}{4} + c_1 \mathrm{e}^{3t} \tag{1}
$$

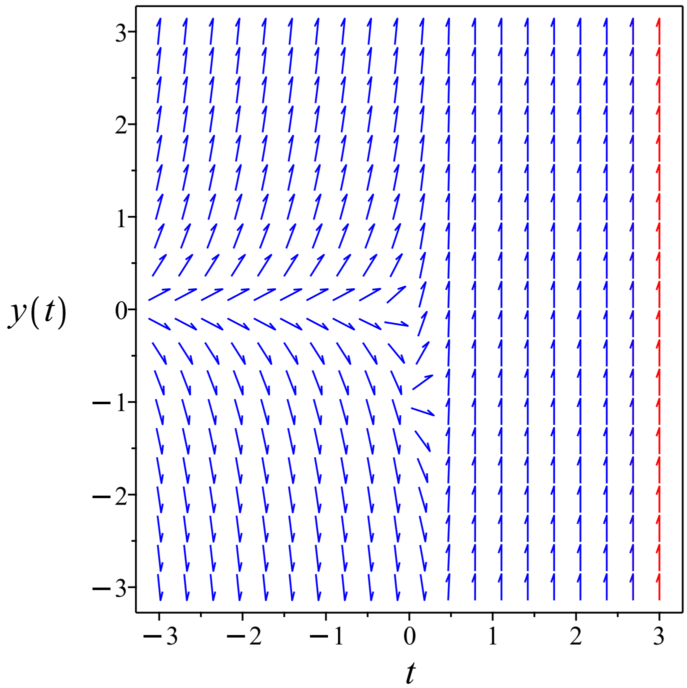


Figure 310: Slope field plot

Verification of solutions

$$y = \frac{e^{7t}}{4} + c_1 e^{3t}$$

Verified OK.

8.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3y + e^{7t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 307: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{3t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{3t}} dy \end{aligned}$$

Which results in

$$S = e^{-3t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = 3y + e^{7t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -3 e^{-3t} y \\ S_y &= e^{-3t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{4t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{4R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{4R}}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-3t} y = \frac{e^{4t}}{4} + c_1$$

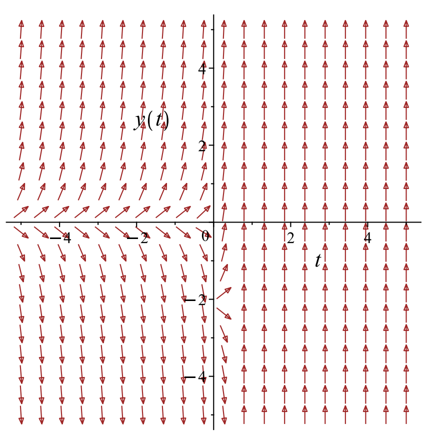
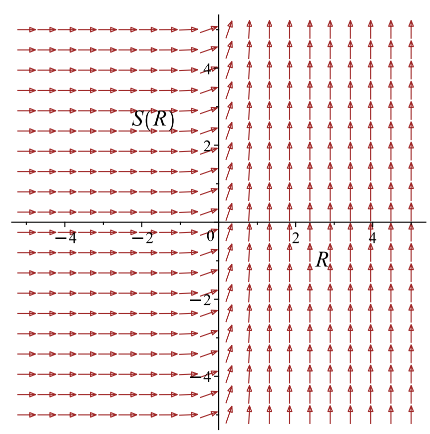
Which simplifies to

$$e^{-3t} y = \frac{e^{4t}}{4} + c_1$$

Which gives

$$y = \frac{(e^{4t} + 4c_1) e^{3t}}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 3y + e^{7t}$ 	$R = t$ $S = e^{-3t} y$	$\frac{dS}{dR} = e^{4R}$ 

Summary
The solution(s) found are the following

$$y = \frac{(e^{4t} + 4c_1) e^{3t}}{4} \tag{1}$$

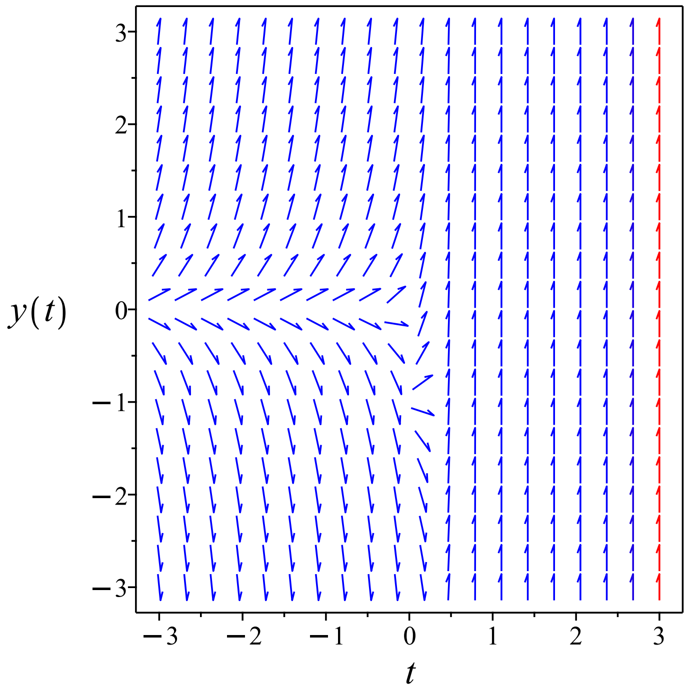


Figure 311: Slope field plot

Verification of solutions

$$y = \frac{(e^{4t} + 4c_1)\, e^{3t}}{4}$$

Verified OK.

8.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(3y + \mathrm{e}^{7t}\right)\mathrm{d}t\\ \left(-3y - \mathrm{e}^{7t}\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$M(t,y) = -3y - \mathrm{e}^{7t}$$
$$N(t,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3y - \mathrm{e}^{7t}\right)\\ &= -3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((-3) - (0)) \\ &= -3 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int -3 \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3t} \\ &= \mathrm{e}^{-3t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-3t}(-3y - \mathrm{e}^{7t}) \\ &= (-3y - \mathrm{e}^{7t})\,\mathrm{e}^{-3t} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-3t}(1) \\ &= \mathrm{e}^{-3t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left((-3y - \mathrm{e}^{7t})\,\mathrm{e}^{-3t}\right) + \left(\mathrm{e}^{-3t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \left(-3y - \mathrm{e}^{7t}\right) \mathrm{e}^{-3t} \,\mathrm{d}t$$

$$\phi = -\frac{\left(\mathrm{e}^{7t} - 4y\right) \mathrm{e}^{-3t}}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-3t} = \mathrm{e}^{-3t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\left(\mathrm{e}^{7t} - 4y\right) \mathrm{e}^{-3t}}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\left(\mathrm{e}^{7t} - 4y\right) \mathrm{e}^{-3t}}{4}$$

The solution becomes

$$y = \frac{\left(\mathrm{e}^{7t}\mathrm{e}^{-3t} + 4c_1\right) \mathrm{e}^{3t}}{4}$$

Summary
The solution(s) found are the following

$$y = \frac{(e^{7t}e^{-3t} + 4c_1)\, e^{3t}}{4} \tag{1}$$

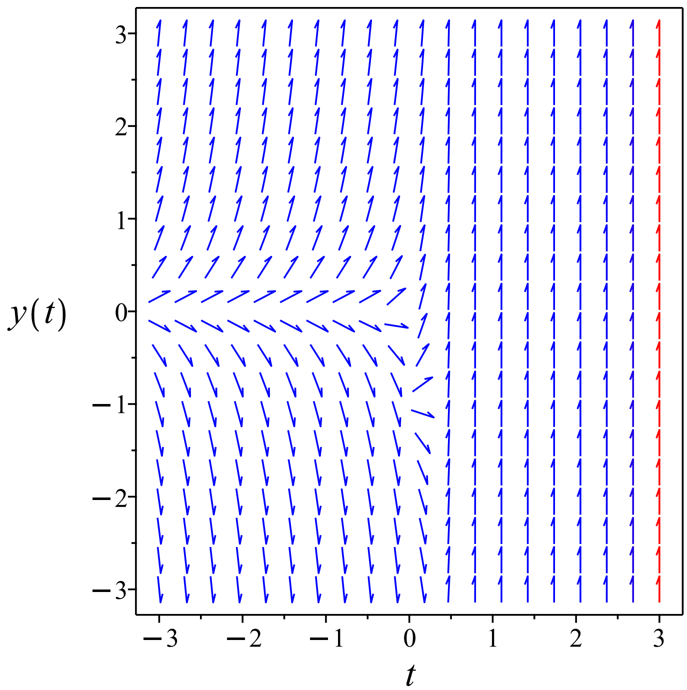


Figure 312: Slope field plot

Verification of solutions

$$y = \frac{(e^{7t}e^{-3t} + 4c_1)\, e^{3t}}{4}$$

Verified OK.

8.10.4 Maple step by step solution

Let's solve

$y' - 3y = e^{7t}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = 3y + e^{7t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y = e^{7t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' - 3y\right) = \mu(t)\, e^{7t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' - 3y\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -3\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = e^{-3t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\, e^{7t} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\, e^{7t} dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t) e^{7t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{-3t}$

  $y = \frac{\int e^{7t} e^{-3t} dt + c_1}{e^{-3t}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{e^{4t}}{4} + c_1}{e^{-3t}}$

- Simplify

  $y = \frac{(e^{4t} + 4c_1) e^{3t}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(t),t)= 3*y(t)+exp(7*t),y(t), singsol=all)
```

$$y(t) = \frac{(e^{4t} + 4c_1) e^{3t}}{4}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 23

```
DSolve[y'[t]==3*y[t]+Exp[7*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{7t}}{4} + c_1 e^{3t}$$

8.11 problem 24

Internal problem ID [13039]
Internal file name [`OUTPUT/11691_Wednesday_November_08_2023_03_28_50_AM_61009845/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{ty}{t^2+1} = 0}$$

8.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{ty}{t^2+1} \end{aligned}$$

Where $f(t) = \frac{t}{t^2+1}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= \frac{t}{t^2+1}\, dt \\
\int \frac{1}{y}\, dy &= \int \frac{t}{t^2+1}\, dt \\
\ln(y) &= \frac{\ln(t^2+1)}{2} + c_1 \\
y &= e^{\frac{\ln(t^2+1)}{2} + c_1} \\
&= c_1\sqrt{t^2+1}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{t^2+1} \tag{1}
$$

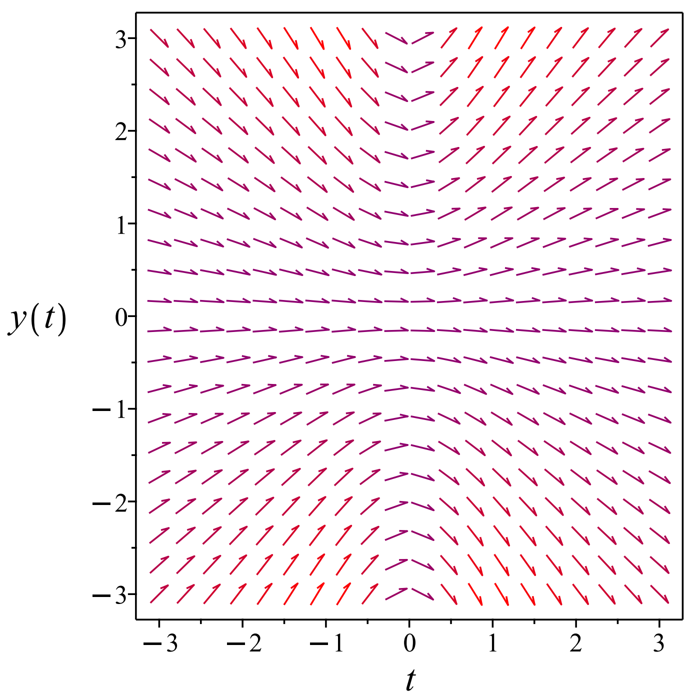


Figure 313: Slope field plot

Verification of solutions

$$
y = c_1\sqrt{t^2+1}
$$

Verified OK.

8.11.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{t}{t^2+1}$$
$$q(t) = 0$$

Hence the ode is

$$y' - \frac{ty}{t^2+1} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{t}{t^2+1}dt}$$
$$= \frac{1}{\sqrt{t^2+1}}$$

The ode becomes

$$\frac{d}{dt}\mu y = 0$$
$$\frac{d}{dt}\left(\frac{y}{\sqrt{t^2+1}}\right) = 0$$

Integrating gives

$$\frac{y}{\sqrt{t^2+1}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{t^2+1}}$ results in

$$y = c_1\sqrt{t^2+1}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{t^2+1} \tag{1}$$

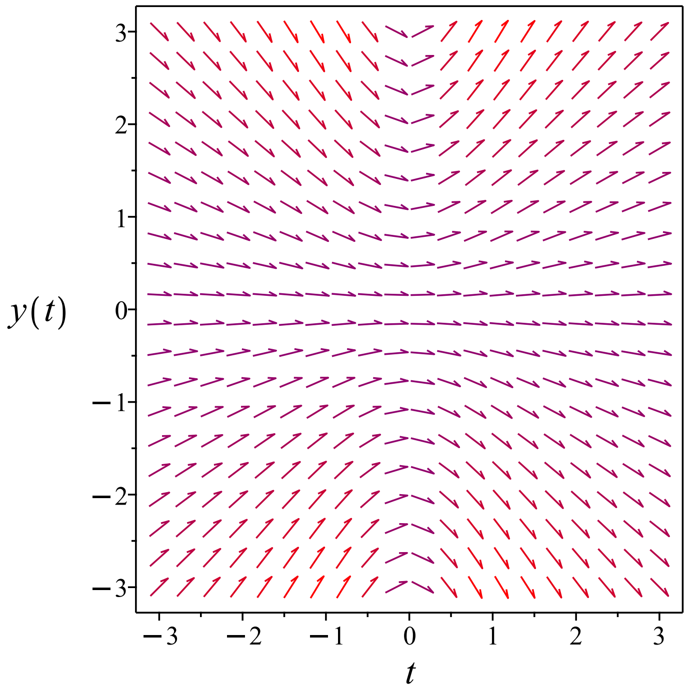


Figure 314: Slope field plot

Verification of solutions

$$y = c_1\sqrt{t^2+1}$$

Verified OK.

8.11.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$u'(t)\,t + u(t) - \frac{t^2 u(t)}{t^2+1} = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(t,u) \\
&= f(t)g(u) \\
&= -\frac{u}{t\,(t^2+1)}
\end{aligned}$$

Where $f(t) = -\frac{1}{t(t^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{t\,(t^2+1)}\,dt \\
\int \frac{1}{u}\,du &= \int -\frac{1}{t\,(t^2+1)}\,dt \\
\ln(u) &= -\ln(t) + \frac{\ln(t^2+1)}{2} + c_2 \\
u &= \mathrm{e}^{-\ln(t)+\frac{\ln(t^2+1)}{2}+c_2} \\
&= c_2\mathrm{e}^{-\ln(t)+\frac{\ln(t^2+1)}{2}}
\end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_2\sqrt{t^2+1}}{t}$$

Therefore the solution y is

$$\begin{aligned}
y &= tu \\
&= c_2\sqrt{t^2+1}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2\sqrt{t^2+1} \tag{1}$$

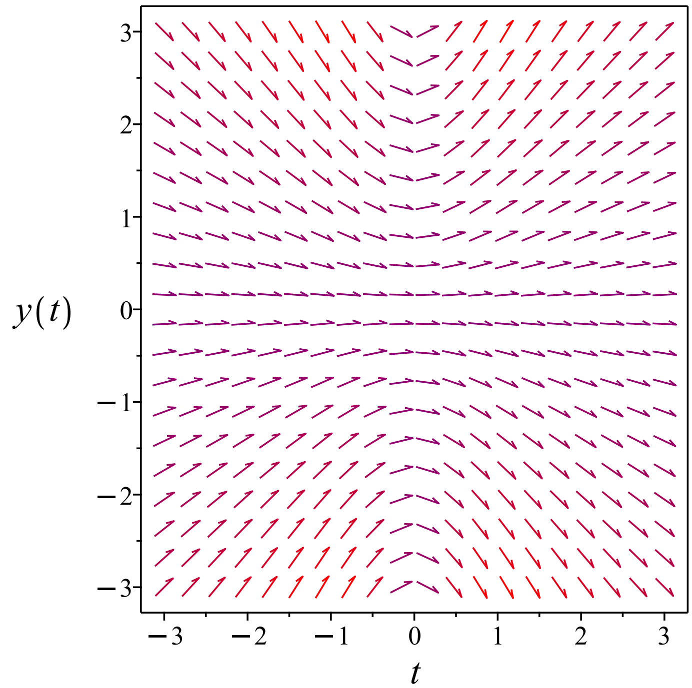


Figure 315: Slope field plot

Verification of solutions

$$y = c_2\sqrt{t^2+1}$$

Verified OK.

8.11.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{ty}{t^2+1}$$
$$y' = \omega(t,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 310: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \sqrt{t^2 + 1}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{t^2+1}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sqrt{t^2+1}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{ty}{t^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{yt}{(t^2+1)^{\frac{3}{2}}} \\ S_y &= \frac{1}{\sqrt{t^2+1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{\sqrt{t^2+1}} = c_1$$

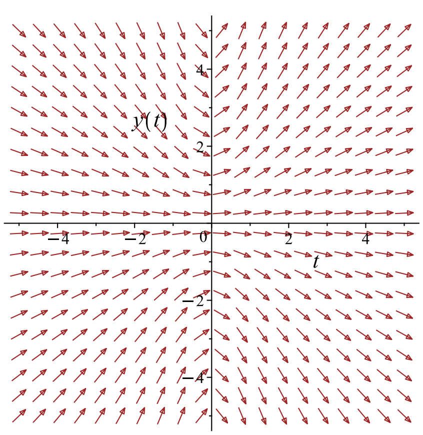
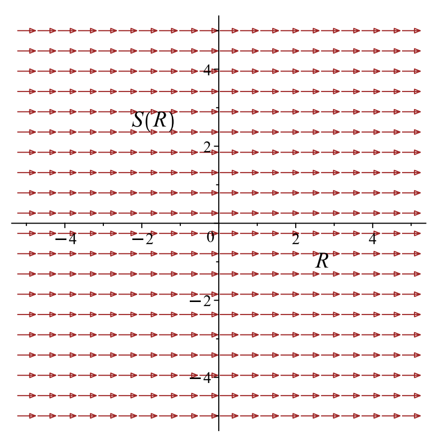
Which simplifies to

$$\frac{y}{\sqrt{t^2+1}} = c_1$$

Which gives

$$y = c_1\sqrt{t^2+1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{ty}{t^2+1}$ 	$R = t$ $S = \frac{y}{\sqrt{t^2+1}}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1\sqrt{t^2+1} \tag{1}$$

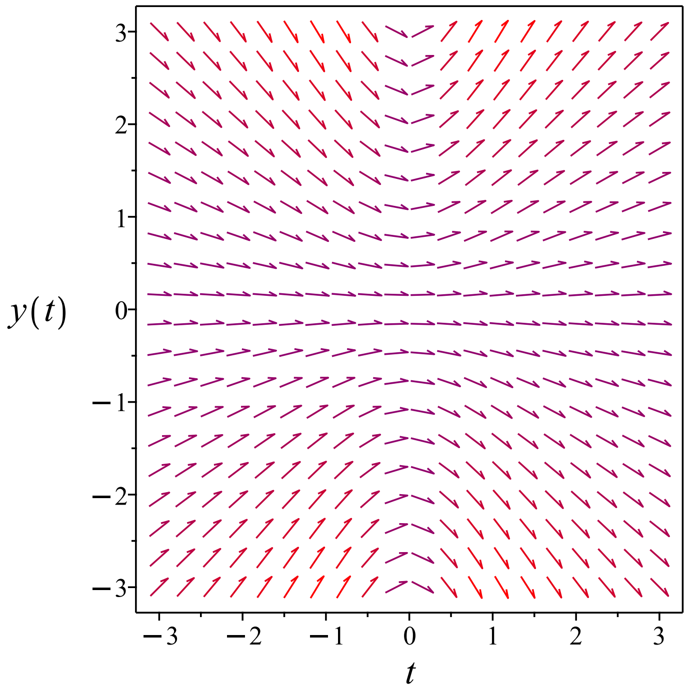


Figure 316: Slope field plot

Verification of solutions

$$y = c_1\sqrt{t^2+1}$$

Verified OK.

8.11.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{t}{t^2+1}\right) \mathrm{d}t \\ \left(-\frac{t}{t^2+1}\right) \mathrm{d}t + \left(\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -\frac{t}{t^2+1} \\ N(t, y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{t}{t^2+1}\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -\frac{t}{t^2+1} \, \mathrm{d}t$$
$$\phi = -\frac{\ln(t^2+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t^2+1)}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t^2+1)}{2} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\frac{\ln(t^2+1)}{2} + c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\ln(t^2+1)}{2} + c_1} \tag{1}$$

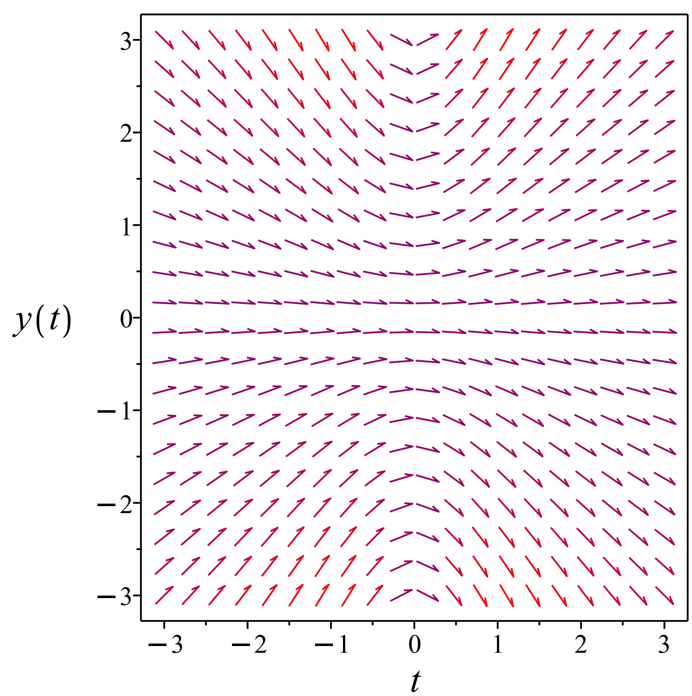


Figure 317: Slope field plot

Verification of solutions

$$y = e^{\frac{\ln(t^2+1)}{2}+c_1}$$

Verified OK.

8.11.6 Maple step by step solution

Let's solve

$y' - \frac{ty}{t^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{t}{t^2+1}$

- Integrate both sides with respect to t

  $\int \frac{y'}{y}dt = \int \frac{t}{t^2+1}dt + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(t^2+1)}{2} + c_1$

- Solve for y

  $y = e^{\frac{\ln(t^2+1)}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(t),t)= t*y(t)/(1+t^2),y(t), singsol=all)
```

$$y(t) = c_1\sqrt{t^2+1}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 22

```
DSolve[y'[t]==t*y[t]/(1+t^2),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1\sqrt{t^2+1}$$
$$y(t) \to 0$$

8.12 problem 25

Internal problem ID [13040]
Internal file name [`OUTPUT/11692_Wednesday_November_08_2023_03_28_51_AM_63651877/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 5y = \sin(3t)}$$

8.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 5$$
$$q(t) = \sin(3t)$$

Hence the ode is

$$y' + 5y = \sin(3t)$$

The integrating factor μ is

$$\mu = e^{\int 5dt}$$
$$= e^{5t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)(\sin(3t))$$
$$\frac{d}{dt}(e^{5t}y) = (e^{5t})(\sin(3t))$$
$$d(e^{5t}y) = (\sin(3t)\,e^{5t})\,dt$$

Integrating gives

$$e^{5t}y = \int \sin(3t)\,e^{5t}\,dt$$
$$e^{5t}y = -\frac{3\cos(3t)\,e^{5t}}{34} + \frac{5\sin(3t)\,e^{5t}}{34} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{5t}$ results in

$$y = e^{-5t}\left(-\frac{3\cos(3t)\,e^{5t}}{34} + \frac{5\sin(3t)\,e^{5t}}{34}\right) + c_1 e^{-5t}$$

which simplifies to

$$y = \frac{5\sin(3t)}{34} - \frac{3\cos(3t)}{34} + c_1 e^{-5t}$$

Summary
The solution(s) found are the following

$$y = \frac{5\sin(3t)}{34} - \frac{3\cos(3t)}{34} + c_1 e^{-5t} \tag{1}$$

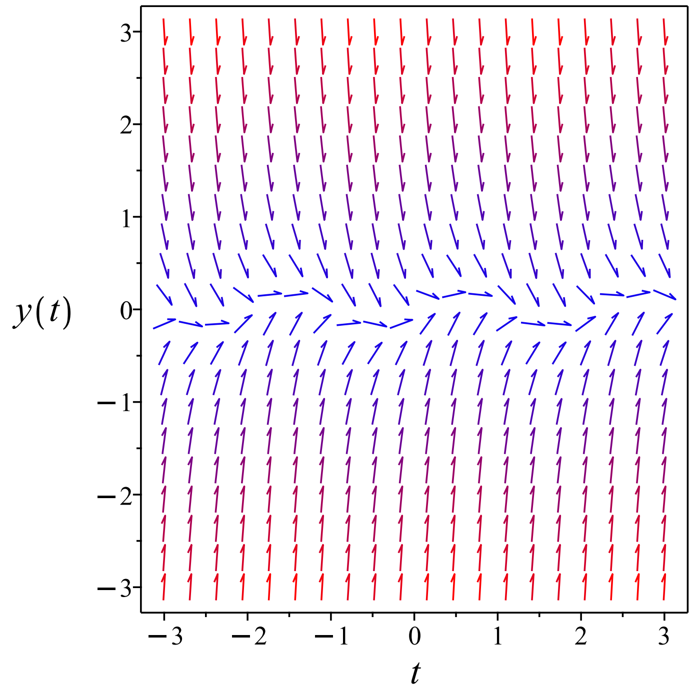


Figure 318: Slope field plot

Verification of solutions

$$y = \frac{5\sin(3t)}{34} - \frac{3\cos(3t)}{34} + c_1 e^{-5t}$$

Verified OK.

8.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -5y + \sin(3t)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 313: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-5t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-5t}} dy \end{aligned}$$

Which results in

$$S = e^{5t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -5y + \sin(3t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 5 e^{5t} y \\ S_y &= e^{5t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(3t) e^{5t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(3R) e^{5R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{e^{5R}(3\cos(3R) - 5\sin(3R))}{34} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{5t}y = c_1 - \frac{e^{5t}(3\cos(3t) - 5\sin(3t))}{34}$$

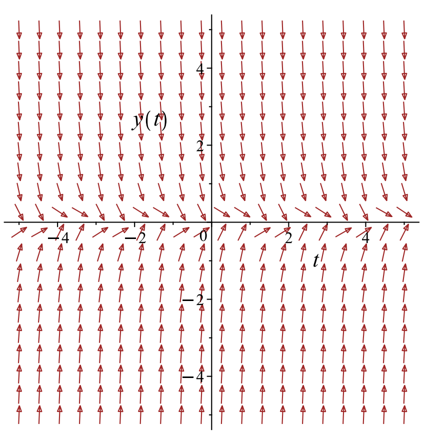
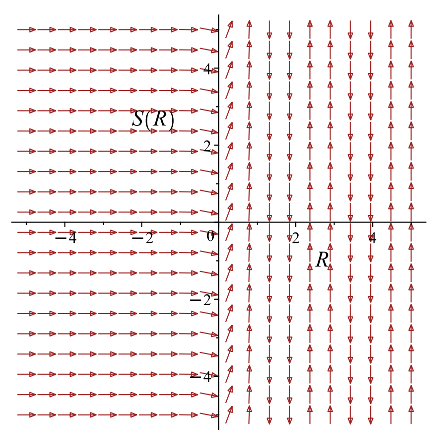
Which simplifies to

$$e^{5t}y = c_1 - \frac{e^{5t}(3\cos(3t) - 5\sin(3t))}{34}$$

Which gives

$$y = \frac{e^{-5t}(5\sin(3t)\,e^{5t} - 3\cos(3t)\,e^{5t} + 34c_1)}{34}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -5y + \sin(3t)$ 	$R = t$ $S = e^{5t}y$	$\frac{dS}{dR} = \sin(3R)\,e^{5R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-5t}(5\sin(3t)\,e^{5t} - 3\cos(3t)\,e^{5t} + 34c_1)}{34} \tag{1}$$

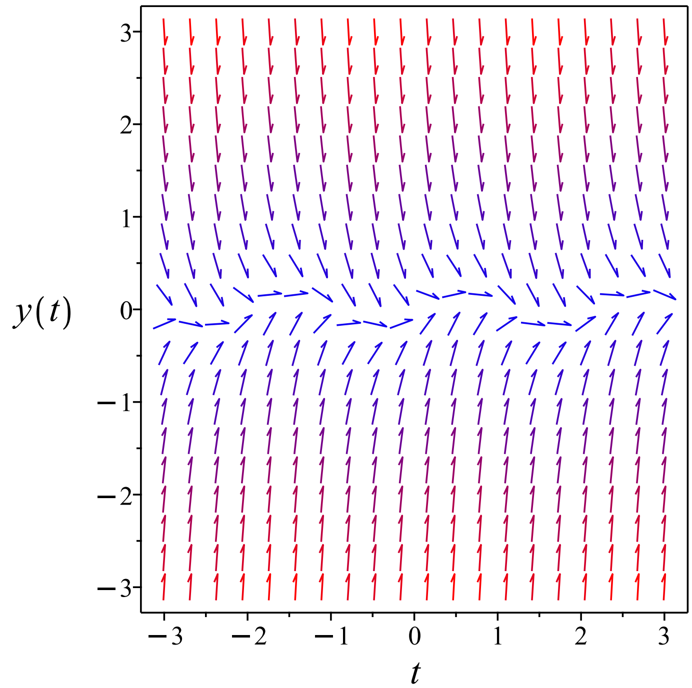


Figure 319: Slope field plot

Verification of solutions

$$y = \frac{e^{-5t}(5\sin(3t)e^{5t} - 3\cos(3t)e^{5t} + 34c_1)}{34}$$

Verified OK.

8.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= (-5y + \sin(3t))\,\mathrm{d}t \\ (5y - \sin(3t))\,\mathrm{d}t + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, y) &= 5y - \sin(3t) \\ N(t, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(5y - \sin(3t)) \\ &= 5 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\ &= 1((5) - (0)) \\ &= 5 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int 5 \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{5t} \\ &= \mathrm{e}^{5t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{5t}(5y - \sin(3t)) \\ &= (5y - \sin(3t))\,\mathrm{e}^{5t} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{5t}(1) \\ &= \mathrm{e}^{5t} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left((5y - \sin(3t))\,\mathrm{e}^{5t}\right) + \left(\mathrm{e}^{5t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int (5y - \sin(3t))\,\mathrm{e}^{5t}\,\mathrm{d}t$$
$$\phi = -\frac{\mathrm{e}^{5t}(-3\cos(3t) + 5\sin(3t) - 34y)}{34} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \mathrm{e}^{5t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{5t}$. Therefore equation (4) becomes

$$\mathrm{e}^{5t} = \mathrm{e}^{5t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{5t}(-3\cos(3t) + 5\sin(3t) - 34y)}{34} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{5t}(-3\cos(3t) + 5\sin(3t) - 34y)}{34}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-5t}(5\sin(3t)\,\mathrm{e}^{5t} - 3\cos(3t)\,\mathrm{e}^{5t} + 34c_1)}{34}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-5t}(5\sin(3t)e^{5t} - 3\cos(3t)e^{5t} + 34c_1)}{34} \tag{1}$$

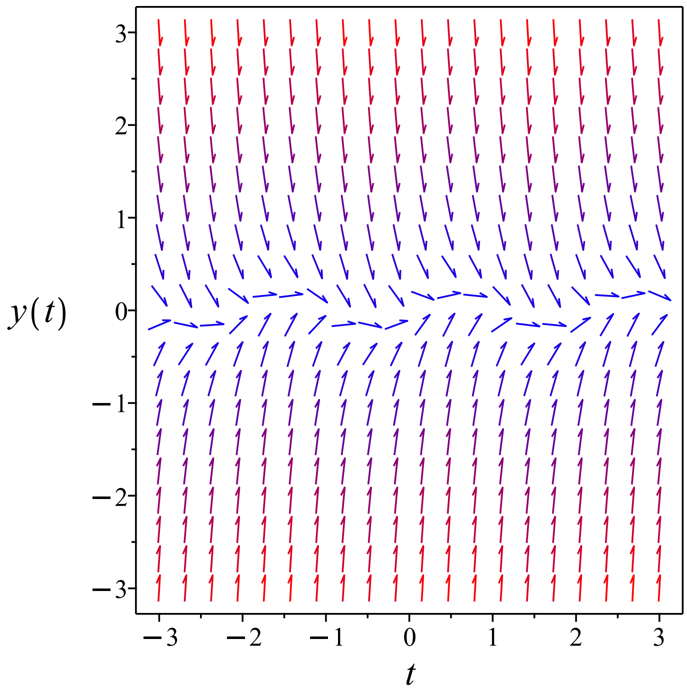


Figure 320: Slope field plot

Verification of solutions

$$y = \frac{e^{-5t}(5\sin(3t)e^{5t} - 3\cos(3t)e^{5t} + 34c_1)}{34}$$

Verified OK.

8.12.4 Maple step by step solution

Let's solve

$y' + 5y = \sin(3t)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -5y + \sin(3t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + 5y = \sin(3t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)(y' + 5y) = \mu(t)\sin(3t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

$\mu(t)(y' + 5y) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

$\mu'(t) = 5\mu(t)$

- Solve to find the integrating factor

$\mu(t) = e^{5t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\sin(3t)\, dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\, y = \int \mu(t)\sin(3t)\, dt + c_1$

- Solve for y

$y = \frac{\int \mu(t)\sin(3t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{5t}$

$y = \frac{\int \sin(3t)e^{5t}dt + c_1}{e^{5t}}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{5\sin(3t)e^{5t}}{34} - \frac{3\cos(3t)e^{5t}}{34} + c_1}{e^{5t}}$

- Simplify

$y = \frac{5\sin(3t)}{34} - \frac{3\cos(3t)}{34} + c_1 e^{-5t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(t),t)= -5*y(t)+sin(3*t),y(t), singsol=all)
```

$$y(t) = -\frac{3\cos(3t)}{34} + \frac{5\sin(3t)}{34} + \mathrm{e}^{-5t}c_1$$

✓ Solution by Mathematica

Time used: 0.165 (sec). Leaf size: 30

```
DSolve[y'[t]==-5*y[t]+Sin[3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{5}{34}\sin(3t) - \frac{3}{34}\cos(3t) + c_1e^{-5t}$$

8.13 problem 26

Internal problem ID [13041]
Internal file name [`OUTPUT/11693_Wednesday_November_08_2023_03_28_52_AM_60644923/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{2y}{1+t} = t}$$

8.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -\frac{2}{1+t}$$
$$q(t) = t$$

Hence the ode is

$$y' - \frac{2y}{1+t} = t$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{1+t}dt}$$
$$= \frac{1}{(1+t)^2}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)(t)$$
$$\frac{d}{dt}\left(\frac{y}{(1+t)^2}\right) = \left(\frac{1}{(1+t)^2}\right)(t)$$
$$d\left(\frac{y}{(1+t)^2}\right) = \left(\frac{t}{(1+t)^2}\right) dt$$

Integrating gives

$$\frac{y}{(1+t)^2} = \int \frac{t}{(1+t)^2}\, dt$$
$$\frac{y}{(1+t)^2} = \ln(1+t) + \frac{1}{1+t} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(1+t)^2}$ results in

$$y = (1+t)^2 \left(\ln(1+t) + \frac{1}{1+t}\right) + c_1(1+t)^2$$

which simplifies to

$$y = (1+t)\left((1+t)\ln(1+t) + c_1 t + c_1 + 1\right)$$

Summary
The solution(s) found are the following

$$y = (1+t)\left((1+t)\ln(1+t) + c_1 t + c_1 + 1\right) \tag{1}$$

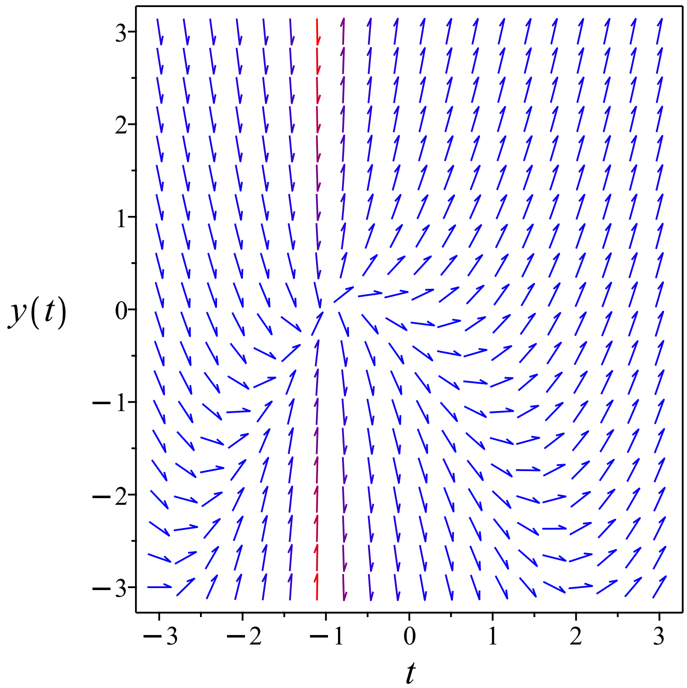


Figure 321: Slope field plot

Verification of solutions

$$y = (1+t)\left((1+t)\ln(1+t) + c_1 t + c_1 + 1\right)$$

Verified OK.

8.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{t^2 + t + 2y}{1+t}$$
$$y' = \omega(t,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 316: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= (1 + t)^2
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{(1+t)^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{(1+t)^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{t^2 + t + 2y}{1+t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{2y}{(1+t)^3} \\ S_y &= \frac{1}{(1+t)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{t}{(1+t)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{(1+R)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(1+R) + \frac{1}{1+R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{(1+t)^2} = \ln(1+t) + \frac{1}{1+t} + c_1$$

Which simplifies to

$$\frac{y}{(1+t)^2} = \ln(1+t) + \frac{1}{1+t} + c_1$$

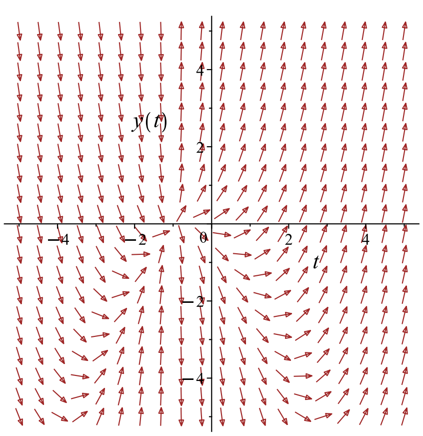
Which gives

$$y = (1+t)\left(\ln(1+t)\,t + c_1 t + \ln(1+t) + c_1 + 1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in t, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>

$\frac{dy}{dt} = \frac{t^2+t+2y}{1+t}$



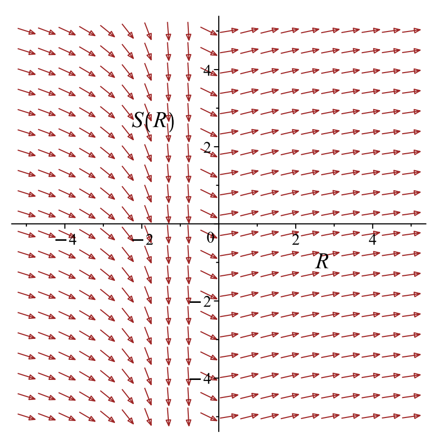
</td><td>

$R = t$

$S = \frac{y}{(1+t)^2}$

</td><td>

$\frac{dS}{dR} = \frac{R}{(1+R)^2}$



</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = (1 + t) (\ln (1 + t) t + c_1 t + \ln (1 + t) + c_1 + 1) \tag{1}$$

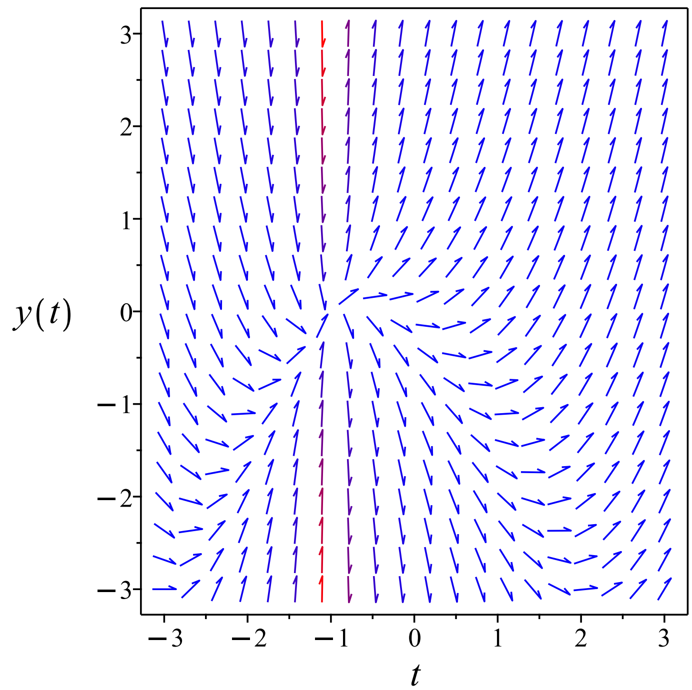


Figure 322: Slope field plot

Verification of solutions

$$y = (1 + t) (\ln (1 + t) t + c_1 t + \ln (1 + t) + c_1 + 1)$$

Verified OK.

8.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(t+\frac{2y}{1+t}\right)\mathrm{d}t\\ \left(-t-\frac{2y}{1+t}\right)\mathrm{d}t+\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-t-\frac{2y}{1+t}\\ N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-t-\frac{2y}{1+t}\right)\\ &=-\frac{2}{1+t}\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(-\frac{2}{1+t}\right) - (0)\right) \\
&= -\frac{2}{1+t}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -\frac{2}{1+t} \, \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(1+t)} \\
&= \frac{1}{(1+t)^2}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(1+t)^2}\left(-t - \frac{2y}{1+t}\right) \\
&= \frac{-t^2 - t - 2y}{(1+t)^3}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(1+t)^2}(1) \\
&= \frac{1}{(1+t)^2}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}t}=0$$
$$\left(\frac{-t^2-t-2y}{(1+t)^3}\right)+\left(\frac{1}{(1+t)^2}\right)\frac{\mathrm{d}y}{\mathrm{d}t}=0$$

The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial\phi}{\partial t}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t=\int\overline{M}\,\mathrm{d}t$$
$$\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t=\int\frac{-t^2-t-2y}{(1+t)^3}\,\mathrm{d}t$$
$$\phi=\frac{y}{(1+t)^2}-\ln(1+t)-\frac{1}{1+t}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=\frac{1}{(1+t)^2}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{1}{(1+t)^2}$. Therefore equation (4) becomes

$$\frac{1}{(1+t)^2}=\frac{1}{(1+t)^2}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{(1+t)^2} - \ln(1+t) - \frac{1}{1+t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{(1+t)^2} - \ln(1+t) - \frac{1}{1+t}$$

The solution becomes

$$y = (1+t)(\ln(1+t)\,t + c_1 t + \ln(1+t) + c_1 + 1)$$

<u>Summary</u>
The solution(s) found are the following

$$y = (1+t)(\ln(1+t)\,t + c_1 t + \ln(1+t) + c_1 + 1) \tag{1}$$

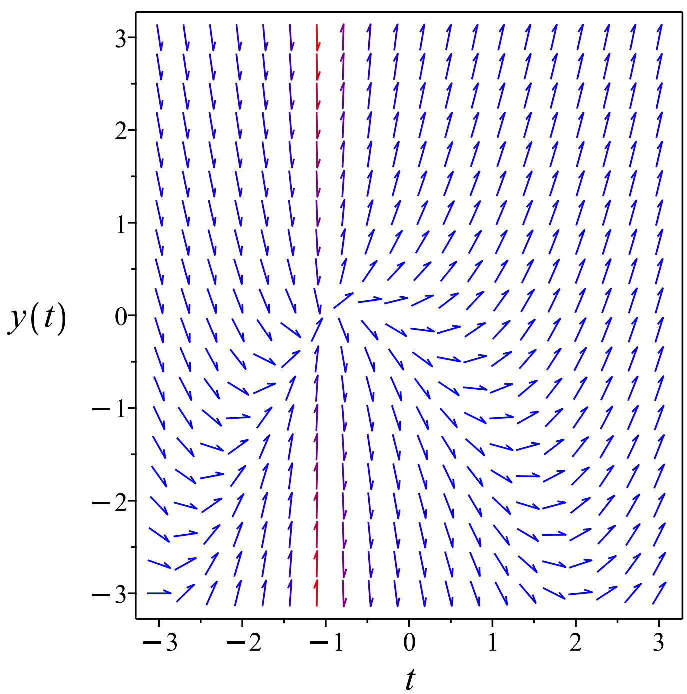


Figure 323: Slope field plot

Verification of solutions

$$y = (1 + t) (\ln (1 + t) t + c_1 t + \ln (1 + t) + c_1 + 1)$$

Verified OK.

8.13.4 Maple step by step solution

Let's solve

$y' - \frac{2y}{1+t} = t$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = t + \frac{2y}{1+t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{1+t} = t$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t) \left(y' - \frac{2y}{1+t}\right) = \mu(t) t$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) y)$

  $\mu(t) \left(y' - \frac{2y}{1+t}\right) = \mu'(t) y + \mu(t) y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\frac{2\mu(t)}{1+t}$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{(1+t)^2}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t) y)\right) dt = \int \mu(t) t dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t) y = \int \mu(t) t dt + c_1$

- Solve for y

  $y = \frac{\int \mu(t) t dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{(1+t)^2}$

$y = (1+t)^2 \left(\int \frac{t}{(1+t)^2} dt + c_1\right)$

- Evaluate the integrals on the rhs

  $y = (1+t)^2 \left(\ln(1+t) + \frac{1}{1+t} + c_1\right)$

- Simplify

  $y = (1+t)\left((1+t)\ln(1+t) + c_1 t + c_1 + 1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(diff(y(t),t)= t+2*y(t)/(1+t),y(t), singsol=all)
```

$$y(t) = (t+1)\left((t+1)\ln(t+1) + c_1 t + c_1 + 1\right)$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 23

```
DSolve[y'[t]==t+2*y[t]/(1+t),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to (t+1)^2 \left(\frac{1}{t+1} + \log(t+1) + c_1\right)$$

8.14 problem 27

Internal problem ID [13042]
Internal file name [OUTPUT/11694_Wednesday_November_08_2023_03_28_52_AM_18688578/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = 3}$$

8.14.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 + 3} dy = t + c_1$$
$$\frac{\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\, y}{3}\right)}{3} = t + c_1$$

Solving for y gives these solutions

$$y_1 = \sqrt{3}\, \tan\left((t + c_1)\sqrt{3}\right)$$

Summary
The solution(s) found are the following

$$y = \sqrt{3}\, \tan\left((t + c_1)\sqrt{3}\right) \tag{1}$$

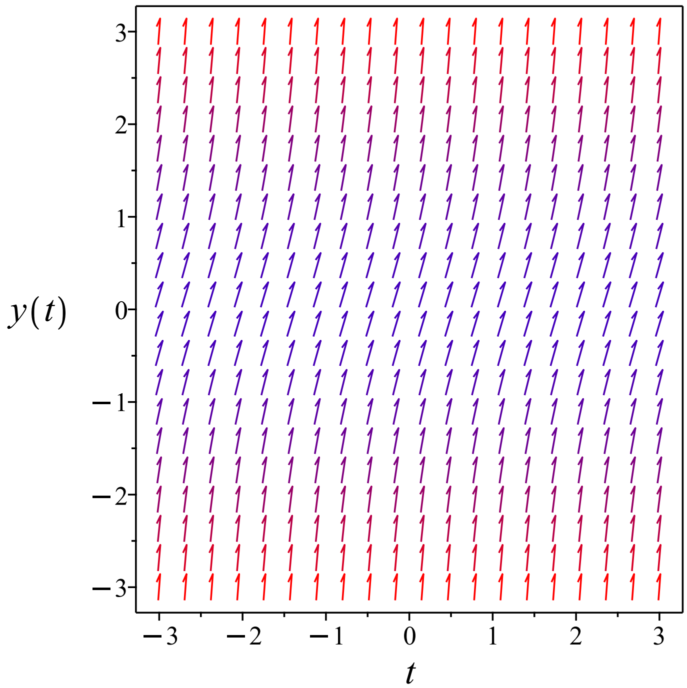


Figure 324: Slope field plot

Verification of solutions

$$y = \sqrt{3}\,\tan\left((t + c_1)\sqrt{3}\right)$$

Verified OK.

8.14.2 Maple step by step solution

Let's solve

$y' - y^2 = 3$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{3+y^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{3+y^2}dt = \int 1dt + c_1$

- Evaluate integral

$$\frac{\sqrt{3}\,\arctan\left(\frac{y\sqrt{3}}{3}\right)}{3} = t + c_1$$

- Solve for y

  $$y = \sqrt{3}\,\tan\left(\left(t + c_1\right)\sqrt{3}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(t),t)= 3+y(t)^2,y(t), singsol=all)
```

$$y(t) = \sqrt{3}\,\tan\left(\left(t + c_1\right)\sqrt{3}\right)$$

✓ Solution by Mathematica

Time used: 0.256 (sec). Leaf size: 48

```
DSolve[y'[t]==3+y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \sqrt{3}\tan\left(\sqrt{3}(t + c_1)\right)$$
$$y(t) \to -i\sqrt{3}$$
$$y(t) \to i\sqrt{3}$$

8.15 problem 28

Internal problem ID [13043]
Internal file name [`OUTPUT/11695_Wednesday_November_08_2023_03_28_53_AM_26392571/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 2y + y^2 = 0}$$

8.15.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 + 2y} dy = \int dt$$
$$-\frac{\ln(y-2)}{2} + \frac{\ln(y)}{2} = t + c_1$$

The above can be written as

$$\left(-\frac{1}{2}\right)(\ln(y-2) - \ln(y)) = t + c_1$$
$$\ln(y-2) - \ln(y) = (-2)(t + c_1)$$
$$= -2t - 2c_1$$

Raising both side to exponential gives

$$e^{\ln(y-2) - \ln(y)} = -2c_1 e^{-2t}$$

Which simplifies to

$$\frac{y-2}{y} = c_2 \mathrm{e}^{-2t}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{-1 + c_2 \mathrm{e}^{-2t}} \tag{1}$$

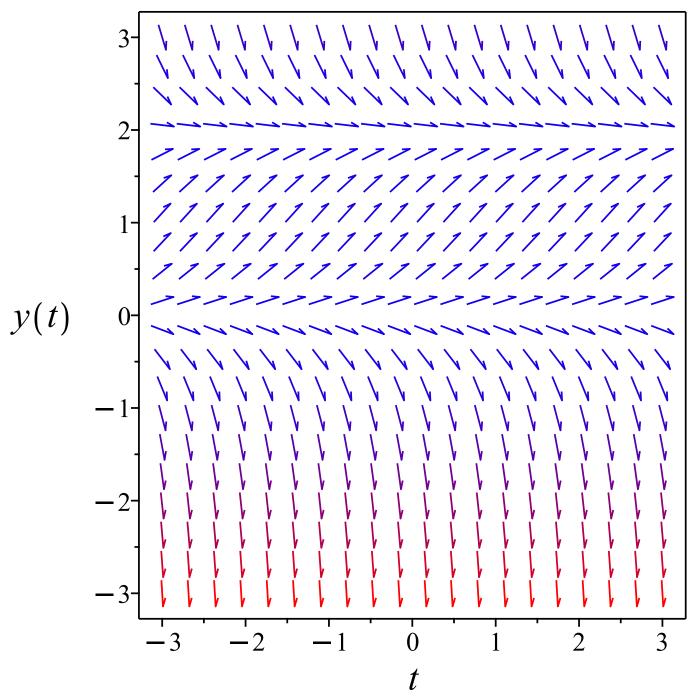


Figure 325: Slope field plot

Verification of solutions

$$y = -\frac{2}{-1 + c_2 \mathrm{e}^{-2t}}$$

Verified OK.

8.15.2 Maple step by step solution

Let's solve

$y' - 2y + y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y-y^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{2y-y^2} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{\ln(y-2)}{2} + \frac{\ln(y)}{2} = t + c_1$

- Solve for y

  $y = \frac{2\,\mathrm{e}^{2t+2c_1}}{-1+\mathrm{e}^{2t+2c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(t),t)= 2*y(t)-y(t)^2,y(t), singsol=all)
```

$$y(t) = \frac{2}{1 + 2\,\mathrm{e}^{-2t} c_1}$$

✓ Solution by Mathematica

Time used: 0.447 (sec). Leaf size: 36

```
DSolve[y'[t]==2*y[t]-y[t]^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{2e^{2t}}{e^{2t} + e^{2c_1}}$$
$$y(t) \to 0$$
$$y(t) \to 2$$

8.16 problem 29

Internal problem ID [13044]
Internal file name [`OUTPUT/11696_Wednesday_November_08_2023_03_28_53_AM_65199612/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 3y = e^{-2t} + t^2}$$

8.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 3$$
$$q(t) = e^{-2t} + t^2$$

Hence the ode is

$$y' + 3y = e^{-2t} + t^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 3dt} \\
&= \mathrm{e}^{3t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\mathrm{e}^{-2t}+t^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{3t}y\right) &= \left(\mathrm{e}^{3t}\right)\left(\mathrm{e}^{-2t}+t^2\right) \\
\mathrm{d}\left(\mathrm{e}^{3t}y\right) &= \left(\left(\mathrm{e}^{2t}t^2+1\right)\mathrm{e}^{t}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{3t}y &= \int \left(\mathrm{e}^{2t}t^2+1\right)\mathrm{e}^{t}\,\mathrm{d}t \\
\mathrm{e}^{3t}y &= \frac{t^2\mathrm{e}^{3t}}{3}-\frac{2t\,\mathrm{e}^{3t}}{9}+\frac{2\,\mathrm{e}^{3t}}{27}+\mathrm{e}^{t}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{3t}$ results in

$$
y = \mathrm{e}^{-3t}\left(\frac{t^2\mathrm{e}^{3t}}{3}-\frac{2t\,\mathrm{e}^{3t}}{9}+\frac{2\,\mathrm{e}^{3t}}{27}+\mathrm{e}^{t}\right)+\mathrm{e}^{-3t}c_1
$$

which simplifies to

$$
y = \frac{t^2}{3}-\frac{2t}{9}+\frac{2}{27}+\mathrm{e}^{-2t}+\mathrm{e}^{-3t}c_1
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{t^2}{3}-\frac{2t}{9}+\frac{2}{27}+\mathrm{e}^{-2t}+\mathrm{e}^{-3t}c_1 \tag{1}
$$

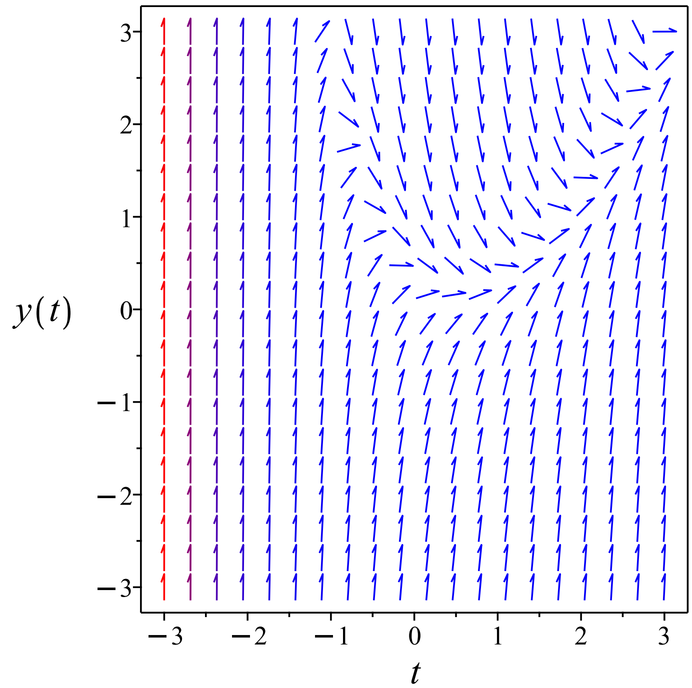


Figure 326: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{t^2}{3} - \frac{2t}{9} + \frac{2}{27} + \mathrm{e}^{-2t} + \mathrm{e}^{-3t} c_1$$

Verified OK.

8.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -3y + \mathrm{e}^{-2t} + t^2$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 321: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= \mathrm{e}^{-3t}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-3t}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{3t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = -3y + \mathrm{e}^{-2t} + t^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 3\,\mathrm{e}^{3t} y \\ S_y &= \mathrm{e}^{3t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t^2 \mathrm{e}^{3t} + \mathrm{e}^{t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 \mathrm{e}^{3R} + \mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2 e^{3R}}{3} - \frac{2R\, e^{3R}}{9} + \frac{2\, e^{3R}}{27} + e^R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{3t} y = \frac{t^2 e^{3t}}{3} - \frac{2t\, e^{3t}}{9} + \frac{2\, e^{3t}}{27} + e^t + c_1$$

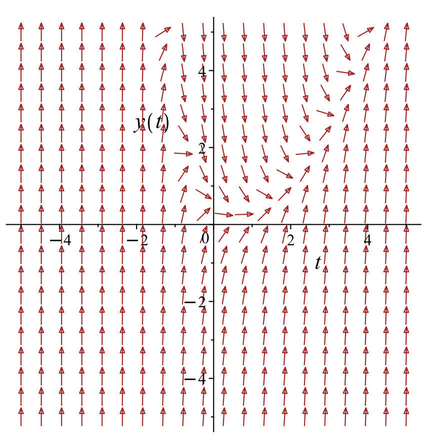
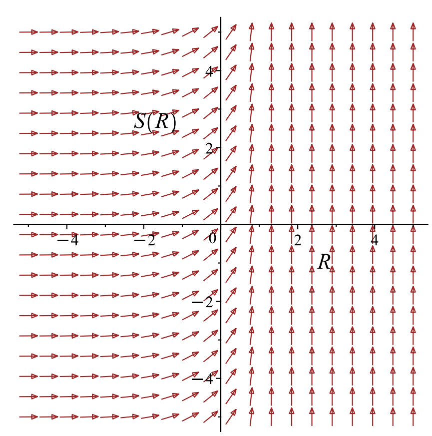
Which simplifies to

$$\frac{(-9t^2 + 6t + 27y - 2)\, e^{3t}}{27} - c_1 - e^t = 0$$

Which gives

$$y = \frac{(9t^2 e^{3t} - 6t\, e^{3t} + 27\, e^t + 2\, e^{3t} + 27c_1)\, e^{-3t}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -3y + e^{-2t} + t^2$ 	$R = t$ $S = e^{3t} y$	$\frac{dS}{dR} = R^2 e^{3R} + e^R$ 

Summary

The solution(s) found are the following

$$y = \frac{(9t^2 e^{3t} - 6t\, e^{3t} + 27\, e^t + 2\, e^{3t} + 27c_1)\, e^{-3t}}{27} \tag{1}$$

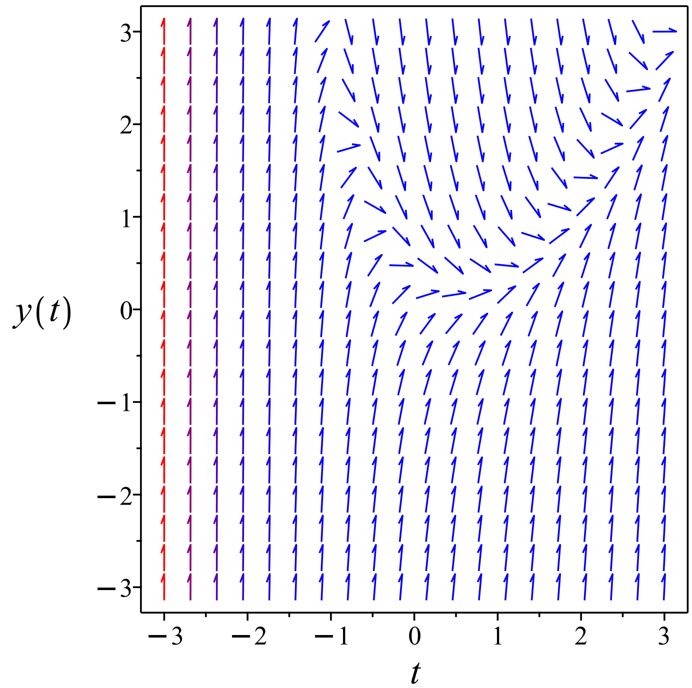


Figure 327: Slope field plot

Verification of solutions

$$y = \frac{(9t^2 e^{3t} - 6t\,e^{3t} + 27\,e^{t} + 2\,e^{3t} + 27c_1)\,e^{-3t}}{27}$$

Verified OK.

8.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-3y + \mathrm{e}^{-2t} + t^2\right)\mathrm{d}t$$
$$\left(3y - \mathrm{e}^{-2t} - t^2\right)\mathrm{d}t + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = 3y - \mathrm{e}^{-2t} - t^2$$
$$N(t, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(3y - \mathrm{e}^{-2t} - t^2\right)$$
$$= 3$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((3) - (0)) \\
&= 3
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int 3 \, \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{3t} \\
&= \mathrm{e}^{3t}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{3t}\left(3y - \mathrm{e}^{-2t} - t^2\right) \\
&= -\mathrm{e}^{t}\left(1 + \left(t^2 - 3y\right)\mathrm{e}^{2t}\right)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{3t}(1) \\
&= \mathrm{e}^{3t}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{t}\left(1 + \left(t^2 - 3y\right)\mathrm{e}^{2t}\right)\right) + \left(\mathrm{e}^{3t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -\mathrm{e}^{t} \left(1 + \left(t^{2} - 3y\right) \mathrm{e}^{2t}\right) \mathrm{d}t$$
$$\phi = \frac{\left(-9t^{2} + 6t + 27y - 2\right) \mathrm{e}^{3t}}{27} - \mathrm{e}^{t} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{3t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{3t}$. Therefore equation (4) becomes

$$\mathrm{e}^{3t} = \mathrm{e}^{3t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\left(-9t^{2} + 6t + 27y - 2\right) \mathrm{e}^{3t}}{27} - \mathrm{e}^{t} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\left(-9t^{2} + 6t + 27y - 2\right) \mathrm{e}^{3t}}{27} - \mathrm{e}^{t}$$

The solution becomes

$$y = \frac{\left(9t^{2} \mathrm{e}^{3t} - 6t \, \mathrm{e}^{3t} + 27 \, \mathrm{e}^{t} + 2 \, \mathrm{e}^{3t} + 27 c_1\right) \mathrm{e}^{-3t}}{27}$$

Summary
The solution(s) found are the following

$$y = \frac{(9t^2 e^{3t} - 6t\,e^{3t} + 27\,e^{t} + 2\,e^{3t} + 27c_1)\,e^{-3t}}{27} \tag{1}$$

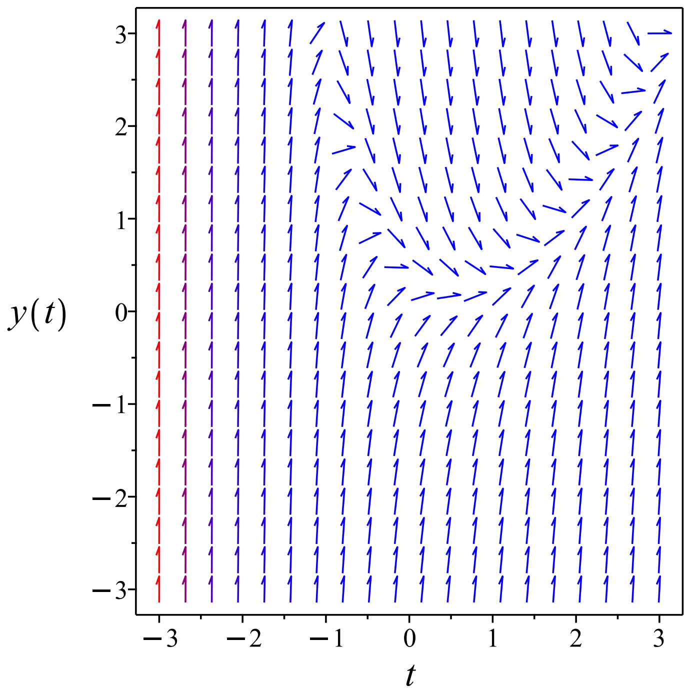


Figure 328: Slope field plot

Verification of solutions

$$y = \frac{(9t^2 e^{3t} - 6t\,e^{3t} + 27\,e^{t} + 2\,e^{3t} + 27c_1)\,e^{-3t}}{27}$$

Verified OK.

8.16.4 Maple step by step solution

Let's solve

$y' + 3y = e^{-2t} + t^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -3y + \mathrm{e}^{-2t} + t^2$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + 3y = \mathrm{e}^{-2t} + t^2$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)\,(y' + 3y) = \mu(t)\,(\mathrm{e}^{-2t} + t^2)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$$\mu(t)\,(y' + 3y) = \mu'(t)\,y + \mu(t)\,y'$$

- Isolate $\mu'(t)$

$$\mu'(t) = 3\mu(t)$$

- Solve to find the integrating factor

$$\mu(t) = \mathrm{e}^{3t}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\,(\mathrm{e}^{-2t} + t^2)\,dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\,y = \int \mu(t)\,(\mathrm{e}^{-2t} + t^2)\,dt + c_1$$

- Solve for y

$$y = \frac{\int \mu(t)(\mathrm{e}^{-2t}+t^2)dt+c_1}{\mu(t)}$$

- Substitute $\mu(t) = \mathrm{e}^{3t}$

$$y = \frac{\int (\mathrm{e}^{-2t}+t^2)\mathrm{e}^{3t}dt+c_1}{\mathrm{e}^{3t}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{(\mathrm{e}^t)^3 t^2}{3} - \frac{2(\mathrm{e}^t)^3 t}{9} + \frac{2(\mathrm{e}^t)^3}{27} + \mathrm{e}^t + c_1}{\mathrm{e}^{3t}}$$

- Simplify

$$y = \frac{\left(\left(t^2 - \frac{2}{3}t + \frac{2}{9}\right)\mathrm{e}^{3t} + 3c_1 + 3\,\mathrm{e}^t\right)\mathrm{e}^{-3t}}{3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(t),t)= -3*y(t)+exp(-2*t)+t^2,y(t), singsol=all)
```

$$y(t) = \frac{t^2}{3} - \frac{2t}{9} + \frac{2}{27} + e^{-2t} + c_1 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.147 (sec). Leaf size: 33

```
DSolve[y'[t]==-3*y[t]+Exp[-2*t]+t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{27}\left(9t^2 - 6t + 2\right) + e^{-2t} + c_1 e^{-3t}$$

8.17 problem 30

Internal problem ID [13045]
Internal file name [`OUTPUT/11697_Wednesday_November_08_2023_03_28_54_AM_79512382/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' + xt = 0}$$

With initial conditions

$$[x(0) = \mathrm{e}]$$

8.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned}p(t) &= t\\ q(t) &= 0\end{aligned}$$

Hence the ode is

$$x' + xt = 0$$

The domain of $p(t) = t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

8.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}x' &= F(t,x)\\ &= f(t)g(x)\\ &= -tx\end{aligned}$$

Where $f(t) = -t$ and $g(x) = x$. Integrating both sides gives

$$\begin{aligned}\frac{1}{x}\,dx &= -t\,dt\\ \int \frac{1}{x}\,dx &= \int -t\,dt\\ \ln(x) &= -\frac{t^2}{2} + c_1\\ x &= \mathrm{e}^{-\frac{t^2}{2}+c_1}\\ &= \mathrm{e}^{-\frac{t^2}{2}} c_1\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} = c_1$$

$$c_1 = \mathrm{e}$$

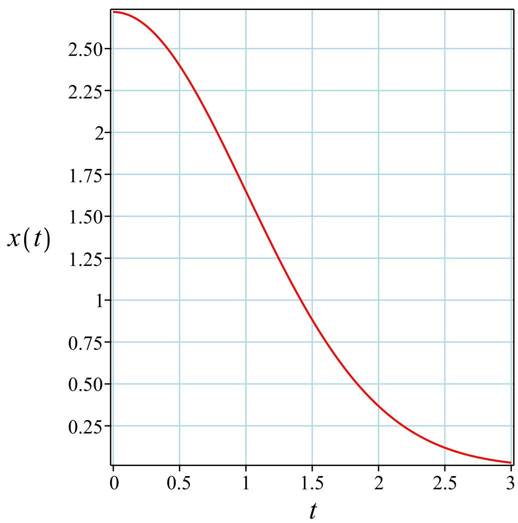
Substituting c_1 found above in the general solution gives

$$x = \mathrm{e}^{-\frac{t^2}{2}} \mathrm{e}$$

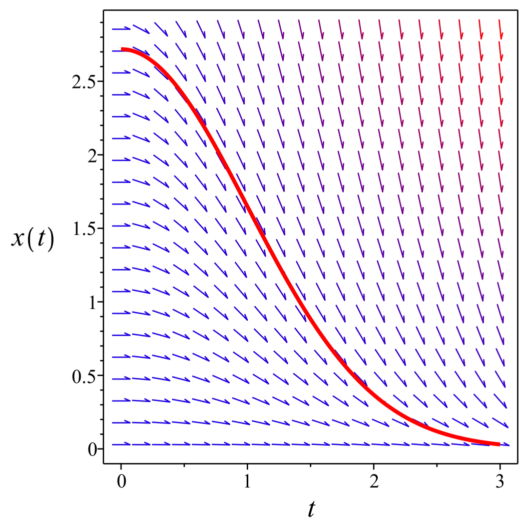
Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-\frac{t^2}{2}} \mathrm{e} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \mathrm{e}^{-\frac{t^2}{2}} \mathrm{e}$$

Verified OK.

8.17.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int t dt} \\ &= \mathrm{e}^{\frac{t^2}{2}} \end{aligned}$$

The ode becomes

$$\frac{d}{dt}\mu x = 0$$

$$\frac{d}{dt}\left(e^{\frac{t^2}{2}} x\right) = 0$$

Integrating gives

$$e^{\frac{t^2}{2}} x = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{t^2}{2}}$ results in

$$x = e^{-\frac{t^2}{2}} c_1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = e$ in the above solution gives an equation to solve for the constant of integration.

$$e = c_1$$

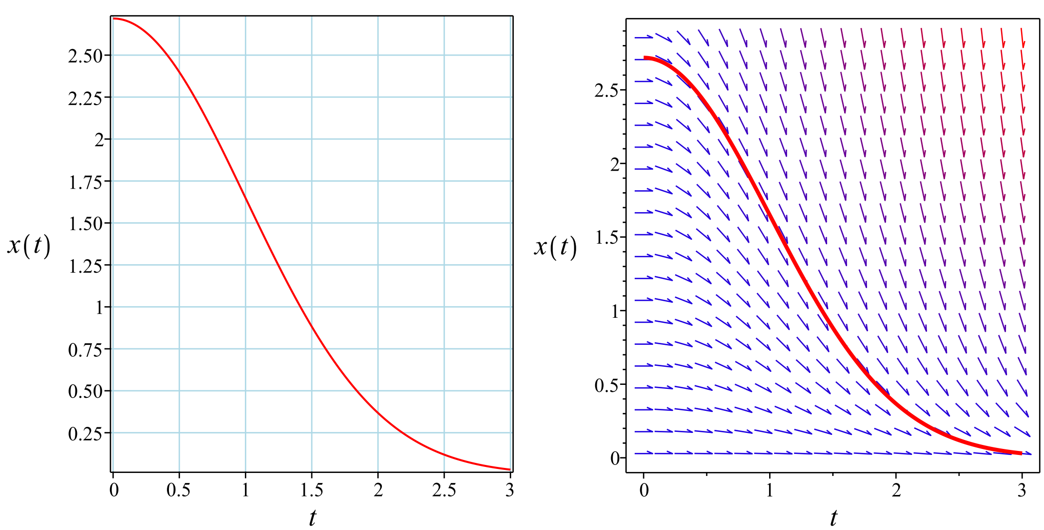
$$c_1 = e$$

Substituting c_1 found above in the general solution gives

$$x = e^{-\frac{t^2}{2}} e$$

Summary
The solution(s) found are the following

$$x = e^{-\frac{t^2}{2}} e \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = e^{-\frac{t^2}{2}} e$$

Verified OK.

8.17.4 Solving as homogeneousTypeD2 ode

Using the change of variables $x = u(t)\, t$ on the above ode results in new ode in $u(t)$

$$u'(t)\, t + u(t) + u(t)\, t^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{u(t^2+1)}{t} \end{aligned}$$

Where $f(t) = -\frac{t^2+1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{t^2+1}{t}\, dt \\ \int \frac{1}{u}\, du &= \int -\frac{t^2+1}{t}\, dt \\ \ln(u) &= -\frac{t^2}{2} - \ln(t) + c_2 \\ u &= e^{-\frac{t^2}{2} - \ln(t) + c_2} \\ &= c_2 e^{-\frac{t^2}{2} - \ln(t)} \end{aligned}$$

Which simplifies to

$$u(t) = \frac{c_2 e^{-\frac{t^2}{2}}}{t}$$

Therefore the solution x is

$$\begin{aligned} x &= tu \\ &= c_2 e^{-\frac{t^2}{2}} \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $x = e$ in the above solution gives an equation to solve for the constant of integration.

$$e = c_2$$

$$c_2 = e$$

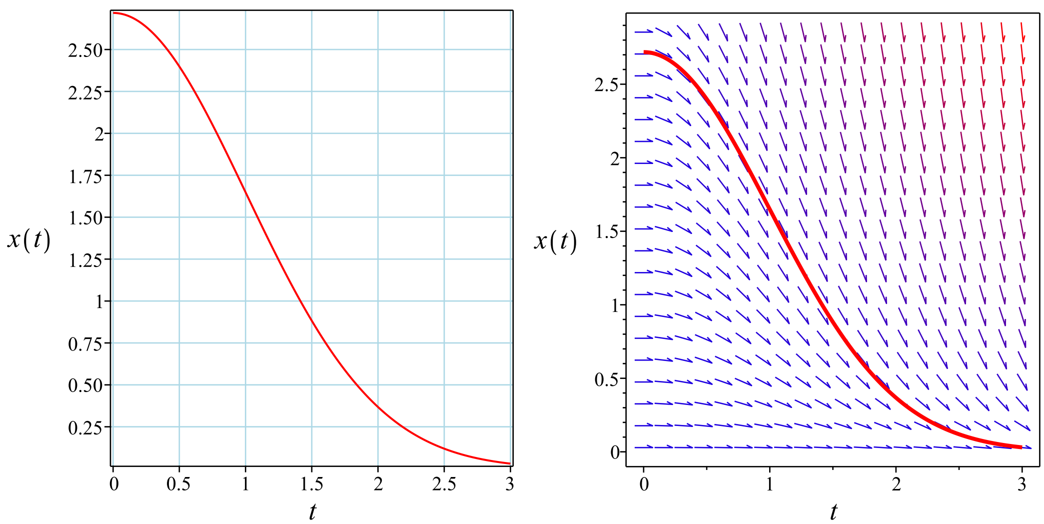
Substituting c_2 found above in the general solution gives

$$x = e^{-\frac{t^2}{2}} e$$

Summary

The solution(s) found are the following

$$x = e^{-\frac{t^2}{2}} e \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = e^{-\frac{t^2}{2}} e$$

Verified OK.

8.17.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -tx$$
$$x' = \omega(t, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 324: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t, x) = 0$$
$$\eta(t, x) = \mathrm{e}^{-\frac{t^2}{2}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{t^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{t^2}{2}} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = -tx$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= t\, e^{\frac{t^2}{2}} x \\ S_x &= e^{\frac{t^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^{\frac{t^2}{2}} x = c_1$$

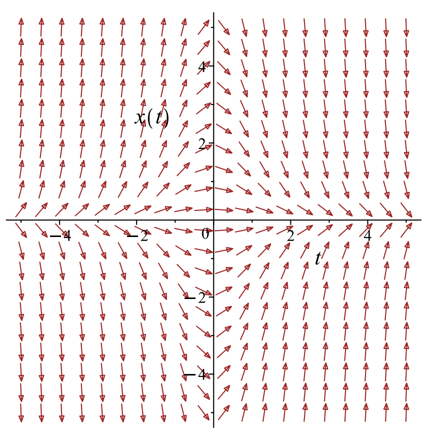
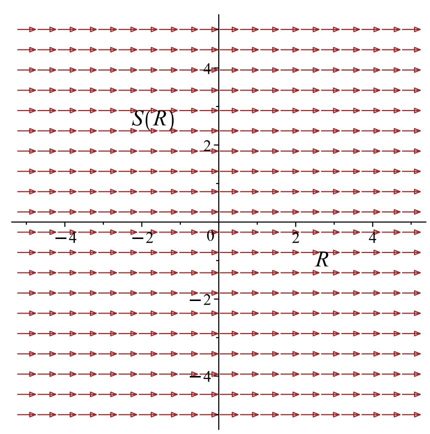
Which simplifies to

$$e^{\frac{t^2}{2}} x = c_1$$

Which gives

$$x = e^{-\frac{t^2}{2}} c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -tx$ 	$R = t$ $S = e^{\frac{t^2}{2}} x$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = e$ in the above solution gives an equation to solve for the constant of integration.

$$e = c_1$$

$$c_1 = e$$

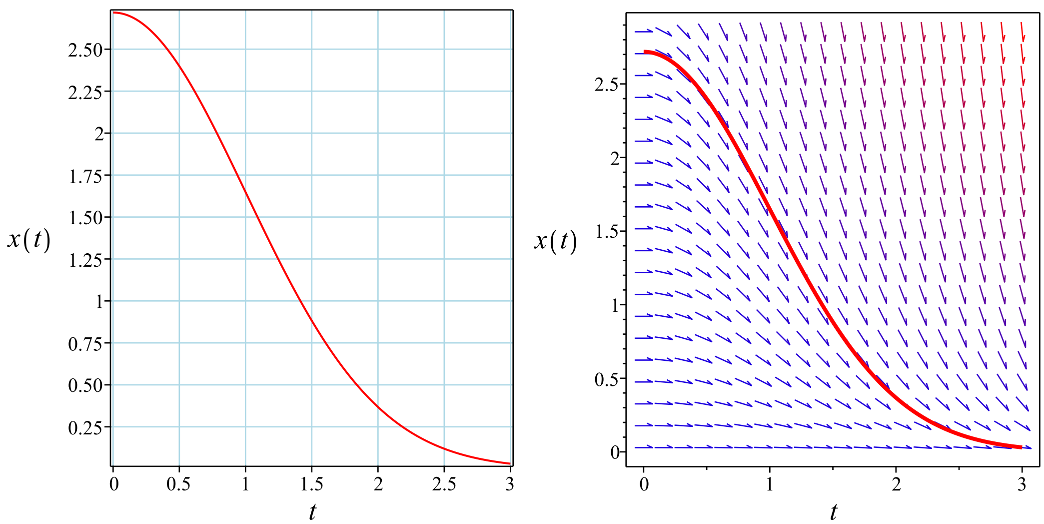
Substituting c_1 found above in the general solution gives

$$x = e^{-\frac{t^2}{2}+1}$$

<u>Summary</u>

The solution(s) found are the following

$$x = e^{-\frac{t^2}{2}+1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$x = e^{-\frac{t^2}{2}+1}$$

Verified OK.

8.17.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{x}\right)\mathrm{d}x&=(t)\,\mathrm{d}t\\ (-t)\,\mathrm{d}t+\left(-\frac{1}{x}\right)\mathrm{d}x&=0 \qquad\qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=-t\\ N(t,x)&=-\frac{1}{x}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x}&=\frac{\partial}{\partial x}(-t)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{t^2}{2} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = -\frac{1}{x}$. Therefore equation (4) becomes

$$
-\frac{1}{x} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = -\frac{1}{x}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x) \, \mathrm{d}x &= \int \left(-\frac{1}{x}\right) \mathrm{d}x \\
f(x) &= -\ln(x) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{t^2}{2} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^2}{2} - \ln(x)$$

The solution becomes

$$x = \mathrm{e}^{-\frac{t^2}{2} - c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = \mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} = \mathrm{e}^{-c_1}$$

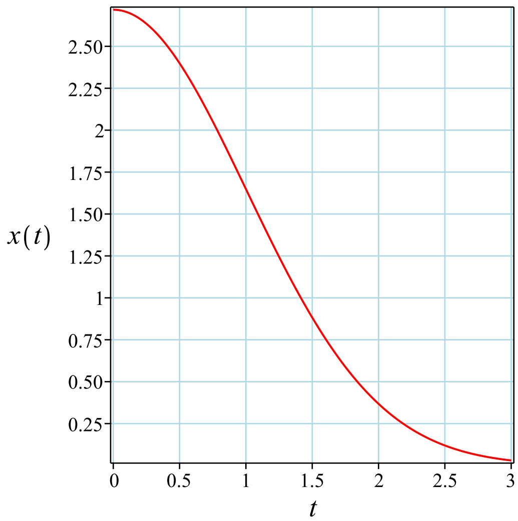
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

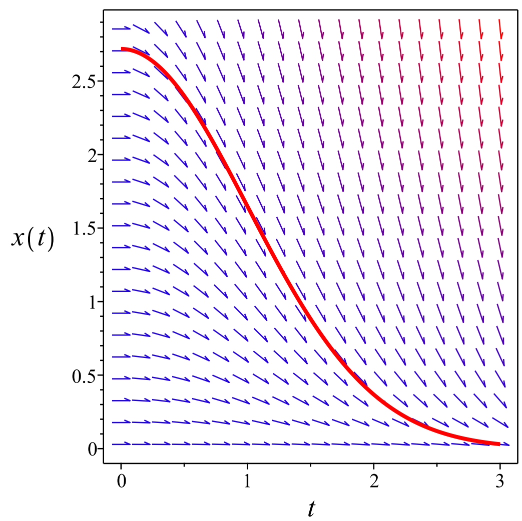
$$x = \mathrm{e}^{-\frac{t^2}{2} + 1}$$

Summary
The solution(s) found are the following

$$x = \mathrm{e}^{-\frac{t^2}{2} + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{-\frac{t^2}{2}+1}$$

Verified OK.

8.17.7 Maple step by step solution

Let's solve

$[x' + xt = 0, x(0) = e]$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x} = -t$

- Integrate both sides with respect to t

  $\int \frac{x'}{x} dt = \int -t dt + c_1$

- Evaluate integral

  $\ln(x) = -\frac{t^2}{2} + c_1$

- Solve for x

  $x = e^{-\frac{t^2}{2}+c_1}$

- Use initial condition $x(0) = \mathrm{e}$

  $\mathrm{e} = \mathrm{e}^{c_1}$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $x = \mathrm{e}^{-\frac{t^2}{2}+1}$

- Solution to the IVP

  $x = \mathrm{e}^{-\frac{t^2}{2}+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(x(t),t)= -t*x(t),x(0) = exp(1)],x(t), singsol=all)
```

$$x(t) = \mathrm{e}^{1-\frac{t^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 16

```
DSolve[{x'[t]==-t*x[t],{x[0]==Exp[1]}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{1-\frac{t^2}{2}}$$

8.18 problem 31

Internal problem ID [13046]
Internal file name [`OUTPUT/11698_Wednesday_November_08_2023_03_28_55_AM_78194016/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = \cos(4t)}$$

With initial conditions

$$[y(0) = 1]$$

8.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = \cos(4t)$$

Hence the ode is

$$y' - 2y = \cos(4t)$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \cos(4t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.18.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-2)dt} \\ &= e^{-2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)(\cos(4t)) \\ \frac{d}{dt}\left(e^{-2t}y\right) &= \left(e^{-2t}\right)(\cos(4t)) \\ d\left(e^{-2t}y\right) &= \left(\cos(4t)e^{-2t}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-2t}y &= \int \cos(4t)e^{-2t}\, dt \\ e^{-2t}y &= -\frac{\cos(4t)e^{-2t}}{10} + \frac{\sin(4t)e^{-2t}}{5} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2t}$ results in

$$y = e^{2t}\left(-\frac{\cos(4t)e^{-2t}}{10} + \frac{\sin(4t)e^{-2t}}{5}\right) + c_1e^{2t}$$

which simplifies to

$$y = c_1e^{2t} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - \frac{1}{10}$$

$$c_1 = \frac{11}{10}$$

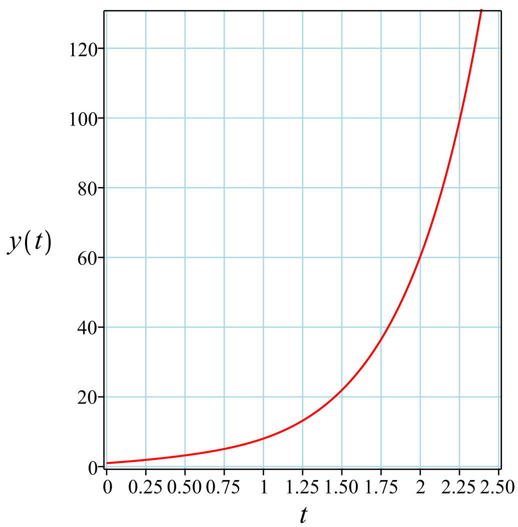
Substituting c_1 found above in the general solution gives

$$y = \frac{11\,\mathrm{e}^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$$

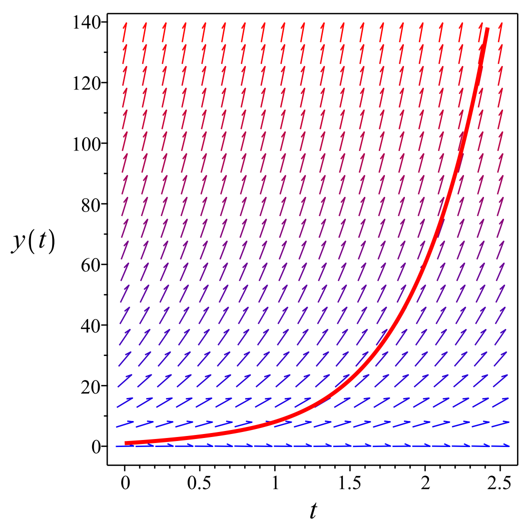
Summary

The solution(s) found are the following

$$y = \frac{11\,\mathrm{e}^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{11\,\mathrm{e}^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$$

Verified OK.

8.18.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2y + \cos(4t)$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 327: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t, y) = 0$$
$$\eta(t, y) = e^{2t} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{2t}} dy \end{aligned}$$

Which results in

$$S = e^{-2t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = 2y + \cos(4t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -2 e^{-2t} y \\ S_y &= e^{-2t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(4t)\,\mathrm{e}^{-2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(4R)\,\mathrm{e}^{-2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{\mathrm{e}^{-2R}(\cos(4R) - 2\sin(4R))}{10} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-2t}y = -\frac{(\cos(4t) - 2\sin(4t))\,\mathrm{e}^{-2t}}{10} + c_1$$

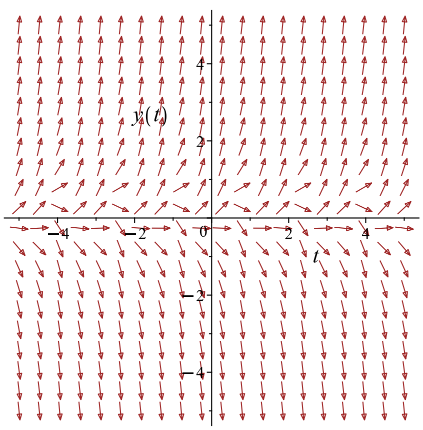
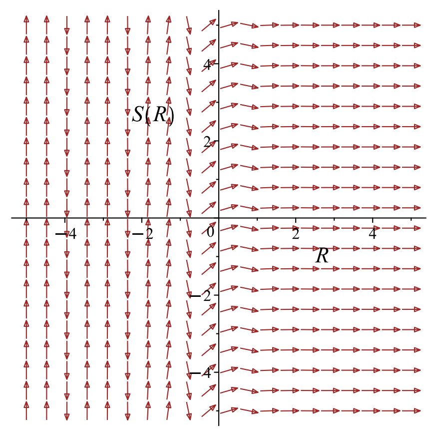
Which simplifies to

$$\mathrm{e}^{-2t}y = -\frac{(\cos(4t) - 2\sin(4t))\,\mathrm{e}^{-2t}}{10} + c_1$$

Which gives

$$y = -\frac{\mathrm{e}^{2t}(\cos(4t)\,\mathrm{e}^{-2t} - 2\sin(4t)\,\mathrm{e}^{-2t} - 10c_1)}{10}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2y + \cos(4t)$ 	$R = t$ $S = \mathrm{e}^{-2t} y$	$\frac{dS}{dR} = \cos(4R)\,\mathrm{e}^{-2R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - \frac{1}{10}$$

$$c_1 = \frac{11}{10}$$

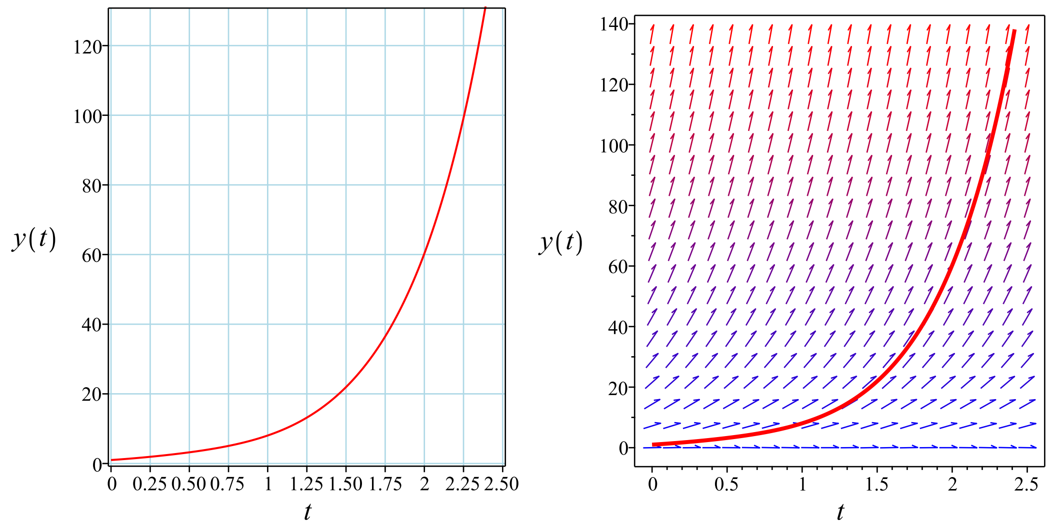
Substituting c_1 found above in the general solution gives

$$y = \frac{11\,\mathrm{e}^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$$

Summary

The solution(s) found are the following

$$y = \frac{11\,\mathrm{e}^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{11\,e^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$$

Verified OK.

8.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}y &= (2y + \cos(4t))\,\mathrm{d}t \\
(-2y - \cos(4t))\,\mathrm{d}t + \mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(t, y) &= -2y - \cos(4t) \\
N(t, y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2y - \cos(4t)) \\
&= -2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \\
&= 1((-2) - (0)) \\
&= -2
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\, \mathrm{d}t} \\ &= e^{\int -2\, \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2t} \\ &= \mathrm{e}^{-2t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-2t}(-2y - \cos(4t)) \\ &= -\mathrm{e}^{-2t}(2y + \cos(4t))\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-2t}(1) \\ &= \mathrm{e}^{-2t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \left(-\mathrm{e}^{-2t}(2y + \cos(4t))\right) + \left(\mathrm{e}^{-2t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int \overline{M}\, \mathrm{d}t \\ \int \frac{\partial \phi}{\partial t}\, \mathrm{d}t &= \int -\mathrm{e}^{-2t}(2y + \cos(4t))\, \mathrm{d}t\end{aligned}$$

$$\phi = \frac{(10y + \cos(4t) - 2\sin(4t))\, \mathrm{e}^{-2t}}{10} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-2t} = \mathrm{e}^{-2t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(10y + \cos(4t) - 2\sin(4t))\,\mathrm{e}^{-2t}}{10} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(10y + \cos(4t) - 2\sin(4t))\,\mathrm{e}^{-2t}}{10}$$

The solution becomes

$$y = -\frac{\mathrm{e}^{2t}(\cos(4t)\,\mathrm{e}^{-2t} - 2\sin(4t)\,\mathrm{e}^{-2t} - 10c_1)}{10}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - \frac{1}{10}$$

$$c_1 = \frac{11}{10}$$

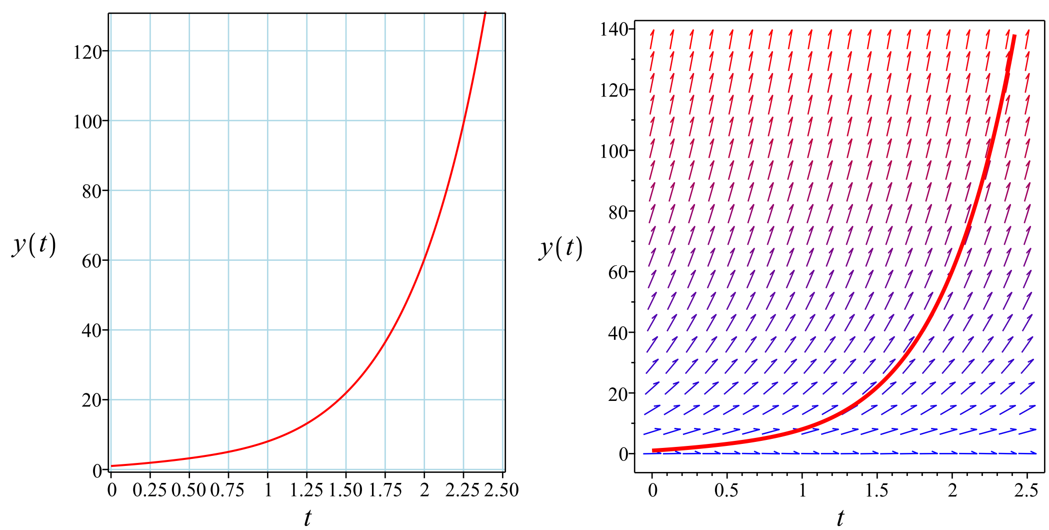
Substituting c_1 found above in the general solution gives

$$y = \frac{11 \,\mathrm{e}^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$$

Summary

The solution(s) found are the following

$$y = \frac{11 \,\mathrm{e}^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{11 \,\mathrm{e}^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$$

Verified OK.

8.18.5 Maple step by step solution

Let's solve

$[y' - 2y = \cos(4t), y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = 2y + \cos(4t)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' - 2y = \cos(4t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)(y' - 2y) = \mu(t)\cos(4t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$\mu(t)(y' - 2y) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

$\mu'(t) = -2\mu(t)$

- Solve to find the integrating factor

$\mu(t) = e^{-2t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int \mu(t)\cos(4t)\, dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\,y = \int \mu(t)\cos(4t)\, dt + c_1$

- Solve for y

$y = \frac{\int \mu(t)\cos(4t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = e^{-2t}$

$y = \frac{\int \cos(4t)e^{-2t}dt + c_1}{e^{-2t}}$

- Evaluate the integrals on the rhs

$y = \frac{-\frac{\cos(4t)e^{-2t}}{10} + \frac{\sin(4t)e^{-2t}}{5} + c_1}{e^{-2t}}$

- Simplify

$y = c_1 e^{2t} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$

- Use initial condition $y(0) = 1$

$1 = c_1 - \frac{1}{10}$

- Solve for c_1

$c_1 = \frac{11}{10}$

- Substitute $c_1 = \frac{11}{10}$ into general solution and simplify

$y = \frac{11\, e^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$

- Solution to the IVP

  $y = \frac{11\,\mathrm{e}^{2t}}{10} + \frac{\sin(4t)}{5} - \frac{\cos(4t)}{10}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve([diff(y(t),t)= 2*y(t)+cos(4*t),y(0) = 1],y(t), singsol=all)
```

$$y(t) = -\frac{\cos(4t)}{10} + \frac{\sin(4t)}{5} + \frac{11\,\mathrm{e}^{2t}}{10}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 29

```
DSolve[{y'[t]==2*y[t]+Cos[4*t],{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{10}\left(11e^{2t} + 2\sin(4t) - \cos(4t)\right)$$

8.19 problem 32

Internal problem ID [13047]
Internal file name [`OUTPUT/11699_Wednesday_November_08_2023_03_28_56_AM_8180774/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = 2\,\mathrm{e}^{3t}}$$

With initial conditions

$$[y(0) = -1]$$

8.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -3$$
$$q(t) = 2\,\mathrm{e}^{3t}$$

Hence the ode is

$$y' - 3y = 2\,\mathrm{e}^{3t}$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2\,\mathrm{e}^{3t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-3)dt} \\ &= \mathrm{e}^{-3t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(2\,\mathrm{e}^{3t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-3t}y\right) &= \left(\mathrm{e}^{-3t}\right)\left(2\,\mathrm{e}^{3t}\right) \\ \mathrm{d}\left(\mathrm{e}^{-3t}y\right) &= 2\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-3t}y &= \int 2\,\mathrm{d}t \\ \mathrm{e}^{-3t}y &= 2t + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-3t}$ results in

$$y = 2t\,\mathrm{e}^{3t} + c_1 \mathrm{e}^{3t}$$

which simplifies to

$$y = \mathrm{e}^{3t}(2t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

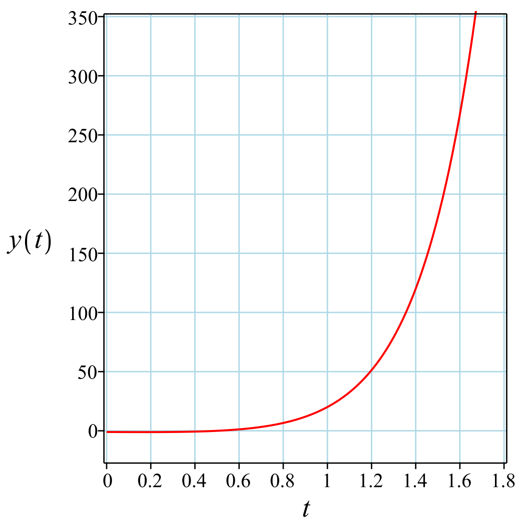
Substituting c_1 found above in the general solution gives

$$y = e^{3t}(2t - 1)$$

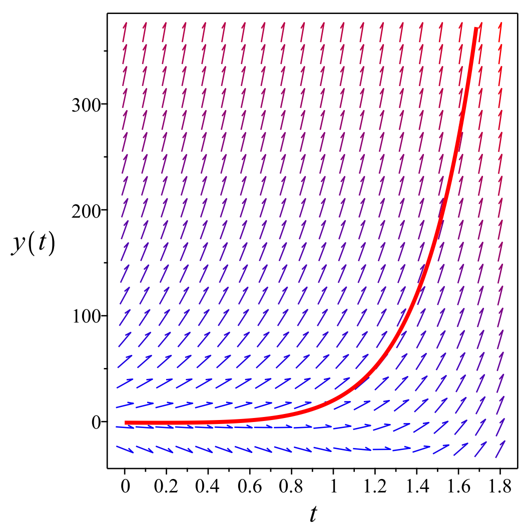
Summary

The solution(s) found are the following

$$y = e^{3t}(2t - 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{3t}(2t - 1)$$

Verified OK.

8.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 3y + 2\,\mathrm{e}^{3t} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 330: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,y) &= 0 \\
\eta(t,y) &= e^{3t}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{3t}} dy
\end{aligned}
$$

Which results in

$$
S = e^{-3t} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$
\omega(t,y) = 3y + 2\,e^{3t}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= -3\,e^{-3t} y \\
S_y &= e^{-3t}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-3t}y = 2t + c_1$$

Which simplifies to

$$e^{-3t}y = 2t + c_1$$

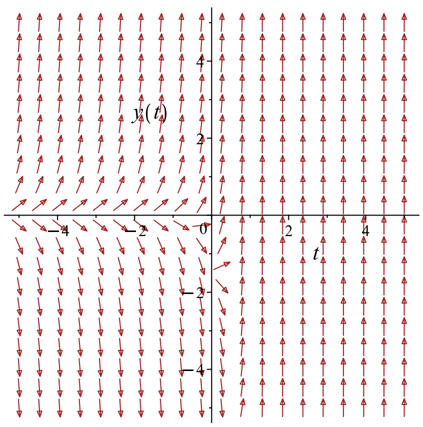
Which gives

$$y = e^{3t}(2t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dt} = 3y + 2\,\mathrm{e}^{3t}$



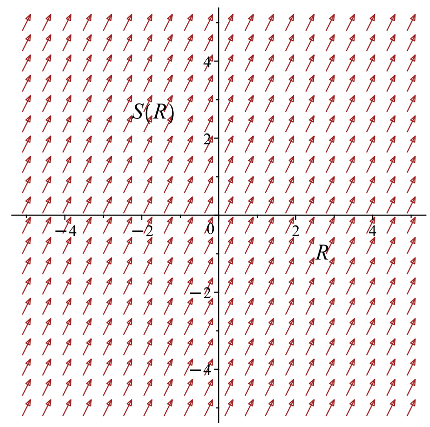
</td><td>

$R = t$

$S = \mathrm{e}^{-3t} y$

</td><td>

$\frac{dS}{dR} = 2$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

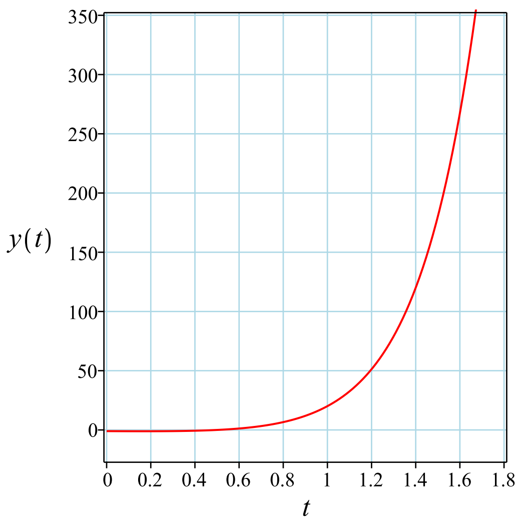
Substituting c_1 found above in the general solution gives

$$y = 2t\,\mathrm{e}^{3t} - \mathrm{e}^{3t}$$

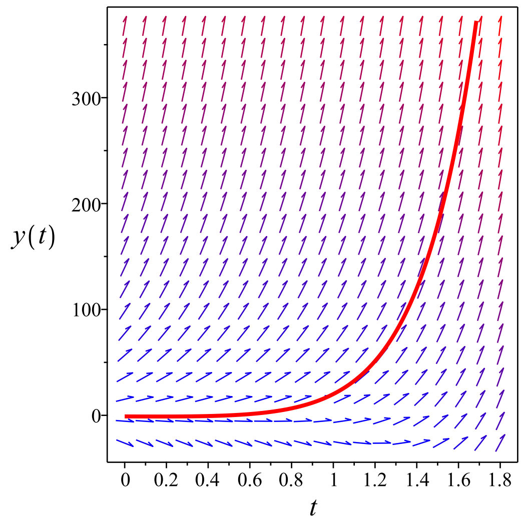
Summary

The solution(s) found are the following

$$y = 2t\,\mathrm{e}^{3t} - \mathrm{e}^{3t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2t\,e^{3t} - e^{3t}$$

Verified OK.

8.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(3y + 2\,\mathrm{e}^{3t}\right)\mathrm{d}t\\ \left(-3y - 2\,\mathrm{e}^{3t}\right)\mathrm{d}t + \mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, y) &= -3y - 2\,\mathrm{e}^{3t}\\ N(t, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3y - 2\,\mathrm{e}^{3t}\right)\\ &= -3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((-3) - (0))\\ &= -3\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dt} \\ &= e^{\int -3 \, dt}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3t} \\ &= \mathrm{e}^{-3t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-3t}(-3y - 2\,\mathrm{e}^{3t}) \\ &= -3\,\mathrm{e}^{-3t} y - 2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-3t}(1) \\ &= \mathrm{e}^{-3t}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ (-3\,\mathrm{e}^{-3t} y - 2) + (\mathrm{e}^{-3t}) \frac{\mathrm{d}y}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\ \int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -3\,\mathrm{e}^{-3t} y - 2 \, \mathrm{d}t\end{aligned}$$

$$\phi = -2t + \mathrm{e}^{-3t} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-3t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-3t} = \mathrm{e}^{-3t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -2t + \mathrm{e}^{-3t} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2t + \mathrm{e}^{-3t} y$$

The solution becomes

$$y = \mathrm{e}^{3t}(2t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

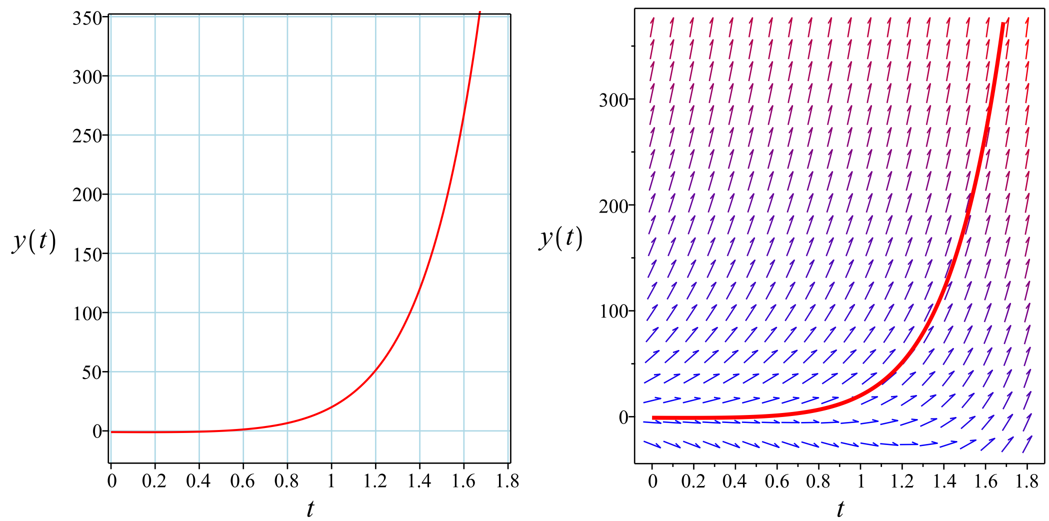
Substituting c_1 found above in the general solution gives

$$y = 2t\,\mathrm{e}^{3t} - \mathrm{e}^{3t}$$

Summary

The solution(s) found are the following

$$y = 2t\,\mathrm{e}^{3t} - \mathrm{e}^{3t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2t\,\mathrm{e}^{3t} - \mathrm{e}^{3t}$$

Verified OK.

8.19.5 Maple step by step solution

Let's solve

$[y' - 3y = 2\,\mathrm{e}^{3t}, y(0) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3y + 2\,\mathrm{e}^{3t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y = 2\,\mathrm{e}^{3t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(y' - 3y\right) = 2\mu(t)\,\mathrm{e}^{3t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$\mu(t)\left(y' - 3y\right) = \mu'(t)\,y + \mu(t)\,y'$

- Isolate $\mu'(t)$

$\mu'(t) = -3\mu(t)$

- Solve to find the integrating factor

$\mu(t) = \mathrm{e}^{-3t}$

- Integrate both sides with respect to t

$\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 2\mu(t)\,\mathrm{e}^{3t} dt + c_1$

- Evaluate the integral on the lhs

$\mu(t)\,y = \int 2\mu(t)\,\mathrm{e}^{3t} dt + c_1$

- Solve for y

$y = \frac{\int 2\mu(t)\mathrm{e}^{3t} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-3t}$

$y = \frac{\int 2\,\mathrm{e}^{3t}\mathrm{e}^{-3t} dt + c_1}{\mathrm{e}^{-3t}}$

- Evaluate the integrals on the rhs

$y = \frac{2t + c_1}{\mathrm{e}^{-3t}}$

- Simplify

$y = \mathrm{e}^{3t}(2t + c_1)$

- Use initial condition $y(0) = -1$

$-1 = c_1$

- Solve for c_1

$c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

$y = \mathrm{e}^{3t}(2t - 1)$

- Solution to the IVP

$y = \mathrm{e}^{3t}(2t - 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(t),t)= 3*y(t)+2*exp(3*t),y(0) = -1],y(t), singsol=all)
```

$$y(t) = (2t - 1)\,e^{3t}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 16

```
DSolve[{y'[t]==3*y[t]+2*Exp[3*t],{y[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{3t}(2t - 1)$$

8.20 problem 33

Internal problem ID [13048]
Internal file name [`OUTPUT/11700_Wednesday_November_08_2023_03_28_56_AM_46356146/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - t^2y^3 - y^3 = 0}$$

With initial conditions

$$\left[y(0) = -\frac{1}{2}\right]$$

8.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^3t^2 + y^3 \end{aligned}$$

The t domain of $f(t, y)$ when $y = -\frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^3 t^2 + y^3\right)\\ &= 3y^2 t^2 + 3y^2\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = -\frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -\frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

8.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(t, y)\\ &= f(t)g(y)\\ &= y^3\left(t^2 + 1\right)\end{aligned}$$

Where $f(t) = t^2 + 1$ and $g(y) = y^3$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^3}\, dy &= t^2 + 1\, dt\\ \int \frac{1}{y^3}\, dy &= \int t^2 + 1\, dt\\ -\frac{1}{2y^2} &= \frac{1}{3}t^3 + t + c_1\end{aligned}$$

Which results in

$$\begin{aligned}y &= -\frac{3}{\sqrt{-6t^3 - 18c_1 - 18t}}\\ y &= \frac{3}{\sqrt{-6t^3 - 18c_1 - 18t}}\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -\frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = \frac{1}{\sqrt{-2c_1}}$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -\frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = -\frac{1}{\sqrt{-2c_1}}$$

$$c_1 = -2$$

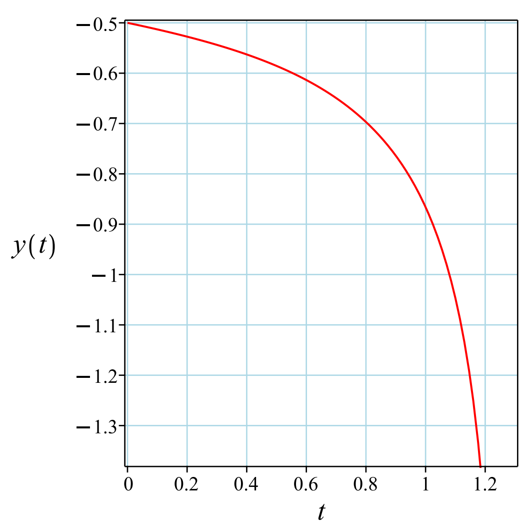
Substituting c_1 found above in the general solution gives

$$y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}}$$

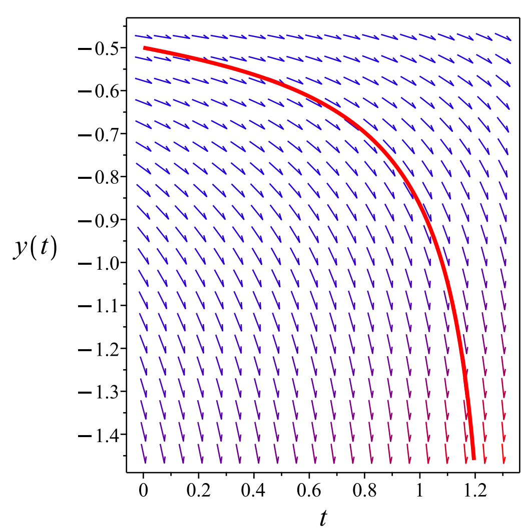
Summary

The solution(s) found are the following

$$y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}}$$

Verified OK.

8.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y^3t^2 + y^3 \\ y' &= \omega(t, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 333: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{1}{t^2 + 1} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{t^2+1}} dt \end{aligned}$$

Which results in

$$S = \frac{1}{3} t^3 + t$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = y^3 t^2 + y^3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= t^2 + 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

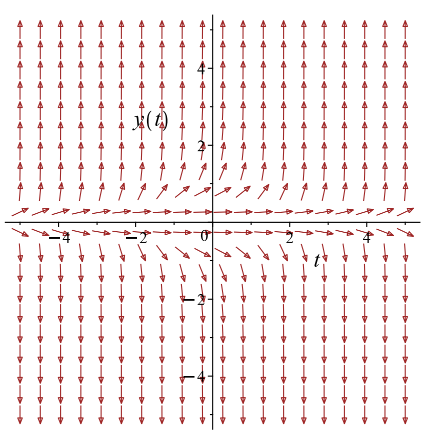
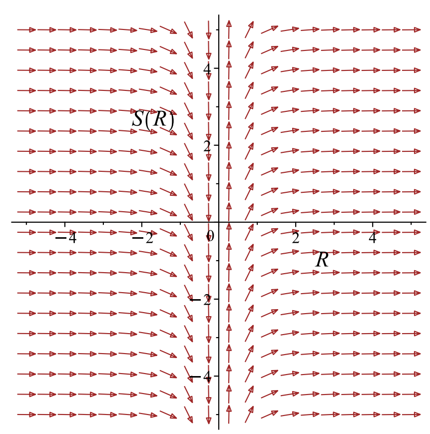
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{1}{3}t^3 + t = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$\frac{1}{3}t^3 + t = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y^3t^2 + y^3$ 	$R = y$ $S = \frac{1}{3}t^3 + t$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -\frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -2 + c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{3}t^3 + t = \frac{4y^2 - 1}{2y^2}$$

The above simplifies to

$$2y^2t^3 + 6t\,y^2 - 12y^2 + 3 = 0$$

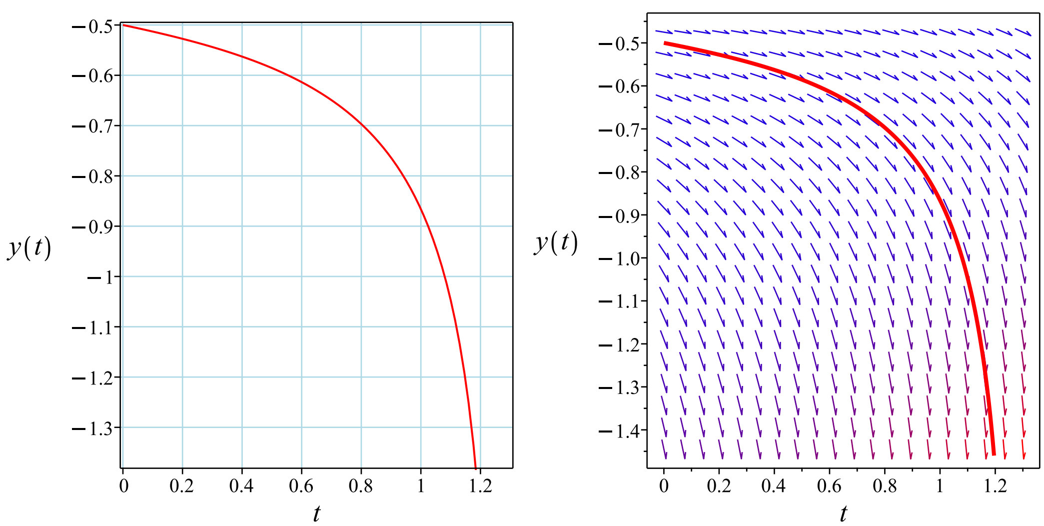
Solving for y from the above gives

$$y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}}$$

Verified OK.

8.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^3}\right)\mathrm{d}y&=\left(t^2+1\right)\mathrm{d}t\\ \left(-t^2-1\right)\mathrm{d}t+\left(\frac{1}{y^3}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, y) &= -t^2 - 1 \\
N(t, y) &= \frac{1}{y^3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-t^2 - 1\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y^3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int M \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -t^2 - 1 \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{1}{3}t^3 - t + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{y^3}$. Therefore equation (4) becomes

$$\frac{1}{y^3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^3}\right)\mathrm{d}y$$
$$f(y) = -\frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} - \frac{1}{2y^2} - t + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} - \frac{1}{2y^2} - t$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -\frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$-\frac{t^3}{3} - \frac{1}{2y^2} - t = -2$$

The above simplifies to

$$-2y^2t^3 - 6t\,y^2 + 12y^2 - 3 = 0$$

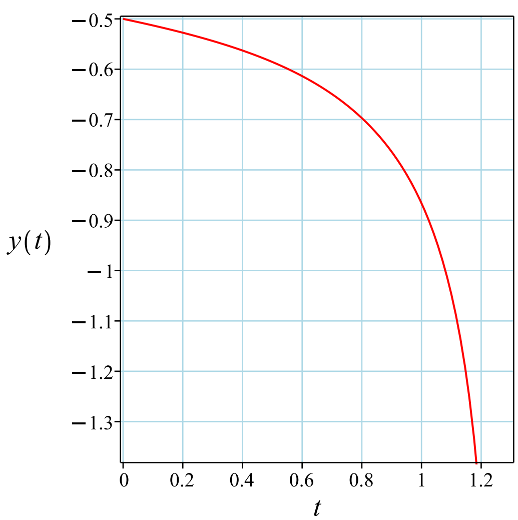
Solving for y from the above gives

$$y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}}$$

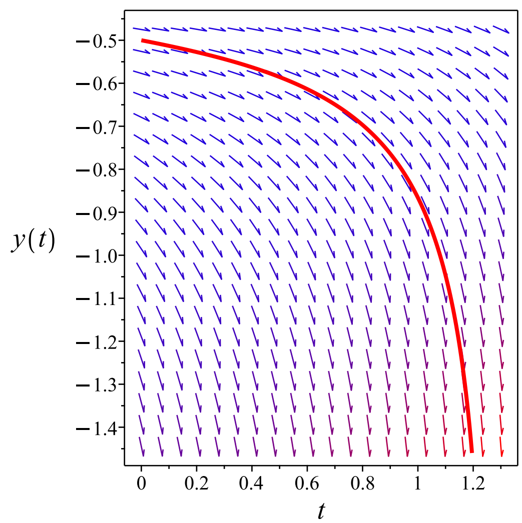
Summary

The solution(s) found are the following

$$y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}}$$

Verified OK.

8.20.5 Maple step by step solution

Let's solve

$\left[y' - t^2 y^3 - y^3 = 0, y(0) = -\frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = t^2 + 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^3} dt = \int (t^2 + 1) \, dt + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = \frac{1}{3} t^3 + t + c_1$

- Solve for y

  $\left\{y = -\frac{3}{\sqrt{-6t^3 - 18c_1 - 18t}}, y = \frac{3}{\sqrt{-6t^3 - 18c_1 - 18t}}\right\}$

- Use initial condition $y(0) = -\frac{1}{2}$

  $-\frac{1}{2} = -\frac{3}{\sqrt{-18c_1}}$

- Solve for c_1

  $c_1 = -2$

- Substitute $c_1 = -2$ into general solution and simplify

  $y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}}$

- Use initial condition $y(0) = -\frac{1}{2}$

  $-\frac{1}{2} = \frac{3}{\sqrt{-18c_1}}$

- Solution does not satisfy initial condition
- Solution to the IVP

  $y = -\frac{3}{\sqrt{-6t^3 - 18t + 36}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 18

```
dsolve([diff(y(t),t)= t^2*y(t)^3+y(t)^3,y(0) = -1/2],y(t), singsol=all)
```

$$y(t) = -\frac{3}{\sqrt{-6t^3 - 18t + 36}}$$

✓ Solution by Mathematica

Time used: 0.319 (sec). Leaf size: 28

```
DSolve[{y'[t]==t^2*y[t]^3+y[t]^3,{y[0]==-1/2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{\sqrt{\frac{3}{2}}}{\sqrt{-t^3 - 3t + 6}}$$

8.21 problem 34

Internal problem ID [13049]
Internal file name [`OUTPUT/11701_Wednesday_November_08_2023_03_28_57_AM_4011091/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 5y = 3\,\mathrm{e}^{-5t}}$$

With initial conditions

$$[y(0) = -2]$$

8.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 3\,\mathrm{e}^{-5t} \end{aligned}$$

Hence the ode is

$$y' + 5y = 3\,\mathrm{e}^{-5t}$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3\,\mathrm{e}^{-5t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.21.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 5dt} \\ &= \mathrm{e}^{5t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(3\,\mathrm{e}^{-5t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{5t}y\right) &= \left(\mathrm{e}^{5t}\right)\left(3\,\mathrm{e}^{-5t}\right) \\ \mathrm{d}\left(\mathrm{e}^{5t}y\right) &= 3\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{5t}y &= \int 3\,\mathrm{d}t \\ \mathrm{e}^{5t}y &= 3t + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{5t}$ results in

$$y = 3t\,\mathrm{e}^{-5t} + c_1\mathrm{e}^{-5t}$$

which simplifies to

$$y = \mathrm{e}^{-5t}(3t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

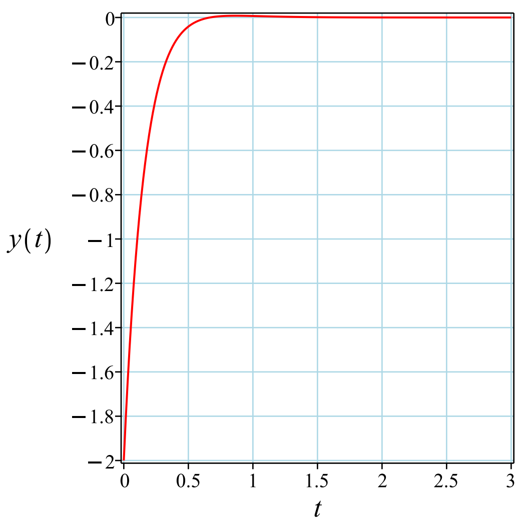
Substituting c_1 found above in the general solution gives

$$y = e^{-5t}(-2 + 3t)$$

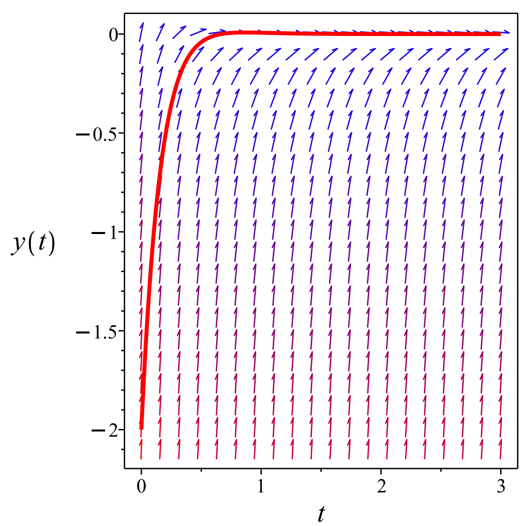
Summary

The solution(s) found are the following

$$y = e^{-5t}(-2 + 3t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-5t}(-2 + 3t)$$

Verified OK.

8.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -5y + 3\,\mathrm{e}^{-5t} \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 336: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,y) = 0$$
$$\eta(t,y) = \mathrm{e}^{-5t} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-5t}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{5t} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -5y + 3\,\mathrm{e}^{-5t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= 5\,\mathrm{e}^{5t} y \\ S_y &= \mathrm{e}^{5t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{5t}y = 3t + c_1$$

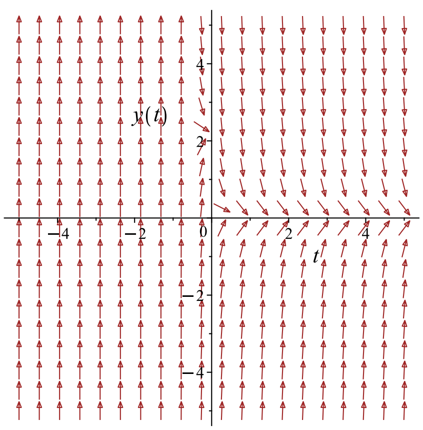
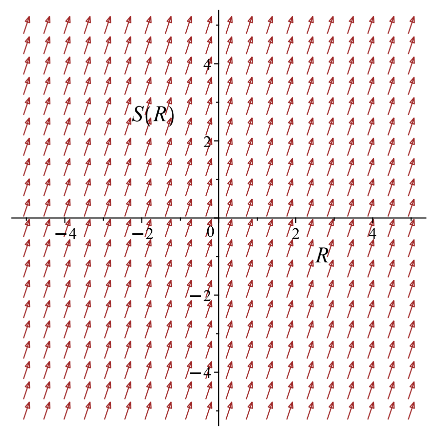
Which simplifies to

$$e^{5t}y = 3t + c_1$$

Which gives

$$y = e^{-5t}(3t + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -5y + 3\,\mathrm{e}^{-5t}$ 	$R = t$ $S = \mathrm{e}^{5t} y$	$\frac{dS}{dR} = 3$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

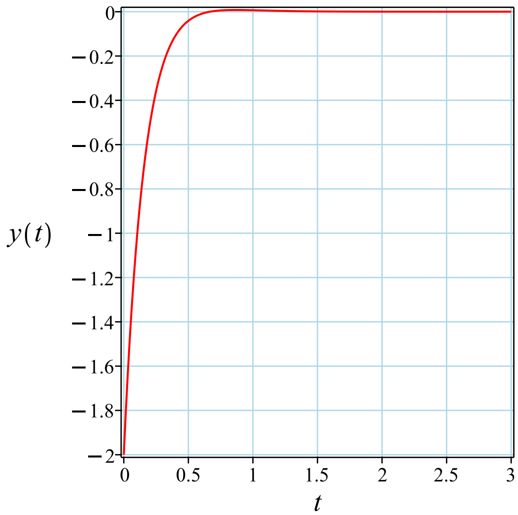
Substituting c_1 found above in the general solution gives

$$y = 3t\,\mathrm{e}^{-5t} - 2\,\mathrm{e}^{-5t}$$

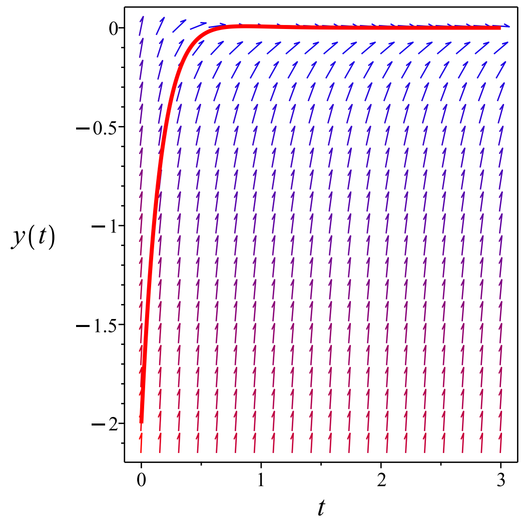
Summary

The solution(s) found are the following

$$y = 3t\,\mathrm{e}^{-5t} - 2\,\mathrm{e}^{-5t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3t\,\mathrm{e}^{-5t} - 2\,\mathrm{e}^{-5t}$$

Verified OK.

8.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-5y + 3\,\mathrm{e}^{-5t}\right)\mathrm{d}t\\ \left(5y - 3\,\mathrm{e}^{-5t}\right)\mathrm{d}t + \mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= 5y - 3\,\mathrm{e}^{-5t}\\ N(t,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(5y - 3\,\mathrm{e}^{-5t}\right)\\ &= 5\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)\\ &= 1((5) - (0))\\ &= 5\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int 5 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{5t} \\
&= \mathrm{e}^{5t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{5t}\left(5y - 3\,\mathrm{e}^{-5t}\right) \\
&= 5\,\mathrm{e}^{5t} y - 3
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{5t}(1) \\
&= \mathrm{e}^{5t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(5\,\mathrm{e}^{5t} y - 3\right) + \left(\mathrm{e}^{5t}\right) \frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int 5\,\mathrm{e}^{5t} y - 3 \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -3t + \mathrm{e}^{5t} y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{5t} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{5t}$. Therefore equation (4) becomes

$$\mathrm{e}^{5t} = \mathrm{e}^{5t} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -3t + \mathrm{e}^{5t}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3t + \mathrm{e}^{5t}y$$

The solution becomes

$$y = \mathrm{e}^{-5t}(3t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

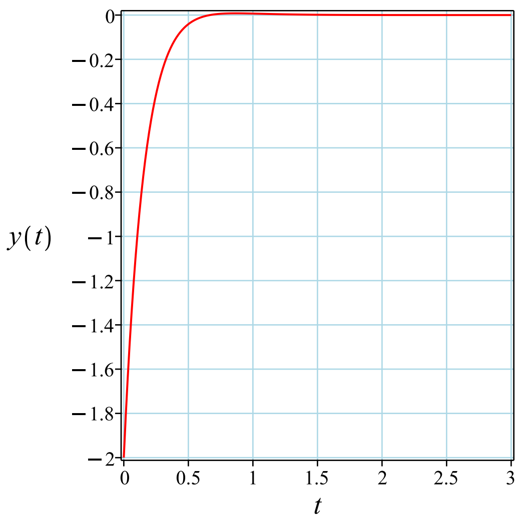
Substituting c_1 found above in the general solution gives

$$y = 3t\,\mathrm{e}^{-5t} - 2\,\mathrm{e}^{-5t}$$

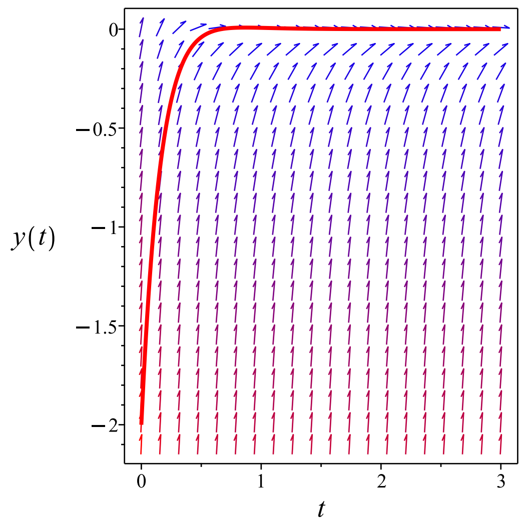
Summary

The solution(s) found are the following

$$y = 3t\,e^{-5t} - 2\,e^{-5t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3t\,e^{-5t} - 2\,e^{-5t}$$

Verified OK.

8.21.5 Maple step by step solution

Let's solve

$[y' + 5y = 3\,e^{-5t}, y(0) = -2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -5y + 3\,e^{-5t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 5y = 3\,e^{-5t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(y'+5y\right)=3\mu(t)\,\mathrm{e}^{-5t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}\left(\mu(t)\,y\right)$

$\mu(t)\left(y'+5y\right)=\mu'(t)\,y+\mu(t)\,y'$

- Isolate $\mu'(t)$

$\mu'(t)=5\mu(t)$

- Solve to find the integrating factor

$\mu(t)=\mathrm{e}^{5t}$

- Integrate both sides with respect to t

$\int\left(\frac{d}{dt}\left(\mu(t)\,y\right)\right)dt=\int 3\mu(t)\,\mathrm{e}^{-5t}dt+c_1$

- Evaluate the integral on the lhs

$\mu(t)\,y=\int 3\mu(t)\,\mathrm{e}^{-5t}dt+c_1$

- Solve for y

$y=\frac{\int 3\mu(t)\mathrm{e}^{-5t}dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=\mathrm{e}^{5t}$

$y=\frac{\int 3\,\mathrm{e}^{5t}\mathrm{e}^{-5t}dt+c_1}{\mathrm{e}^{5t}}$

- Evaluate the integrals on the rhs

$y=\frac{3t+c_1}{\mathrm{e}^{5t}}$

- Simplify

$y=\mathrm{e}^{-5t}(3t+c_1)$

- Use initial condition $y(0)=-2$

$-2=c_1$

- Solve for c_1

$c_1=-2$

- Substitute $c_1=-2$ into general solution and simplify

$y=\mathrm{e}^{-5t}(-2+3t)$

- Solution to the IVP

$y=\mathrm{e}^{-5t}(-2+3t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(y(t),t)+5*y(t)= 3*exp(-5*t),y(0) = -2],y(t), singsol=all)
```

$$y(t) = (-2 + 3t)\,\mathrm{e}^{-5t}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 16

```
DSolve[{y'[t]+5*y[t]== 3*Exp[-5*t],{y[0]==-2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-5t}(3t - 2)$$

8.22 problem 35

Internal problem ID [13050]
Internal file name [`OUTPUT/11702_Wednesday_November_08_2023_03_28_58_AM_47061977/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - 2ty = 3t\,e^{t^2}}$$

With initial conditions

$$[y(0) = 1]$$

8.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2t$$
$$q(t) = 3t\,e^{t^2}$$

Hence the ode is

$$y' - 2ty = 3t\,\mathrm{e}^{t^2}$$

The domain of $p(t) = -2t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3t\,\mathrm{e}^{t^2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.22.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -2t dt} \\ &= \mathrm{e}^{-t^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(3t\,\mathrm{e}^{t^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t^2} y\right) &= \left(\mathrm{e}^{-t^2}\right)\left(3t\,\mathrm{e}^{t^2}\right) \\ \mathrm{d}\left(\mathrm{e}^{-t^2} y\right) &= (3t)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-t^2} y &= \int 3t\,\mathrm{d}t \\ \mathrm{e}^{-t^2} y &= \frac{3t^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t^2}$ results in

$$y = \frac{3t^2 \mathrm{e}^{t^2}}{2} + c_1 \mathrm{e}^{t^2}$$

which simplifies to

$$y = \mathrm{e}^{t^2}\left(\frac{3t^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

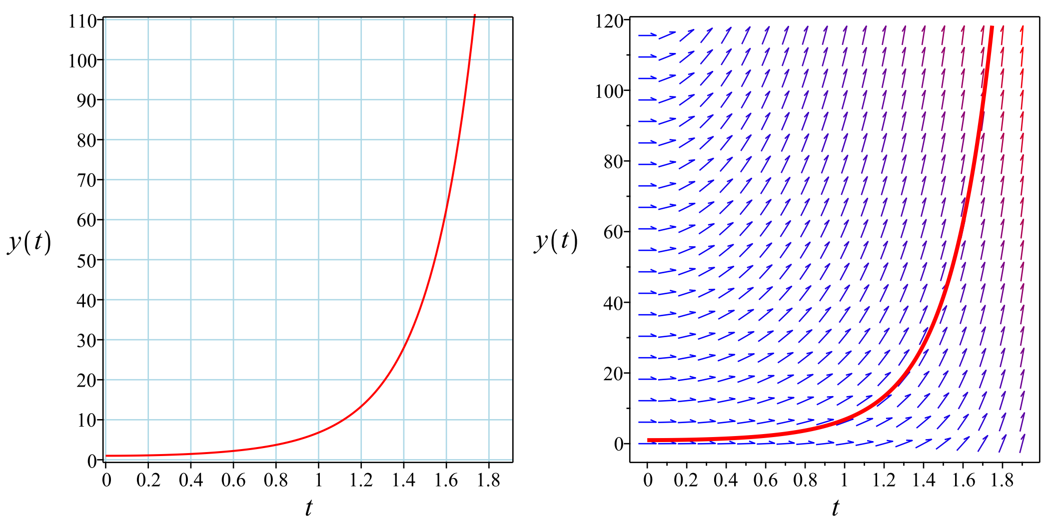
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{t^2}(3t^2 + 2)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{t^2}(3t^2 + 2)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{t^2}(3t^2 + 2)}{2}$$

Verified OK.

8.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2ty + 3t\,e^{t^2}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 339: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, y) &= 0 \\ \eta(t, y) &= e^{t^2}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{t^2}} dy\end{aligned}$$

Which results in

$$S = e^{-t^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = 2ty + 3t\, e^{t^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1 \\ R_y &= 0 \\ S_t &= -2\, e^{-t^2} t y \\ S_y &= e^{-t^2}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$e^{-t^2} y = \frac{3t^2}{2} + c_1$$

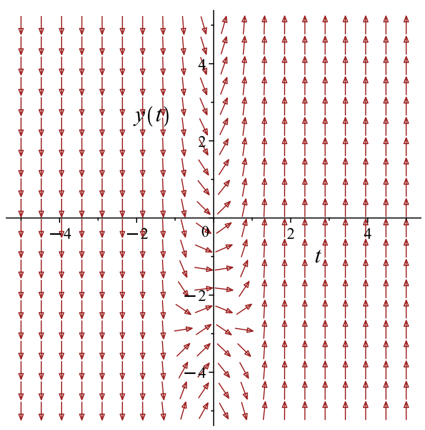
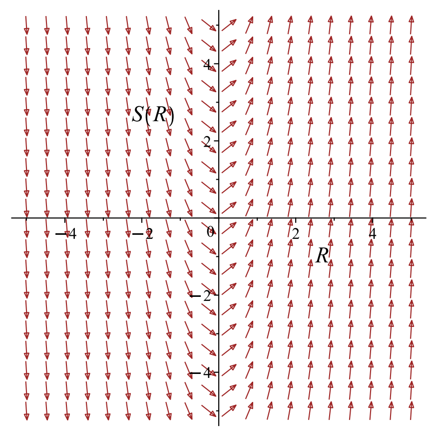
Which simplifies to

$$e^{-t^2} y = \frac{3t^2}{2} + c_1$$

Which gives

$$y = \frac{e^{t^2}(3t^2 + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 2ty + 3t\,\mathrm{e}^{t^2}$ 	$R = t$ $S = \mathrm{e}^{-t^2} y$	$\frac{dS}{dR} = 3R$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

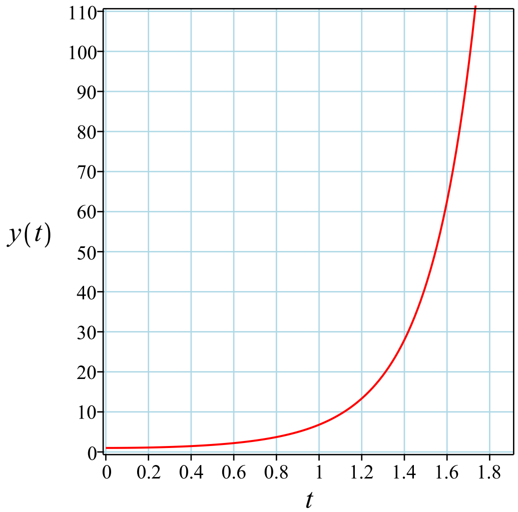
Substituting c_1 found above in the general solution gives

$$y = \frac{3t^2 \mathrm{e}^{t^2}}{2} + \mathrm{e}^{t^2}$$

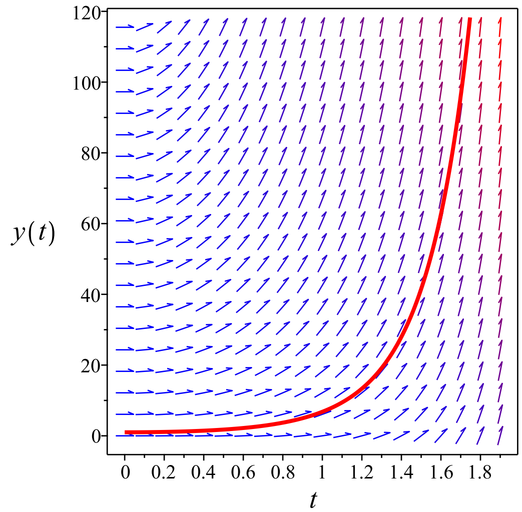
Summary

The solution(s) found are the following

$$y = \frac{3t^2 \mathrm{e}^{t^2}}{2} + \mathrm{e}^{t^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3t^2 e^{t^2}}{2} + e^{t^2}$$

Verified OK.

8.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(2ty+3t\,\mathrm{e}^{t^2}\right)\mathrm{d}t$$

$$\left(-2ty-3t\,\mathrm{e}^{t^2}\right)\mathrm{d}t+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-2ty-3t\,\mathrm{e}^{t^2}\\N(t,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-2ty-3t\,\mathrm{e}^{t^2}\right)\\&=-2t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t}&=\frac{\partial}{\partial t}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)\\&=1((-2t)-(0))\\&=-2t\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -2t \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-t^2} \\
&= \mathrm{e}^{-t^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-t^2} \left(-2ty - 3t\,\mathrm{e}^{t^2}\right) \\
&= -2\,\mathrm{e}^{-t^2} ty - 3t
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-t^2}(1) \\
&= \mathrm{e}^{-t^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left(-2\,\mathrm{e}^{-t^2} ty - 3t\right) + \left(\mathrm{e}^{-t^2}\right) \frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -2\,\mathrm{e}^{-t^2} ty - 3t \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = -\frac{3t^2}{2} + \mathrm{e}^{-t^2} y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-t^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{-t^2} = \mathrm{e}^{-t^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3t^2}{2} + \mathrm{e}^{-t^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3t^2}{2} + \mathrm{e}^{-t^2} y$$

The solution becomes

$$y = \frac{\mathrm{e}^{t^2}(3t^2 + 2c_1)}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

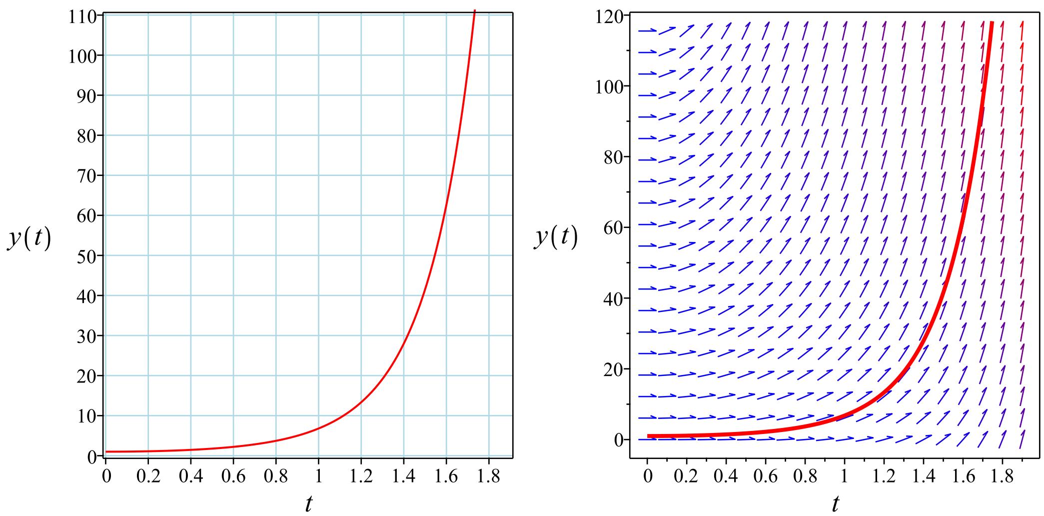
Substituting c_1 found above in the general solution gives

$$y = \frac{3t^2 e^{t^2}}{2} + e^{t^2}$$

Summary

The solution(s) found are the following

$$y = \frac{3t^2 e^{t^2}}{2} + e^{t^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{3t^2 e^{t^2}}{2} + e^{t^2}$$

Verified OK.

8.22.5 Maple step by step solution

Let's solve

$\left[y' - 2ty = 3t\, e^{t^2}, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2ty + 3t\,\mathrm{e}^{t^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2ty = 3t\,\mathrm{e}^{t^2}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(y' - 2ty) = 3\mu(t)\,t\,\mathrm{e}^{t^2}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

  $\mu(t)\,(y' - 2ty) = \mu'(t)\,y + \mu(t)\,y'$
- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)\,t$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t^2}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,y)\right) dt = \int 3\mu(t)\,t\,\mathrm{e}^{t^2} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,y = \int 3\mu(t)\,t\,\mathrm{e}^{t^2} dt + c_1$
- Solve for y

  $y = \frac{\int 3\mu(t) t\,\mathrm{e}^{t^2} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-t^2}$

  $y = \frac{\int 3t\,\mathrm{e}^{t^2}\mathrm{e}^{-t^2} dt + c_1}{\mathrm{e}^{-t^2}}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{3t^2}{2} + c_1}{\mathrm{e}^{-t^2}}$
- Simplify

  $y = \frac{\mathrm{e}^{t^2}(3t^2 + 2c_1)}{2}$
- Use initial condition $y(0) = 1$

  $1 = c_1$
- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{e^{t^2}(3t^2+2)}{2}$

- Solution to the IVP

  $y = \frac{e^{t^2}(3t^2+2)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(t),t)= 2*t*y(t)+3*t*exp(t^2),y(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{(3t^2+2)\,e^{t^2}}{2}$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 21

```
DSolve[{y'[t]== 2*t*y[t]+3*t*Exp[t^2],{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{t^2}(3t^2+2)$$

8.23 problem 36

Internal problem ID [13051]
Internal file name [`OUTPUT/11703_Wednesday_November_08_2023_03_28_59_AM_25959753/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{(1+t)^2}{(y+1)^2} = 0}$$

With initial conditions

$$[y(0) = 0]$$

8.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= \frac{(1+t)^2}{(y+1)^2} \end{aligned}$$

The t domain of $f(t,y)$ when $y=0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t,y)$ when $t=0$ is

$$\{y < -1 \lor -1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{(1+t)^2}{(y+1)^2}\right) \\ &= -\frac{2(1+t)^2}{(y+1)^3} \end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y=0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t=0$ is

$$\{y < -1 \lor -1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

8.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{(1+t)^2}{(y+1)^2} \end{aligned}$$

Where $f(t) = (1+t)^2$ and $g(y) = \frac{1}{(y+1)^2}$. Integrating both sides gives

$$\frac{1}{\frac{1}{(y+1)^2}}\, dy = (1+t)^2\, dt$$

$$\int \frac{1}{\frac{1}{(y+1)^2}}\, dy = \int (1+t)^2\, dt$$

$$\frac{(y+1)^3}{3} = \frac{(1+t)^3}{3} + c_1$$

Which results in

$$y = \left(t^3 + 3t^2 + 3c_1 + 3t + 1\right)^{\frac{1}{3}} - 1$$

$$y = -\frac{\left(t^3 + 3t^2 + 3c_1 + 3t + 1\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(t^3 + 3t^2 + 3c_1 + 3t + 1\right)^{\frac{1}{3}}}{2} - 1$$

$$y = -\frac{\left(t^3 + 3t^2 + 3c_1 + 3t + 1\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(t^3 + 3t^2 + 3c_1 + 3t + 1\right)^{\frac{1}{3}}}{2} - 1$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{(3c_1 + 1)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(3c_1 + 1)^{\frac{1}{3}}}{2} - 1$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{(3c_1 + 1)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(3c_1 + 1)^{\frac{1}{3}}}{2} - 1$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = (3c_1 + 1)^{\frac{1}{3}} - 1$$

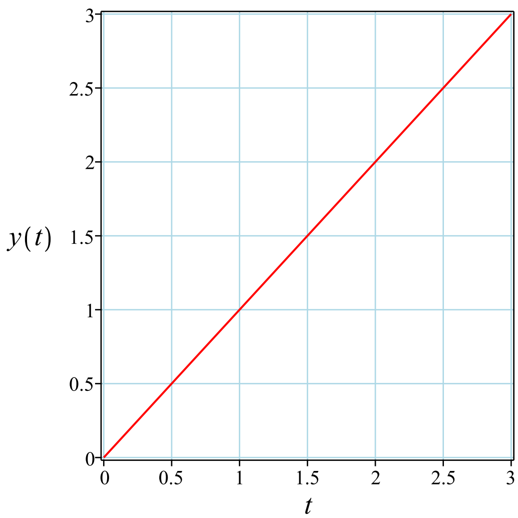
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

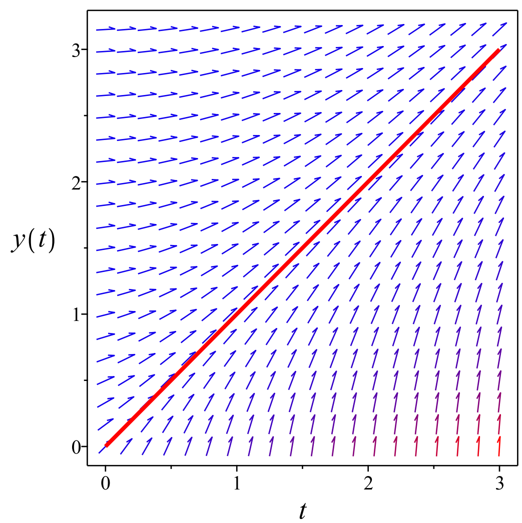
$$y = (t^3 + 3t^2 + 3t + 1)^{\frac{1}{3}} - 1$$

Summary
The solution(s) found are the following

$$y = (t^3 + 3t^2 + 3t + 1)^{\frac{1}{3}} - 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (t^3 + 3t^2 + 3t + 1)^{\frac{1}{3}} - 1$$

Verified OK.

8.23.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{(1+t)^2}{(y+1)^2} \tag{1}$$

Which becomes

$$(y^2 + 2y + 1)\,dy = ((1+t)^2)\,dt \tag{2}$$

But the RHS is complete differential because

$$\left((1+t)^2\right) dt = d\left(\frac{1}{3}t^3 + t^2 + t\right)$$

Hence (2) becomes

$$\left(y^2 + 2y + 1\right) dy = d\left(\frac{1}{3}t^3 + t^2 + t\right)$$

Integrating both sides gives gives these solutions

$$
\begin{aligned}
y &= \left(t^3 + 3t^2 + 3c_1 + 3t\right)^{\frac{1}{3}} - 1 + c_1 \\
y &= -\frac{\left(t^3 + 3t^2 + 3c_1 + 3t\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(t^3 + 3t^2 + 3c_1 + 3t\right)^{\frac{1}{3}}}{2} - 1 + c_1 \\
y &= -\frac{\left(t^3 + 3t^2 + 3c_1 + 3t\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(t^3 + 3t^2 + 3c_1 + 3t\right)^{\frac{1}{3}}}{2} - 1 + c_1
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{c_1^{\frac{1}{3}} 3^{\frac{1}{3}}}{2} - \frac{i 3^{\frac{5}{6}} c_1^{\frac{1}{3}}}{2} - 1 + c_1$$

$$c_1 = 1 + \frac{i\left(\frac{\left(108+36\sqrt{13}\right)^{\frac{1}{3}}}{6} - \frac{6\left(-\frac{3^{\frac{1}{3}}}{6} - \frac{i3^{\frac{5}{6}}}{6}\right)}{\left(108+36\sqrt{13}\right)^{\frac{1}{3}}}\right)3^{\frac{5}{6}}}{2} + \frac{\left(\frac{\left(108+36\sqrt{13}\right)^{\frac{1}{3}}}{6} - \frac{6\left(-\frac{3^{\frac{1}{3}}}{6} - \frac{i3^{\frac{5}{6}}}{6}\right)}{\left(108+36\sqrt{13}\right)^{\frac{1}{3}}}\right)3^{\frac{1}{3}}}{2}$$

Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{c_1^{\frac{1}{3}} 3^{\frac{1}{3}}}{2} + \frac{i 3^{\frac{5}{6}} c_1^{\frac{1}{3}}}{2} - 1 + c_1$$

$$c_1 = 1 - \frac{i\left(\frac{\left(108+36\sqrt{13}\right)^{\frac{1}{3}}}{6} - \frac{6\left(\frac{i3^{\frac{5}{6}}}{6} - \frac{3^{\frac{1}{3}}}{6}\right)}{\left(108+36\sqrt{13}\right)^{\frac{1}{3}}}\right)3^{\frac{5}{6}}}{2} + \frac{\left(\frac{\left(108+36\sqrt{13}\right)^{\frac{1}{3}}}{6} - \frac{6\left(\frac{i3^{\frac{5}{6}}}{6} - \frac{3^{\frac{1}{3}}}{6}\right)}{\left(108+36\sqrt{13}\right)^{\frac{1}{3}}}\right)3^{\frac{1}{3}}}{2}$$

Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1^{\frac{1}{3}} 3^{\frac{1}{3}} - 1 + c_1$$

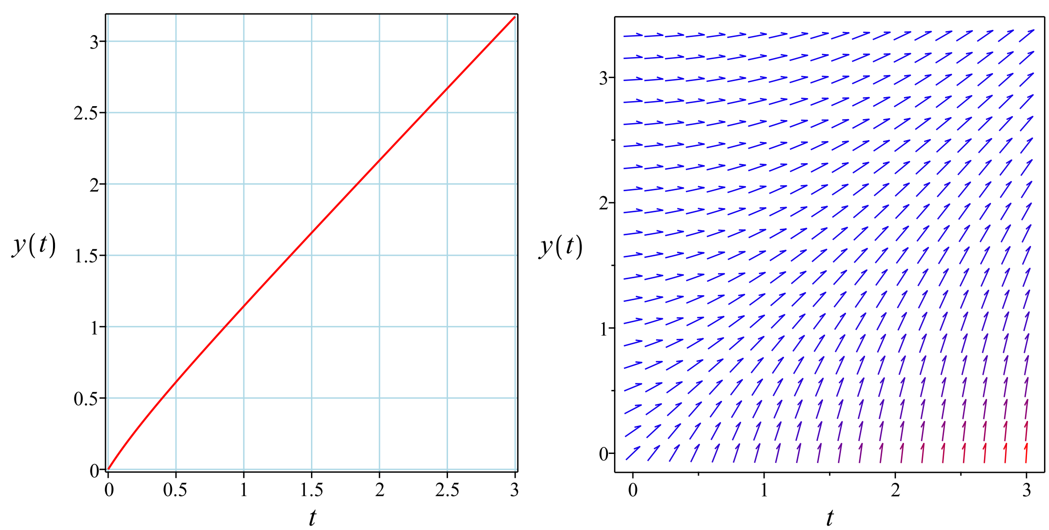
$$c_1 = -3^{\frac{1}{3}} \left(\frac{\left(108 + 36\sqrt{13}\right)^{\frac{1}{3}}}{6} - \frac{2\, 3^{\frac{1}{3}}}{\left(108 + 36\sqrt{13}\right)^{\frac{1}{3}}} \right) + 1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-3^{\frac{1}{3}} \left(108 + 36\sqrt{13}\right)^{\frac{7}{9}} + 12\, 3^{\frac{2}{3}} \left(108 + 36\sqrt{13}\right)^{\frac{1}{9}} + 6 \left(t^3 \left(108 + 36\sqrt{13}\right)^{\frac{1}{3}} - \frac{3^{\frac{1}{3}} \left(108+36\sqrt{13}\right)^{\frac{2}{3}}}{2} + 3t^2 \left(108 + \right.\right.}{6 \left(108 + 36\sqrt{13}\right)^{\frac{4}{9}}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{-3^{\frac{1}{3}} \left(108 + 36\sqrt{13}\right)^{\frac{7}{9}} + 12\, 3^{\frac{2}{3}} \left(108 + 36\sqrt{13}\right)^{\frac{1}{9}} + 6 \left(t^3 \left(108 + 36\sqrt{13}\right)^{\frac{1}{3}} - \frac{3^{\frac{1}{3}} \left(108+36\sqrt{13}\right)^{\frac{2}{3}}}{2} + 3t^2 \left(108 + 36 \right.\right.}{6 \left(108 + 36\sqrt{13}\right)^{\frac{4}{9}}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

y

$$= \frac{-3^{\frac{1}{3}}\left(108+36\sqrt{13}\right)^{\frac{7}{9}}+12\,3^{\frac{2}{3}}\left(108+36\sqrt{13}\right)^{\frac{1}{9}}+6\left(t^{3}\left(108+36\sqrt{13}\right)^{\frac{1}{3}}-\frac{3^{\frac{1}{3}}\left(108+36\sqrt{13}\right)^{\frac{2}{3}}}{2}+3t^{2}\left(108+36\right.\right.}{6\left(108+36\sqrt{13}\right)^{\frac{4}{9}}}$$

Verified OK.

8.23.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = t + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{(1+X+x_0)^2}{(Y(X)+y_0+1)^2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X^2}{Y(X)^2}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{X^2}{Y^2} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^n f(X,Y)$$

In this case, it can be seen that both $M = X^2$ and $N = Y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{du}{dX}X + u = \frac{1}{u^2}$$
$$\frac{du}{dX} = \frac{\frac{1}{u(X)^2} - u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{1}{u(X)^2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)u(X)^2 X + u(X)^3 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^3 - 1}{u^2 X} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^3-1}{u^2}$. Integrating both sides gives

$$\frac{1}{\frac{u^3-1}{u^2}}\,du = -\frac{1}{X}\,dX$$
$$\int \frac{1}{\frac{u^3-1}{u^2}}\,du = \int -\frac{1}{X}\,dX$$
$$\frac{\ln(u^3 - 1)}{3} = -\ln(X) + c_2$$

Raising both side to exponential gives

$$\left(u^3 - 1\right)^{\frac{1}{3}} = e^{-\ln(X) + c_2}$$

Which simplifies to

$$\left(u^3 - 1\right)^{\frac{1}{3}} = \frac{c_3}{X}$$

Which simplifies to

$$\left(u(X)^3 - 1\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{X}$$

The solution is

$$\left(u(X)^3-1\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\left(\frac{Y(X)^3}{X^3}-1\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\left(\frac{Y(X)^3-X^3}{X^3}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y+y_0 \\ X &= t+x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y-1 \\ X &= t-1 \end{aligned}$$

Then the solution in y becomes

$$\left(\frac{(y+1)^3-(1+t)^3}{(1+t)^3}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{1+t}$$

Initial conditions are used to solve for c_2. Substituting $t=0$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_3 \mathrm{e}^{c_2}$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

8.23.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{(1+t)^2}{(y+1)^2} \\
y' &= \omega(t,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 342: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{1}{(1+t)^2} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dt \\
&= \int \frac{1}{\frac{1}{(1+t)^2}} dt
\end{aligned}
$$

Which results in

$$
S = \frac{(1+t)^3}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}
$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$
\omega(t, y) = \frac{(1+t)^2}{(y+1)^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 0 \\
R_y &= 1 \\
S_t &= (1+t)^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = (y+1)^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = (R+1)^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{(R+1)^3}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$
\frac{(1+t)^3}{3} = \frac{(y+1)^3}{3} + c_1
$$

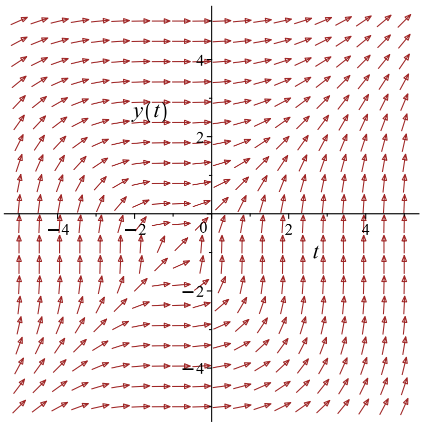
Which simplifies to

$$
\frac{(1+t)^3}{3} = \frac{(y+1)^3}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dt} = \frac{(1+t)^2}{(y+1)^2}$



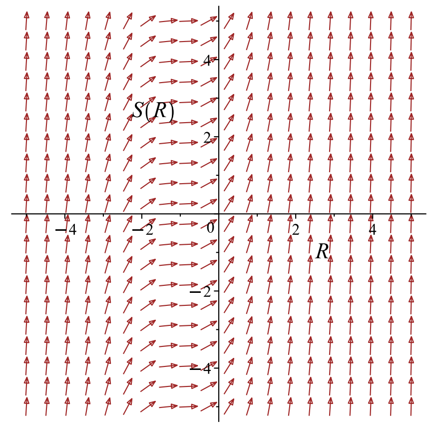
</td><td>

$R = y$

$S = \frac{(1+t)^3}{3}$

</td><td>

$\frac{dS}{dR} = (R+1)^2$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{3} = c_1 + \frac{1}{3}$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{(1+t)^3}{3} = \frac{(y+1)^3}{3}$$

Summary

The solution(s) found are the following

$$\frac{(1+t)^3}{3} = \frac{(y+1)^3}{3} \tag{1}$$

Verification of solutions

$$\frac{(1+t)^3}{3} = \frac{(y+1)^3}{3}$$

Verified OK.

8.23.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left((y+1)^2\right)\mathrm{d}y&=\left((1+t)^2\right)\mathrm{d}t\\ \left(-(1+t)^2\right)\mathrm{d}t+\left((y+1)^2\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-(1+t)^2\\ N(t,y)&=(y+1)^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-(1+t)^2\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left((y+1)^2\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -(1+t)^2\,\mathrm{d}t\\ \phi &= -\frac{(1+t)^3}{3} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (y+1)^2$. Therefore equation (4) becomes

$$(y+1)^2 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = (y+1)^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left((y+1)^2\right)\mathrm{d}y$$

$$f(y) = \frac{(y+1)^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(1+t)^3}{3} + \frac{(y+1)^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(1+t)^3}{3} + \frac{(y+1)^3}{3}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-\frac{(1+t)^3}{3} + \frac{(y+1)^3}{3} = 0$$

<u>Summary</u>

The solution(s) found are the following

$$-\frac{(1+t)^3}{3} + \frac{(y+1)^3}{3} = 0 \tag{1}$$

Verification of solutions

$$-\frac{(1+t)^3}{3} + \frac{(y+1)^3}{3} = 0$$

Verified OK.
The solution

$$\frac{(1+t)^3}{3} = \frac{(y+1)^3}{3}$$

can be simplified to

$$(1+t)^3 = (y+1)^3$$

8.23.7 Maple step by step solution

Let's solve

$\left[y' - \frac{(1+t)^2}{(y+1)^2} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(y+1)^2 = (1+t)^2$

- Integrate both sides with respect to t

  $\int y'(y+1)^2 dt = \int (1+t)^2 dt + c_1$

- Evaluate integral

  $\frac{(y+1)^3}{3} = \frac{(1+t)^3}{3} + c_1$

- Solve for y

  $y = (t^3 + 3t^2 + 3c_1 + 3t + 1)^{\frac{1}{3}} - 1$

- Use initial condition $y(0) = 0$

  $0 = (3c_1 + 1)^{\frac{1}{3}} - 1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = \left((1+t)^3\right)^{\frac{1}{3}} - 1$

- Solution to the IVP

$$y = \left((1+t)^3\right)^{\frac{1}{3}} - 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 5

```
dsolve([diff(y(t),t)= (t+1)^2/(y(t)+1)^2,y(0) = 0],y(t), singsol=all)
```

$$y(t) = t$$

✓ Solution by Mathematica

Time used: 0.805 (sec). Leaf size: 16

```
DSolve[{y'[t]== (t+1)^2/(y[t]+1)^2,{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \sqrt[3]{(t+1)^3} - 1$$

8.24 problem 37

Internal problem ID [13052]
Internal file name [`OUTPUT/11704_Wednesday_November_08_2023_03_29_00_AM_45043933/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2ty^2 - 3t^2y^2 = 0}$$

With initial conditions

$$[y(1) = -1]$$

8.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= 3y^2t^2 + 2t\,y^2 \end{aligned}$$

The t domain of $f(t,y)$ when $y=-1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The y domain of $f(t,y)$ when $t=1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3y^2t^2 + 2t\,y^2\right) \\ &= 6y\,t^2 + 4ty \end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y=-1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t=1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

8.24.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \left(3t^2 + 2t\right) y^2 \end{aligned}$$

Where $f(t) = 3t^2 + 2t$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = 3t^2 + 2t\,dt$$

$$\int \frac{1}{y^2}\,dy = \int 3t^2 + 2t\,dt$$

$$-\frac{1}{y} = t^3 + t^2 + c_1$$

Which results in

$$y = -\frac{1}{t^3 + t^2 + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{c_1 + 2}$$

$$c_1 = -1$$

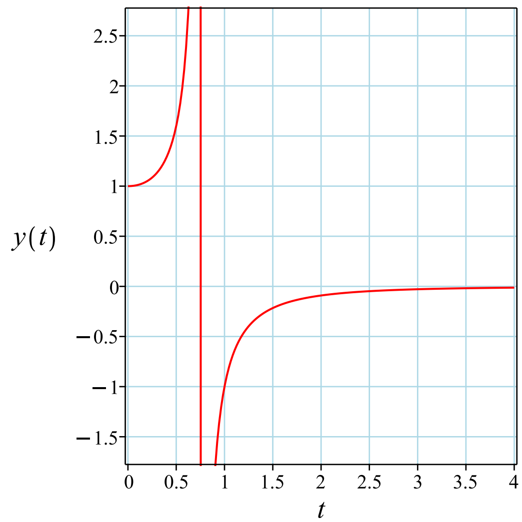
Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{t^3 + t^2 - 1}$$

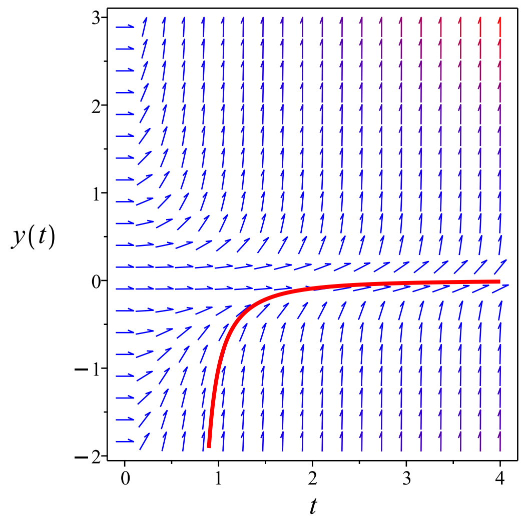
Summary

The solution(s) found are the following

$$y = -\frac{1}{t^3 + t^2 - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Verified OK.

8.24.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3y^2t^2 + 2t\,y^2$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2\xi_y - \omega_t\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 345: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{1}{3t^2 + 2t} \\
\eta(t, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{1}{3t^2+2t}} dt \end{aligned}$$

Which results in

$$S = t^3 + t^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y) S_y}{R_t + \omega(t, y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = 3y^2t^2 + 2t\,y^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= 3t^2 + 2t \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$t^2(1+t) = -\frac{1}{y} + c_1$$

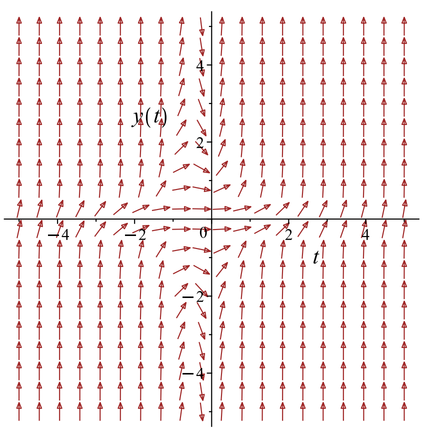
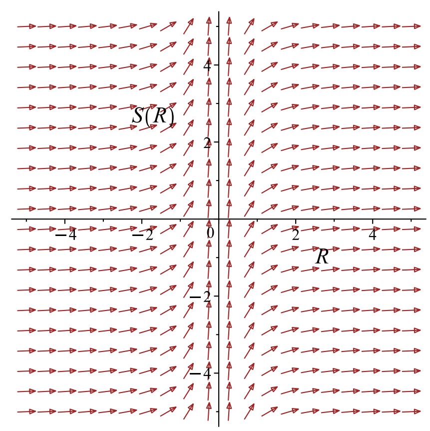
Which simplifies to

$$t^2(1+t) = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{1}{-t^3 - t^2 + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = 3y^2t^2 + 2t\,y^2$ 	$R = y$ $S = t^2(1+t)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{1}{-2 + c_1}$$

$$c_1 = 1$$

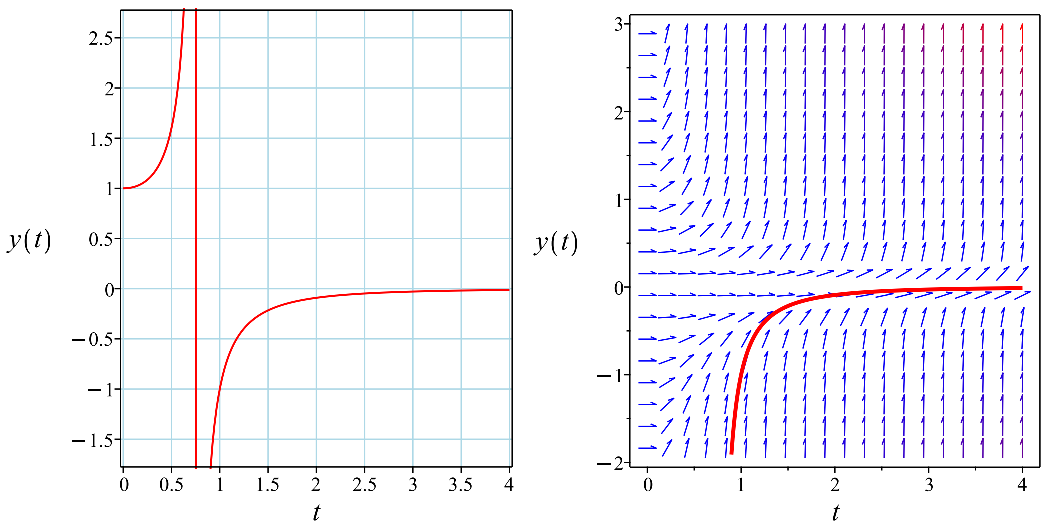
Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{t^3 + t^2 - 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Verified OK.

8.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2}\right)\mathrm{d}y=\left(3t^2+2t\right)\mathrm{d}t$$
$$\left(-3t^2-2t\right)\mathrm{d}t+\left(\frac{1}{y^2}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t,y)=-3t^2-2t$$
$$N(t,y)=\frac{1}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-3t^2 - 2t)$$
$$= 0$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{y^2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -3t^2 - 2t\,\mathrm{d}t$$
$$\phi = -t^3 - t^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2}\right)\mathrm{d}y$$
$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -t^3 - t^2 - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t^3 - t^2 - \frac{1}{y}$$

The solution becomes

$$y = -\frac{1}{t^3 + t^2 + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{c_1 + 2}$$

$$c_1 = -1$$

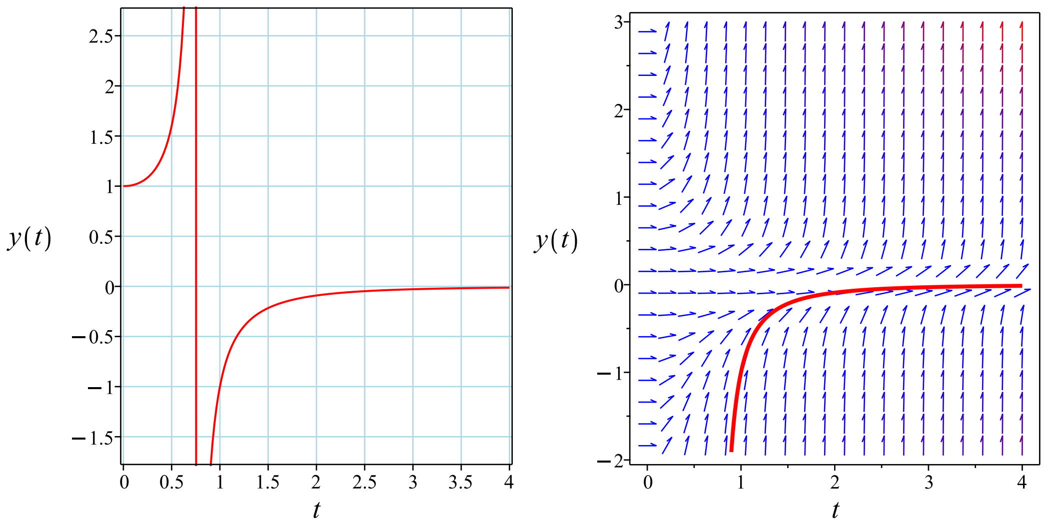
Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{t^3 + t^2 - 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Verified OK.

8.24.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= 3y^2t^2 + 2t\,y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 3y^2t^2 + 2t\,y^2$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = 0$ and $f_2(t) = 3t^2 + 2t$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(3t^2 + 2t)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(t) - (f_2' + f_1f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 6t + 2 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(3t^2 + 2t\right) u''(t) - (6t + 2)\, u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 + t^2(1 + t)\, c_2$$

The above shows that

$$u'(t) = c_2t(3t + 2)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2t(3t + 2)}{(3t^2 + 2t)\,(c_1 + t^2\,(1 + t)\, c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{t^3 + t^2 + c_3}$$

Initial conditions are used to solve for c_3. Substituting $t = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{c_3 + 2}$$

$$c_3 = -1$$

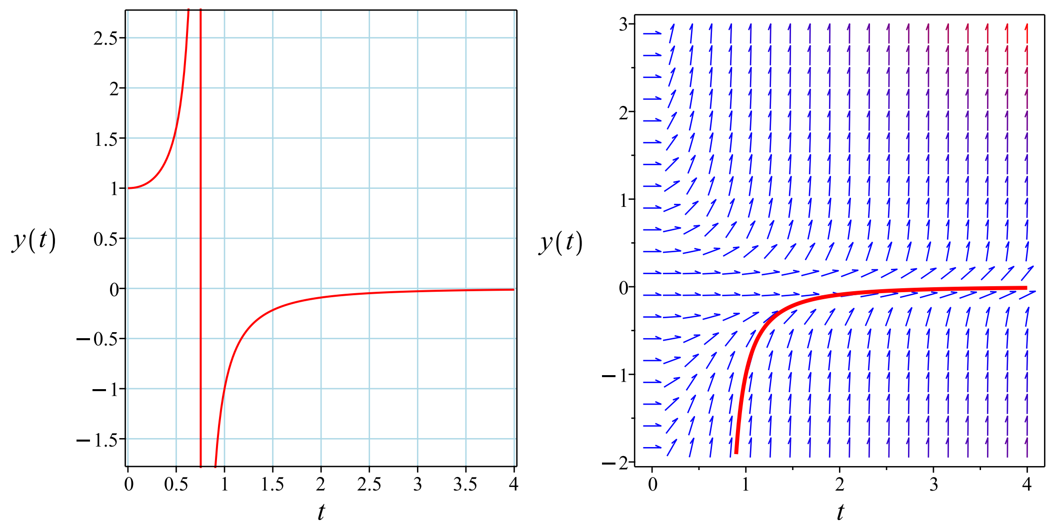
Substituting c_3 found above in the general solution gives

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{t^3 + t^2 - 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{t^3 + t^2 - 1}$$

Verified OK.

8.24.6 Maple step by step solution

Let's solve

$[y' - 2ty^2 - 3t^2y^2 = 0, y(1) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = t(3t+2)$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2} dt = \int t(3t+2)\, dt + c_1$

- Evaluate integral

  $-\frac{1}{y} = t^3 + t^2 + c_1$

- Solve for y

  $y = -\frac{1}{t^3+t^2+c_1}$

- Use initial condition $y(1) = -1$

  $-1 = -\frac{1}{c_1+2}$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = -\frac{1}{t^3+t^2-1}$

- Solution to the IVP

  $y = -\frac{1}{t^3+t^2-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(t),t)= 2*t*y(t)^2+3*t^2*y(t)^2,y(1) = -1],y(t), singsol=all)
```

$$y(t) = -\frac{1}{t^3+t^2-1}$$

✓ Solution by Mathematica

Time used: 0.222 (sec). Leaf size: 17

```
DSolve[{y'[t]== 2*t*y[t]^2+3*t^2*y[t]^2,{y[1]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{t^3 + t^2 - 1}$$

8.25 problem 38

Internal problem ID [13053]
Internal file name [`OUTPUT/11705_Wednesday_November_08_2023_03_29_01_AM_7767521/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 = 1}$$

With initial conditions

$$[y(0) = 1]$$

8.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= -y^2 + 1 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-y^2+1)\\ &= -2y\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.25.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{-y^2+1}dy &= t + c_1\\ \operatorname{arctanh}(y) &= t + c_1\end{aligned}$$

Solving for y gives these solutions

$$y_1 = \tanh(t + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

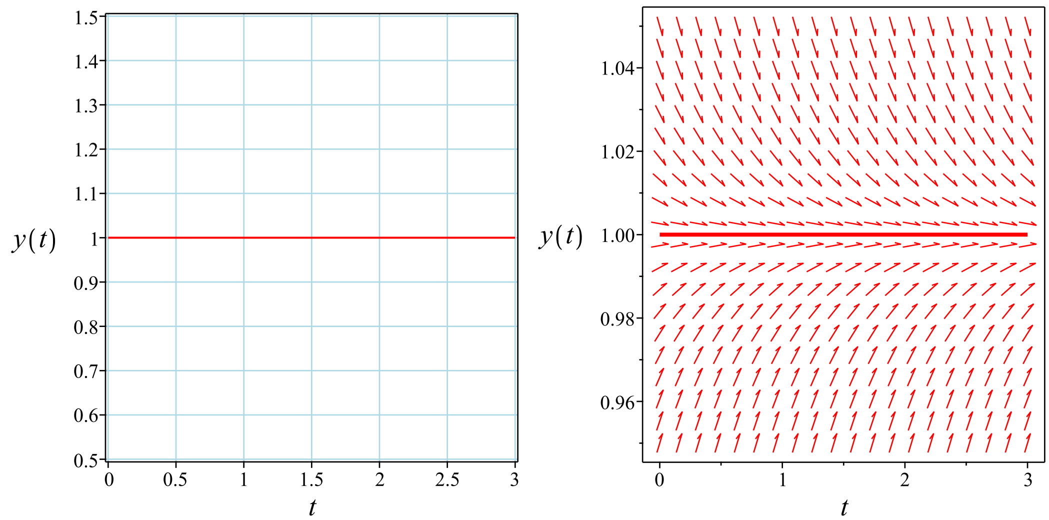
$$1 = \tanh(c_1)$$

Unable to solve for constant of integration. Since $\lim_{c_1\to\infty}$ gives $y = \tanh(t + c_1) = y = 1$ and this result satisfies the given initial condition.

Summary

The solution(s) found are the following

$$y = 1$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 1$$

Verified OK.

8.25.3 Maple step by step solution

Let's solve

$[y' + y^2 = 1, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1-y^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{1-y^2} dt = \int 1 dt + c_1$

- Evaluate integral

  $\operatorname{arctanh}(y) = t + c_1$

- Solve for y

  $y = \tanh(t + c_1)$

- Use initial condition $y(0) = 1$

  $1 = \tanh(c_1)$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(t),t)= 1-y(t)^2,y(0) = 1],y(t), singsol=all)
```

$$y(t) = 1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[t]== 1-y[t]^2,{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 1$$

8.26 problem 39

Internal problem ID [13054]
Internal file name [`OUTPUT/11706_Wednesday_November_08_2023_03_29_02_AM_91067722/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{t^2}{y + y t^3} = 0}$$

With initial conditions

$$[y(0) = -2]$$

8.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(t,y)\\
&= \frac{t^2}{y\,(t^3+1)}
\end{aligned}
$$

The t domain of $f(t,y)$ when $y=-2$ is

$$
\{t<-1\lor -1<t\}
$$

And the point $t_0=0$ is inside this domain. The y domain of $f(t,y)$ when $t=0$ is

$$
\{-\infty<y<\infty\}
$$

And the point $y_0=-2$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{t^2}{y\,(t^3+1)}\right)\\
&= -\frac{t^2}{y^2\,(t^3+1)}
\end{aligned}
$$

The t domain of $\frac{\partial f}{\partial y}$ when $y=-2$ is

$$
\{t<-1\lor -1<t\}
$$

And the point $t_0=0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t=0$ is

$$
\{-\infty<y<\infty\}
$$

And the point $y_0=-2$ is inside this domain. Therefore solution exists and is unique.

8.26.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(t,y)\\
&= f(t)g(y)\\
&= \frac{t^2}{y\,(t^3+1)}
\end{aligned}
$$

Where $f(t) = \frac{t^2}{t^3+1}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = \frac{t^2}{t^3+1}\,dt$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int \frac{t^2}{t^3+1}\,dt$$

$$\frac{y^2}{2} = \frac{\ln\left(t^3+1\right)}{3} + c_1$$

Which results in

$$y = \frac{\sqrt{6\ln\left(t^3+1\right)+18c_1}}{3}$$

$$y = -\frac{\sqrt{6\ln\left(t^3+1\right)+18c_1}}{3}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -\sqrt{c_1}\,\sqrt{2}$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\sqrt{6\ln\left(t^3+1\right)+36}}{3}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

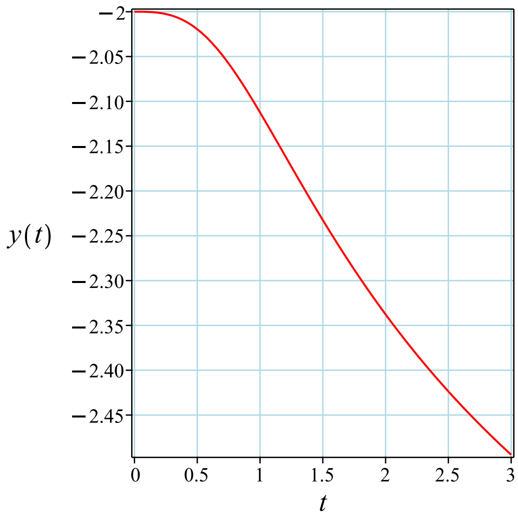
$$-2 = \sqrt{c_1}\,\sqrt{2}$$

Warning: Unable to solve for constant of integration.

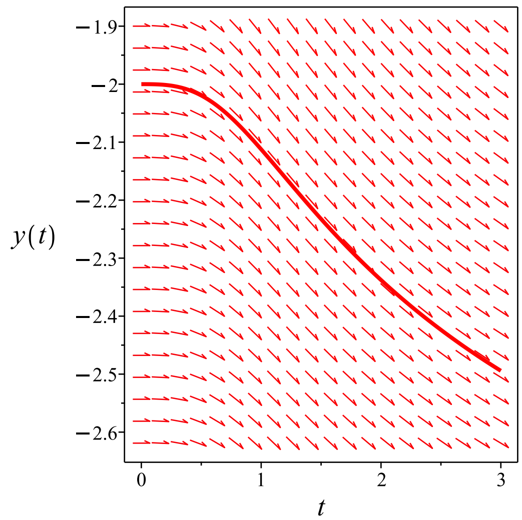
Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{6\ln\left(t^3+1\right)+}}{3}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\sqrt{6 \ln (t^3 + 1) + 36}}{3}$$

Verified OK.

8.26.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{t^2}{y (t^3 + 1)}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 349: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= \frac{t^3 + 1}{t^2} \\
\eta(t, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{t^3+1}{t^2}} dt \end{aligned}$$

Which results in

$$S = \frac{\ln(t^3+1)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{t^2}{y(t^3+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_y &= 1 \\ S_t &= \frac{t^2}{(t^2-t+1)(1+t)} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

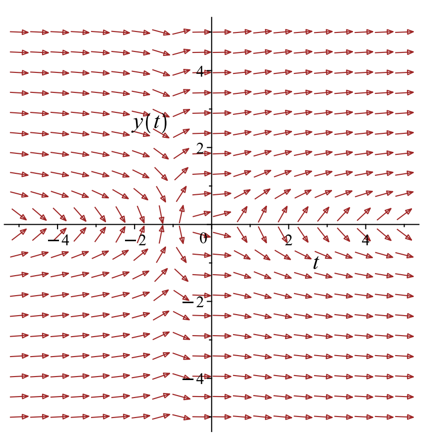
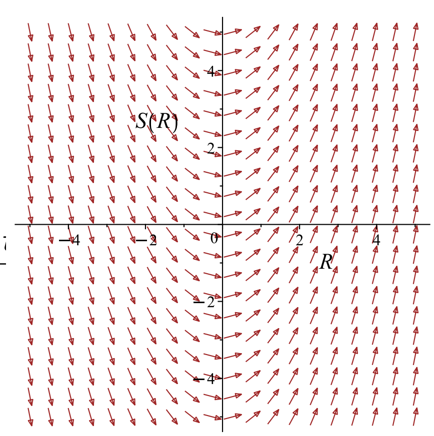
To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{\ln(1+t)}{3} + \frac{\ln(t^2 - t + 1)}{3} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{\ln(1+t)}{3} + \frac{\ln(t^2 - t + 1)}{3} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{t^2}{y(t^3+1)}$ 	$R = y$ $S = \frac{\ln(1+t)}{3} + \frac{\ln(t^2 -}{3}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 2$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(1+t)}{3} + \frac{\ln(t^2-t+1)}{3} = \frac{y^2}{2} - 2$$

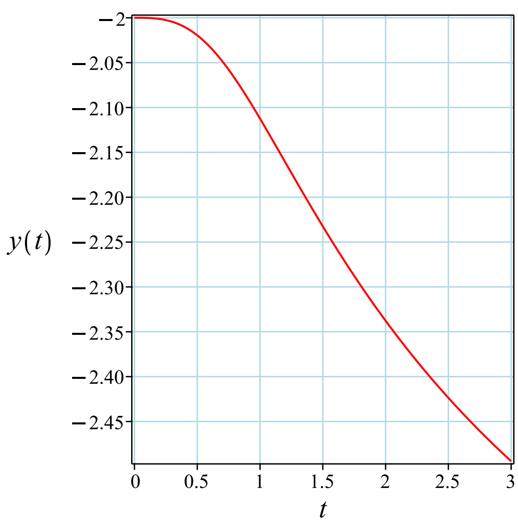
Solving for y from the above gives

$$y = -\frac{\sqrt{36 + 6\ln(1+t) + 6\ln(t^2-t+1)}}{3}$$

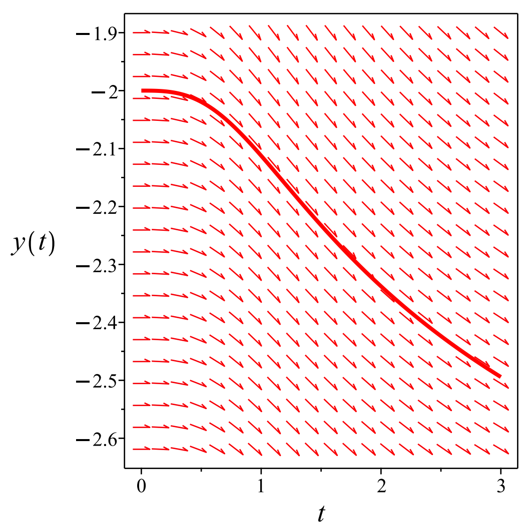
Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{36 + 6\ln(1+t) + 6\ln(t^2-t+1)}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{36 + 6\ln(1+t) + 6\ln(t^2-t+1)}}{3}$$

Verified OK.

8.26.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t + N(t,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y)\,\mathrm{d}y &= \left(\frac{t^2}{t^3+1}\right)\mathrm{d}t\\ \left(-\frac{t^2}{t^3+1}\right)\mathrm{d}t + (y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y) &= -\frac{t^2}{t^3+1}\\ N(t,y) &= y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{t^2}{t^3+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{t^2}{t^3+1}\,\mathrm{d}t\end{aligned}$$

$$\phi = -\frac{\ln(t^3+1)}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (y) \,\mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(t^3 + 1)}{3} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(t^3 + 1)}{3} + \frac{y^2}{2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

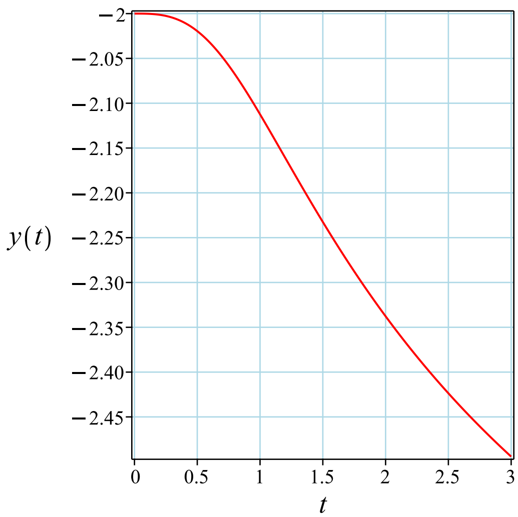
$$-\frac{\ln(t^3 + 1)}{3} + \frac{y^2}{2} = 2$$

Solving for y from the above gives

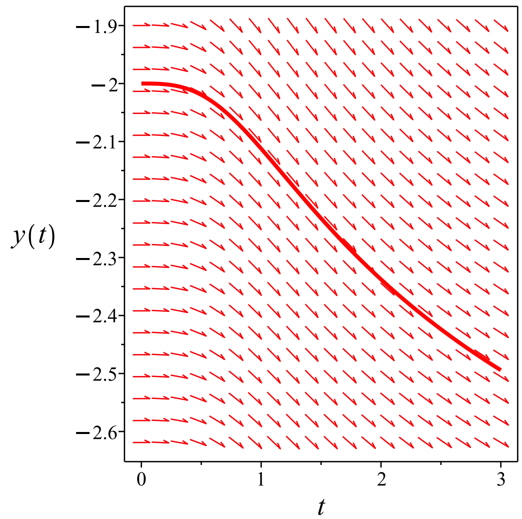
$$y = -\frac{\sqrt{6\ln(t^3 + 1) + 36}}{3}$$

Summary
The solution(s) found are the following

$$y = -\frac{\sqrt{6 \ln \left(t^{3}+1\right)+36}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{6 \ln \left(t^{3}+1\right)+36}}{3}$$

Verified OK.

8.26.5 Maple step by step solution

Let's solve
$\left[y' - \frac{t^2}{y+y t^3} = 0, y(0) = -2\right]$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $y' y = \frac{t^2}{(t^2-t+1)(1+t)}$
- Integrate both sides with respect to t

$$\int y'y dt = \int \frac{t^2}{(t^2-t+1)(1+t)} dt + c_1$$

- Evaluate integral

$$\frac{y^2}{2} = \frac{\ln((1+t)(t^2-t+1))}{3} + c_1$$

- Solve for y

$$\left\{y = -\frac{\sqrt{18c_1+6\ln((1+t)(t^2-t+1))}}{3}, y = \frac{\sqrt{18c_1+6\ln((1+t)(t^2-t+1))}}{3}\right\}$$

- Use initial condition $y(0) = -2$

$$-2 = -\frac{\sqrt{18}\,\sqrt{c_1}}{3}$$

- Solve for c_1

$$c_1 = 2$$

- Substitute $c_1 = 2$ into general solution and simplify

$$y = -\frac{\sqrt{6\ln((1+t)(t^2-t+1))+36}}{3}$$

- Use initial condition $y(0) = -2$

$$-2 = \frac{\sqrt{18}\,\sqrt{c_1}}{3}$$

- Solution does not satisfy initial condition
- Solution to the IVP

$$y = -\frac{\sqrt{6\ln((1+t)(t^2-t+1))+36}}{3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(y(t),t)= t^2/(y(t)+t^3*y(t)),y(0) = -2],y(t), singsol=all)
```

$$y(t) = -\frac{\sqrt{36 + 6 \ln\left(t^3 + 1\right)}}{3}$$

✓ Solution by Mathematica

Time used: 0.195 (sec). Leaf size: 26

```
DSolve[{y'[t]== t^2/(y[t]+t^3*y[t]),{y[0]==-2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\sqrt{\frac{2}{3}}\sqrt{\log\left(t^3 + 1\right) + 6}$$

8.27 problem 40

Internal problem ID [13055]
Internal file name [`OUTPUT/11707_Wednesday_November_08_2023_03_29_03_AM_54714494/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 + 2y = 1}$$

With initial conditions

$$[y(0) = 2]$$

8.27.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= y^2 - 2y + 1 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - 2y + 1) \\ &= 2y - 2\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

8.27.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{y^2 - 2y + 1} dy &= t + c_1 \\ -\frac{1}{y - 1} &= t + c_1\end{aligned}$$

Solving for y gives these solutions

$$y_1 = \frac{c_1 + t - 1}{t + c_1}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{-1 + c_1}{c_1}$$

$$c_1 = -1$$

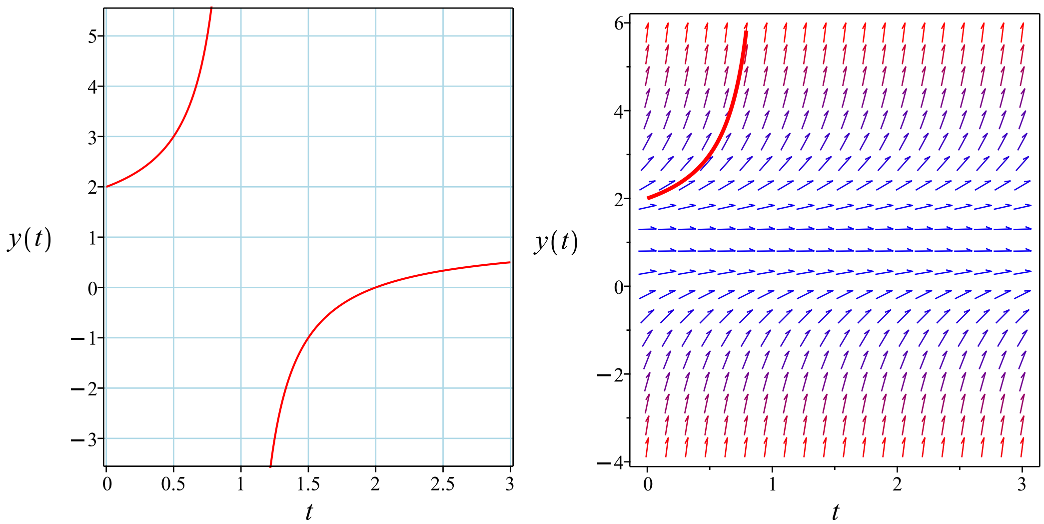
Substituting c_1 found above in the general solution gives

$$y = \frac{-2 + t}{t - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-2 + t}{t - 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{-2+t}{t-1}$$

Verified OK.

8.27.3 Maple step by step solution

Let's solve

$[y' - y^2 + 2y = 1, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-2y+1} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y^2-2y+1} dt = \int 1 dt + c_1$

- Evaluate integral

  $-\frac{1}{y-1} = t + c_1$

- Solve for y

$y = \frac{c_1+t-1}{t+c_1}$

- Use initial condition $y(0) = 2$

  $2 = \frac{-1+c_1}{c_1}$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = \frac{-2+t}{t-1}$

- Solution to the IVP

  $y = \frac{-2+t}{t-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([diff(y(t),t)= y(t)^2-2*y(t)+1,y(0) = 2],y(t), singsol=all)
```

$$y(t) = \frac{t-2}{t-1}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 14

```
DSolve[{y'[t]== y[t]^2-2*y[t]+1,{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{t-2}{t-1}$$

8.28 problem 43

Internal problem ID [13056]
Internal file name [`OUTPUT/11708_Wednesday_November_08_2023_03_29_03_AM_89585763/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - (y-2)\,(y+1-\cos(t)) = 0}$$

8.28.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= -(y-2)\,(-y+\cos(t)-1) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\cos(t)\,y + y^2 + 2\cos(t) - y - 2$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 2\cos(t) - 2$, $f_1(t) = -\cos(t) - 1$ and $f_2(t) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= -\cos(t) - 1 \\ f_2^2 f_0 &= 2\cos(t) - 2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(t) - (-\cos(t) - 1)\, u'(t) + (2\cos(t) - 2)\, u(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(t) = c_1 \mathrm{e}^{-2t+\pi} + i c_2 \mathrm{e}^{-2t} \left(\int \mathrm{e}^{3t - \frac{3\pi}{2} - \sin(t)} dt \right)$$

The above shows that

$$u'(t) = -2c_1 \mathrm{e}^{-2t+\pi} - 2 i c_2 \mathrm{e}^{-2t} \left(\int \mathrm{e}^{3t - \frac{3\pi}{2} - \sin(t)} dt \right) + i c_2 \mathrm{e}^{t - \frac{3\pi}{2} - \sin(t)}$$

Using the above in (1) gives the solution

$$y = -\frac{-2c_1 \mathrm{e}^{-2t+\pi} - 2 i c_2 \mathrm{e}^{-2t} \left(\int \mathrm{e}^{3t - \frac{3\pi}{2} - \sin(t)} dt \right) + i c_2 \mathrm{e}^{t - \frac{3\pi}{2} - \sin(t)}}{c_1 \mathrm{e}^{-2t+\pi} + i c_2 \mathrm{e}^{-2t} \left(\int \mathrm{e}^{3t - \frac{3\pi}{2} - \sin(t)} dt \right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2 i c_3 \mathrm{e}^{-2t+\pi} - 2\, \mathrm{e}^{-2t} \left(\int \mathrm{e}^{3t - \frac{3\pi}{2} - \sin(t)} dt \right) + \mathrm{e}^{t - \frac{3\pi}{2} - \sin(t)}}{i c_3 \mathrm{e}^{-2t+\pi} - \mathrm{e}^{-2t} \left(\int \mathrm{e}^{3t - \frac{3\pi}{2} - \sin(t)} dt \right)}$$

Summary

The solution(s) found are the following

$$y = \frac{2ic_3 e^{-2t+\pi} - 2\,e^{-2t}\left(\int e^{3t-\frac{3\pi}{2}-\sin(t)}dt\right) + e^{t-\frac{3\pi}{2}-\sin(t)}}{ic_3 e^{-2t+\pi} - e^{-2t}\left(\int e^{3t-\frac{3\pi}{2}-\sin(t)}dt\right)} \tag{1}$$

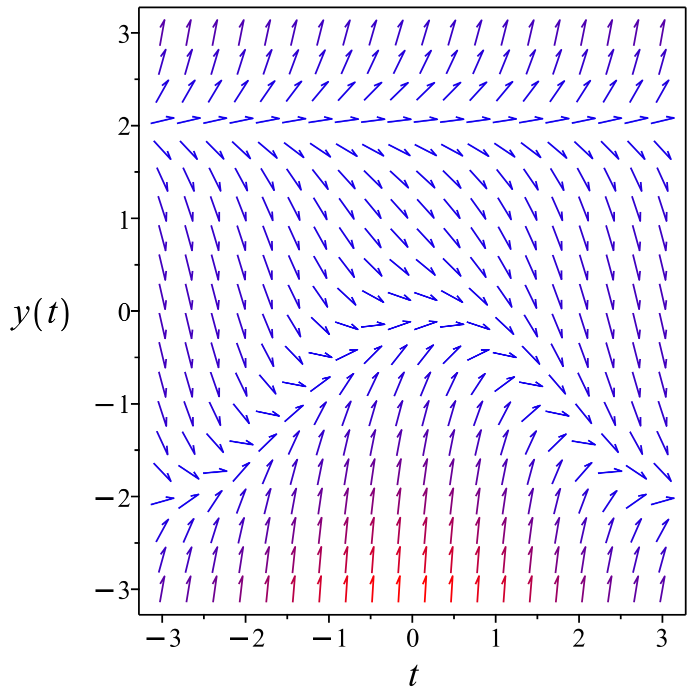


Figure 360: Slope field plot

Verification of solutions

$$y = \frac{2ic_3 e^{-2t+\pi} - 2\,e^{-2t}\left(\int e^{3t-\frac{3\pi}{2}-\sin(t)}dt\right) + e^{t-\frac{3\pi}{2}-\sin(t)}}{ic_3 e^{-2t+\pi} - e^{-2t}\left(\int e^{3t-\frac{3\pi}{2}-\sin(t)}dt\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-cos(x)-1)*(diff(y(x), x))+(2
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Kummer
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 81

```
dsolve(diff(y(t),t)=(y(t)-2)*(y(t)+1-cos(t)),y(t), singsol=all)
```

$$y(t) = \frac{-2c_1 \mathrm{e}^{-2t}\left(\int \mathrm{e}^{-\frac{3\pi}{2}+3t-\sin(t)}dt\right) + c_1 \mathrm{e}^{t-\frac{3\pi}{2}-\sin(t)} + 2i\mathrm{e}^{-2t+\pi}}{-c_1 \mathrm{e}^{-2t}\left(\int \mathrm{e}^{-\frac{3\pi}{2}+3t-\sin(t)}dt\right) + i\mathrm{e}^{-2t+\pi}}$$

✓ Solution by Mathematica

Time used: 3.379 (sec). Leaf size: 224

```
DSolve[y'[t]==(y[t]-2)*(y[t]+1-Cos[t]),y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{-2\int_1^{e^{it}} e^{\frac{i\left(K[1]^2-1\right)}{2K[1]}} K[1]^{-1-3i}dK[1] + ie^{\frac{1}{2}ie^{-it}\left(-1+e^{2it}\right)}\left(e^{it}\right)^{-3i} - 2c_1}{\int_1^{e^{it}} e^{\frac{i\left(K[1]^2-1\right)}{2K[1]}} K[1]^{-1-3i}dK[1] + c_1}$$

$$y(t) \to 2$$

$$y(t) \to 2 - \frac{ie^{\frac{1}{2}ie^{-it}\left(-1+e^{2it}\right)}\left(e^{it}\right)^{-3i}}{\int_1^{e^{it}} e^{\frac{i\left(K[1]^2-1\right)}{2K[1]}} K[1]^{-1-3i}dK[1]}$$

8.29 problem 44

Internal problem ID [13057]
Internal file name [`OUTPUT/11709_Wednesday_November_08_2023_03_29_05_AM_81720901/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind"**

Maple gives the following as the ode type

`[_Abel]`

Unable to solve or complete the solution.

$$\boxed{y' - (y-1)(y-2)\left(y - e^{\frac{t}{2}}\right) = 0}$$

8.29.1 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2 + f_3(t)y^3$$

Comparing the above to given ODE which is

$$y' = y^3 + \left(-3 - e^{\frac{t}{2}}\right)y^2 + \left(2 + 3\,e^{\frac{t}{2}}\right)y - 2\,e^{\frac{t}{2}} \tag{1}$$

Therefore

$$\begin{aligned}
f_0(t) &= -2\,e^{\frac{t}{2}} \\
f_1(t) &= 2 + 3\,e^{\frac{t}{2}} \\
f_2(t) &= -3 - e^{\frac{t}{2}} \\
f_3(t) &= 1
\end{aligned}$$

Since $f_2(t) = -3 - e^{\frac{t}{2}}$ is not zero, then the first step is to apply the following transformation to remove f_2. Let $y = u(t) - \frac{f_2}{3f_3}$ or

$$
\begin{aligned}
y &= u(t) - \left(\frac{-3 - e^{\frac{t}{2}}}{3}\right) \\
&= u(t) + 1 + \frac{e^{\frac{t}{2}}}{3}
\end{aligned}
$$

The above transformation applied to (1) gives a new ODE as

$$
u'(t) = -\frac{e^{\frac{t}{2}}}{2} + u(t)^3 - u(t) + u(t)\,e^{\frac{t}{2}} - \frac{u(t)\,e^{t}}{3} + \frac{e^{t}}{3} - \frac{2\,e^{\frac{3t}{2}}}{27} \tag{2}
$$

This is Abel first kind ODE, it has the form

$$
u'(t) = f_0(t) + f_1(t)u(t) + f_2(t)u(t)^2 + f_3(t)u(t)^3
$$

Comparing the above to given ODE which is

$$
u'(t) = u(t)^3 + \left(-1 - \frac{e^{t}}{3} + e^{\frac{t}{2}}\right) u(t) - \frac{e^{\frac{t}{2}}}{2} + \frac{e^{t}}{3} - \frac{2\,e^{\frac{3t}{2}}}{27} \tag{1}
$$

Therefore

$$
\begin{aligned}
f_0(t) &= -\frac{e^{\frac{t}{2}}}{2} + \frac{e^{t}}{3} - \frac{2\,e^{\frac{3t}{2}}}{27} \\
f_1(t) &= -1 - \frac{e^{t}}{3} + e^{\frac{t}{2}} \\
f_2(t) &= 0 \\
f_3(t) &= 1
\end{aligned}
$$

Since $f_2(t) = 0$ then we check the Abel invariant to see if it depends on t or not. The Abel invariant is given by

$$
-\frac{f_1^3}{f_0^2 f_3}
$$

Which when evaluating gives

$$
-\frac{\left(\frac{e^{\frac{t}{2}}}{4} - \frac{e^{t}}{3} + \frac{e^{\frac{3t}{2}}}{9} + 3\left(-\frac{e^{\frac{t}{2}}}{2} + \frac{e^{t}}{3} - \frac{2\,e^{\frac{3t}{2}}}{27}\right)\left(-1 - \frac{e^{t}}{3} + e^{\frac{t}{2}}\right)\right)^3}{27\left(-\frac{e^{\frac{t}{2}}}{2} + \frac{e^{t}}{3} - \frac{2\,e^{\frac{3t}{2}}}{27}\right)^5}
$$

Since the Abel invariant depends on t then unable to solve this ode at this time.

Unable to complete the solution now.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1/(exp(y(x)+x-2*exp((1/2)*x)-1)-1), y(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying inverse_Riccati
      trying an equivalence to an Abel ODE
      differential order: 1; trying a linearization to 2nd order
      --- trying a change of variables {x -> y(x), y(x) -> x}
      differential order: 1; trying a linearization to 2nd order
      trying 1st order ODE linearizable_by_differentiation
      --- Trying Lie symmetry methods, 1st order ---
      `, `-> Computing symmetries using: way = 3
      `, `-> Computing symmetries using: way = 4
      `, `-> Computing symmetries using: way = 5
      trying symmetry patterns for 1st order ODEs
      -> trying a symmetry pattern of the form [F(x)*G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)*G(y)]
      -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
      -> trying a symmetry pattern of the form [F(x),G(x)]
      -> trying a symmetry pattern of the form [F(y),G(y)]
      -> trying a symmetry pattern of the form [F(x)+G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)+G(y)]
```

✗ Solution by Maple

```
dsolve(diff(y(t),t)=(y(t)-1)*(y(t)-2)*(y(t)-exp(t/2)),y(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[t]==(y[t]-1)*(y[t]-2)*(y[t]-Exp[t/2]),y[t],t,IncludeSingularSolutions -> True]
```

Timed out

8.30 problem 45

Internal problem ID [13058]
Internal file name [`OUTPUT/11710_Wednesday_November_08_2023_03_29_08_AM_40993183/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - t^2 y - y = t^2 + 1}$$

8.30.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= f(t)g(y) \\ &= \left(t^2+1\right)(y+1) \end{aligned}$$

Where $f(t) = t^2 + 1$ and $g(y) = y + 1$. Integrating both sides gives

$$\frac{1}{y+1}\,dy = t^2 + 1\,dt$$

$$\int \frac{1}{y+1}\, dy = \int t^2 + 1\, dt$$

$$\ln(y+1) = \frac{1}{3}t^3 + t + c_1$$

Raising both side to exponential gives

$$y + 1 = e^{\frac{1}{3}t^3 + t + c_1}$$

Which simplifies to

$$y + 1 = c_2 e^{\frac{1}{3}t^3 + t}$$

Summary

The solution(s) found are the following

$$y = c_2 e^{\frac{1}{3}t^3 + t + c_1} - 1 \tag{1}$$

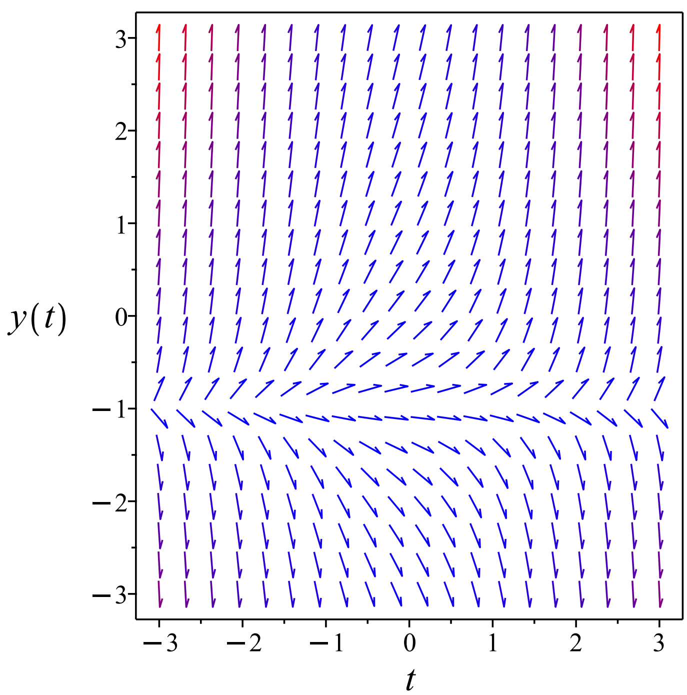


Figure 361: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{1}{3}t^3 + t + c_1} - 1$$

Verified OK.

8.30.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned}p(t) &= -t^2 - 1\\ q(t) &= t^2 + 1\end{aligned}$$

Hence the ode is

$$y' + \left(-t^2 - 1\right) y = t^2 + 1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-t^2-1)dt}\\ &= \mathrm{e}^{-\frac{1}{3}t^3-t}\end{aligned}$$

Which simplifies to

$$\mu = \mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(t^2+1\right)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}} y\right) &= \left(\mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}}\right)\left(t^2+1\right)\\ \mathrm{d}\left(\mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}} y\right) &= \left(\left(t^2+1\right)\mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}}\right)\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}} y &= \int \left(t^2+1\right)\mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}}\,\mathrm{d}t\\ \mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}} y &= -\mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}}$ results in

$$y = -\mathrm{e}^{\frac{t\left(t^2+3\right)}{3}}\mathrm{e}^{-\frac{t\left(t^2+3\right)}{3}} + c_1\mathrm{e}^{\frac{t\left(t^2+3\right)}{3}}$$

which simplifies to

$$y = -1 + c_1 e^{\frac{t(t^2+3)}{3}}$$

Summary

The solution(s) found are the following

$$y = -1 + c_1 e^{\frac{t(t^2+3)}{3}} \tag{1}$$

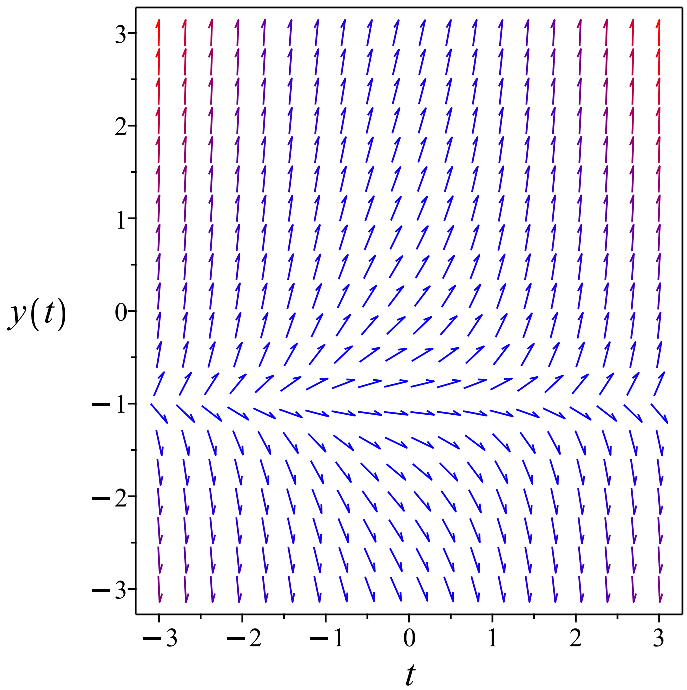


Figure 362: Slope field plot

Verification of solutions

$$y = -1 + c_1 e^{\frac{t(t^2+3)}{3}}$$

Verified OK.

8.30.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y\,t^2 + t^2 + y + 1 \\
y' &= \omega(t, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 353: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t, y) = 0$$
$$\eta(t, y) = e^{\frac{1}{3}t^3+t} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{1}{3}t^3+t}} dy\end{aligned}$$

Which results in

$$S = e^{-\frac{1}{3}t^3-t}y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, y)S_y}{R_t + \omega(t, y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t, y)$ is the right hand side of the original ode given by

$$\omega(t, y) = y\,t^2 + t^2 + y + 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_t &= 1 \\ R_y &= 0 \\ S_t &= -\left(t^2+1\right) e^{-\frac{t\left(t^2+3\right)}{3}} y \\ S_y &= e^{-\frac{t\left(t^2+3\right)}{3}}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (t^2+1)\,\mathrm{e}^{-\frac{t(t^2+3)}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R^2+1)\,\mathrm{e}^{-\frac{R(R^2+3)}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-\frac{R(R^2+3)}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\mathrm{e}^{-\frac{t(t^2+3)}{3}} y = -\mathrm{e}^{-\frac{t(t^2+3)}{3}} + c_1$$

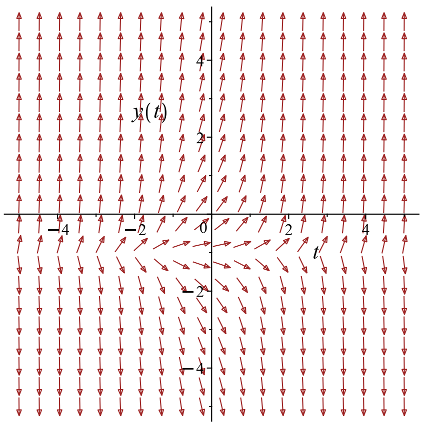
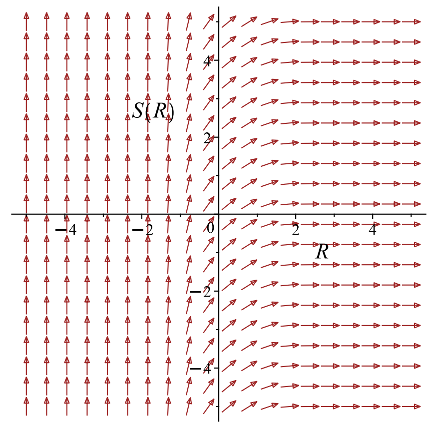
Which simplifies to

$$\mathrm{e}^{-\frac{t(t^2+3)}{3}} y = -\mathrm{e}^{-\frac{t(t^2+3)}{3}} + c_1$$

Which gives

$$y = -\left(\mathrm{e}^{-\frac{t(t^2+3)}{3}} - c_1\right)\mathrm{e}^{\frac{t(t^2+3)}{3}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = y\,t^2 + t^2 + y + 1$ 	$R = t$ $S = \mathrm{e}^{-\frac{t(t^2+3)}{3}} y$	$\frac{dS}{dR} = (R^2+1)\,\mathrm{e}^{-\frac{R(R^2+3)}{3}}$ 

Summary

The solution(s) found are the following

$$y = -\left(\mathrm{e}^{-\frac{t(t^2+3)}{3}} - c_1\right) \mathrm{e}^{\frac{t(t^2+3)}{3}} \tag{1}$$

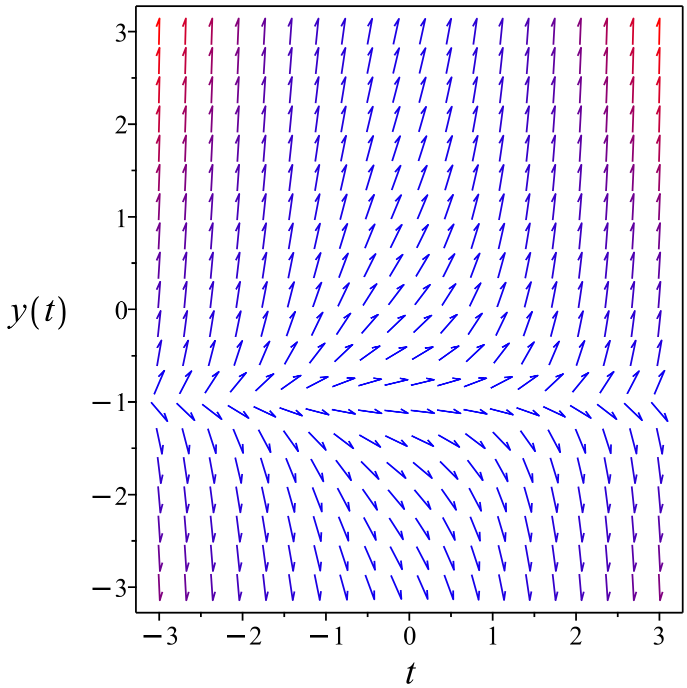


Figure 363: Slope field plot

Verification of solutions

$$y = -\left(e^{-\frac{t(t^2+3)}{3}} - c_1\right) e^{\frac{t(t^2+3)}{3}}$$

Verified OK.

8.30.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, y) \, \mathrm{d}t + N(t, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y+1}\right) \mathrm{d}y = \left(t^2 + 1\right) \mathrm{d}t$$
$$\left(-t^2 - 1\right) \mathrm{d}t + \left(\frac{1}{y+1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, y) = -t^2 - 1$$
$$N(t, y) = \frac{1}{y+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-t^2 - 1\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{y+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t, y)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int M \,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t &= \int -t^2 - 1 \,\mathrm{d}t \\
\phi &= -\frac{1}{3}t^3 - t + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+1}$. Therefore equation (4) becomes

$$
\frac{1}{y+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y+1}\right) \mathrm{d}y \\
f(y) &= \ln(y+1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{t^3}{3} - t + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{t^3}{3} - t + \ln(y+1)$$

The solution becomes

$$y = \mathrm{e}^{\frac{1}{3}t^3+t+c_1} - 1$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\frac{1}{3}t^3+t+c_1} - 1 \tag{1}$$

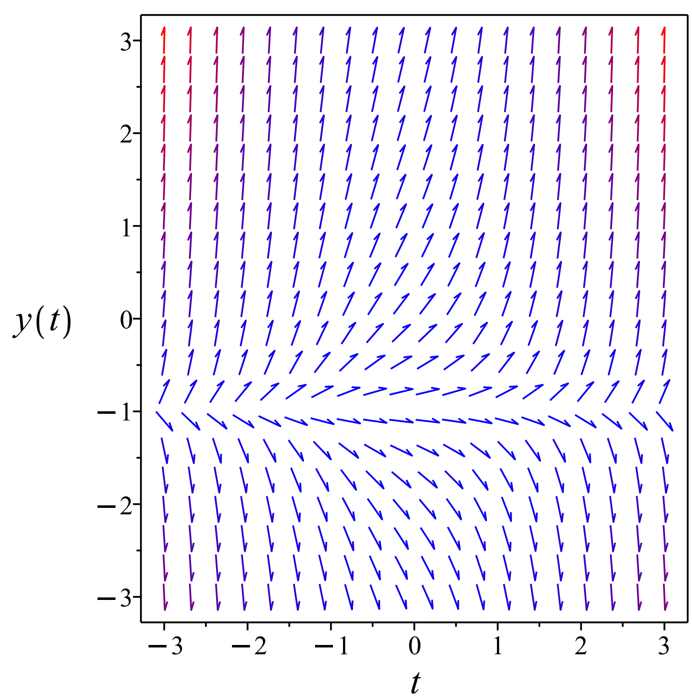


Figure 364: Slope field plot

Verification of solutions

$$y = e^{\frac{1}{3}t^3+t+c_1} - 1$$

Verified OK.

8.30.5 Maple step by step solution

Let's solve

$y' - t^2 y - y = t^2 + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1} = t^2 + 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{y+1} dt = \int (t^2 + 1) dt + c_1$

- Evaluate integral

  $\ln(y + 1) = \frac{1}{3}t^3 + t + c_1$

- Solve for y

  $y = e^{\frac{1}{3}t^3+t+c_1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(t),t)=t^2*y(t)+1+y(t)+t^2,y(t), singsol=all)
```

$$y(t) = -1 + e^{\frac{t(t^2+3)}{3}} c_1$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 26

```
DSolve[y'[t]==t^2*y[t]+1+y[t]+t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -1 + c_1 e^{\frac{t^3}{3}+t}$$
$$y(t) \to -1$$

8.31 problem 46

Internal problem ID [13059]
Internal file name [`OUTPUT/11711_Wednesday_November_08_2023_03_29_09_AM_67191522/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{2y+1}{t} = 0}$$

8.31.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t,y) \\ &= f(t)g(y) \\ &= \frac{2y+1}{t} \end{aligned}$$

Where $f(t) = \frac{1}{t}$ and $g(y) = 2y + 1$. Integrating both sides gives

$$\frac{1}{2y+1}\, dy = \frac{1}{t}\, dt$$
$$\int \frac{1}{2y+1}\, dy = \int \frac{1}{t}\, dt$$
$$\frac{\ln(2y+1)}{2} = \ln(t) + c_1$$

Raising both side to exponential gives

$$\sqrt{2y+1} = \mathrm{e}^{\ln(t)+c_1}$$

Which simplifies to

$$\sqrt{2y+1} = c_2 t$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_2^2 t^2 \mathrm{e}^{2c_1}}{2} - \frac{1}{2} \tag{1}$$

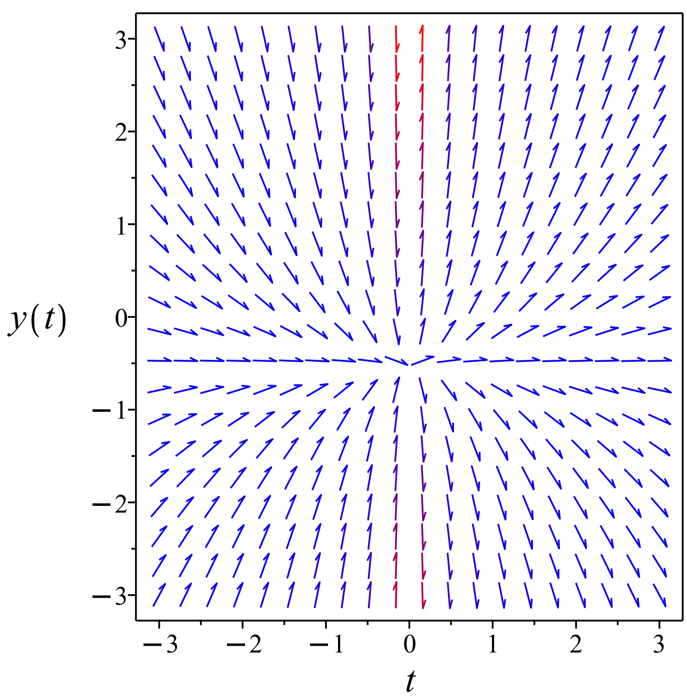


Figure 365: Slope field plot

Verification of solutions

$$y = \frac{c_2^2 t^2 \mathrm{e}^{2c_1}}{2} - \frac{1}{2}$$

Verified OK.

8.31.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= -\frac{2}{t} \\ q(t) &= \frac{1}{t} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{t} = \frac{1}{t}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{t}dt} \\ &= \frac{1}{t^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{1}{t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(\frac{1}{t}\right) \\ \mathrm{d}\left(\frac{y}{t^2}\right) &= \frac{1}{t^3}\,\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{t^2} &= \int \frac{1}{t^3}\,\mathrm{d}t \\ \frac{y}{t^2} &= -\frac{1}{2t^2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$y = -\frac{1}{2} + t^2 c_1$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2} + t^2 c_1 \tag{1}$$

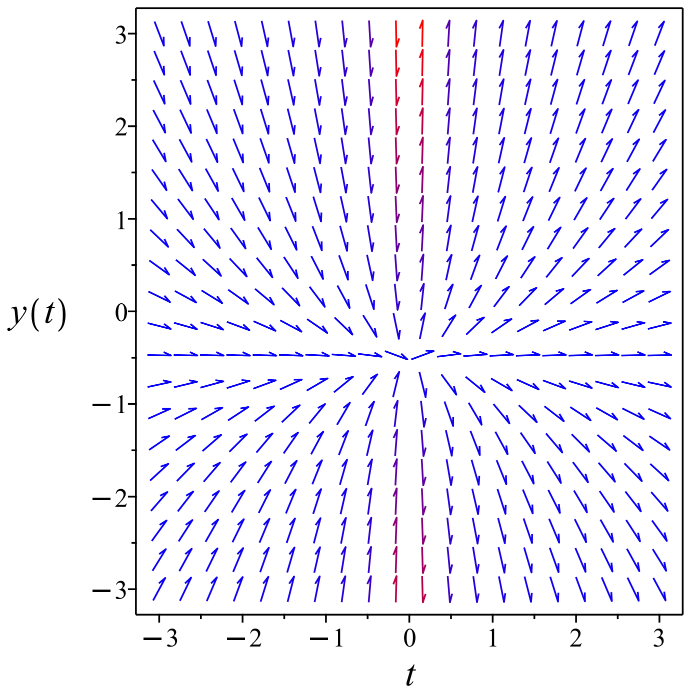


Figure 366: Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + t^2 c_1$$

Verified OK.

8.31.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = t + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 2y_0 + 1}{X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = -\frac{1}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{2Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{du}{dX}X + u = 2u$$
$$\frac{du}{dX} = \frac{u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X-u(X)=0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= \frac{u}{X}\end{aligned}$$

Where $f(X)=\frac{1}{X}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{1}{X}\,dX\\ \int\frac{1}{u}\,du &= \int\frac{1}{X}\,dX\\ \ln(u) &= \ln(X)+c_2\\ u &= e^{\ln(X)+c_2}\\ &= c_2X\end{aligned}$$

Now u in the above solution is replaced back by Y using $u=\frac{Y}{X}$ which results in the solution

$$Y(X)=X^2c_2$$

Using the solution for $Y(X)$

$$Y(X)=X^2c_2$$

And replacing back terms in the above solution using

$$\begin{aligned}Y &= y+y_0\\ X &= t+x_0\end{aligned}$$

Or

$$\begin{aligned}Y &= y-\frac{1}{2}\\ X &= t\end{aligned}$$

Then the solution in y becomes

$$y+\frac{1}{2}=c_2t^2$$

Summary
The solution(s) found are the following

$$y + \frac{1}{2} = c_2 t^2 \tag{1}$$

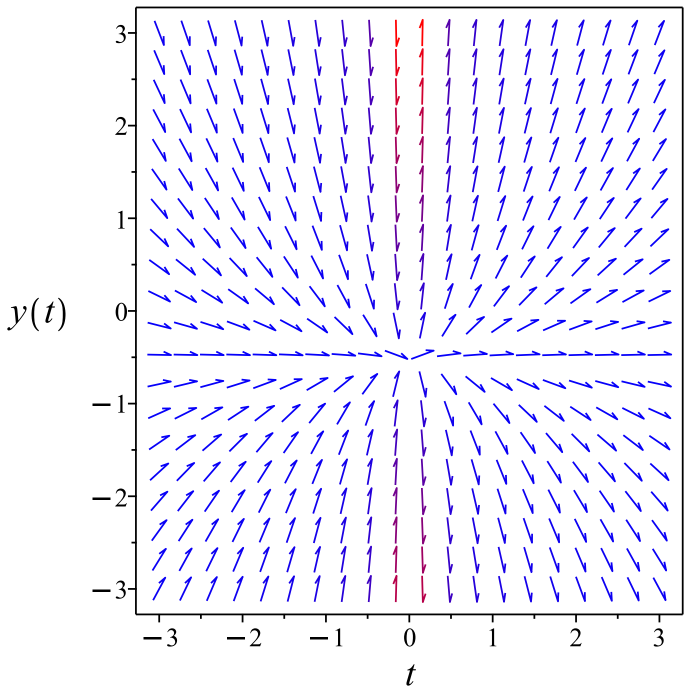


Figure 367: Slope field plot

Verification of solutions

$$y + \frac{1}{2} = c_2 t^2$$

Verified OK.

8.31.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2y + 1}{t}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 356: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, y) &= 0 \\
\eta(t, y) &= t^2 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{t^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{t^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = \frac{2y + 1}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= -\frac{2y}{t^3} \\ S_y &= \frac{1}{t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{t^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$\frac{y}{t^2} = -\frac{1}{2t^2} + c_1$$

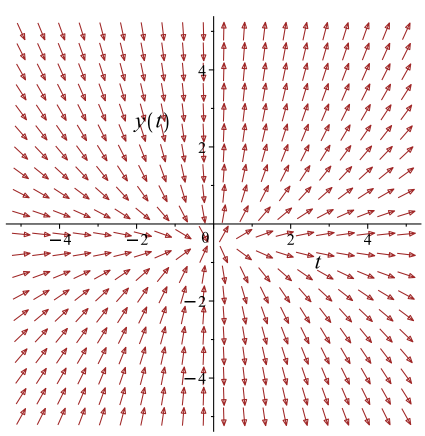
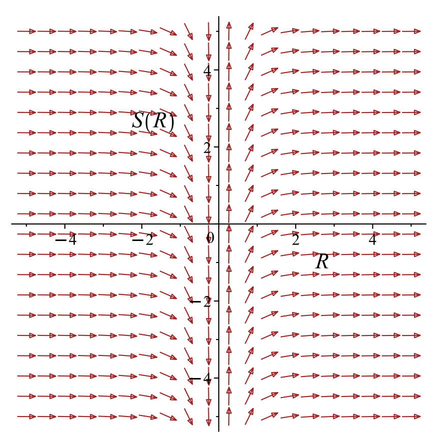
Which simplifies to

$$\frac{y}{t^2} = -\frac{1}{2t^2} + c_1$$

Which gives

$$y = -\frac{1}{2} + t^2 c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = \frac{2y+1}{t}$ 	$R = t$ $S = \frac{y}{t^2}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{2} + t^2 c_1 \tag{1}$$

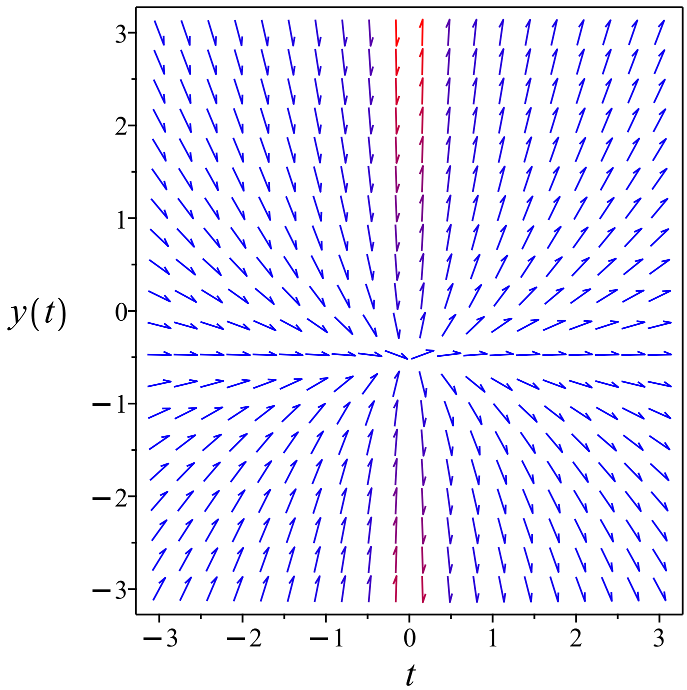


Figure 368: Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + t^2 c_1$$

Verified OK.

8.31.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2y+1}\right)\mathrm{d}y&=\left(\frac{1}{t}\right)\mathrm{d}t\\ \left(-\frac{1}{t}\right)\mathrm{d}t+\left(\frac{1}{2y+1}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=-\frac{1}{t}\\ N(t,y)&=\frac{1}{2y+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{t}\right)\\ &=0\end{aligned}$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2y+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t, y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, dt = \int M \, dt$$
$$\int \frac{\partial \phi}{\partial t} \, dt = \int -\frac{1}{t} \, dt$$
$$\phi = -\ln(t) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y+1}$. Therefore equation (4) becomes

$$\frac{1}{2y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y+1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(\frac{1}{2y+1}\right) dy$$
$$f(y) = \frac{\ln(2y+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(t) + \frac{\ln(2y+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) + \frac{\ln(2y+1)}{2}$$

The solution becomes

$$y = \frac{e^{2c_1}t^2}{2} - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{2c_1}t^2}{2} - \frac{1}{2} \tag{1}$$

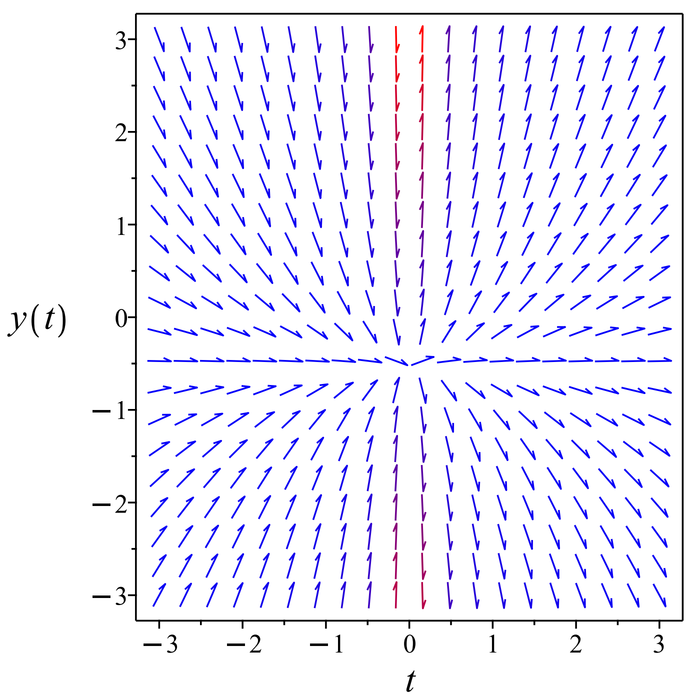


Figure 369: Slope field plot

Verification of solutions

$$y = \frac{e^{2c_1} t^2}{2} - \frac{1}{2}$$

Verified OK.

8.31.6 Maple step by step solution

Let's solve

$y' - \frac{2y+1}{t} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y+1} = \frac{1}{t}$

- Integrate both sides with respect to t

  $\int \frac{y'}{2y+1} dt = \int \frac{1}{t} dt + c_1$

- Evaluate integral

  $\frac{\ln(2y+1)}{2} = \ln(t) + c_1$

- Solve for y

  $y = \frac{e^{2c_1} t^2}{2} - \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(t),t)=(2*y(t)+1)/t,y(t), singsol=all)
```

$$y(t) = -\frac{1}{2} + c_1 t^2$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 22

```
DSolve[y'[t]==(2*y[t]+1)/t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{1}{2} + c_1 t^2$$
$$y(t) \to -\frac{1}{2}$$

8.32 problem 47

Internal problem ID [13060]
Internal file name [`OUTPUT/11712_Wednesday_November_08_2023_03_29_10_AM_98042492/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 1. First-Order Differential Equations. Review Exercises for chapter 1. page 136
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' + y^2 = 3}$$

With initial conditions

$$[y(0) = 0]$$

8.32.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= -y^2 + 3 \end{aligned}$$

The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-y^2+3)\\ &= -2y\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

8.32.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{-y^2+3}dy &= t + c_1\\ \frac{\sqrt{3}\,\operatorname{arctanh}\left(\frac{\sqrt{3}\,y}{3}\right)}{3} &= t + c_1\end{aligned}$$

Solving for y gives these solutions

$$y_1 = \sqrt{3}\,\tanh\left((t+c_1)\sqrt{3}\right)$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,c_1} - \sqrt{3}}{\mathrm{e}^{2\sqrt{3}\,c_1} + 1}$$

$$c_1 = 0$$

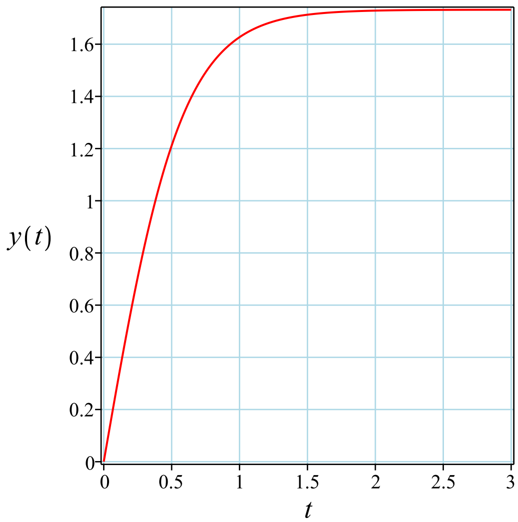
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,t} - \sqrt{3}}{\mathrm{e}^{2\sqrt{3}\,t} + 1}$$

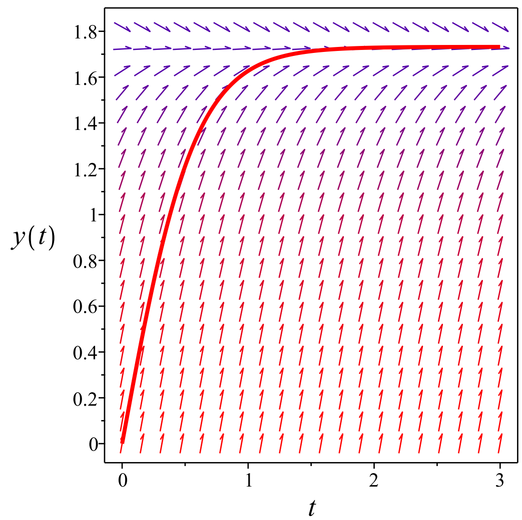
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,t} - \sqrt{3}}{\mathrm{e}^{2\sqrt{3}\,t} + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{3}\,e^{2\sqrt{3}\,t} - \sqrt{3}}{e^{2\sqrt{3}\,t} + 1}$$

Verified OK.

8.32.3 Maple step by step solution

Let's solve

$[y' + y^2 = 3, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{3-y^2} = 1$

- Integrate both sides with respect to t

  $\int \frac{y'}{3-y^2} dt = \int 1 dt + c_1$

- Evaluate integral

  $\frac{\sqrt{3}\,\mathrm{arctanh}\left(\frac{y\sqrt{3}}{3}\right)}{3} = t + c_1$

- Solve for y

$$y = \sqrt{3}\,\tanh\left((t + c_1)\sqrt{3}\right)$$

- Use initial condition $y(0) = 0$

$$0 = \sqrt{3}\,\tanh\left(\sqrt{3}\,c_1\right)$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \sqrt{3}\,\tanh\left(\sqrt{3}\,t\right)$$

- Solution to the IVP

$$y = \sqrt{3}\,\tanh\left(\sqrt{3}\,t\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 14

```
dsolve([diff(y(t),t)=3-y(t)^2,y(0) = 0],y(t), singsol=all)
```

$$y(t) = \sqrt{3}\,\tanh\left(\sqrt{3}\,t\right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 37

```
DSolve[{y'[t]==3-y[t]^2,{y[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\sqrt{3}\left(e^{2\sqrt{3}t} - 1\right)}{e^{2\sqrt{3}t} + 1}$$

9 Chapter 3. Linear Systems. Exercises section 3.1. page 258

9.1 problem 1

Internal problem ID [13061]
Internal file name [`OUTPUT/11713_Wednesday_November_08_2023_04_49_52_AM_64086885/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) - y \\
y' &= x(t) - y
\end{aligned}
$$

9.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} (1+t)\,c_1 - tc_2 \\ tc_1 + (1-t)\,c_2 \end{bmatrix} \\ &= \begin{bmatrix} (c_1 - c_2)\,t + c_1 \\ (c_1 - c_2)\,t + c_2 \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

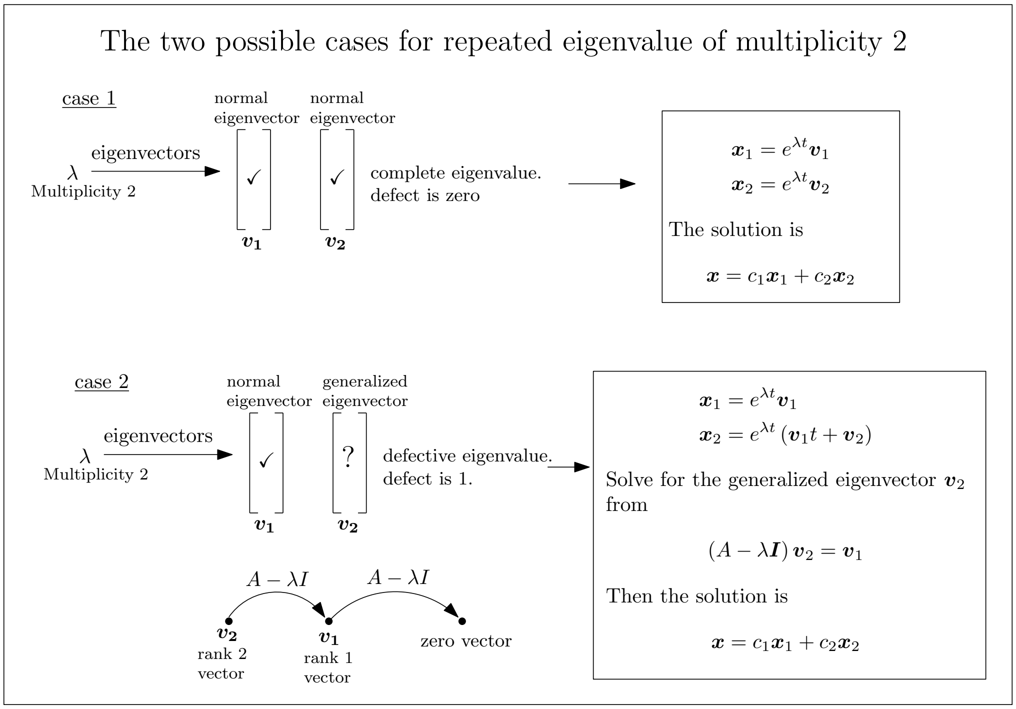


Figure 371: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) 1 \\
&= \begin{bmatrix} t+2 \\ 1+t \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t+2 \\ 1+t \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 + c_2(t+2) \\ c_2 t + c_1 + c_2 \end{bmatrix}
$$

The following is the phase plot of the system.

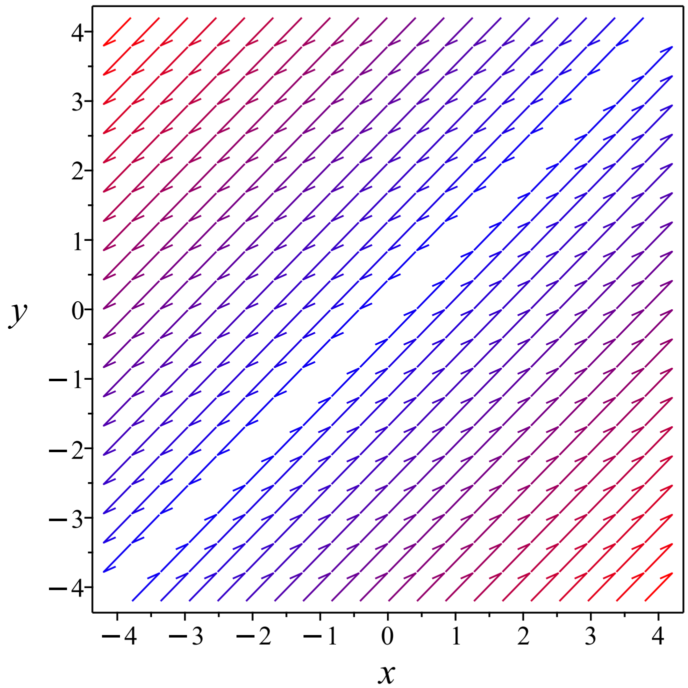


Figure 372: Phase plot

9.1.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - y, y' = x(t) - y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

  $$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = c_1, y = c_1\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(x(t),t)=x(t)-y(t),diff(y(t),t)=x(t)-y(t)],singsol=all)
```

$$x(t) = c_1 t + c_2$$
$$y(t) = c_1 t - c_1 + c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 32

```
DSolve[{x'[t]==x[t]-y[t],y'[t]==x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1(t+1) - c_2 t$$
$$y(t) \to (c_1 - c_2)t + c_2$$

9.2 problem 2

Internal problem ID [13062]
Internal file name [`OUTPUT/11714_Wednesday_November_08_2023_04_49_55_AM_14524618/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 2x(t) - y \\ y' &= 0 \end{aligned}$$

9.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & -\frac{e^{2t}}{2} + \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & -\frac{e^{2t}}{2}+\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 + \left(-\frac{e^{2t}}{2}+\frac{1}{2}\right) c_2 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-c_2)e^{2t}}{2}+\frac{c_2}{2} \\ c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -1 \\ 0 & -\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(2-\lambda)(-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix}\frac{t}{2}\\ t\end{bmatrix}=\begin{bmatrix}\frac{t}{2}\\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}\frac{t}{2}\\ t\end{bmatrix}=t\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix}\frac{t}{2}\\ t\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix}\frac{t}{2}\\ t\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}$$

Considering the eigenvalue $\lambda_2=2$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}-(2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}0 & -1\\ 0 & -2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}0 & -1 & 0\\ 0 & -2 & 0\end{array}\right]$$

$$R_2=R_2-2R_1\Longrightarrow\left[\begin{array}{cc|c}0 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{2t}$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{0}$$
$$= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{0}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{2t} + \frac{c_2}{2} \\ c_2 \end{bmatrix}$$

The following is the phase plot of the system.

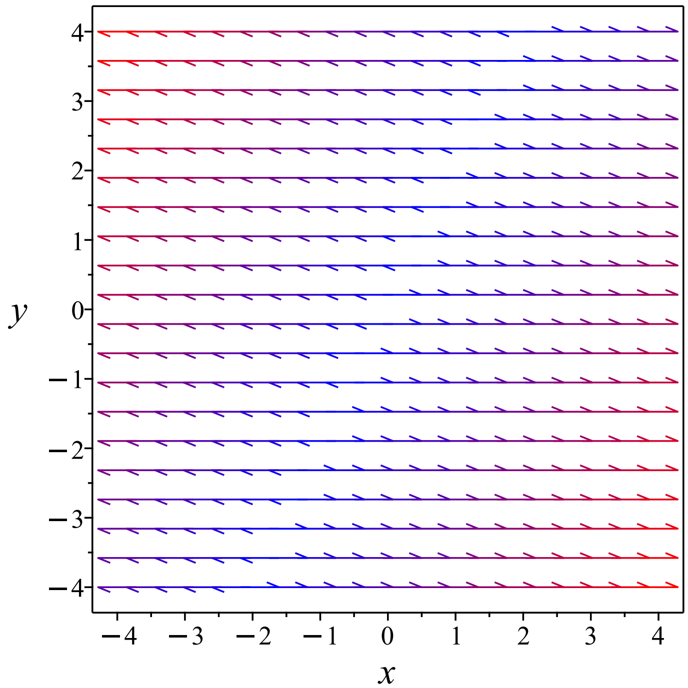


Figure 373: Phase plot

9.2.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) - y, y' = 0]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = e^{2t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_2 e^{2t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{c_1}{2} \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 e^{2t} + \frac{c_1}{2} \\ c_1 \end{bmatrix}$$

- Solution to the system of ODEs

  $\left\{x(t) = c_2 e^{2t} + \frac{c_1}{2}, y = c_1\right\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve([diff(x(t),t)=2*x(t)-y(t),diff(y(t),t)=0],singsol=all)
```

$$x(t) = \frac{c_2}{2} + c_1 e^{2t}$$
$$y(t) = c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 32

```
DSolve[{x'[t]==2*x[t]-y[t],y'[t]==0},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \left(c_1 - \frac{c_2}{2}\right) e^{2t} + \frac{c_2}{2}$$
$$y(t) \to c_2$$

9.3 problem 3

Internal problem ID [13063]
Internal file name [`OUTPUT/11715_Wednesday_November_08_2023_04_49_55_AM_47063498/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) \\
y' &= 2x(t) + y
\end{aligned}
$$

9.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} & 0 \\ 2t\,\mathrm{e}^{t} & \mathrm{e}^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{t} & 0 \\ 2t\,\mathrm{e}^{t} & \mathrm{e}^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} c_1 \\ 2t\,\mathrm{e}^{t} c_1 + \mathrm{e}^{t} c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} c_1 \\ \mathrm{e}^{t}(2c_1 t + c_2) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(1-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	2	1	Yes	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

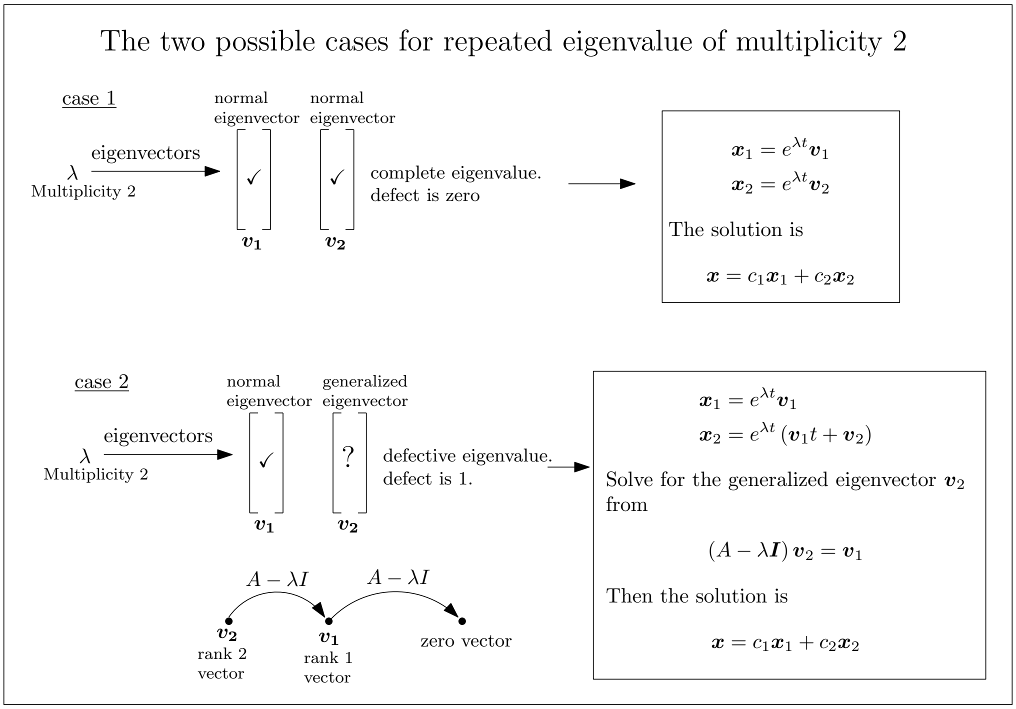


Figure 374: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\
&= \begin{bmatrix} 0 \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right) \mathrm{e}^{t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} \\ \mathrm{e}^{t}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} \\ \mathrm{e}^{t}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_2 \mathrm{e}^{t}}{2} \\ \mathrm{e}^{t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

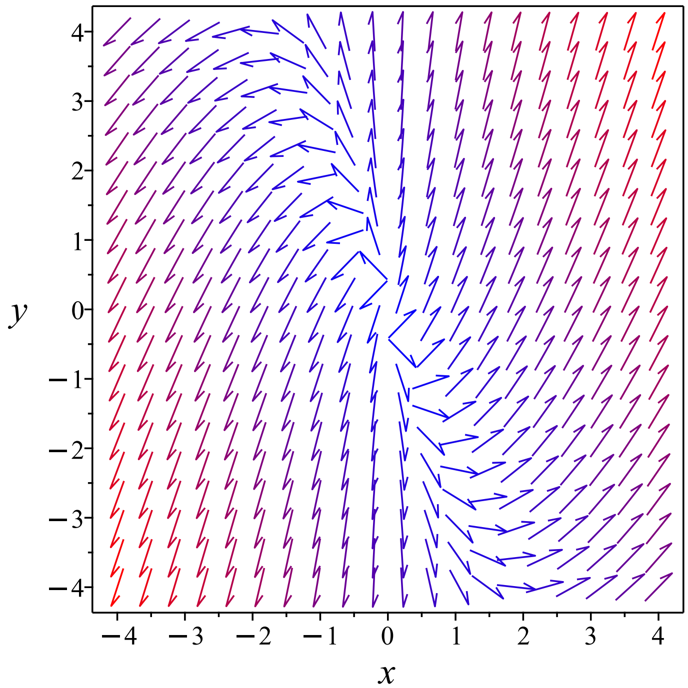


Figure 375: Phase plot

9.3.3 Maple step by step solution

Let's solve

$[x'(t) = x(t), y' = 2x(t) + y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

  $$\overrightarrow{x}_1(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

  $$\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $$\lambda\, \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $$\lambda\, \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

  $$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{x}_2(t) = \mathrm{e}^t \cdot \left(t \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^t \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^t \cdot \left(t \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \mathrm{e}^t (c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = 0, y = \mathrm{e}^t (c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 21

```
dsolve([diff(x(t),t)=x(t),diff(y(t),t)=2*x(t)+y(t)],singsol=all)
```

$$x(t) = c_2 \mathrm{e}^t$$
$$y(t) = (2c_2 t + c_1)\,\mathrm{e}^t$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[{x'[t]==x[t],y'[t]==2*x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^t$$
$$y(t) \to e^t(2c_1 t + c_2)$$

9.4 problem 4

Internal problem ID [13064]
Internal file name [`OUTPUT/11716_Wednesday_November_08_2023_04_49_55_AM_40386715/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t) + 2y$$
$$y' = 2x(t) - y$$

9.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{4t}+1)e^{-3t}}{2} & \frac{(e^{4t}-1)e^{-3t}}{2} \\ \frac{(e^{4t}-1)e^{-3t}}{2} & \frac{(e^{4t}+1)e^{-3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(e^{4t}+1)e^{-3t}}{2} & \frac{(e^{4t}-1)e^{-3t}}{2} \\ \frac{(e^{4t}-1)e^{-3t}}{2} & \frac{(e^{4t}+1)e^{-3t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(e^{4t}+1)e^{-3t}c_1}{2} + \frac{(e^{4t}-1)e^{-3t}c_2}{2} \\ \frac{(e^{4t}-1)e^{-3t}c_1}{2} + \frac{(e^{4t}+1)e^{-3t}c_2}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{((c_1+c_2)e^{4t}+c_1-c_2)e^{-3t}}{2} \\ \frac{e^{-3t}((c_1+c_2)e^{4t}-c_1+c_2)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the

corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\left(-c_2 \mathrm{e}^{4t} + c_1\right) \mathrm{e}^{-3t} \\ \left(c_2 \mathrm{e}^{4t} + c_1\right) \mathrm{e}^{-3t} \end{bmatrix}
$$

The following is the phase plot of the system.

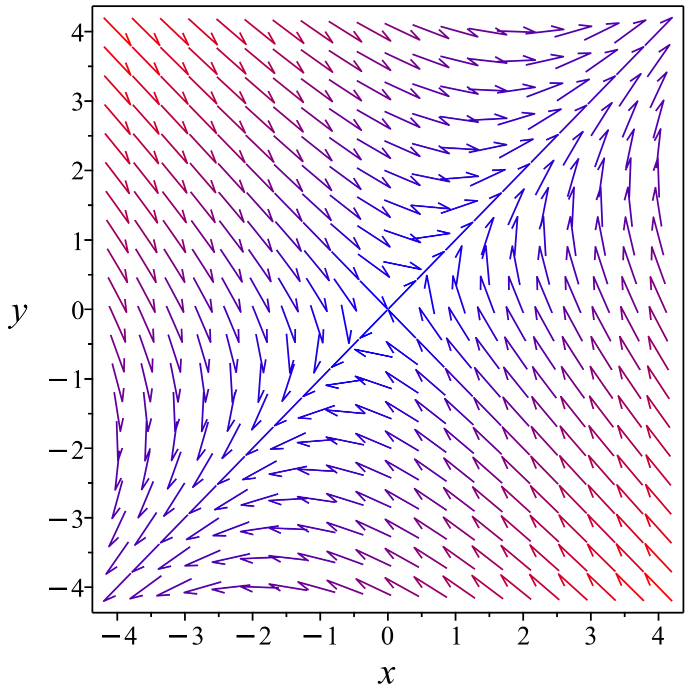


Figure 376: Phase plot

9.4.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + 2y, y' = 2x(t) - y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = e^{-3t} c_1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -(-c_2 e^{4t} + c_1) e^{-3t} \\ (c_2 e^{4t} + c_1) e^{-3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = -(-c_2 e^{4t} + c_1) e^{-3t}, y = (c_2 e^{4t} + c_1) e^{-3t}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)=-x(t)+2*y(t),diff(y(t),t)=2*x(t)-y(t)],singsol=all)
```

$$x(t) = c_1 e^{t} + c_2 e^{-3t}$$
$$y(t) = c_1 e^{t} - c_2 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 68

```
DSolve[{x'[t]==-x[t]+2*y[t],y'[t]==2*x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to \frac{1}{2} e^{-3t} \left(c_1 \left(e^{4t} + 1\right) + c_2 \left(e^{4t} - 1\right)\right)$$
$$y(t) \to \frac{1}{2} e^{-3t} \left(c_1 \left(e^{4t} - 1\right) + c_2 \left(e^{4t} + 1\right)\right)$$

9.5 problem 5

Internal problem ID [13065]
Internal file name [`OUTPUT/11720_Sunday_December_03_2023_07_15_15_PM_64086885/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + y \\
y' &= x(t) + y
\end{aligned}
$$

9.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{\left(5+\sqrt{5}\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} - \frac{e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(-5+\sqrt{5}\right)}{10} & -\frac{\left(-e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}}{5} \\ -\frac{\left(-e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}}{5} & \frac{\left(5-\sqrt{5}\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} + \frac{e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(5+\sqrt{5}\right)}{10} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{(5+\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(-5+\sqrt{5})}{10} & -\frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}}{5} \\ -\frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}}{5} & \frac{(5-\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(5+\sqrt{5})}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{(5+\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(-5+\sqrt{5})}{10}\right) c_1 - \frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}\, c_2}{5} \\ -\frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}\, c_1}{5} + \left(\frac{(5-\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(5+\sqrt{5})}{10}\right) c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{((c_1+2c_2)\sqrt{5}+5c_1)\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}((c_1+2c_2)\sqrt{5}-5c_1)}{10} \\ \frac{((2c_1-c_2)\sqrt{5}+5c_2)\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{\left((c_1-\frac{c_2}{2})\sqrt{5}-\frac{5c_2}{2}\right)e^{-\frac{(\sqrt{5}-3)t}{2}}}{5} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} - \frac{\sqrt{5}}{2}$	1	real eigenvalue
$\frac{3}{2} + \frac{\sqrt{5}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} - \left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{1}{2} + \frac{\sqrt{5}}{2} & 1\\ 1 & \frac{\sqrt{5}}{2} - \frac{1}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{1}{2} + \frac{\sqrt{5}}{2} & 1 & 0\\ 1 & \frac{\sqrt{5}}{2} - \frac{1}{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2}+\frac{\sqrt{5}}{2}} \implies \left[\begin{array}{cc|c} \frac{1}{2}+\frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}+\frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{\sqrt{5}+1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 & 0 \\ 1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2}-\frac{\sqrt{5}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{\sqrt{5}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number

of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{3}{2}+\frac{\sqrt{5}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1\end{array}\right]$
$\frac{3}{2}-\frac{\sqrt{5}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{3}{2}+\frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \\
&= \left[\begin{array}{c}\frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1\end{array}\right] e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $\frac{3}{2}-\frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \\
&= \left[\begin{array}{c}\frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1\end{array}\right] e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \frac{e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t}}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \end{array}\right]
$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1\left(\sqrt{5}+1\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{c_2 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-1\right)}{2} \\ c_1 e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

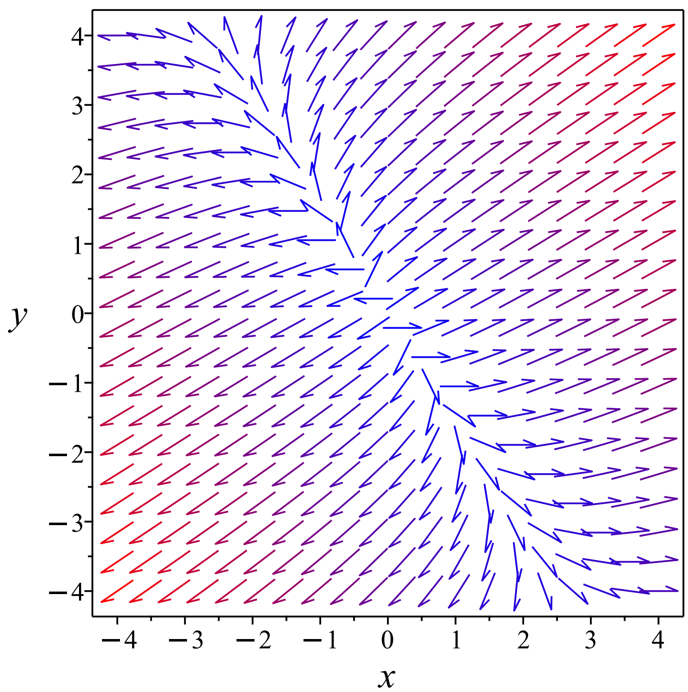


Figure 377: Phase plot

9.5.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + y, y' = x(t) + y]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{3}{2} + \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{3}{2} + \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$
- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{c_2\left(\sqrt{5}+1\right)\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-1\right)}{2} \\ c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{c_2\left(\sqrt{5}+1\right)\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-1\right)}{2}, y = c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=2*x(t)+y(t),diff(y(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}$$
$$y(t) = \frac{c_1 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}\sqrt{5}}{2} - \frac{c_2 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\sqrt{5}}{2} - \frac{c_1 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{c_2 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}}{2}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 145

```
DSolve[{x'[t]==2*x[t]+y[t],y'[t]==x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{10}e^{-\frac{1}{2}\left(\sqrt{5}-3\right)t}\left(c_1\left(\left(5+\sqrt{5}\right)e^{\sqrt{5}t} + 5 - \sqrt{5}\right) + 2\sqrt{5}c_2\left(e^{\sqrt{5}t} - 1\right)\right)$$
$$y(t) \to \frac{1}{10}e^{-\frac{1}{2}\left(\sqrt{5}-3\right)t}\left(2\sqrt{5}c_1\left(e^{\sqrt{5}t} - 1\right) - c_2\left(\left(\sqrt{5}-5\right)e^{\sqrt{5}t} - 5 - \sqrt{5}\right)\right)$$

9.6 problem 6

Internal problem ID [13066]
Internal file name [`OUTPUT/11721_Sunday_December_03_2023_07_15_18_PM_40386715/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3y$$
$$y' = 3\pi y - \frac{x(t)}{3}$$

9.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -\frac{1}{3} & 3\pi \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\frac{3\left(\pi+\frac{\sqrt{9\pi^2-4}}{3}\right)e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}}{2} - \frac{3\,e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}\left(\pi-\frac{\sqrt{9\pi^2-4}}{3}\right)}{2}}{\sqrt{9\pi^2-4}} & -\frac{3\left(-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}+e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}\right)}{\sqrt{9\pi^2-4}} \\ \frac{-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}+e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}}{3\sqrt{9\pi^2-4}} & -\frac{3\left(\left(\pi-\frac{\sqrt{9\pi^2-4}}{3}\right)e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}\left(\pi+\cdots\right)\right)}{2\sqrt{9\pi^2-4}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \left[\begin{array}{cc} \frac{\frac{3\left(\pi+\frac{\sqrt{9\pi^2-4}}{3}\right)e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}}{2}-\frac{3\,e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}\left(\pi-\frac{\sqrt{9\pi^2-4}}{3}\right)}{2}}{\sqrt{9\pi^2-4}} & -\frac{3\left(-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}+e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}\right)}{\sqrt{9\pi^2-4}} \\ \frac{-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}+e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}}{3\sqrt{9\pi^2-4}} & -\frac{3\left(\left(\pi-\frac{\sqrt{9\pi^2-4}}{3}\right)e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}\left(\pi-\right)\right)}{2\sqrt{9\pi^2-4}} \end{array}\right.$$

$$= \left[\begin{array}{c} \frac{3\left(\left(\pi+\frac{\sqrt{9\pi^2-4}}{3}\right)e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}\left(\pi-\frac{\sqrt{9\pi^2-4}}{3}\right)\right)c_1}{2\sqrt{9\pi^2-4}}-\frac{3\left(-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}+e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}\right)c_2}{\sqrt{9\pi^2-4}} \\ \frac{\left(-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}+e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}\right)c_1}{3\sqrt{9\pi^2-4}}-\frac{3\left(\left(\pi-\frac{\sqrt{9\pi^2-4}}{3}\right)e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}\left(\pi+\frac{\sqrt{9\pi^2-4}}{3}\right)\right)c_2}{2\sqrt{9\pi^2-4}} \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{\frac{3\left(c_1\pi+\frac{c_1\sqrt{9\pi^2-4}}{3}-2c_2\right)e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}}{2}-\frac{3\,e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}\left(c_1\pi-\frac{c_1\sqrt{9\pi^2-4}}{3}-2c_2\right)}{2}}{\sqrt{9\pi^2-4}} \\ -\frac{3\left(\left(c_2\pi-\frac{c_2\sqrt{9\pi^2-4}}{3}-\frac{2c_1}{9}\right)e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}-e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}\left(c_2\pi+\frac{c_2\sqrt{9\pi^2-4}}{3}-\frac{2c_1}{9}\right)\right)}{2\sqrt{9\pi^2-4}} \end{array}\right]$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\left[\begin{array}{c} x'(t) \\ y' \end{array}\right] = \left[\begin{array}{cc} 0 & 3 \\ -\frac{1}{3} & 3\pi \end{array}\right]\left[\begin{array}{c} x(t) \\ y \end{array}\right]$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\left[\begin{array}{cc} 0 & 3 \\ -\frac{1}{3} & 3\pi \end{array}\right]-\lambda\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)=0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 3 \\ -\frac{1}{3} & 3\pi - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$-3\pi\lambda + \lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3\pi}{2} - \frac{\sqrt{9\pi^2 - 4}}{2}$$
$$\lambda_2 = \frac{3\pi}{2} + \frac{\sqrt{9\pi^2 - 4}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3\pi}{2} + \frac{\sqrt{9\pi^2-4}}{2}$	1	real eigenvalue
$\frac{3\pi}{2} - \frac{\sqrt{9\pi^2-4}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3\pi}{2} - \frac{\sqrt{9\pi^2-4}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 3 \\ -\frac{1}{3} & 3\pi \end{bmatrix} - \left(\frac{3\pi}{2} - \frac{\sqrt{9\pi^2 - 4}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3\pi}{2} + \frac{\sqrt{9\pi^2-4}}{2} & 3 \\ -\frac{1}{3} & \frac{3\pi}{2} + \frac{\sqrt{9\pi^2-4}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3\pi}{2} + \frac{\sqrt{9\pi^2-4}}{2} & 3 & 0 \\ -\frac{1}{3} & \frac{3\pi}{2} + \frac{\sqrt{9\pi^2-4}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{-\frac{9\pi}{2} + \frac{3\sqrt{9\pi^2-4}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{3\pi}{2} + \frac{\sqrt{9\pi^2-4}}{2} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2} & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{6t}{3\pi-\sqrt{9\pi^2-4}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{6t}{3\pi-\sqrt{9\pi^2-4}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6t}{3\pi-\sqrt{9\pi^2-4}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{6t}{3\pi-\sqrt{9\pi^2-4}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{6}{3\pi-\sqrt{9\pi^2-4}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{6t}{3\pi-\sqrt{9\pi^2-4}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{3\pi-\sqrt{9\pi^2-4}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3\pi}{2} + \frac{\sqrt{9\pi^2-4}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 3 \\ -\frac{1}{3} & 3\pi \end{bmatrix} - \left(\frac{3\pi}{2} + \frac{\sqrt{9\pi^2-4}}{2} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2} & 3 \\ -\frac{1}{3} & \frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2} & 3 & 0 \\ -\frac{1}{3} & \frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{-\frac{9\pi}{2} - \frac{3\sqrt{9\pi^2-4}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{3\pi}{2} - \frac{\sqrt{9\pi^2-4}}{2} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3\pi}{2} - \frac{\sqrt{9\pi^2-4}}{2} & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{6t}{3\pi+\sqrt{9\pi^2-4}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{6t}{3\pi+\sqrt{9\pi^2-4}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6t}{3\pi+\sqrt{9\pi^2-4}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{6t}{3\pi+\sqrt{9\pi^2-4}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{6}{3\pi+\sqrt{9\pi^2-4}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{6t}{3\pi+\sqrt{9\pi^2-4}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{3\pi+\sqrt{9\pi^2-4}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{6t}{3\pi+\sqrt{9\pi^2-4}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{3\pi+\sqrt{9\pi^2-4}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2}$	1	1	No	$\left[\begin{array}{c} \frac{3}{\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2}} \\ 1 \end{array}\right]$
$\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2}$	1	1	No	$\left[\begin{array}{c} \frac{3}{\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2}\right)t} \\
&= \left[\begin{array}{c} \frac{3}{\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2}} \\ 1 \end{array}\right] e^{\left(\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2}\right)t} \\
&= \left[\begin{array}{c} \frac{3}{\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2}} \\ 1 \end{array}\right] e^{\left(\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \frac{3\,\mathrm{e}^{\left(\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2}\right)t}}{\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2}} \\ \mathrm{e}^{\left(\frac{3\pi}{2}-\frac{\sqrt{9\pi^2-4}}{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{3\,\mathrm{e}^{\left(\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2}\right)t}}{\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2}} \\ \mathrm{e}^{\left(\frac{3\pi}{2}+\frac{\sqrt{9\pi^2-4}}{2}\right)t} \end{array}\right]
$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{9c_1\left(\pi+\frac{\sqrt{9\pi^2-4}}{3}\right)e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}}{2} + \frac{9\,e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}c_2\left(\pi-\frac{\sqrt{9\pi^2-4}}{3}\right)}{2} \\ c_1e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}} + c_2e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

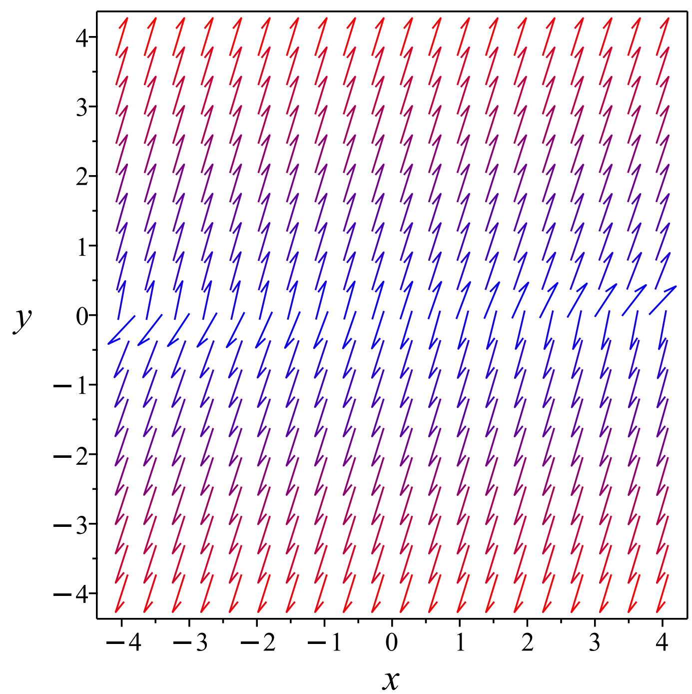


Figure 378: Phase plot

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 120

```
dsolve([diff(x(t),t)=3*y(t),diff(y(t),t)=3*Pi*y(t)-1/3*x(t)],singsol=all)
```

$$x(t) = c_1e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}} + c_2e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}}$$

$$y(t) = \left(\frac{\pi}{2}+\frac{\sqrt{9\pi^2-4}}{6}\right)c_2e^{\frac{\left(3\pi+\sqrt{9\pi^2-4}\right)t}{2}} + \left(\frac{\pi}{2}-\frac{\sqrt{9\pi^2-4}}{6}\right)c_1e^{\frac{\left(3\pi-\sqrt{9\pi^2-4}\right)t}{2}}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 233

```
DSolve[{x'[t]==3*y[t],y'[t]==3*Pi*y[t]-1/3*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to \frac{e^{-\frac{1}{2}\left(\sqrt{9\pi^2-4}-3\pi\right)t}\left(\sqrt{9\pi^2-4}c_1\left(e^{\sqrt{9\pi^2-4}t}+1\right)-3\pi c_1\left(e^{\sqrt{9\pi^2-4}t}-1\right)+6c_2\left(e^{\sqrt{9\pi^2-4}t}-1\right)\right)}{2\sqrt{9\pi^2-4}}$$

$$y(t) \to \frac{e^{-\frac{1}{2}\left(\sqrt{9\pi^2-4}-3\pi\right)t}\left(3c_2\left(3\pi\left(e^{\sqrt{9\pi^2-4}t}-1\right)+\sqrt{9\pi^2-4}\left(e^{\sqrt{9\pi^2-4}t}+1\right)\right)-2c_1\left(e^{\sqrt{9\pi^2-4}t}-1\right)\right)}{6\sqrt{9\pi^2-4}}$$

9.7 problem 7

Internal problem ID [13067]
Internal file name [`OUTPUT/11722_Sunday_December_03_2023_07_15_19_PM_78487314/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
p'(t) &= 3p(t) - 2q(t) - 7r(t) \\
q'(t) &= -2p(t) + 6r(t) \\
r'(t) &= \frac{73q(t)}{100} + 2r(t)
\end{aligned}
$$

9.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as Warning. Unable to find the matrix exponential.

9.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} p'(t) \\ q'(t) \\ r'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & \frac{73}{100} & 2 \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & \frac{73}{100} & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 & -7 \\ -2 & -\lambda & 6 \\ 0 & \frac{73}{100} & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 - \frac{119}{50}\lambda + \frac{273}{25} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{\left(31130 + 6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30} + \frac{1607}{15\left(31130 + 6i\sqrt{895302429}\right)^{\frac{1}{3}}} + \frac{5}{3}$$

$$\lambda_2 = -\frac{\left(31130 + 6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60} - \frac{1607}{30\left(31130 + 6i\sqrt{895302429}\right)^{\frac{1}{3}}} + \frac{5}{3} + \frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30} - \frac{1}{}\right.}{2}$$

$$\lambda_3 = -\frac{\left(31130 + 6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60} - \frac{1607}{30\left(31130 + 6i\sqrt{895302429}\right)^{\frac{1}{3}}} + \frac{5}{3} - \frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30} - \frac{1}{}\right.}{2}$$

This table summarises the above result

eigenvalue	algel
$-\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60}-\frac{1607}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}\right)}{2}$	1
$-\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60}-\frac{1607}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}+\frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}\right)}{2}$	1
$\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}$	1

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & \frac{73}{100} & 2 \end{array}\right]-\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{}{15}\right.\right.$$

$$\left[\begin{array}{ccc} \frac{-\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}+40\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}-3214}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} & -2 & \\ -2 & \frac{-\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}-50\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}-3214}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} & \\ 0 & \frac{73}{100} & \frac{-(}{} \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\begin{bmatrix} \frac{4}{3}-\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} & -2 \\ -2 & -\frac{5}{3}-\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} \\ 0 & \frac{73}{100} \end{bmatrix}$$

$$R_2 = R_2 + \frac{2R_1}{\frac{4}{3}-\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}} \Longrightarrow \begin{bmatrix} \frac{-\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}+40\left(31130+6i\sqrt{89530\cdots}\right.}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 = R_3 - \frac{73\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}\left(\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}-40\left(31130\cdots\right.\right.}{20\left(-i\sqrt{895302429}\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}-10i\sqrt{895302429}-138\left(31130+6i\sqrt{895302\cdots}\right.\right.}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{-\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}+40\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}-3214}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} & -2 \\ 0 & \frac{-i\sqrt{895302429}\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}-10i\sqrt{895302429}-138\left(31130\cdots\right.}{5\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}\left(\left(31130+6i\sqrt{89530242\cdots}\right.\right.} \\ 0 & 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = -\frac{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}} t\left(7i\sqrt{895302429}\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}+43\right.}{\left(i\sqrt{895302429}\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}+10i\sqrt{895302429}+138\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}+\right.} \right.$

Hence the solution is

$$\left[\begin{array}{c} -\frac{30\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}} t\left(7 \mathrm{I}\sqrt{895302429}\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+430 \mathrm{I}\sqrt{895302429}+2766\left(31130+6 \mathrm{I}\sqrt{89530\right.\right.}{\left(\mathrm{I}\sqrt{895302429}\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+10 \mathrm{I}\sqrt{895302429}+138\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+10545\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+17735\right.} \\ \frac{60t\left(3 \mathrm{I}\sqrt{895302429}+15\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+1607\left(31130+6 \mathrm{I}\sqrt{8953024\right.\right.}{\mathrm{I}\sqrt{895302429}\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+10 \mathrm{I}\sqrt{895302429}+138\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+10545\left(\right.} \\ t \end{array}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} -\frac{30\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}} t\left(7 \mathrm{I}\sqrt{895302429}\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+430 \mathrm{I}\sqrt{895302429}+2766\left(31130+6 \mathrm{I}\sqrt{89530\right.\right.}{\left(\mathrm{I}\sqrt{895302429}\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+10 \mathrm{I}\sqrt{895302429}+138\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+10545\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+17735\right.} \\ \frac{60t\left(3 \mathrm{I}\sqrt{895302429}+15\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+1607\left(31130+6 \mathrm{I}\sqrt{8953024\right.\right.}{\mathrm{I}\sqrt{895302429}\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+10 \mathrm{I}\sqrt{895302429}+138\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+10545\left(\right.} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} -\frac{30\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}} t\left(7 \mathrm{I}\sqrt{895302429}\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+430 \mathrm{I}\sqrt{895302429}+2766\left(31130+6 \mathrm{I}\sqrt{89530\right.\right.}{\left(\mathrm{I}\sqrt{895302429}\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+10 \mathrm{I}\sqrt{895302429}+138\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+10545\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+17735\right.} \\ \frac{60t\left(3 \mathrm{I}\sqrt{895302429}+15\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+1607\left(31130+6 \mathrm{I}\sqrt{8953024\right.\right.}{\mathrm{I}\sqrt{895302429}\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+10 \mathrm{I}\sqrt{895302429}+138\left(31130+6 \mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+10545\left(\right.} \\ t \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} -\frac{30\left(31130+6\,\mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}t\left(7\,\mathrm{I}\sqrt{895302429}\left(31130+6\,\mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+430\,\mathrm{I}\sqrt{895302429}+2766\left(31130+6\,\mathrm{I}\sqrt{89530\right.\right.}}{\left(\mathrm{I}\sqrt{895302429}\left(31130+6\,\mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+10\,\mathrm{I}\sqrt{895302429}+138\left(31130+6\,\mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+10545\left(31130+6\,\mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+17735\right.} \\ \frac{60t\left(3\,\mathrm{I}\sqrt{895302429}+15\left(31130+6\,\mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+1607\left(31130+6\,\mathrm{I}\sqrt{8953024\right.\right.}}{\mathrm{I}\sqrt{895302429}\left(31130+6\,\mathrm{I}\sqrt{895302429}\right)^{\frac{1}{3}}+10\,\mathrm{I}\sqrt{895302429}+138\left(31130+6\,\mathrm{I}\sqrt{895302429}\right)^{\frac{2}{3}}+10545\left(\right.} \\ t \end{array}\right.$$

Considering the eigenvalue $\lambda_2 = -\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60} - \frac{1607}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} + \frac{5}{3} - \frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30} - \frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}\right)}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & \frac{73}{100} & 2 \end{array}\right] - \left(-\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60} - \frac{16}{30\left(31130+6\right.}\right.\right.$$

$$\left[\begin{array}{ccc} \frac{4}{3}+\frac{\sqrt{3214}\cos\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)}{30}-\frac{\sqrt{3214}\sin\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)\sqrt{3}}{30} & -2 & \\ -2 & -\frac{5}{3}+\frac{\sqrt{3214}\cos\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)}{30}-\frac{\sqrt{3}}{} & \\ 0 & \frac{73}{100} & \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\left[\begin{array}{c} \frac{4}{3}+\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60}+\frac{1607}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}\right)}{2} \\ -2 \quad\quad -\frac{5}{3}+ \\ 0 \end{array}\right.$$

$$R_2 = R_2 + \frac{2R_1}{\frac{4}{3}+\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60}+\frac{1607}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}\right.}{2}}$$

$$R_3 = R_3 - \frac{73\left(3214+80\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}+\left(1+i\sqrt{3}\right)\left(31130+\right.\right.}{20\left(\left(-i\sqrt{895302429}+10545i\sqrt{3}-3\sqrt{298434143}-10545\right)\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}-177\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{ccc} \frac{4}{3}+\frac{\sqrt{3214}\cos\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)}{30}-\frac{\sqrt{3214}\sin\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)\sqrt{3}}{30} & & \\ 0 & & \frac{\left(-i\sqrt{895302429}+10545i\sqrt{3}-3\sqrt{298434143}-10545\right)}{5\left(3214+80\left(31130+6i\sqrt{8\right.\right.}} \\ 0 & & \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{30t\left(25i\left(31130+6i\sqrt{3}\sqrt{298434143}\right)^{\frac{2}{3}}\sqrt{3}\sqrt{298434143}+3649\right.}{\left(5i\left(31130+6i\sqrt{3}\sqrt{298434143}\right)^{\frac{2}{3}}\sqrt{3}\sqrt{298434143}+299208i\left(31130+6i\sqrt{3}\sqrt{298434143}\right)^{\frac{2}{3}}\sqrt{3}-6\right.}\right.$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

Expression too large to displayExpression too large to display

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

Considering the eigenvalue $\lambda_3 = -\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60} - \frac{1607}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} + \frac{5}{3} + \frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}\right)}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & \frac{73}{100} & 2 \end{array}\right] - \left(-\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60} - \frac{16}{30\left(31130+6i\right.}\right.\right.$$

$$\left[\begin{array}{ccc} \frac{4}{3} + \frac{\sqrt{3214}\cos\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)}{30} + \frac{\sqrt{3214}\sin\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)\sqrt{3}}{30} & -2 \\ -2 & -\frac{5}{3} + \frac{\sqrt{3214}\cos\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)}{30} + \frac{\sqrt{3}}{} \\ 0 & \frac{73}{100} \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc} \frac{4}{3} + \frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60} + \frac{1607}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30} - \frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}\right)}{2} & \\ -2 & -\frac{5}{3} + \\ 0 & \end{array}\right.$$

$$R_2 = R_2 + \frac{2R_1}{\frac{4}{3} + \frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{60} + \frac{1607}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30} - \frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}\right.}{2}}$$

$$R_3 = R_3 - \frac{73\left(-3214 - 80\left(31130 + 6i\sqrt{3}\sqrt{298434143}\right)^{\frac{1}{3}} + \left(i\sqrt{3} - 1\right)\left(3113\right.\right.}{20\left(1773516 + \left(31130 + 6i\sqrt{3}\sqrt{298434143}\right)^{\frac{1}{3}}\left(10545 + i\left(10545 + \sqrt{298434143}\right)\sqrt{3} - 3\sqrt{298\right.\right.}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{4}{3} + \frac{\sqrt{3214}\cos\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)}{30} + \frac{\sqrt{3214}\sin\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)\sqrt{3}}{30} & \\ 0 & \frac{1773516+\left(31130+6i\sqrt{3}\sqrt{298434143}\right)^{\frac{1}{3}}\left(10545+i\left(1\right.\right.}{5\left(-3214-80\left(31130+6i\sqrt{\right.\right.}} \\ 0 & \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{30t\left(10947\sqrt{298434143}\left(31130+6i\sqrt{3}\sqrt{298434143}\right)^{\frac{1}{3}}+3649i\right.}{\left(40+\sqrt{3214}\left(\sin\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)\sqrt{3}+\cos\left(\frac{\arctan\left(\frac{3\sqrt{895302429}}{15565}\right)}{3}\right)\right)\right)\left(15\left(31130+6\right.\right.} \right.$

Hence the solution is

Expression too large to display

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

Expression too large to displayExpression too large to display

Let $t = 1$ the eigenvector becomes

Expression too large to display

Which is normalized to

Expression too large to display

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	alge
$\frac{\left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{1607}{15 \left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}$	
$-\frac{\left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}{60}-\frac{1607}{30 \left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}+\frac{i \sqrt{3}\, \left(\frac{\left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15 \left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}\right)}{2}$	
$-\frac{\left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}{60}-\frac{1607}{30 \left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{i \sqrt{3}\, \left(\frac{\left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1607}{15 \left(31130+6 i \sqrt{895302429}\right)^{\frac{1}{3}}}\right)}{2}$	

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\left[\begin{array}{c} p(t) \\ q(t) \\ r(t) \end{array}\right] = c_1 \left[\begin{array}{c}
\frac{711\,\mathrm{e}^{\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)t}\left(\frac{7\left(31130+6i\sqrt{895302429}\right)\cdots}{30}\cdots\right.}{\left(200\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)^{2}-\frac{1111\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}-\frac{1785377}{15\left(31130+6i\sqrt{895302429}\right)^{\cdots}}\cdots\right.} \\
-\frac{100\,\mathrm{e}^{\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)t}\left(7\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\cdots}}\cdots\right.\right.}{200\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)^{2}-\frac{1111\left(31130+6i\sqrt{89530242\cdots}\right.}{30}\cdots} \\
\mathrm{e}^{\left(\frac{\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}{30}+\frac{1607}{15\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}+\cdots\right.}
\end{array}\right.
$$

Which becomes

Expression too large to display

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 1006

```
dsolve([diff(p(t),t)=3*p(t)-2*q(t)-7*r(t),diff(q(t),t)=-2*p(t)+6*r(t),diff(r(t),t)=73/100*q(
```

$p(t) =$

$$-\frac{\left(-i\sqrt{3}\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}+\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}+96420i\sqrt{3}\left(31130+6i\sqrt{89530242\right.}}{}$$

$$+\frac{\left(-i\sqrt{3}\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}-\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}+96420i\sqrt{3}\left(31130+6i\sqrt{89530242\right.}}{}$$

$$+\frac{\left(\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}-5114\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}-180i\sqrt{895302429}-96420\left(31130+\right.\right.}{2400\left(31130+6i\sqrt{89530242\right.}}$$

$$q(t) = c_1 e^{-\frac{\left(i\sqrt{3}\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}-3214i\sqrt{3}+\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}-100\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}+3214\right)t}{60\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}}$$

$$+ c_2 e^{\frac{\left(i\sqrt{3}\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}-3214i\sqrt{3}-\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}+100\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}-3214\right)t}{60\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}}$$

$$+ c_3 e^{\frac{\left(\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}+50\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}+3214\right)t}{30\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}}}$$

$r(t)$

$$= \frac{\left(i\sqrt{3}\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}-\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}+32140i\sqrt{3}\left(31130+6i\sqrt{895302429}\right)^{\frac{1}{3}}\right.}{}$$

$$-\frac{\left(i\sqrt{3}\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}+\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}+32140i\sqrt{3}\left(31130+6i\sqrt{895302429}\right)\right.}{}$$

$$+\frac{\left(\left(31130+6i\sqrt{895302429}\right)^{\frac{4}{3}}-3114\left(31130+6i\sqrt{895302429}\right)^{\frac{2}{3}}+60i\sqrt{895302429}+32140\left(31130+6\right.\right.}{7200\left(31130+6i\sqrt{89530242\right.}}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 602

```
DSolve[{p'[t]==3*p[t]-2*q[t]-7*r[t],q'[t]==-2*p[t]+6*r[t],r'[t]==73/100*q[t]+2*r[t]},{p[t],q
```

$$p(t) \to -100c_2\text{RootSum}\left[\#1^3 - 500\#1^2 - 23800\#1 + 10920000\&, \frac{2\#1e^{\frac{\#1t}{100}} + 111e^{\frac{\#1t}{100}}}{3\#1^2 - 1000\#1 - 23800}\&\right] - 100c_3\text{RootSum}\left[\#1^3 - 500\#1^2 - 23800\#1 + 10920000\&, \frac{7\#1e^{\frac{\#1t}{100}} + 1200e^{\frac{\#1t}{100}}}{3\#1^2 - 1000\#1 - 23800}\&\right] + c_1\text{RootSum}\left[\#1^3 - 500\#1^2 - 23800\#1 + 10920000\&, \frac{\#1^2e^{\frac{\#1t}{100}} - 200\#1e^{\frac{\#1t}{100}} - 43800e^{\frac{\#1t}{100}}}{3\#1^2 - 1000\#1 - 23800}\&\right]$$

$$q(t) \to -200c_1\text{RootSum}\left[\#1^3 - 500\#1^2 - 23800\#1 + 10920000\&, \frac{\#1e^{\frac{\#1t}{100}} - 200e^{\frac{\#1t}{100}}}{3\#1^2 - 1000\#1 - 23800}\&\right] + 200c_3\text{RootSum}\left[\#1^3 - 500\#1^2 - 23800\#1 + 10920000\&, \frac{3\#1e^{\frac{\#1t}{100}} - 200e^{\frac{\#1t}{100}}}{3\#1^2 - 1000\#1 - 23800}\&\right] + c_2\text{RootSum}\left[\#1^3 - 500\#1^2 - 23800\#1 + 10920000\&, \frac{\#1^2e^{\frac{\#1t}{100}} - 500\#1e^{\frac{\#1t}{100}} + 60000e^{\frac{\#1t}{100}}}{3\#1^2 - 1000\#1 - 23800}\&\right]$$

$$r(t) \to -14600c_1\text{RootSum}\left[\#1^3 - 500\#1^2 - 23800\#1 + 10920000\&, \frac{e^{\frac{\#1t}{100}}}{3\#1^2 - 1000\#1 - 23800}\&\right] + 73c_2\text{RootSum}\left[\#1^3 - 500\#1^2 - 23800\#1 + 10920000\&, \frac{\#1e^{\frac{\#1t}{100}} - 300e^{\frac{\#1t}{100}}}{3\#1^2 - 1000\#1 - 23800}\&\right] + c_3\text{RootSum}\left[\#1^3 - 500\#1^2 - 23800\#1 + 10920000\&, \frac{\#1^2e^{\frac{\#1t}{100}} - 300\#1e^{\frac{\#1t}{100}} - 40000e^{\frac{\#1t}{100}}}{3\#1^2 - 1000\#1 - 23800}\&\right]$$

9.8 problem 8

Internal problem ID [13068]
Internal file name [`OUTPUT/11723_Sunday_December_03_2023_07_16_06_PM_91365529/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -3x(t) + 2\pi y \\
y' &= 4x(t) - y
\end{aligned}
$$

9.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 2\pi \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{\left(\sqrt{1+8\pi}+1\right) e^{-\left(2+\sqrt{1+8\pi}\right)t} + e^{\left(-2+\sqrt{1+8\pi}\right)t}\left(-1+\sqrt{1+8\pi}\right)}{2\sqrt{1+8\pi}} & \frac{\pi\left(e^{\left(-2+\sqrt{1+8\pi}\right)t} - e^{-\left(2+\sqrt{1+8\pi}\right)t}\right)}{\sqrt{1+8\pi}} \\ \frac{2\,e^{\left(-2+\sqrt{1+8\pi}\right)t} - 2\,e^{-\left(2+\sqrt{1+8\pi}\right)t}}{\sqrt{1+8\pi}} & \frac{\left(-1+\sqrt{1+8\pi}\right) e^{-\left(2+\sqrt{1+8\pi}\right)t} + e^{\left(-2+\sqrt{1+8\pi}\right)t}\left(\sqrt{1+8\pi}+1\right)}{2\sqrt{1+8\pi}} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{(\sqrt{1+8\pi}+1)e^{-(2+\sqrt{1+8\pi})t}+e^{(-2+\sqrt{1+8\pi})t}(-1+\sqrt{1+8\pi})}{2\sqrt{1+8\pi}} & \frac{\pi\left(e^{(-2+\sqrt{1+8\pi})t}-e^{-(2+\sqrt{1+8\pi})t}\right)}{\sqrt{1+8\pi}} \\ \frac{2e^{(-2+\sqrt{1+8\pi})t}-2e^{-(2+\sqrt{1+8\pi})t}}{\sqrt{1+8\pi}} & \frac{(-1+\sqrt{1+8\pi})e^{-(2+\sqrt{1+8\pi})t}+e^{(-2+\sqrt{1+8\pi})t}(\sqrt{1+8\pi}+1)}{2\sqrt{1+8\pi}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((\sqrt{1+8\pi}+1)e^{-(2+\sqrt{1+8\pi})t}+e^{(-2+\sqrt{1+8\pi})t}(-1+\sqrt{1+8\pi})\right)c_1}{2\sqrt{1+8\pi}} + \frac{\pi\left(e^{(-2+\sqrt{1+8\pi})t}-e^{-(2+\sqrt{1+8\pi})t}\right)c_2}{\sqrt{1+8\pi}} \\ \frac{2\left(e^{(-2+\sqrt{1+8\pi})t}-e^{-(2+\sqrt{1+8\pi})t}\right)c_1}{\sqrt{1+8\pi}} + \frac{\left((-1+\sqrt{1+8\pi})e^{-(2+\sqrt{1+8\pi})t}+e^{(-2+\sqrt{1+8\pi})t}(\sqrt{1+8\pi}+1)\right)c_2}{2\sqrt{1+8\pi}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(-c_2\pi+\frac{c_1\sqrt{1+8\pi}}{2}+\frac{c_1}{2}\right)e^{-(2+\sqrt{1+8\pi})t}+e^{(-2+\sqrt{1+8\pi})t}\left(c_2\pi+\frac{c_1\sqrt{1+8\pi}}{2}-\frac{c_1}{2}\right)}{\sqrt{1+8\pi}} \\ \frac{2\left(\frac{c_2\sqrt{1+8\pi}}{4}-c_1-\frac{c_2}{4}\right)e^{-(2+\sqrt{1+8\pi})t}+2e^{(-2+\sqrt{1+8\pi})t}\left(\frac{c_2\sqrt{1+8\pi}}{4}+c_1+\frac{c_2}{4}\right)}{\sqrt{1+8\pi}} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 2\pi \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 2\pi \\ 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 2\pi \\ 4 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 8\pi + 4\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2 + \sqrt{1 + 8\pi}$$
$$\lambda_2 = -2 - \sqrt{1 + 8\pi}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-2 + \sqrt{1 + 8\pi}$	1	real eigenvalue
$-2 - \sqrt{1 + 8\pi}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2 - \sqrt{1 + 8\pi}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 2\pi \\ 4 & -1 \end{bmatrix} - \left(-2 - \sqrt{1 + 8\pi}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 + \sqrt{1 + 8\pi} & 2\pi \\ 4 & \sqrt{1 + 8\pi} + 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 + \sqrt{1 + 8\pi} & 2\pi & 0 \\ 4 & \sqrt{1 + 8\pi} + 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{-1 + \sqrt{1 + 8\pi}} \Longrightarrow \left[\begin{array}{cc|c} -1 + \sqrt{1 + 8\pi} & 2\pi & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 + \sqrt{1 + 8\pi} & 2\pi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2\pi t}{-1+\sqrt{1+8\pi}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2\pi t}{-1+\sqrt{1+8\pi}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2\pi t}{-1+\sqrt{1+8\pi}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2\pi t}{-1+\sqrt{1+8\pi}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2\pi}{-1+\sqrt{1+8\pi}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2\pi t}{-1+\sqrt{1+8\pi}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2\pi}{-1+\sqrt{1+8\pi}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2\pi t}{-1+\sqrt{1+8\pi}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2\pi}{-1+\sqrt{1+8\pi}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2 + \sqrt{1+8\pi}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 2\pi \\ 4 & -1 \end{bmatrix} - \left(-2 + \sqrt{1+8\pi}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 - \sqrt{1+8\pi} & 2\pi \\ 4 & 1 - \sqrt{1+8\pi} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 - \sqrt{1+8\pi} & 2\pi & 0 \\ 4 & 1 - \sqrt{1+8\pi} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{-1-\sqrt{1+8\pi}} \Longrightarrow \left[\begin{array}{cc|c} -1-\sqrt{1+8\pi} & 2\pi & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-\sqrt{1+8\pi} & 2\pi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2\pi t}{\sqrt{1+8\pi}+1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2\pi t}{\sqrt{1+8\pi}+1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2\pi t}{\sqrt{1+8\pi}+1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2\pi t}{\sqrt{1+8\pi}+1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2\pi}{\sqrt{1+8\pi}+1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2\pi t}{\sqrt{1+8\pi}+1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2\pi}{\sqrt{1+8\pi}+1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2\pi t}{\sqrt{1+8\pi}+1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2\pi}{\sqrt{1+8\pi}+1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-2+\sqrt{1+8\pi}$	1	1	No	$\begin{bmatrix} \frac{2\pi}{\sqrt{1+8\pi}+1} \\ 1 \end{bmatrix}$
$-2-\sqrt{1+8\pi}$	1	1	No	$\begin{bmatrix} \frac{2\pi}{1-\sqrt{1+8\pi}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-2+\sqrt{1+8\pi}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-2+\sqrt{1+8\pi}\right)t} \\
&= \begin{bmatrix} \frac{2\pi}{\sqrt{1+8\pi}+1} \\ 1 \end{bmatrix} e^{\left(-2+\sqrt{1+8\pi}\right)t}
\end{aligned}
$$

Since eigenvalue $-2-\sqrt{1+8\pi}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-2-\sqrt{1+8\pi}\right)t} \\
&= \begin{bmatrix} \frac{2\pi}{1-\sqrt{1+8\pi}} \\ 1 \end{bmatrix} e^{\left(-2-\sqrt{1+8\pi}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(-2+\sqrt{1+8\pi}\right)t}\pi}{\sqrt{1+8\pi}+1} \\ \mathrm{e}^{\left(-2+\sqrt{1+8\pi}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(-2-\sqrt{1+8\pi}\right)t}\pi}{1-\sqrt{1+8\pi}} \\ \mathrm{e}^{\left(-2-\sqrt{1+8\pi}\right)t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{c_2\left(\sqrt{1+8\pi}+1\right)\mathrm{e}^{-\left(2+\sqrt{1+8\pi}\right)t}}{4} + \frac{c_1 \mathrm{e}^{\left(-2+\sqrt{1+8\pi}\right)t}\left(-1+\sqrt{1+8\pi}\right)}{4} \\ c_1 \mathrm{e}^{\left(-2+\sqrt{1+8\pi}\right)t} + c_2 \mathrm{e}^{-\left(2+\sqrt{1+8\pi}\right)t} \end{bmatrix}
$$

The following is the phase plot of the system.

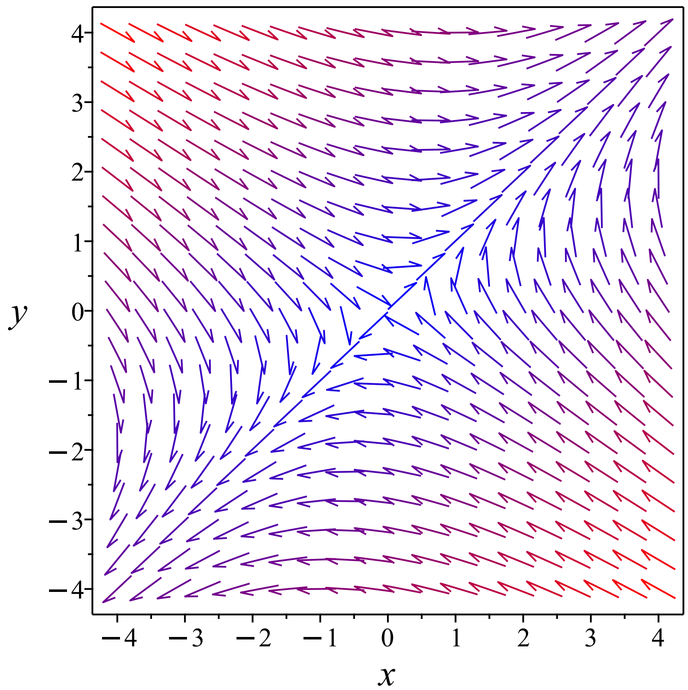


Figure 379: Phase plot

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 119

```
dsolve([diff(x(t),t)=-3*x(t)+2*Pi*y(t),diff(y(t),t)=4*x(t)-y(t)],singsol=all)
```

$$x(t) = c_1 e^{-\left(2+\sqrt{1+8\pi}\right)t} + c_2 e^{\left(-2+\sqrt{1+8\pi}\right)t}$$
$$y(t) = -\frac{c_1 e^{-\left(2+\sqrt{1+8\pi}\right)t}\sqrt{1+8\pi} - c_2 e^{\left(-2+\sqrt{1+8\pi}\right)t}\sqrt{1+8\pi} - c_1 e^{-\left(2+\sqrt{1+8\pi}\right)t} - c_2 e^{\left(-2+\sqrt{1+8\pi}\right)t}}{2\pi}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 189

```
DSolve[{x'[t]==-3*x[t]+2*Pi*y[t],y'[t]==4*x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{e^{-\left(\left(2+\sqrt{1+8\pi}\right)t\right)}\left(c_1\left(\left(\sqrt{1+8\pi}-1\right)e^{2\sqrt{1+8\pi}t}+1+\sqrt{1+8\pi}\right)+2\pi c_2\left(e^{2\sqrt{1+8\pi}t}-1\right)\right)}{2\sqrt{1+8\pi}}$$

$$y(t) \to \frac{e^{-\left(\left(2+\sqrt{1+8\pi}\right)t\right)}\left(4c_1\left(e^{2\sqrt{1+8\pi}t}-1\right)+c_2\left(\left(1+\sqrt{1+8\pi}\right)e^{2\sqrt{1+8\pi}t}-1+\sqrt{1+8\pi}\right)\right)}{2\sqrt{1+8\pi}}$$

9.9 problem 9

Internal problem ID [13069]
Internal file name [`OUTPUT/11724_Sunday_December_03_2023_07_16_07_PM_53256305/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \beta y \\
y' &= \gamma x(t) - y
\end{aligned}
$$

9.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ \gamma & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{\left(1+\sqrt{4\gamma\beta+1}\right)e^{\frac{\left(-1+\sqrt{4\gamma\beta+1}\right)t}{2}}+e^{-\frac{\left(1+\sqrt{4\gamma\beta+1}\right)t}{2}}\left(-1+\sqrt{4\gamma\beta+1}\right)}{2\sqrt{4\gamma\beta+1}} & \frac{\beta\left(e^{\frac{\left(-1+\sqrt{4\gamma\beta+1}\right)t}{2}}-e^{-\frac{\left(1+\sqrt{4\gamma\beta+1}\right)t}{2}}\right)}{\sqrt{4\gamma\beta+1}} \\ \frac{\gamma\left(e^{\frac{\left(-1+\sqrt{4\gamma\beta+1}\right)t}{2}}-e^{-\frac{\left(1+\sqrt{4\gamma\beta+1}\right)t}{2}}\right)}{\sqrt{4\gamma\beta+1}} & \frac{\left(-1+\sqrt{4\gamma\beta+1}\right)e^{\frac{\left(-1+\sqrt{4\gamma\beta+1}\right)t}{2}}+e^{-\frac{\left(1+\sqrt{4\gamma\beta+1}\right)t}{2}}\left(1+\sqrt{4\gamma\beta+1}\right)}{2\sqrt{4\gamma\beta+1}} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{cc}
\frac{(1+\sqrt{4\gamma\beta+1})\,\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}+\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}(-1+\sqrt{4\gamma\beta+1})}{2\sqrt{4\gamma\beta+1}} & \frac{\beta\left(\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}-\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}\right)}{\sqrt{4\gamma\beta+1}} \\
\frac{\gamma\left(\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}-\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}\right)}{\sqrt{4\gamma\beta+1}} & \frac{(-1+\sqrt{4\gamma\beta+1})\,\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}+\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}(1+\sqrt{4\gamma\beta+1})}{2\sqrt{4\gamma\beta+1}}
\end{array}\right] \\
&= \left[\begin{array}{c}
\frac{\left((1+\sqrt{4\gamma\beta+1})\,\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}+\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}(-1+\sqrt{4\gamma\beta+1})\right)c_1}{2\sqrt{4\gamma\beta+1}}+\frac{\beta\left(\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}-\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}\right)c_2}{\sqrt{4\gamma\beta+1}} \\
\frac{\gamma\left(\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}-\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}\right)c_1}{\sqrt{4\gamma\beta+1}}+\frac{\left((-1+\sqrt{4\gamma\beta+1})\,\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}+\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}(1+\sqrt{4\gamma\beta+1})\right)c_2}{2\sqrt{4\gamma\beta+1}}
\end{array}\right] \\
&= \left[\begin{array}{c}
\frac{\left(\beta c_2+\frac{c_1\sqrt{4\gamma\beta+1}}{2}+\frac{c_1}{2}\right)\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}-\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}\left(\beta c_2-\frac{c_1\sqrt{4\gamma\beta+1}}{2}+\frac{c_1}{2}\right)}{\sqrt{4\gamma\beta+1}} \\
\frac{\left(\gamma c_1+\frac{c_2\sqrt{4\gamma\beta+1}}{2}-\frac{c_2}{2}\right)\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}}-\left(\gamma c_1-\frac{c_2\sqrt{4\gamma\beta+1}}{2}-\frac{c_2}{2}\right)\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}}{\sqrt{4\gamma\beta+1}}
\end{array}\right]
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ \gamma & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & \beta \\ \gamma & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & \beta \\ \gamma & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$-\gamma\beta + \lambda^2 + \lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{4\gamma\beta+1}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{\sqrt{4\gamma\beta+1}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2} + \frac{\sqrt{4\gamma\beta+1}}{2}$	1	real eigenvalue
$-\frac{1}{2} - \frac{\sqrt{4\gamma\beta+1}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2} - \frac{\sqrt{4\gamma\beta+1}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & \beta \\ \gamma & -1 \end{bmatrix} - \left(-\frac{1}{2} - \frac{\sqrt{4\gamma\beta+1}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{2} + \frac{\sqrt{4\gamma\beta+1}}{2} & \beta \\ \gamma & -\frac{1}{2} + \frac{\sqrt{4\gamma\beta+1}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} + \frac{\sqrt{4\gamma\beta+1}}{2} & \beta & 0 \\ \gamma & -\frac{1}{2} + \frac{\sqrt{4\gamma\beta+1}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{\gamma R_1}{\frac{1}{2} + \frac{\sqrt{4\gamma\beta+1}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} + \frac{\sqrt{4\gamma\beta+1}}{2} & \beta & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2} & \beta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2\beta t}{1+\sqrt{4\gamma\beta+1}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2\beta t}{1+\sqrt{4\gamma\beta+1}} \\ t \end{bmatrix}=\begin{bmatrix} -\frac{2\beta t}{1+\sqrt{4\gamma\beta+1}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2\beta t}{1+\sqrt{4\gamma\beta+1}} \\ t \end{bmatrix}=t\begin{bmatrix} -\frac{2\beta}{1+\sqrt{4\gamma\beta+1}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2\beta t}{1+\sqrt{4\gamma\beta+1}} \\ t \end{bmatrix}=\begin{bmatrix} -\frac{2\beta}{1+\sqrt{4\gamma\beta+1}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & \beta \\ \gamma & -1 \end{bmatrix}-\left(-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2} & \beta \\ \gamma & -\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2} & \beta & 0 \\ \gamma & -\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{\gamma R_1}{\frac{1}{2} - \frac{\sqrt{4\gamma\beta+1}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} - \frac{\sqrt{4\gamma\beta+1}}{2} & \beta & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} - \frac{\sqrt{4\gamma\beta+1}}{2} & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2\beta t}{-1+\sqrt{4\gamma\beta+1}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2\beta t}{-1+\sqrt{4\gamma\beta+1}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2\beta t}{-1+\sqrt{4\gamma\beta+1}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2\beta t}{-1+\sqrt{4\gamma\beta+1}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2\beta}{-1+\sqrt{4\gamma\beta+1}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2\beta t}{-1+\sqrt{4\gamma\beta+1}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2\beta}{-1+\sqrt{4\gamma\beta+1}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}$	1	1	No	$\left[\begin{array}{c} \frac{\beta}{-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}} \\ 1 \end{array}\right]$
$-\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2}$	1	1	No	$\left[\begin{array}{c} \frac{\beta}{-\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}\right)t} \\
&= \left[\begin{array}{c} \frac{\beta}{-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}} \\ 1 \end{array}\right] e^{\left(-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $-\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2}\right)t} \\
&= \left[\begin{array}{c} \frac{\beta}{-\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2}} \\ 1 \end{array}\right] e^{\left(-\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \frac{\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}\right)t}\beta}{-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{4\gamma\beta+1}}{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2}\right)t}\beta}{-\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{4\gamma\beta+1}}{2}\right)t} \end{array}\right]
$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1(1+\sqrt{4\gamma\beta+1})\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}} - c_2\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}}(-1+\sqrt{4\gamma\beta+1})}{2\gamma} \\ c_1\mathrm{e}^{\frac{(-1+\sqrt{4\gamma\beta+1})t}{2}} + c_2\mathrm{e}^{-\frac{(1+\sqrt{4\gamma\beta+1})t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 119

```
dsolve([diff(x(t),t)=beta*y(t),diff(y(t),t)=gamma*x(t)-y(t)],singsol=all)
```

$$x(t) = c_1\mathrm{e}^{\frac{(-1+\sqrt{4\beta\gamma+1})t}{2}} + c_2\mathrm{e}^{-\frac{(1+\sqrt{4\beta\gamma+1})t}{2}}$$

$$y(t) = \frac{\left(-\frac{1}{2}+\frac{\sqrt{4\beta\gamma+1}}{2}\right)c_1\mathrm{e}^{\frac{(-1+\sqrt{4\beta\gamma+1})t}{2}}}{\beta} + \frac{\left(-\frac{\mathrm{e}^{-\frac{(1+\sqrt{4\beta\gamma+1})t}{2}}\sqrt{4\beta\gamma+1}}{2} - \frac{\mathrm{e}^{-\frac{(1+\sqrt{4\beta\gamma+1})t}{2}}}{2}\right)c_2}{\beta}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 202

```
DSolve[{x'[t]==\[Beta]*y[t],y'[t]==\[Gamma]*x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolution
```

$$x(t) \to \frac{e^{-\frac{1}{2}t(\sqrt{4\beta\gamma+1}+1)}\left(c_1\left(\sqrt{4\beta\gamma+1}+\left(\sqrt{4\beta\gamma+1}+1\right)e^{t\sqrt{4\beta\gamma+1}}-1\right)+2\beta c_2\left(e^{t\sqrt{4\beta\gamma+1}}-1\right)\right)}{2\sqrt{4\beta\gamma+1}}$$

$$y(t) \to \frac{e^{-\frac{1}{2}t(\sqrt{4\beta\gamma+1}+1)}\left(2\gamma c_1\left(e^{t\sqrt{4\beta\gamma+1}}-1\right)+c_2\left(\sqrt{4\beta\gamma+1}+\left(\sqrt{4\beta\gamma+1}-1\right)e^{t\sqrt{4\beta\gamma+1}}+1\right)\right)}{2\sqrt{4\beta\gamma+1}}$$

9.10 problem 24

Internal problem ID [13070]
Internal file name [`OUTPUT/11725_Sunday_December_03_2023_07_16_07_PM_4428554/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 2y \\ y' &= x(t) + y \end{aligned}$$

With initial conditions

$$[x(0) = -2, y(0) = -1]$$

9.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,e^{-t}}{3} + \frac{e^{2t}}{3} & \frac{2\,e^{2t}}{3} - \frac{2\,e^{-t}}{3} \\ \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{e^{-t}}{3} + \frac{2\,e^{2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} & \frac{2\,\mathrm{e}^{2t}}{3}-\frac{2\,\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{-t}}{3}+\frac{2\,\mathrm{e}^{2t}}{3} \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2\,\mathrm{e}^{-t}}{3}-\frac{4\,\mathrm{e}^{2t}}{3} \\ -\frac{4\,\mathrm{e}^{2t}}{3}+\frac{\mathrm{e}^{-t}}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -2\,e^{-t} \\ e^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} - 2c_2 e^{-t} \\ c_1 e^{2t} + c_2 e^{-t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = -2 \\ y(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 - 2c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -\frac{4}{3} \\ c_2 = \frac{1}{3} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{2\,e^{-t}}{3} - \frac{4\,e^{2t}}{3} \\ -\frac{4\,e^{2t}}{3} + \frac{e^{-t}}{3} \end{bmatrix}$$

The following is the phase plot of the system.

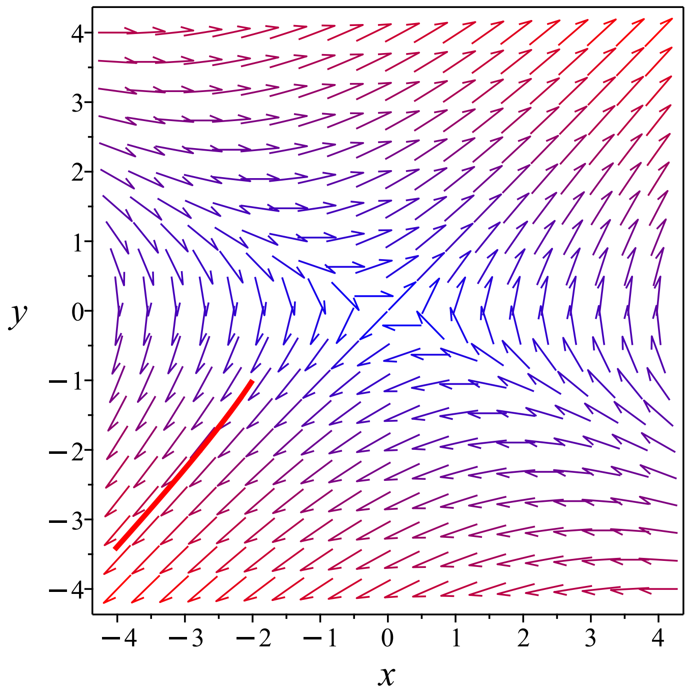
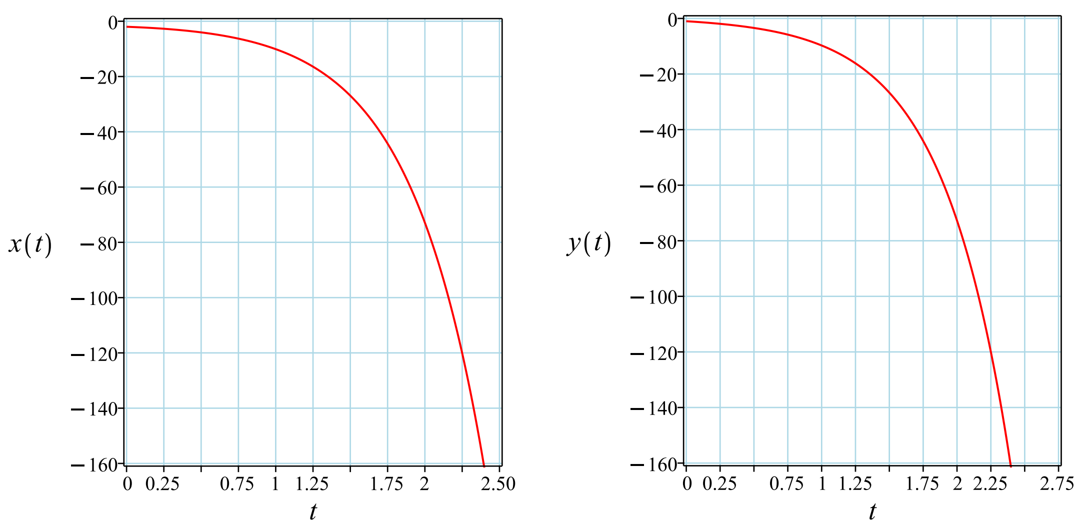


Figure 380: Phase plot

The following are plots of each solution.



✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = 2*y(t), diff(y(t),t) = x(t)+y(t), x(0) = -2, y(0) = -1], singsol=all)
```

$$x(t) = -\frac{2\,\mathrm{e}^{-t}}{3} - \frac{4\,\mathrm{e}^{2t}}{3}$$
$$y(t) = \frac{\mathrm{e}^{-t}}{3} - \frac{4\,\mathrm{e}^{2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 44

```
DSolve[{x'[t]==2*y[t],y'[t]==x[t]+y[t]},{x[0]==-2,y[0]==-1},{x[t],y[t]},t,IncludeSingularSol
```

$$x(t) \to -\frac{2}{3}e^{-t}\left(2e^{3t}+1\right)$$
$$y(t) \to \frac{1}{3}e^{-t}\left(1-4e^{3t}\right)$$

9.11 problem 25

Internal problem ID [13071]
Internal file name [OUTPUT/11726_Sunday_December_03_2023_07_16_08_PM_81532362/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) - y \\
y' &= x(t) + 3y
\end{aligned}
$$

With initial conditions

$$
[x(0) = 0, y(0) = 2]
$$

9.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{2t}(1-t) & -\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}t & \mathrm{e}^{2t}(1+t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{x}_0\\ &= \begin{bmatrix} e^{2t}(1-t) & -e^{2t}t \\ e^{2t}t & e^{2t}(1+t) \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}\\ &= \begin{bmatrix} -2\,e^{2t}t \\ 2\,e^{2t}(1+t) \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

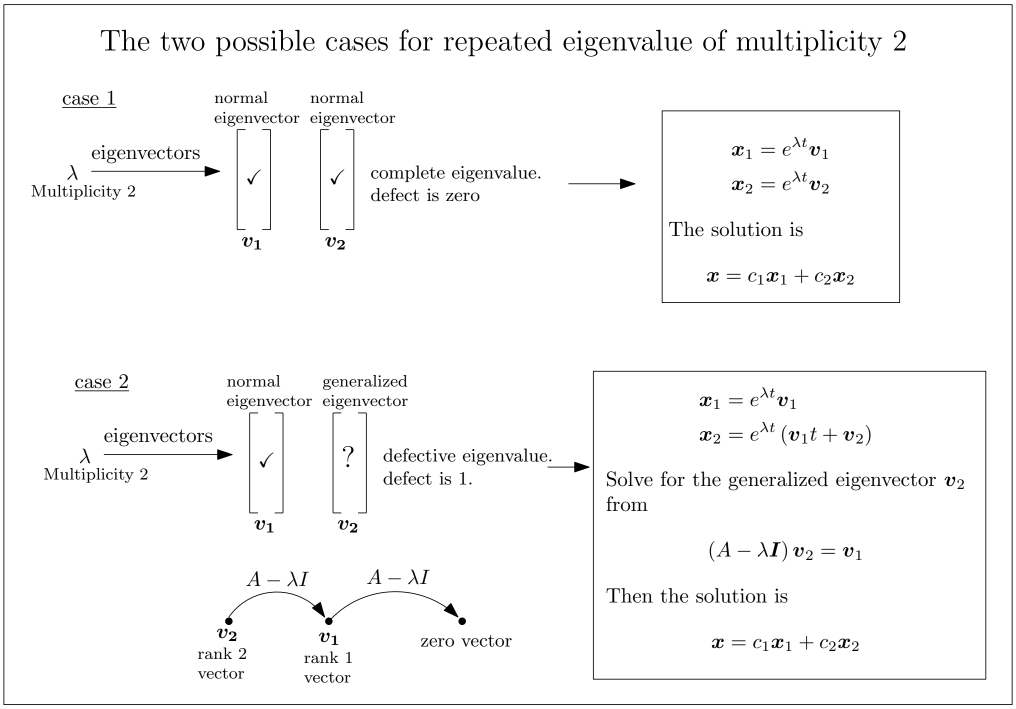


Figure 381: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\ &= \begin{bmatrix} -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(1+t) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{2t}(-t c_2 - c_1) \\ \mathrm{e}^{2t}(t c_2 + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 0 \\ c_2 = 2 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -2\,\mathrm{e}^{2t} t \\ \mathrm{e}^{2t}(2t + 2) \end{bmatrix}$$

The following is the phase plot of the system.

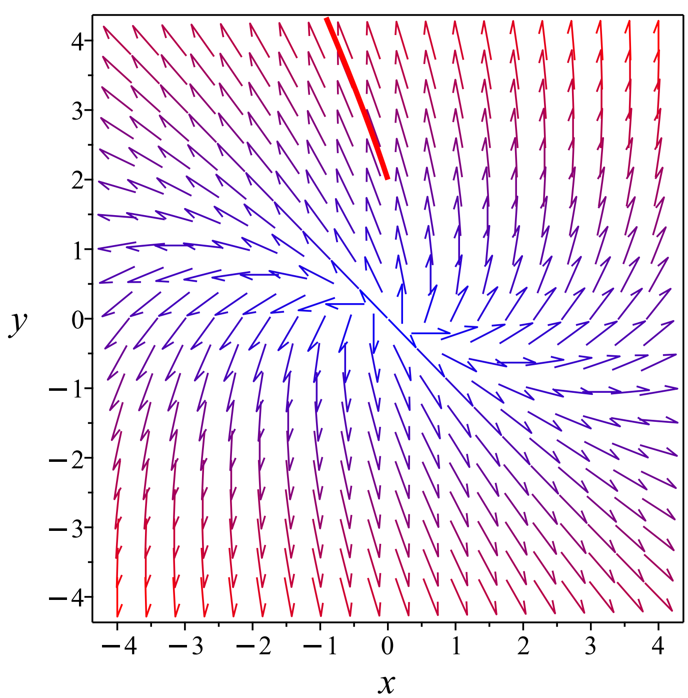
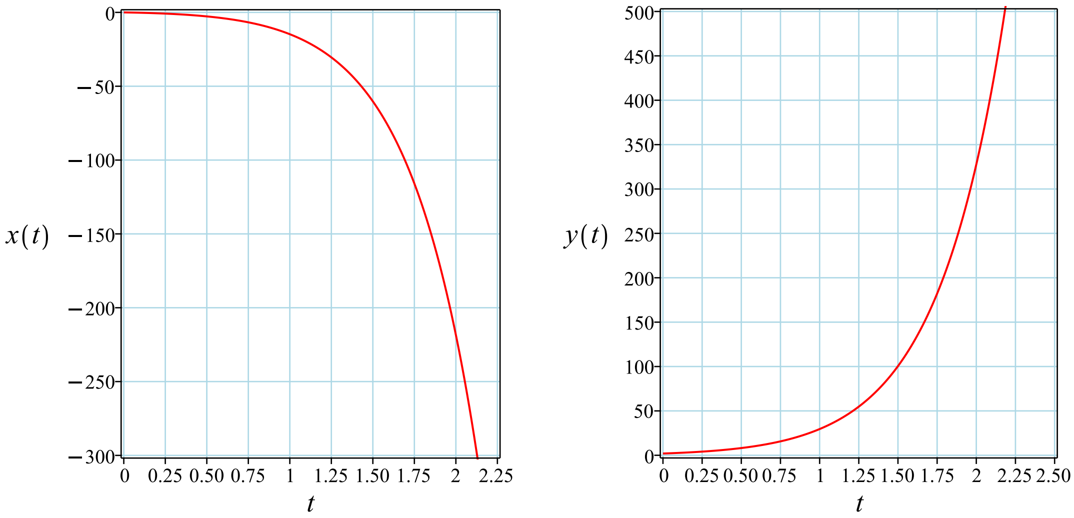


Figure 382: Phase plot

The following are plots of each solution.



✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve([diff(x(t),t) = x(t)-y(t), diff(y(t),t) = x(t)+3*y(t), x(0) = 0, y(0) = 2], singsol=a
```

$$x(t) = -2\,e^{2t}t$$
$$y(t) = -e^{2t}(-2t-2)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 26

```
DSolve[{x'[t]==x[t]-y[t],y'[t]==x[t]+3*y[t]},{x[0]==0,y[0]==2},{x[t],y[t]},t,IncludeSingular
```

$$x(t) \to -2e^{2t}t$$
$$y(t) \to 2e^{2t}(t+1)$$

9.12 problem 26

Internal problem ID [13072]
Internal file name [OUTPUT/11727_Sunday_December_03_2023_07_16_08_PM_80783439/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) - y \\
y' &= 2x(t) - 5y
\end{aligned}
$$

With initial conditions

$$[x(0) = 2, y(0) = 3]$$

9.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\mathrm{e}^{-4t} + 2\,\mathrm{e}^{-3t} & -\mathrm{e}^{-3t} + \mathrm{e}^{-4t} \\ 2\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-4t} & 2\,\mathrm{e}^{-4t} - \mathrm{e}^{-3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} -\mathrm{e}^{-4t}+2\,\mathrm{e}^{-3t} & -\mathrm{e}^{-3t}+\mathrm{e}^{-4t} \\ 2\,\mathrm{e}^{-3t}-2\,\mathrm{e}^{-4t} & 2\,\mathrm{e}^{-4t}-\mathrm{e}^{-3t} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-4t}+\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}+2\,\mathrm{e}^{-4t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -1 \\ 2 & -5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 7\lambda + 12 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
-4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -4 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-4t}$$
$$= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-4t}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{-4t}}{2} \\ e^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-3t} \\ e^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-4t}}{2} + c_2 e^{-3t} \\ c_1 e^{-4t} + c_2 e^{-3t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 2 \\ y(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{2} + c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 2 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} e^{-4t} + e^{-3t} \\ e^{-3t} + 2\,e^{-4t} \end{bmatrix}$$

The following is the phase plot of the system.

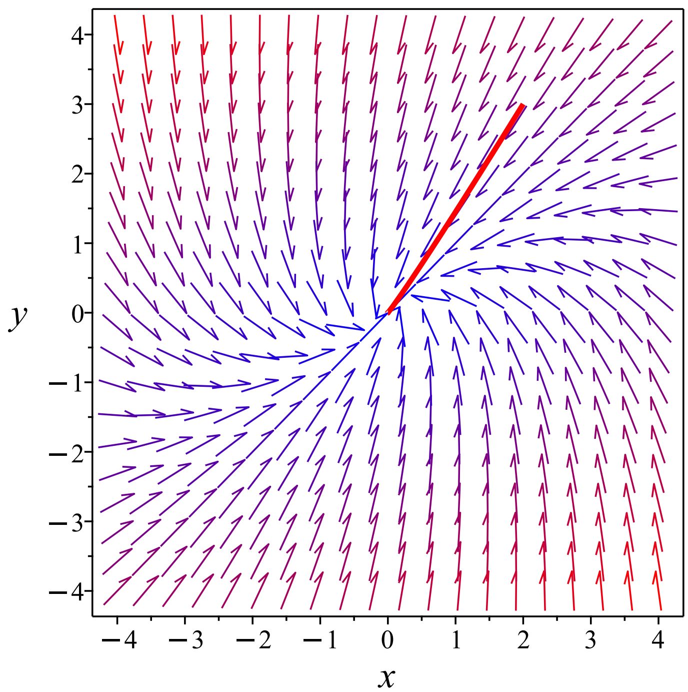
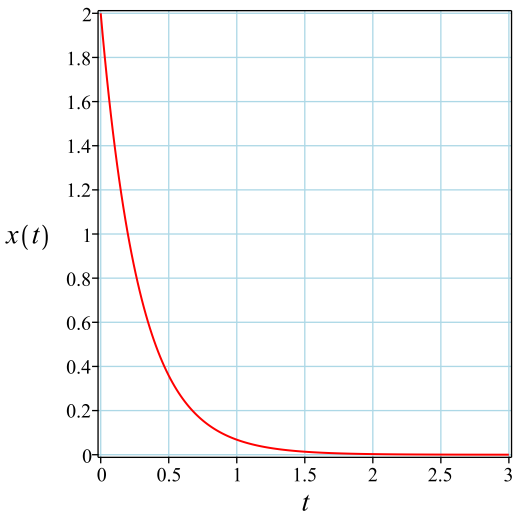
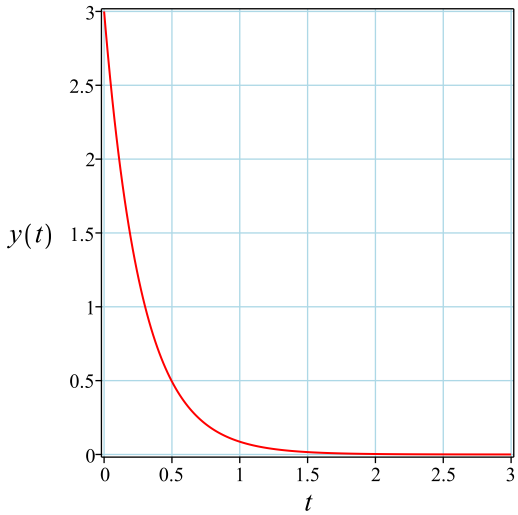


Figure 383: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 28

```
dsolve([diff(x(t),t) = -2*x(t)-y(t), diff(y(t),t) = 2*x(t)-5*y(t), x(0) = 2, y(0) = 3], sing
```

$$x(t) = e^{-4t} + e^{-3t}$$
$$y(t) = 2\,e^{-4t} + e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 30

```
DSolve[{x'[t]==-2*x[t]-y[t],y'[t]==2*x[t]-5*y[t]},{x[0]==2,y[0]==3},{x[t],y[t]},t,IncludeSin
```

$$x(t) \to e^{-4t}(e^t + 1)$$
$$y(t) \to e^{-4t}(e^t + 2)$$

9.13 problem 28

Internal problem ID [13073]
Internal file name [`OUTPUT/11728_Sunday_December_03_2023_07_16_08_PM_97356223/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) - 3y \\
y' &= 3x(t) - 2y
\end{aligned}
$$

With initial conditions

$$[x(0) = 2, y(0) = 3]$$

9.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-2t}\cos(3t) & -\mathrm{e}^{-2t}\sin(3t) \\ \mathrm{e}^{-2t}\sin(3t) & \mathrm{e}^{-2t}\cos(3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-2t}\cos(3t) & -e^{-2t}\sin(3t) \\ e^{-2t}\sin(3t) & e^{-2t}\cos(3t) \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2\,e^{-2t}\cos(3t) - 3\,e^{-2t}\sin(3t) \\ 2\,e^{-2t}\sin(3t) + 3\,e^{-2t}\cos(3t) \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(2\cos(3t) - 3\sin(3t)) \\ e^{-2t}(2\sin(3t) + 3\cos(3t)) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 13 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2 + 3i$$
$$\lambda_2 = -2 - 3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-2 - 3i$	1	complex eigenvalue
$-2 + 3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2 - 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} - (-2 - 3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3i & -3 \\ 3 & 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3i & -3 & 0 \\ 3 & 3i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 3i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3i & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2 + 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} - (-2+3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -3i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3i & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-2+3i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-2-3i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{(-2+3i)t} \\ \mathrm{e}^{(-2+3i)t} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{(-2-3i)t} \\ \mathrm{e}^{(-2-3i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -i\left(c_2\mathrm{e}^{(-2-3i)t} - c_1\mathrm{e}^{(-2+3i)t}\right) \\ c_1\mathrm{e}^{(-2+3i)t} + c_2\mathrm{e}^{(-2-3i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 2 \\ y(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} i(c_1 - c_2) \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{3}{2} - i \\ c_2 = \frac{3}{2} + i \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -i\left(\left(\frac{3}{2} + i\right)\mathrm{e}^{(-2-3i)t} + \left(-\frac{3}{2} + i\right)\mathrm{e}^{(-2+3i)t}\right) \\ \left(\frac{3}{2} - i\right)\mathrm{e}^{(-2+3i)t} + \left(\frac{3}{2} + i\right)\mathrm{e}^{(-2-3i)t} \end{bmatrix}$$

The following is the phase plot of the system.

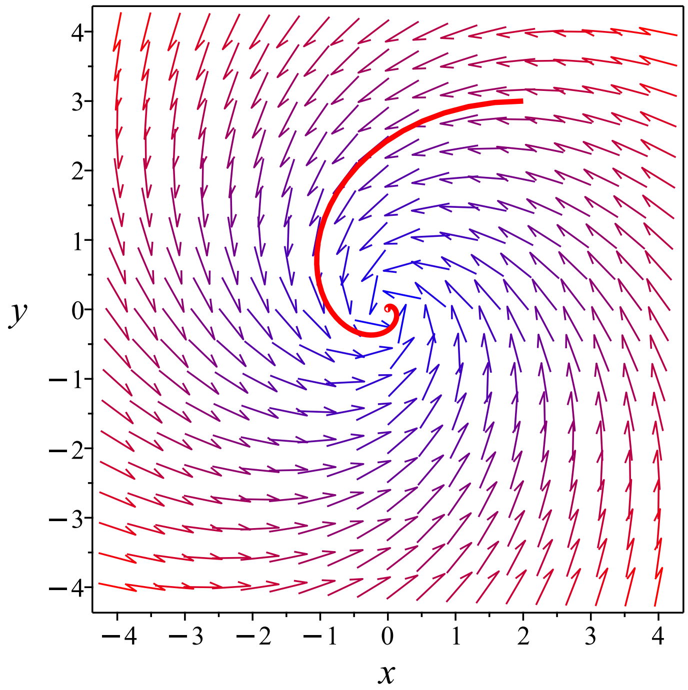


Figure 384: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
dsolve([diff(x(t),t) = -2*x(t)-3*y(t), diff(y(t),t) = 3*x(t)-2*y(t), x(0) = 2, y(0) = 3], si
```

$$x(t) = e^{-2t}(-3\sin(3t) + 2\cos(3t))$$
$$y(t) = -e^{-2t}(-3\cos(3t) - 2\sin(3t))$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 46

```
DSolve[{x'[t]==-2*x[t]-3*y[t],y'[t]==3*x[t]-2*y[t]},{x[0]==2,y[0]==3},{x[t],y[t]},t,IncludeS
```

$$x(t) \to e^{-2t}(2\cos(3t) - 3\sin(3t))$$
$$y(t) \to e^{-2t}(2\sin(3t) + 3\cos(3t))$$

9.14 problem 29

Internal problem ID [13074]
Internal file name [`OUTPUT/11729_Sunday_December_03_2023_07_16_09_PM_44482920/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + 3y \\
y' &= x(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = 2, y(0) = 3]
$$

9.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{e^{-t}}{4} + \frac{3\,e^{3t}}{4} & \frac{3\,e^{3t}}{4} - \frac{3\,e^{-t}}{4} \\ \frac{e^{3t}}{4} - \frac{e^{-t}}{4} & \frac{3\,e^{-t}}{4} + \frac{e^{3t}}{4} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$
$$= \begin{bmatrix} \frac{e^{-t}}{4} + \frac{3\,e^{3t}}{4} & \frac{3\,e^{3t}}{4} - \frac{3\,e^{-t}}{4} \\ \frac{e^{3t}}{4} - \frac{e^{-t}}{4} & \frac{3\,e^{-t}}{4} + \frac{e^{3t}}{4} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{7\,e^{-t}}{4} + \frac{15\,e^{3t}}{4} \\ \frac{5\,e^{3t}}{4} + \frac{7\,e^{-t}}{4} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

9.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 3 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 3\,e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 3c_1 e^{3t} - c_2 e^{-t} \\ c_1 e^{3t} + c_2 e^{-t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 2 \\ y(0) = 3 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3c_1 - c_2 \\ c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$
\begin{bmatrix} c_1 = \frac{5}{4} \\ c_2 = \frac{7}{4} \end{bmatrix}
$$

Substituting these constants back in original solution in Eq. (1) gives

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{7\,e^{-t}}{4} + \frac{15\,e^{3t}}{4} \\ \frac{5\,e^{3t}}{4} + \frac{7\,e^{-t}}{4} \end{bmatrix}
$$

The following is the phase plot of the system.

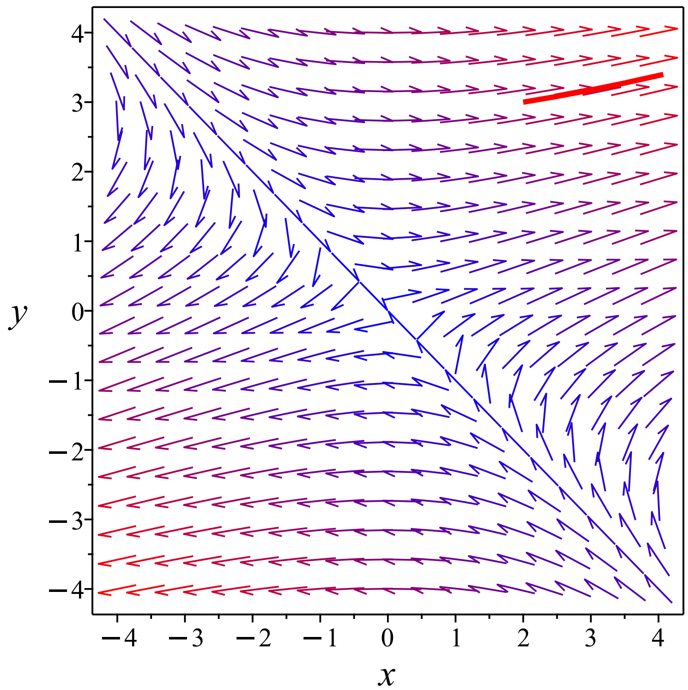
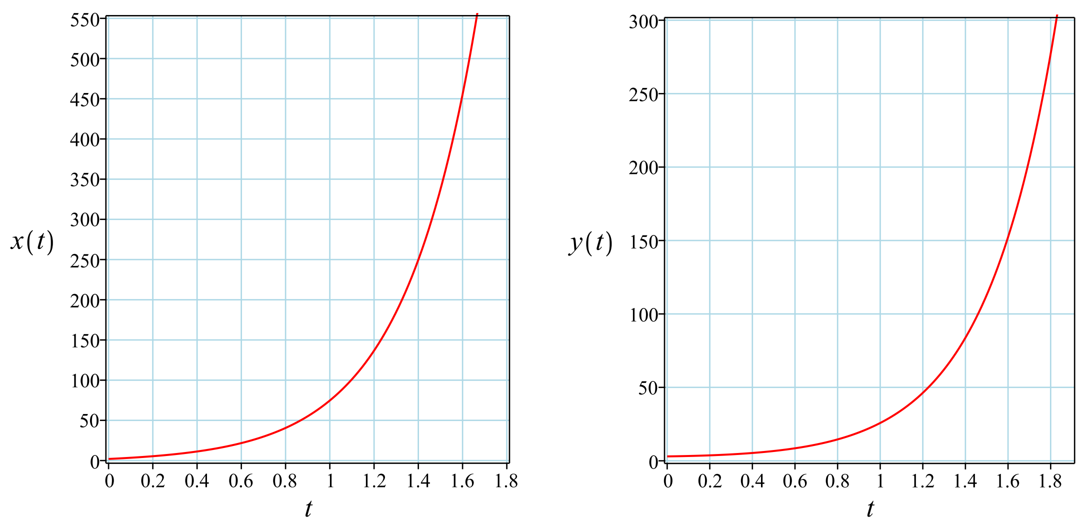


Figure 385: Phase plot

The following are plots of each solution.



✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = 2*x(t)+3*y(t), diff(y(t),t) = x(t), x(0) = 2, y(0) = 3], singsol=all)
```

$$x(t) = \frac{15\,e^{3t}}{4} - \frac{7\,e^{-t}}{4}$$
$$y(t) = \frac{5\,e^{3t}}{4} + \frac{7\,e^{-t}}{4}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 44

```
DSolve[{x'[t]==2*x[t]+3*y[t],y'[t]==x[t]},{x[0]==2,y[0]==3},{x[t],y[t]},t,IncludeSingularSol
```

$$x(t) \to \frac{1}{4}e^{-t}\left(15e^{4t} - 7\right)$$
$$y(t) \to \frac{1}{4}e^{-t}\left(5e^{4t} + 7\right)$$

9.15 problem 34

Internal problem ID [13075]
Internal file name [OUTPUT/11730_Sunday_December_03_2023_07_16_09_PM_26869848/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.1. page 258
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 1 \\
y' &= x(t)
\end{aligned}
$$

9.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} c_1 \\ tc_1 + c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \int \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt \\ &= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} t \\ -\frac{t^2}{2} \end{bmatrix} \\ &= \begin{bmatrix} t \\ \frac{t^2}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} c_1 + t \\ tc_1 + c_2 + \frac{1}{2}t^2 \end{bmatrix} \end{aligned}$$

9.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

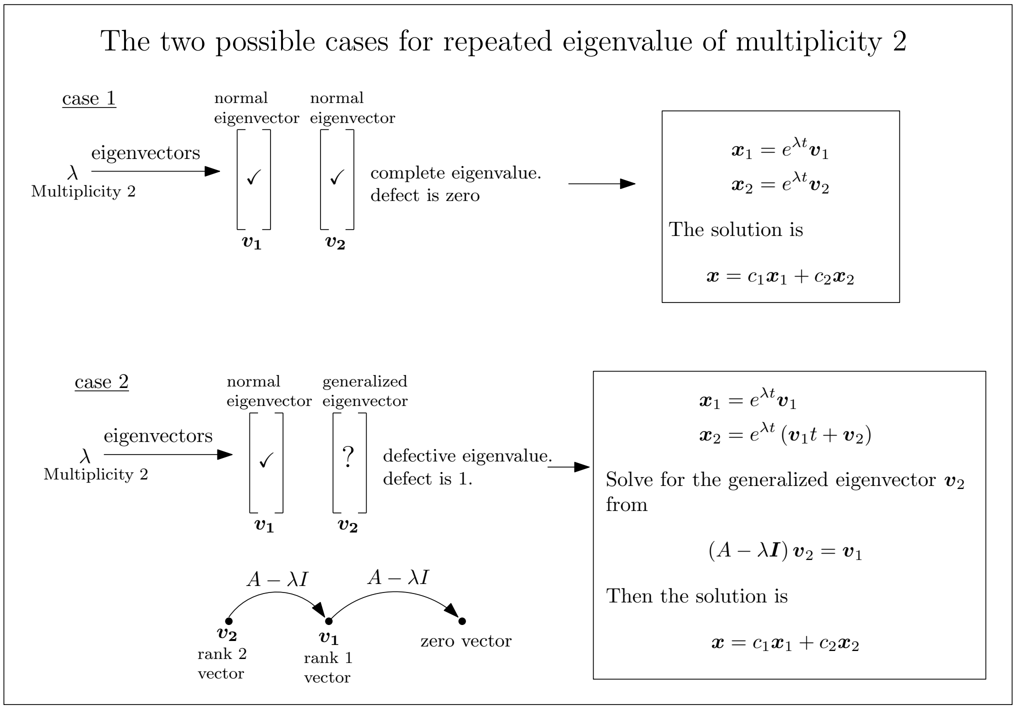


Figure 386: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) 1 \\
&= \begin{bmatrix} 1 \\ 1+t \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1+t \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1+t \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$\Phi^{-1} = \begin{bmatrix} -1-t & 1 \\ 1 & 0 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 0 & 1 \\ 1 & 1+t \end{bmatrix} \int \begin{bmatrix} -1-t & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 1+t \end{bmatrix} \int \begin{bmatrix} -1-t \\ 1 \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 1+t \end{bmatrix} \begin{bmatrix} -t-\frac{1}{2}t^2 \\ t \end{bmatrix} \\
&= \begin{bmatrix} t \\ \frac{t^2}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2(1+t) \end{bmatrix} + \begin{bmatrix} t \\ \frac{t^2}{2} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 + t \\ c_1 + c_2 t + c_2 + \frac{1}{2}t^2 \end{bmatrix}$$

The following is the phase plot of the system.

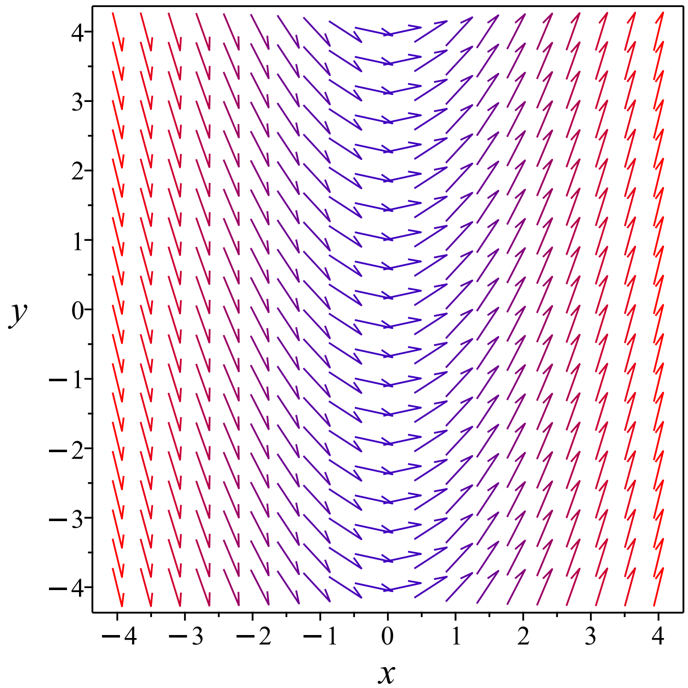


Figure 387: Phase plot

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve([diff(x(t),t)=1,diff(y(t),t)=x(t)],singsol=all)
```

$$x(t) = c_2 + t$$
$$y(t) = c_2 t + \frac{1}{2}t^2 + c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 26

```
DSolve[{x'[t]==1,y'[t]==x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to t + c_1$$
$$y(t) \to \frac{t^2}{2} + c_1 t + c_2$$

10 Chapter 3. Linear Systems. Exercises section 3.2. page 277

10.1 problem 1

Internal problem ID [13076]
Internal file name [`OUTPUT/11731_Sunday_December_03_2023_07_16_10_PM_52240061/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3x(t) \\
y' &= -2y
\end{aligned}
$$

10.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{3t} & 0 \\ 0 & \mathrm{e}^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}c_1 \\ e^{-2t}c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 0 \\ 0 & -2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(3-\lambda)(-2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 0 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-2t}$$
$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

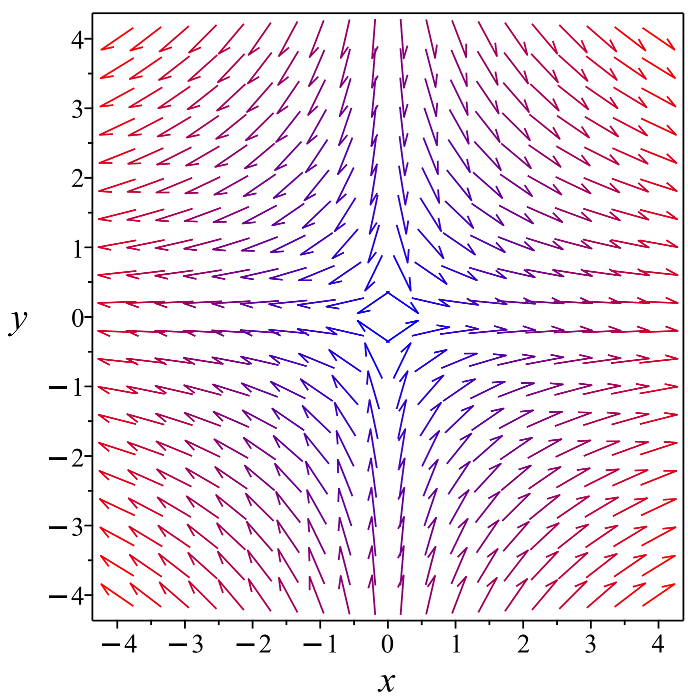


Figure 388: Phase plot

10.1.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t), y' = -2y]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$\vec{x}'(t) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- System to solve

$\vec{x}'(t) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$\left[\left[-2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$

- Consider eigenpair

$\left[-2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$

- Solution to homogeneous system from eigenpair

$\vec{x}_1 = e^{-2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{3t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 e^{3t} \\ c_1 e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = c_2 e^{3t}, y = c_1 e^{-2t}\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve([diff(x(t),t)=3*x(t),diff(y(t),t)=-2*y(t)],singsol=all)
```

$$x(t) = c_2 e^{3t}$$
$$y(t) = c_1 e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 32

```
DSolve[{x'[t]==3*x[t],y'[t]==-2*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 e^{3t}$$
$$y(t) \to c_2 - \frac{2}{3} c_1 \left(e^{3t} - 1\right)$$

10.2 problem 2

Internal problem ID [13077]
Internal file name [`OUTPUT/11732_Sunday_December_03_2023_07_16_10_PM_54236071/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -4x(t) - 2y \\
y' &= -x(t) - 3y
\end{aligned}
$$

10.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-5t}}{3} + \frac{\mathrm{e}^{-2t}}{3} & -\frac{2\,\mathrm{e}^{-2t}}{3} + \frac{2\,\mathrm{e}^{-5t}}{3} \\ -\frac{\mathrm{e}^{-2t}}{3} + \frac{\mathrm{e}^{-5t}}{3} & \frac{\mathrm{e}^{-5t}}{3} + \frac{2\,\mathrm{e}^{-2t}}{3} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,e^{-5t}}{3}+\frac{e^{-2t}}{3} & -\frac{2\,e^{-2t}}{3}+\frac{2\,e^{-5t}}{3} \\ -\frac{e^{-2t}}{3}+\frac{e^{-5t}}{3} & \frac{e^{-5t}}{3}+\frac{2\,e^{-2t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,e^{-5t}}{3}+\frac{e^{-2t}}{3}\right)c_1+\left(-\frac{2\,e^{-2t}}{3}+\frac{2\,e^{-5t}}{3}\right)c_2 \\ \left(-\frac{e^{-2t}}{3}+\frac{e^{-5t}}{3}\right)c_1+\left(\frac{e^{-5t}}{3}+\frac{2\,e^{-2t}}{3}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1+2c_2)e^{-5t}}{3}+\frac{e^{-2t}(c_1-2c_2)}{3} \\ \frac{(c_1+c_2)e^{-5t}}{3}-\frac{e^{-2t}(c_1-2c_2)}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & -2 \\ -1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 7\lambda + 10 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-2t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-5t}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-2t} + 2c_2 \mathrm{e}^{-5t} \\ c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-5t} \end{bmatrix}$$

The following is the phase plot of the system.

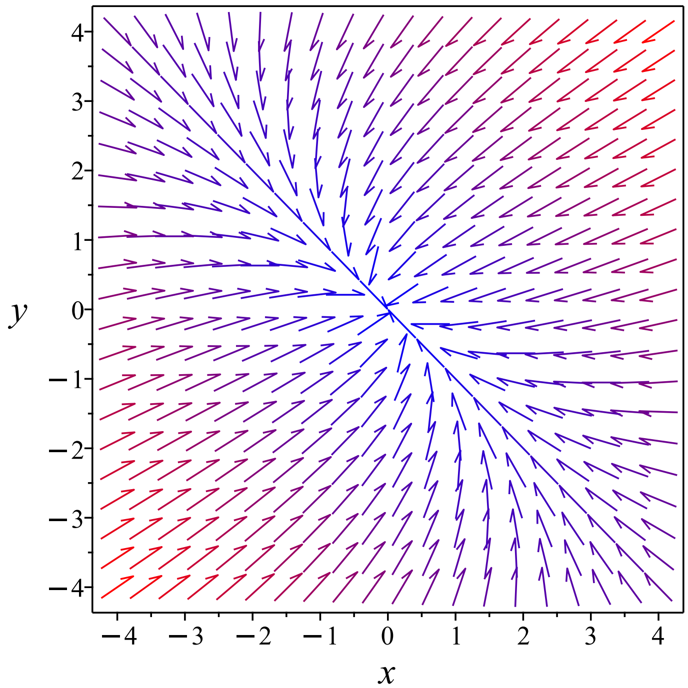


Figure 389: Phase plot

10.2.3 Maple step by step solution

Let's solve

$[x'(t) = -4x(t) - 2y, y' = -x(t) - 3y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_1 = e^{-5t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_2 = e^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

- Substitute solutions into the general solution

  $$\vec{x} = c_1 e^{-5t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 2c_1 \mathrm{e}^{-5t} - c_2 \mathrm{e}^{-2t} \\ c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = 2c_1 \mathrm{e}^{-5t} - c_2 \mathrm{e}^{-2t}, y = c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{-2t}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=-4*x(t)-2*y(t),diff(y(t),t)=-x(t)-3*y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{-2t}$$
$$y(t) = \frac{c_1 \mathrm{e}^{-5t}}{2} - c_2 \mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 71

```
DSolve[{x'[t]==-4*x[t]-2*y[t],y'[t]==-x[t]-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{3} e^{-5t} \left(c_1 \left(e^{3t} + 2\right) - 2c_2 \left(e^{3t} - 1\right)\right)$$
$$y(t) \to \frac{1}{3} e^{-5t} \left(c_1 \left(-e^{3t}\right) + 2c_2 e^{3t} + c_1 + c_2\right)$$

10.3 problem 3

Internal problem ID [13078]
Internal file name [`OUTPUT/11733_Sunday_December_03_2023_07_16_10_PM_11568325/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -5x(t) - 2y \\ y' &= -x(t) - 4y \end{aligned}$$

10.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-6t}}{3} + \frac{\mathrm{e}^{-3t}}{3} & -\frac{2\,\mathrm{e}^{-3t}}{3} + \frac{2\,\mathrm{e}^{-6t}}{3} \\ -\frac{\mathrm{e}^{-3t}}{3} + \frac{\mathrm{e}^{-6t}}{3} & \frac{\mathrm{e}^{-6t}}{3} + \frac{2\,\mathrm{e}^{-3t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,e^{-6t}}{3} + \frac{e^{-3t}}{3} & -\frac{2\,e^{-3t}}{3} + \frac{2\,e^{-6t}}{3} \\ -\frac{e^{-3t}}{3} + \frac{e^{-6t}}{3} & \frac{e^{-6t}}{3} + \frac{2\,e^{-3t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,e^{-6t}}{3} + \frac{e^{-3t}}{3}\right) c_1 + \left(-\frac{2\,e^{-3t}}{3} + \frac{2\,e^{-6t}}{3}\right) c_2 \\ \left(-\frac{e^{-3t}}{3} + \frac{e^{-6t}}{3}\right) c_1 + \left(\frac{e^{-6t}}{3} + \frac{2\,e^{-3t}}{3}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1+2c_2)e^{-6t}}{3} + \frac{e^{-3t}(c_1-2c_2)}{3} \\ \frac{(c_1+c_2)e^{-6t}}{3} - \frac{e^{-3t}(c_1-2c_2)}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -5-\lambda & -2 \\ -1 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 9\lambda + 18 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = -6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
-6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix} - (-6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -1 & -1 & 0 \end{array} \right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
-6	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-3t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$

Since eigenvalue -6 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-6t}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-6t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{-6t} \\ \mathrm{e}^{-6t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-3t} + 2c_2 \mathrm{e}^{-6t} \\ c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-6t} \end{bmatrix}$$

The following is the phase plot of the system.

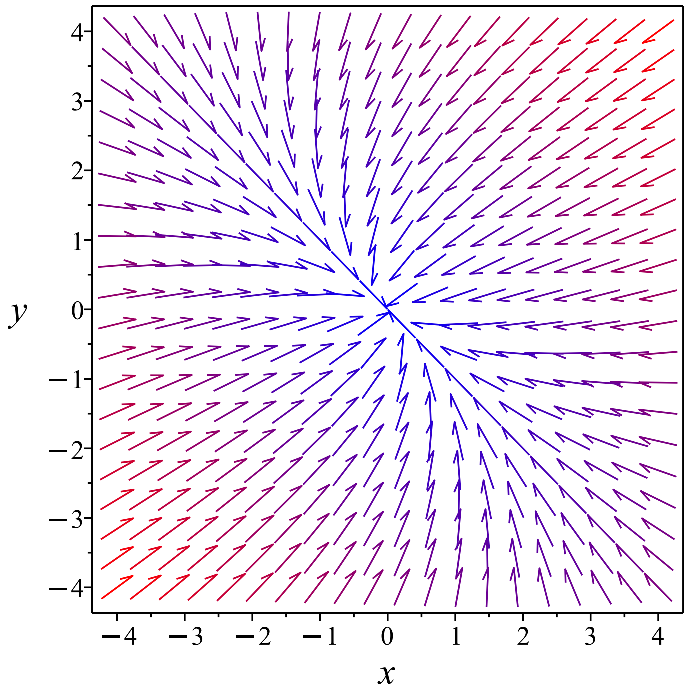


Figure 390: Phase plot

10.3.3 Maple step by step solution

Let's solve

$[x'(t) = -5x(t) - 2y, y' = -x(t) - 4y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-6, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-6, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-6t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-6t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 2c_1 e^{-6t} - c_2 e^{-3t} \\ c_1 e^{-6t} + c_2 e^{-3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = 2c_1 e^{-6t} - c_2 e^{-3t}, y = c_1 e^{-6t} + c_2 e^{-3t}\}$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=-5*x(t)-2*y(t),diff(y(t),t)=-x(t)-4*y(t)],singsol=all)
```

$$x(t) = e^{-6t} c_1 + c_2 e^{-3t}$$
$$y(t) = \frac{e^{-6t} c_1}{2} - c_2 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 71

```
DSolve[{x'[t]==-5*x[t]-2*y[t],y'[t]==-x[t]-4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{3} e^{-6t} \left(c_1 \left(e^{3t} + 2\right) - 2c_2 \left(e^{3t} - 1\right)\right)$$
$$y(t) \to \frac{1}{3} e^{-6t} \left(c_1 \left(-e^{3t}\right) + 2c_2 e^{3t} + c_1 + c_2\right)$$

10.4 problem 4

Internal problem ID [13079]
Internal file name [`OUTPUT/11734_Sunday_December_03_2023_07_16_11_PM_47153869/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + y \\
y' &= -x(t) + 4y
\end{aligned}
$$

10.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{3t}(1-t) & t\,\mathrm{e}^{3t} \\ -t\,\mathrm{e}^{3t} & \mathrm{e}^{3t}(1+t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t}(1-t) & t\,e^{3t} \\ -t\,e^{3t} & e^{3t}(1+t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(1-t)\,c_1 + t\,e^{3t}c_2 \\ -t\,e^{3t}c_1 + e^{3t}(1+t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(-tc_1 + c_2t + c_1) \\ e^{3t}(-tc_1 + c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

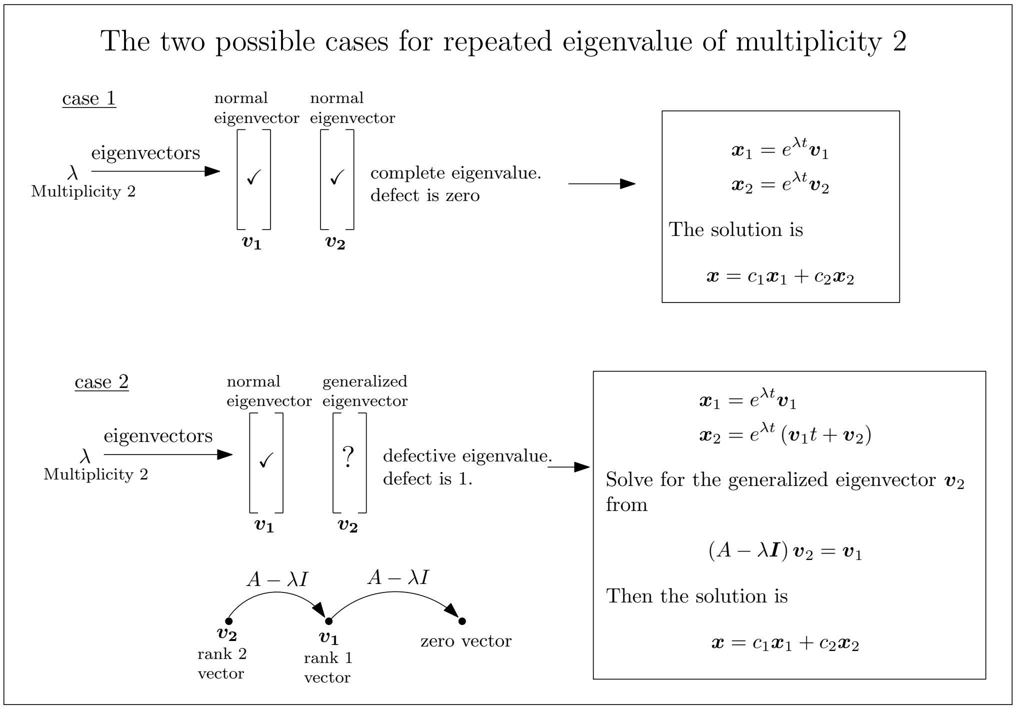


Figure 391: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 3. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{3t} \\
&= \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \mathrm{e}^{3t} \\
&= \begin{bmatrix} t\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} t\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{3t}(t c_2 + c_1) \\ \mathrm{e}^{3t}(t c_2 + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

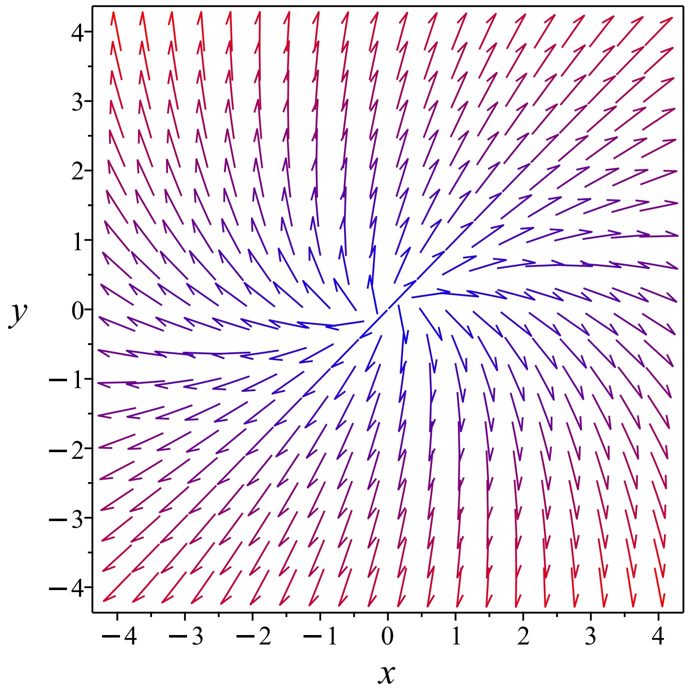


Figure 392: Phase plot

10.4.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + y, y' = -x(t) + 4y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

- Rewrite the system as

  $$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

  $$\vec{x}_1(t) = e^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

  $$\vec{x}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\left[\begin{array}{cc} 2 & 1 \\ -1 & 4 \end{array}\right]-3\cdot\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c} -1 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 3

$$\vec{x}_2(t)=\mathrm{e}^{3t}\cdot\left(t\cdot\left[\begin{array}{c} 1 \\ 1 \end{array}\right]+\left[\begin{array}{c} -1 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\vec{x}=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x}=c_1\mathrm{e}^{3t}\cdot\left[\begin{array}{c} 1 \\ 1 \end{array}\right]+c_2\mathrm{e}^{3t}\cdot\left(t\cdot\left[\begin{array}{c} 1 \\ 1 \end{array}\right]+\left[\begin{array}{c} -1 \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right]=\left[\begin{array}{c} \mathrm{e}^{3t}((t-1)\,c_2+c_1) \\ \mathrm{e}^{3t}(c_2t+c_1) \end{array}\right]$$

- Solution to the system of ODEs

$$\{x(t)=\mathrm{e}^{3t}((t-1)\,c_2+c_1)\,,y=\mathrm{e}^{3t}(c_2t+c_1)\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve([diff(x(t),t)=2*x(t)+1*y(t),diff(y(t),t)=-x(t)+4*y(t)],singsol=all)
```

$$x(t)=\mathrm{e}^{3t}(c_2t+c_1)$$
$$y(t)=\mathrm{e}^{3t}(c_2t+c_1+c_2)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 44

```
DSolve[{x'[t]==2*x[t]+1*y[t],y'[t]==-x[t]+4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to e^{3t}(c_1(-t) + c_2 t + c_1)$$
$$y(t) \to e^{3t}((c_2 - c_1)t + c_2)$$

10.5 problem 5

Internal problem ID [13080]
Internal file name [`OUTPUT/11735_Sunday_December_03_2023_07_16_11_PM_4736820/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -\frac{x(t)}{2} \\
y' &= x(t) - \frac{y}{2}
\end{aligned}
$$

10.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} & 0 \\ \mathrm{e}^{-\frac{t}{2}} t & \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-\frac{t}{2}} & 0 \\ e^{-\frac{t}{2}}t & e^{-\frac{t}{2}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{t}{2}}c_1 \\ e^{-\frac{t}{2}}tc_1 + e^{-\frac{t}{2}}c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{t}{2}}c_1 \\ e^{-\frac{t}{2}}(c_1 t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\frac{1}{2} - \lambda & 0 \\ 1 & -\frac{1}{2} - \lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\frac{1}{2}-\lambda)(-\frac{1}{2}-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-\frac{1}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-\frac{1}{2}$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-\frac{1}{2} & 0\\ 1 & -\frac{1}{2}\end{bmatrix}-\left(-\frac{1}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}0 & 0 & 0\\ 1 & 0 & 0\end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2}$	2	1	Yes	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue $-\frac{1}{2}$ is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

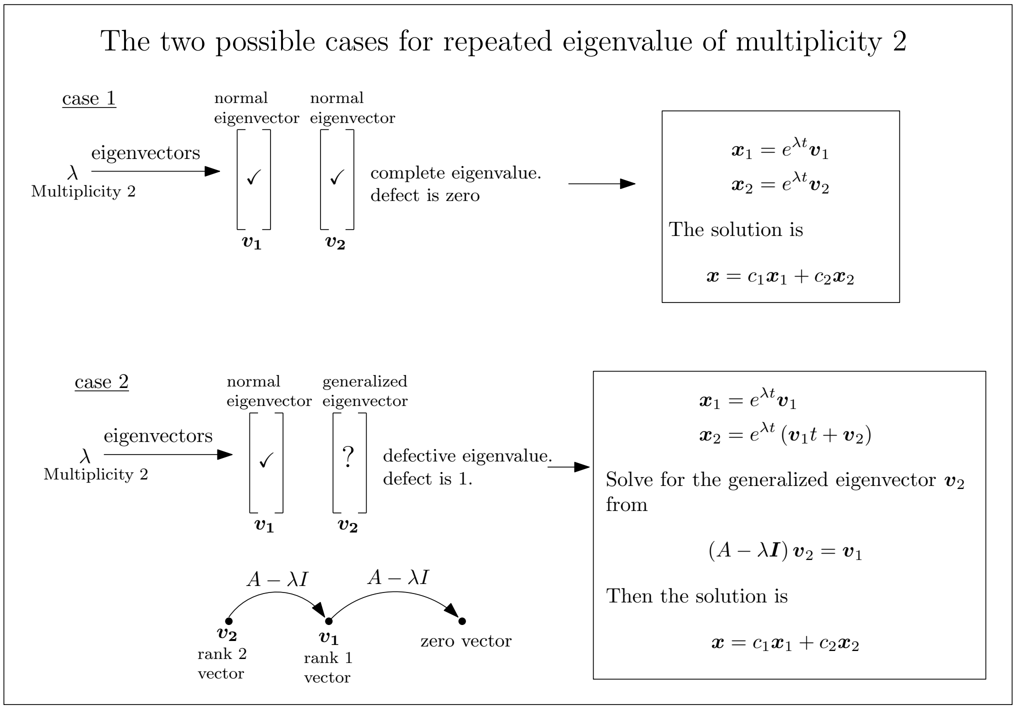


Figure 393: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} - \left(-\frac{1}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue $-\frac{1}{2}$. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{-\frac{t}{2}} \\
&= \begin{bmatrix} 0 \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-\frac{t}{2}} \\
&= \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} \\ \mathrm{e}^{-\frac{t}{2}}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} \\ \mathrm{e}^{-\frac{t}{2}}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{-\frac{t}{2}} \\ \mathrm{e}^{-\frac{t}{2}}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

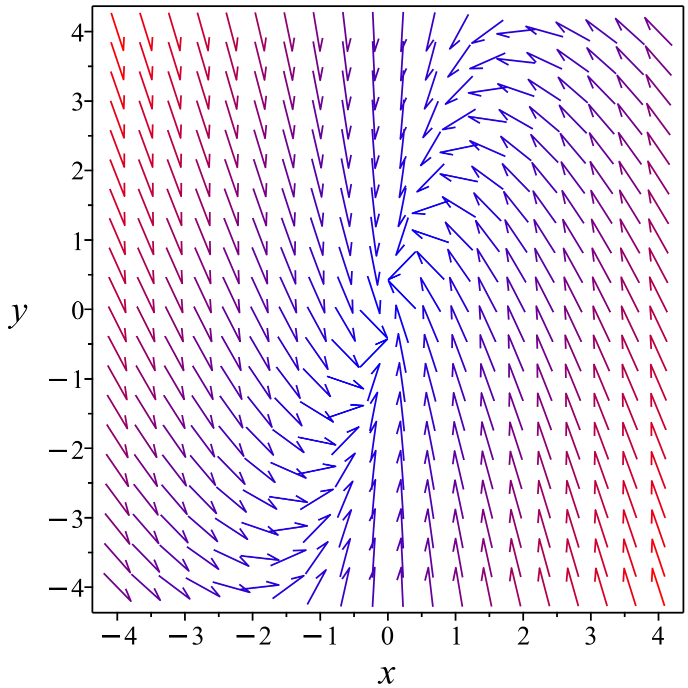


Figure 394: Phase plot

10.5.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -\frac{x(t)}{2}, y' = x(t) - \frac{y}{2}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[-\frac{1}{2}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $\left[-\frac{1}{2}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$

- First solution from eigenvalue $-\frac{1}{2}$

  $\overrightarrow{x}_1(t) = e^{-\frac{t}{2}} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -\frac{1}{2}$ is the eigenvalue, an

  $\overrightarrow{x}_2(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $-\frac{1}{2}$

$$\left(\left[\begin{array}{cc}-\frac{1}{2} & 0 \\ 1 & -\frac{1}{2}\end{array}\right]--\frac{1}{2}\cdot\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}0 \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}0 \\ 0\end{array}\right]$$

- Second solution from eigenvalue $-\frac{1}{2}$

$$\overrightarrow{x}_2(t)=\mathrm{e}^{-\frac{t}{2}}\cdot\left(t\cdot\left[\begin{array}{c}0 \\ 1\end{array}\right]+\left[\begin{array}{c}0 \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x}=c_1\overrightarrow{x}_1(t)+c_2\overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x}=c_1\mathrm{e}^{-\frac{t}{2}}\cdot\left[\begin{array}{c}0 \\ 1\end{array}\right]+c_2\mathrm{e}^{-\frac{t}{2}}\cdot\left(t\cdot\left[\begin{array}{c}0 \\ 1\end{array}\right]+\left[\begin{array}{c}0 \\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right]=\left[\begin{array}{c}0 \\ \mathrm{e}^{-\frac{t}{2}}(c_2t+c_1)\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t)=0, y=\mathrm{e}^{-\frac{t}{2}}(c_2t+c_1)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve([diff(x(t),t)=-1/2*x(t),diff(y(t),t)=x(t)-1/2*y(t)],singsol=all)
```

$$x(t)=c_2\mathrm{e}^{-\frac{t}{2}}$$
$$y(t)=(c_2t+c_1)\,\mathrm{e}^{-\frac{t}{2}}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 33

```
DSolve[{x'[t]==-1/2*x[t],y'[t]==x[t]-1/2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tru
```

$$x(t) \rightarrow c_1 e^{-t/2}$$
$$y(t) \rightarrow e^{-t/2}(c_1 t + c_2)$$

10.6 problem 6

Internal problem ID [13081]
Internal file name [`OUTPUT/11736_Sunday_December_03_2023_07_16_11_PM_35031517/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 5x(t) + 4y \\
y' &= 9x(t)
\end{aligned}
$$

10.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(9\,\mathrm{e}^{13t}+4)\mathrm{e}^{-4t}}{13} & \frac{4(\mathrm{e}^{13t}-1)\mathrm{e}^{-4t}}{13} \\ \frac{9(\mathrm{e}^{13t}-1)\mathrm{e}^{-4t}}{13} & \frac{(4\,\mathrm{e}^{13t}+9)\mathrm{e}^{-4t}}{13} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(9\,\mathrm{e}^{13t}+4)\,\mathrm{e}^{-4t}}{13} & \frac{4(\mathrm{e}^{13t}-1)\,\mathrm{e}^{-4t}}{13} \\ \frac{9(\mathrm{e}^{13t}-1)\,\mathrm{e}^{-4t}}{13} & \frac{(4\,\mathrm{e}^{13t}+9)\,\mathrm{e}^{-4t}}{13} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(9\,\mathrm{e}^{13t}+4)\,\mathrm{e}^{-4t}c_1}{13} + \frac{4(\mathrm{e}^{13t}-1)\,\mathrm{e}^{-4t}c_2}{13} \\ \frac{9(\mathrm{e}^{13t}-1)\,\mathrm{e}^{-4t}c_1}{13} + \frac{(4\,\mathrm{e}^{13t}+9)\,\mathrm{e}^{-4t}c_2}{13} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-4t}((9c_1+4c_2)\mathrm{e}^{13t}+4c_1-4c_2)}{13} \\ \frac{9\left(\left(c_1+\frac{4c_2}{9}\right)\mathrm{e}^{13t}+c_2-c_1\right)\mathrm{e}^{-4t}}{13} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & 4 \\ 9 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & 4 \\ 9 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda - 36 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$
$$\lambda_2 = 9$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue
9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 4 \\ 9 & 0 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 9 & 4 \\ 9 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 9 & 4 & 0 \\ 9 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 9 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 9 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{4t}{9}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{4t}{9} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{9} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{9} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{9} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{4t}{9} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{9} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{4t}{9} \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 4 \\ 9 & 0 \end{bmatrix} - (9) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 4 \\ 9 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 9 & -9 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{9R_1}{4} \implies \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	1	1	No	$\begin{bmatrix} -\frac{4}{9} \\ 1 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-4t} \\
&= \begin{bmatrix} -\frac{4}{9} \\ 1 \end{bmatrix} e^{-4t}
\end{aligned}
$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{9t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{9t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{4\,\mathrm{e}^{-4t}}{9} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{9t} \\ \mathrm{e}^{9t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{-4t} \mathrm{e}^{13t} - \frac{4 c_1 \mathrm{e}^{-4t}}{9} \\ \left(c_2 \mathrm{e}^{13t} + c_1\right) \mathrm{e}^{-4t} \end{bmatrix}
$$

The following is the phase plot of the system.

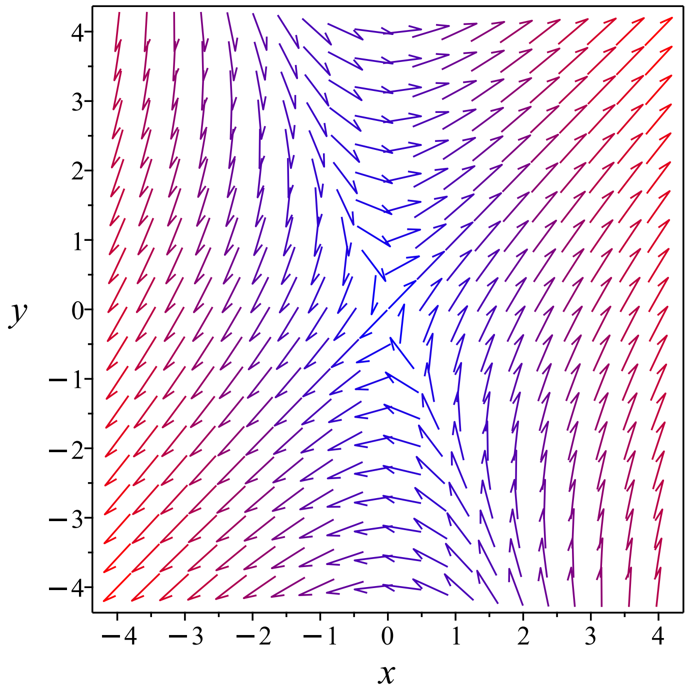


Figure 395: Phase plot

10.6.3 Maple step by step solution

Let's solve

$[x'(t) = 5x(t) + 4y, y' = 9x(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 5 & 4 \\ 9 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 5 & 4 \\ 9 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & 4 \\ 9 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \left[\begin{array}{c} -\frac{4}{9} \\ 1 \end{array}\right]\right], \left[9, \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-4, \left[\begin{array}{c} -\frac{4}{9} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = e^{-4t} \cdot \left[\begin{array}{c} -\frac{4}{9} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[9, \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = e^{9t} \cdot \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 e^{-4t} \cdot \left[\begin{array}{c} -\frac{4}{9} \\ 1 \end{array}\right] + c_2 e^{9t} \cdot \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{-4t} \mathrm{e}^{13t} - \frac{4c_1 \mathrm{e}^{-4t}}{9} \\ \left(c_2 \mathrm{e}^{13t} + c_1\right) \mathrm{e}^{-4t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = c_2 \mathrm{e}^{-4t} \mathrm{e}^{13t} - \frac{4c_1 \mathrm{e}^{-4t}}{9}, y = \left(c_2 \mathrm{e}^{13t} + c_1\right) \mathrm{e}^{-4t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=5*x(t)+4*y(t),diff(y(t),t)=9*x(t)],singsol=all)
```

$$x(t) = -\frac{4\,\mathrm{e}^{-4t} c_1}{9} + c_2 \mathrm{e}^{9t}$$
$$y(t) = \mathrm{e}^{-4t} c_1 + c_2 \mathrm{e}^{9t}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 74

```
DSolve[{x'[t]==5*x[t]+4*y[t],y'[t]==9*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{13} e^{-4t} \left(c_1 \left(9e^{13t} + 4\right) + 4c_2 \left(e^{13t} - 1\right)\right)$$
$$y(t) \to \frac{1}{13} e^{-4t} \left(9c_1 \left(e^{13t} - 1\right) + c_2 \left(4e^{13t} + 9\right)\right)$$

10.7 problem 7

Internal problem ID [13082]
Internal file name [`OUTPUT/11737_Sunday_December_03_2023_07_16_12_PM_37067417/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3x(t) + 4y$$
$$y' = x(t)$$

10.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{5} + \frac{4\,e^{4t}}{5} & \frac{4\,e^{4t}}{5} - \frac{4\,e^{-t}}{5} \\ \frac{e^{4t}}{5} - \frac{e^{-t}}{5} & \frac{4\,e^{-t}}{5} + \frac{e^{4t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-t}}{5}+\frac{4e^{4t}}{5} & \frac{4e^{4t}}{5}-\frac{4e^{-t}}{5} \\ \frac{e^{4t}}{5}-\frac{e^{-t}}{5} & \frac{4e^{-t}}{5}+\frac{e^{4t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-t}}{5}+\frac{4e^{4t}}{5}\right)c_1+\left(\frac{4e^{4t}}{5}-\frac{4e^{-t}}{5}\right)c_2 \\ \left(\frac{e^{4t}}{5}-\frac{e^{-t}}{5}\right)c_1+\left(\frac{4e^{-t}}{5}+\frac{e^{4t}}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-4c_2+c_1)e^{-t}}{5}+\frac{4e^{4t}(c_1+c_2)}{5} \\ \frac{(4c_2-c_1)e^{-t}}{5}+\frac{e^{4t}(c_1+c_2)}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 4 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 4t\}$

Hence the solution is

$$\begin{bmatrix} 4t \\ t \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 4t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t}\\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{4t}\\ &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{4t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 4\,\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-t} + 4c_2 \mathrm{e}^{4t} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t} \end{bmatrix}$$

The following is the phase plot of the system.

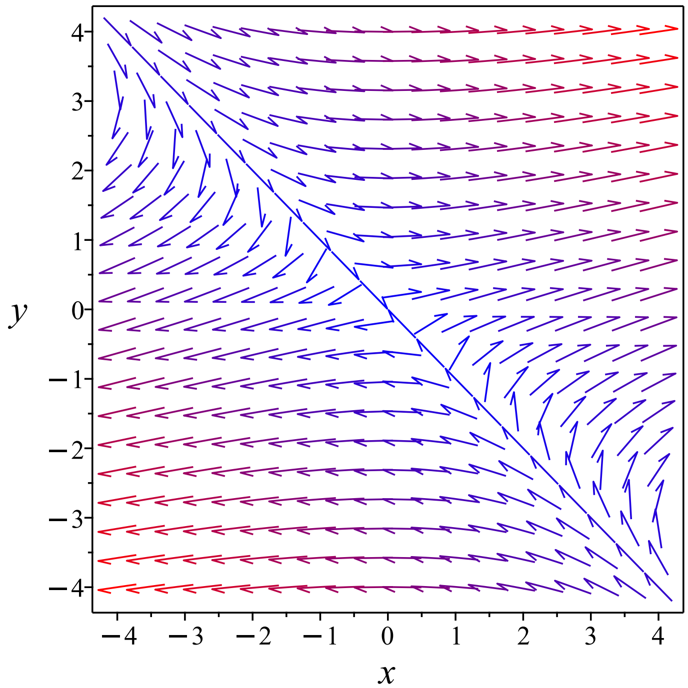


Figure 396: Phase plot

10.7.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) + 4y, y' = x(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{4t} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{4t} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 e^{-t} + 4c_2 e^{4t} \\ c_1 e^{-t} + c_2 e^{4t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = -c_1 e^{-t} + 4c_2 e^{4t}, y = c_1 e^{-t} + c_2 e^{4t}\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=3*x(t)+4*y(t),diff(y(t),t)=1*x(t)],singsol=all)
```

$$x(t) = 4c_1 e^{4t} - c_2 e^{-t}$$
$$y(t) = c_1 e^{4t} + c_2 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 71

```
DSolve[{x'[t]==3*x[t]+4*y[t],y'[t]==1*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{5} e^{-t} \left(c_1 \left(4e^{5t} + 1\right) + 4c_2 \left(e^{5t} - 1\right)\right)$$
$$y(t) \to \frac{1}{5} e^{-t} \left(c_1 \left(e^{5t} - 1\right) + c_2 \left(e^{5t} + 4\right)\right)$$

10.8 problem 8

Internal problem ID [13083]
Internal file name [`OUTPUT/11738_Sunday_December_03_2023_07_16_12_PM_94234493/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) - y \\
y' &= -x(t) + y
\end{aligned}
$$

10.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{\left(5+\sqrt{5}\right)\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} - \frac{\mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(-5+\sqrt{5}\right)}{10} & \frac{\left(-\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}+\mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}}{5} \\ \frac{\left(-\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}+\mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}}{5} & \frac{\left(5-\sqrt{5}\right)\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} + \frac{\mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(5+\sqrt{5}\right)}{10} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{\left(5+\sqrt{5}\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} - \frac{e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(-5+\sqrt{5}\right)}{10} & \frac{\left(-e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}}{5} \\ \frac{\left(-e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}}{5} & \frac{\left(5-\sqrt{5}\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} + \frac{e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(5+\sqrt{5}\right)}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{\left(5+\sqrt{5}\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} - \frac{e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(-5+\sqrt{5}\right)}{10}\right)c_1 + \frac{\left(-e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}\,c_2}{5} \\ \frac{\left(-e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}\,c_1}{5} + \left(\frac{\left(5-\sqrt{5}\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} + \frac{e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(5+\sqrt{5}\right)}{10}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left((c_1-2c_2)\sqrt{5}+5c_1\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} - \frac{e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left((c_1-2c_2)\sqrt{5}-5c_1\right)}{10} \\ \frac{\left((-2c_1-c_2)\sqrt{5}+5c_2\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} + \frac{\left((c_1+\frac{c_2}{2})\sqrt{5}+\frac{5c_2}{2}\right)e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} - \frac{\sqrt{5}}{2}$	1	real eigenvalue
$\frac{3}{2} + \frac{\sqrt{5}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & -1 \\ -1 & \frac{\sqrt{5}}{2} - \frac{1}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} + \frac{\sqrt{5}}{2} & -1 & 0 \\ -1 & \frac{\sqrt{5}}{2} - \frac{1}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\frac{1}{2}+\frac{\sqrt{5}}{2}} \implies \left[\begin{array}{cc|c} \frac{1}{2}+\frac{\sqrt{5}}{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}+\frac{\sqrt{5}}{2} & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{\sqrt{5}+1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & -1 \\ -1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{\sqrt{5}}{2} & -1 & 0 \\ -1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\frac{1}{2}-\frac{\sqrt{5}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{\sqrt{5}}{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{\sqrt{5}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number

of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{3}{2}+\frac{\sqrt{5}}{2}$	1	1	No	$\left[\begin{array}{c}-\frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1\end{array}\right]$
$\frac{3}{2}-\frac{\sqrt{5}}{2}$	1	1	No	$\left[\begin{array}{c}-\frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{3}{2}+\frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \\
&= \left[\begin{array}{c}-\frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1\end{array}\right] e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $\frac{3}{2}-\frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \\
&= \left[\begin{array}{c}-\frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1\end{array}\right] e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t}}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \end{array}\right]
$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{c_1(\sqrt{5}+1)e^{\frac{(3+\sqrt{5})t}{2}}}{2} + \frac{c_2 e^{-\frac{(\sqrt{5}-3)t}{2}}(\sqrt{5}-1)}{2} \\ c_1 e^{\frac{(3+\sqrt{5})t}{2}} + c_2 e^{-\frac{(\sqrt{5}-3)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

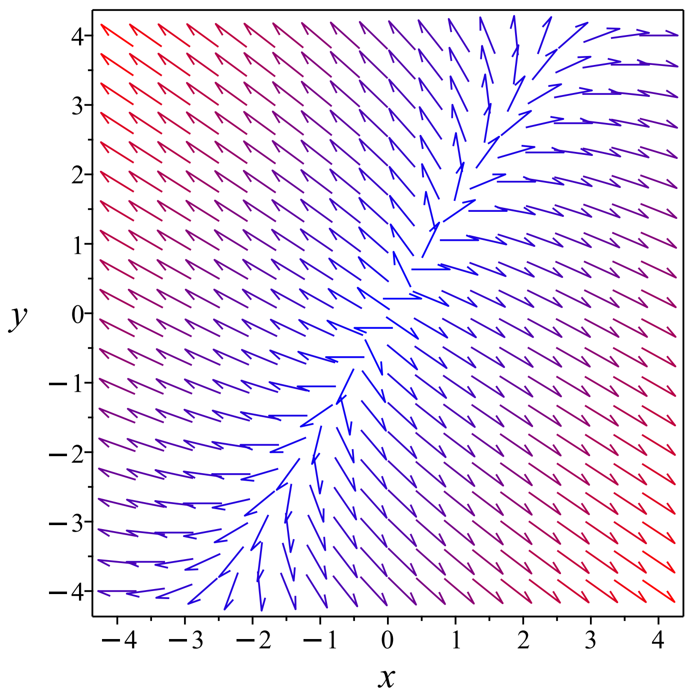


Figure 397: Phase plot

10.8.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) - y, y' = -x(t) + y]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} -\frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{3}{2} + \frac{\sqrt{5}}{2}, \begin{bmatrix} -\frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} -\frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} -\frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{3}{2} + \frac{\sqrt{5}}{2}, \begin{bmatrix} -\frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} -\frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs
$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$
- Substitute solutions into the general solution
$$\overrightarrow{x} = c_1 e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} -\frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix} + c_2 e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} -\frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}$$
- Substitute in vector of dependent variables
$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{c_2\left(\sqrt{5}+1\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} + \frac{c_1 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-1\right)}{2} \\ c_1 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 e^{\frac{\left(3+\sqrt{5}\right)t}{2}} \end{bmatrix}$$
- Solution to the system of ODEs
$$\left\{x(t) = -\frac{c_2\left(\sqrt{5}+1\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} + \frac{c_1 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-1\right)}{2}, y = c_1 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 e^{\frac{\left(3+\sqrt{5}\right)t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=2*x(t)-y(t),diff(y(t),t)=-1*x(t)+y(t)],singsol=all)
```

$$x(t) = c_1 e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}$$
$$y(t) = -\frac{c_1 e^{\frac{\left(3+\sqrt{5}\right)t}{2}}\sqrt{5}}{2} + \frac{c_2 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\sqrt{5}}{2} + \frac{c_1 e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} + \frac{c_2 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}}{2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 144

```
DSolve[{x'[t]==2*x[t]-y[t],y'[t]==-1*x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to \frac{1}{10}e^{-\frac{1}{2}\left(\sqrt{5}-3\right)t}\left(c_1\left(\left(5+\sqrt{5}\right)e^{\sqrt{5}t}+5-\sqrt{5}\right)-2\sqrt{5}c_2\left(e^{\sqrt{5}t}-1\right)\right)$$
$$y(t) \to -\frac{1}{10}e^{-\frac{1}{2}\left(\sqrt{5}-3\right)t}\left(2\sqrt{5}c_1\left(e^{\sqrt{5}t}-1\right)+c_2\left(\left(\sqrt{5}-5\right)e^{\sqrt{5}t}-5-\sqrt{5}\right)\right)$$

10.9 problem 9

Internal problem ID [13084]
Internal file name [`OUTPUT/11739_Sunday_December_03_2023_07_16_13_PM_73874009/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2x(t) + y$$
$$y' = x(t) + y$$

10.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(5+\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(-5+\sqrt{5})}{10} & -\frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}}{5} \\ -\frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}}{5} & \frac{(5-\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(5+\sqrt{5})}{10} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{(5+\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(-5+\sqrt{5})}{10} & -\frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}}{5} \\ -\frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}}{5} & \frac{(5-\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(5+\sqrt{5})}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{(5+\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(-5+\sqrt{5})}{10}\right)c_1 - \frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}\,c_2}{5} \\ -\frac{\left(-e^{\frac{(3+\sqrt{5})t}{2}} + e^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}\,c_1}{5} + \left(\frac{(5-\sqrt{5})\,e^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}(5+\sqrt{5})}{10}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((c_1+2c_2)\sqrt{5}+5c_1\right)e^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{e^{-\frac{(\sqrt{5}-3)t}{2}}\left((c_1+2c_2)\sqrt{5}-5c_1\right)}{10} \\ \frac{\left((2c_1-c_2)\sqrt{5}+5c_2\right)e^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{\left((c_1-\frac{c_2}{2})\sqrt{5}-\frac{5c_2}{2}\right)e^{-\frac{(\sqrt{5}-3)t}{2}}}{5} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} - \frac{\sqrt{5}}{2}$	1	real eigenvalue
$\frac{3}{2} + \frac{\sqrt{5}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} - \left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{1}{2} + \frac{\sqrt{5}}{2} & 1\\ 1 & \frac{\sqrt{5}}{2} - \frac{1}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{1}{2} + \frac{\sqrt{5}}{2} & 1 & 0\\ 1 & \frac{\sqrt{5}}{2} - \frac{1}{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2}+\frac{\sqrt{5}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}+\frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}+\frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{\sqrt{5}+1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}+1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 & 0 \\ 1 & -\frac{1}{2}-\frac{\sqrt{5}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{1}{2}-\frac{\sqrt{5}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{\sqrt{5}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{\sqrt{5}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number

of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{3}{2} + \frac{\sqrt{5}}{2}$	1	1	No	$\begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \\ 1 \end{bmatrix}$
$\frac{3}{2} - \frac{\sqrt{5}}{2}$	1	1	No	$\begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{3}{2} + \frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)t} \\
&= \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \\ 1 \end{bmatrix} e^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $\frac{3}{2} - \frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t} \\
&= \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix} e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)t}}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \\ e^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t}}{-\frac{1}{2} - \frac{\sqrt{5}}{2}} \\ e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t} \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1\left(\sqrt{5}+1\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{c_2 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-1\right)}{2} \\ c_1 e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

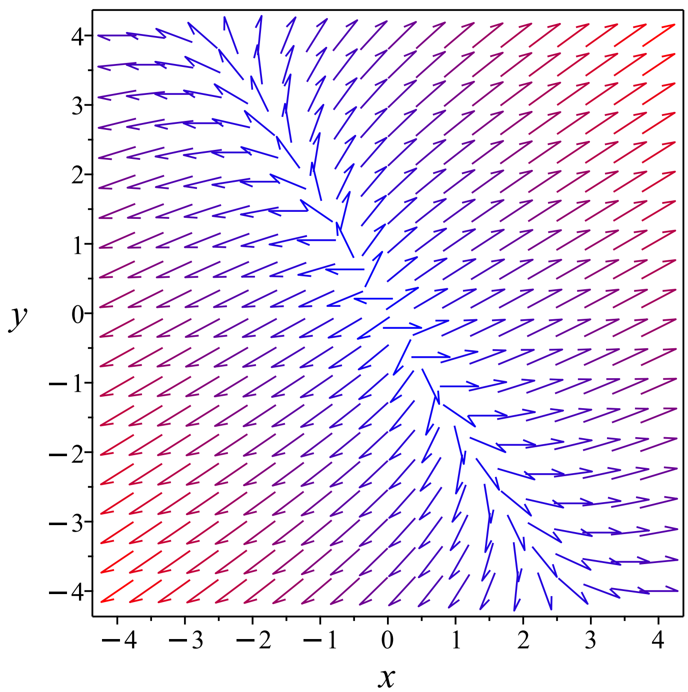


Figure 398: Phase plot

10.9.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + y, y' = x(t) + y]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{3}{2} + \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{-\frac{1}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{3}{2} + \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{1}{2}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{-\frac{1}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\frac{\sqrt{5}}{2}-\frac{1}{2}} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{c_2\left(\sqrt{5}+1\right)\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-1\right)}{2} \\ c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{c_2\left(\sqrt{5}+1\right)\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-1\right)}{2}, y = c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=2*x(t)+y(t),diff(y(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}$$
$$y(t) = \frac{c_1 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}\sqrt{5}}{2} - \frac{c_2 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\sqrt{5}}{2} - \frac{c_1 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{c_2 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}}{2}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 145

```
DSolve[{x'[t]==2*x[t]+y[t],y'[t]==x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{10} e^{-\frac{1}{2}\left(\sqrt{5}-3\right)t}\left(c_1\left(\left(5+\sqrt{5}\right)e^{\sqrt{5}t} + 5 - \sqrt{5}\right) + 2\sqrt{5}c_2\left(e^{\sqrt{5}t} - 1\right)\right)$$
$$y(t) \to \frac{1}{10} e^{-\frac{1}{2}\left(\sqrt{5}-3\right)t}\left(2\sqrt{5}c_1\left(e^{\sqrt{5}t} - 1\right) - c_2\left(\left(\sqrt{5}-5\right)e^{\sqrt{5}t} - 5 - \sqrt{5}\right)\right)$$

10.10 problem 10

Internal problem ID [13085]
Internal file name [`OUTPUT/11740_Sunday_December_03_2023_07_16_13_PM_70773648/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t) - 2y$$
$$y' = x(t) - 4y$$

10.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -e^{-3t} + 2\,e^{-2t} & -2\,e^{-2t} + 2\,e^{-3t} \\ e^{-2t} - e^{-3t} & 2\,e^{-3t} - e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-2t} & -2\,\mathrm{e}^{-2t} + 2\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-2t} - \mathrm{e}^{-3t} & 2\,\mathrm{e}^{-3t} - \mathrm{e}^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-2t}\right) c_1 + \left(-2\,\mathrm{e}^{-2t} + 2\,\mathrm{e}^{-3t}\right) c_2 \\ \left(\mathrm{e}^{-2t} - \mathrm{e}^{-3t}\right) c_1 + \left(2\,\mathrm{e}^{-3t} - \mathrm{e}^{-2t}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-c_1 + 2c_2)\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-2t}(-c_2 + c_1) \\ (-c_1 + 2c_2)\,\mathrm{e}^{-3t} + \mathrm{e}^{-2t}(-c_2 + c_1) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -1-\lambda & -2 \\ 1 & -4-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 5\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-3t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-2t}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-3t} + 2c_2 \mathrm{e}^{-2t} \\ c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

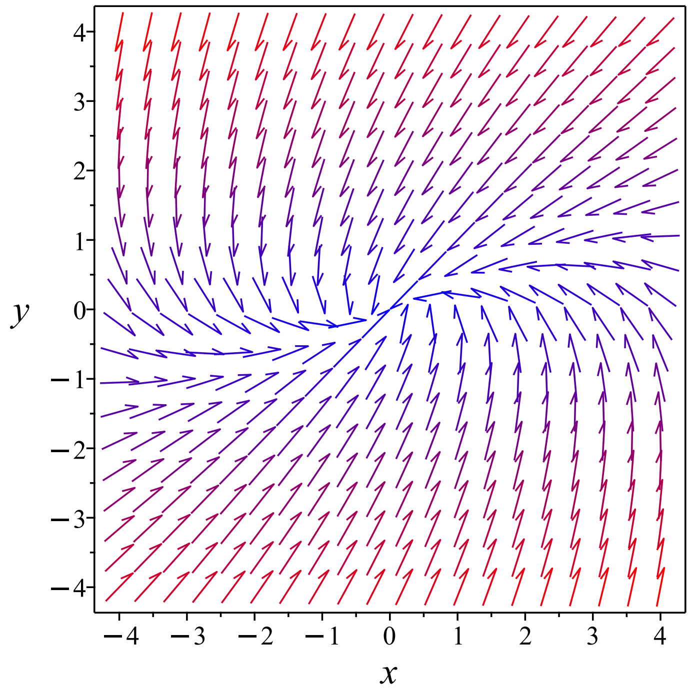


Figure 399: Phase plot

10.10.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) - 2y, y' = x(t) - 4y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[-2, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = \mathrm{e}^{-3t} c_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-2t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} e^{-3t}c_1 + 2c_2e^{-2t} \\ e^{-3t}c_1 + c_2e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs
  $\{x(t) = e^{-3t}c_1 + 2c_2e^{-2t}, y = e^{-3t}c_1 + c_2e^{-2t}\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=-x(t)-2*y(t),diff(y(t),t)=x(t)-4*y(t)],singsol=all)
```

$$x(t) = c_1e^{-2t} + c_2e^{-3t}$$
$$y(t) = \frac{c_1e^{-2t}}{2} + c_2e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 58

```
DSolve[{x'[t]==-x[t]-2*y[t],y'[t]==x[t]-4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to e^{-3t}(c_1(2e^t - 1) - 2c_2(e^t - 1))$$
$$y(t) \to e^{-3t}(c_1(e^t - 1) - c_2(e^t - 2))$$

10.11 problem 11 (a)

Internal problem ID [13086]
Internal file name [`OUTPUT/11741_Sunday_December_03_2023_07_16_13_PM_83227821/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 11 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -2x(t) - 2y$$
$$y' = -2x(t) + y$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

10.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{5t}+4)e^{-3t}}{5} & -\frac{2(e^{5t}-1)e^{-3t}}{5} \\ -\frac{2(e^{5t}-1)e^{-3t}}{5} & \frac{(4\,e^{5t}+1)e^{-3t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{(e^{5t}+4)e^{-3t}}{5} & -\frac{2(e^{5t}-1)e^{-3t}}{5} \\ -\frac{2(e^{5t}-1)e^{-3t}}{5} & \frac{(4e^{5t}+1)e^{-3t}}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(e^{5t}+4)e^{-3t}}{5} \\ -\frac{2(e^{5t}-1)e^{-3t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{2t}$$
$$= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{2t}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{2t}}{2} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,e^{-3t} \\ e^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{(c_1 e^{5t} - 4c_2)e^{-3t}}{2} \\ (c_1 e^{5t} + c_2)\,e^{-3t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{c_1}{2} + 2c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -\frac{2}{5} \\ c_2 = \frac{2}{5} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\left(-\frac{2\,e^{5t}}{5}-\frac{8}{5}\right)e^{-3t}}{2} \\ \left(-\frac{2\,e^{5t}}{5}+\frac{2}{5}\right)e^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

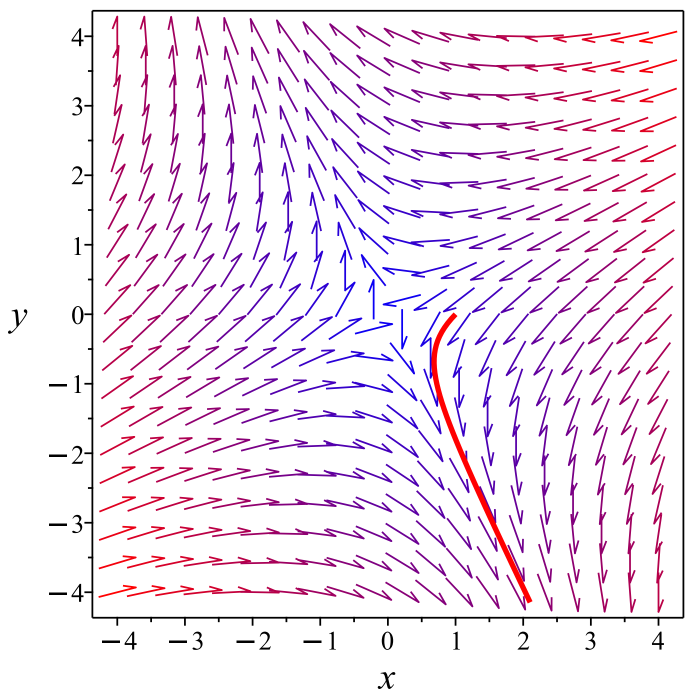
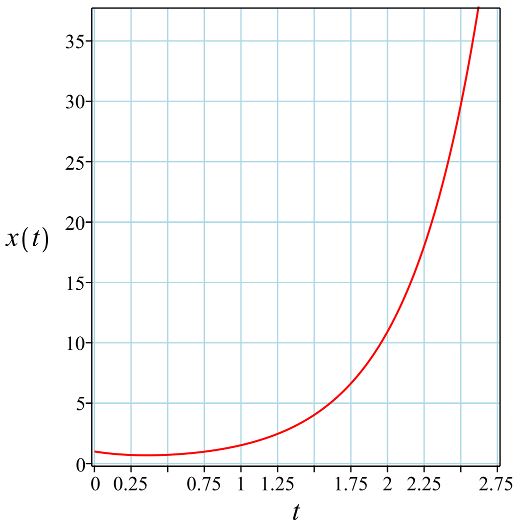
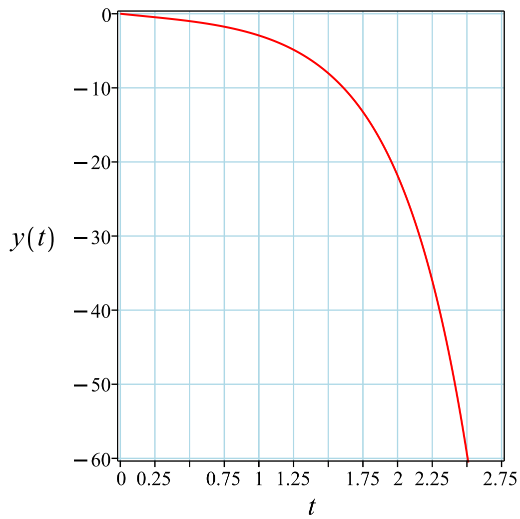


Figure 400: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = -2*x(t)-2*y(t), diff(y(t),t) = -2*x(t)+y(t), x(0) = 1, y(0) = 0], sin
```

$$x(t) = \frac{e^{2t}}{5} + \frac{4\,e^{-3t}}{5}$$
$$y(t) = -\frac{2\,e^{2t}}{5} + \frac{2\,e^{-3t}}{5}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 40

```
DSolve[{x'[t]==-2*x[t]-2*y[t],y'[t]==-2*x[t]+y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to \frac{1}{5}e^{-3t}\left(e^{5t}+4\right)$$
$$y(t) \to -\frac{2}{5}e^{-3t}\left(e^{5t}-1\right)$$

10.12 problem 11 (b)

Internal problem ID [13087]
Internal file name [`OUTPUT/11742_Sunday_December_03_2023_07_16_14_PM_48038817/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 11 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) - 2y \\
y' &= -2x(t) + y
\end{aligned}
$$

With initial conditions

$$
[x(0) = 0, y(0) = 1]
$$

10.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{(e^{5t}+4)e^{-3t}}{5} & -\frac{2(e^{5t}-1)e^{-3t}}{5} \\ -\frac{2(e^{5t}-1)e^{-3t}}{5} & \frac{(4\,e^{5t}+1)e^{-3t}}{5} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{(e^{5t}+4)e^{-3t}}{5} & -\frac{2(e^{5t}-1)e^{-3t}}{5} \\ -\frac{2(e^{5t}-1)e^{-3t}}{5} & \frac{(4e^{5t}+1)e^{-3t}}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2(e^{5t}-1)e^{-3t}}{5} \\ \frac{(4e^{5t}+1)e^{-3t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-3t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{2t}}{2} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,e^{-3t} \\ e^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{(c_1 e^{5t} - 4c_2)e^{-3t}}{2} \\ (c_1 e^{5t} + c_2)\,e^{-3t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{c_1}{2} + 2c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{4}{5} \\ c_2 = \frac{1}{5} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\left(\frac{4\,e^{5t}}{5}-\frac{4}{5}\right)e^{-3t}}{2} \\ \left(\frac{1}{5}+\frac{4\,e^{5t}}{5}\right)e^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

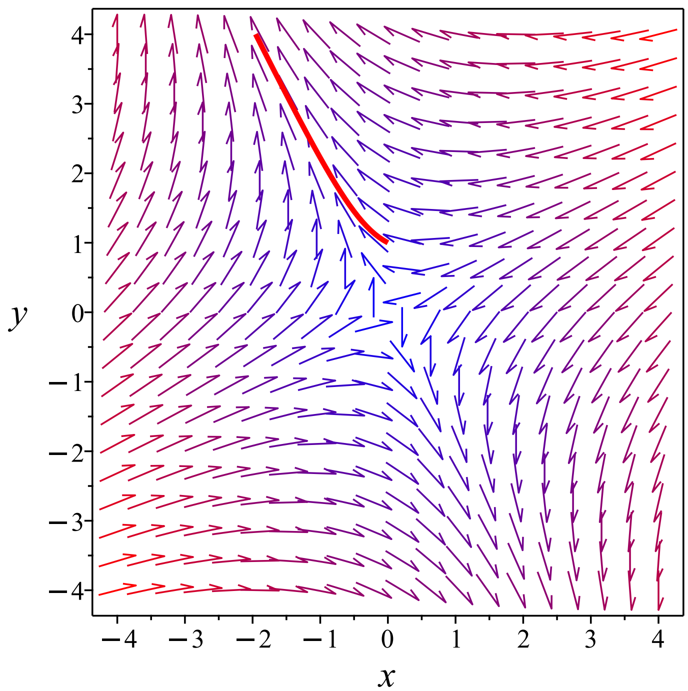
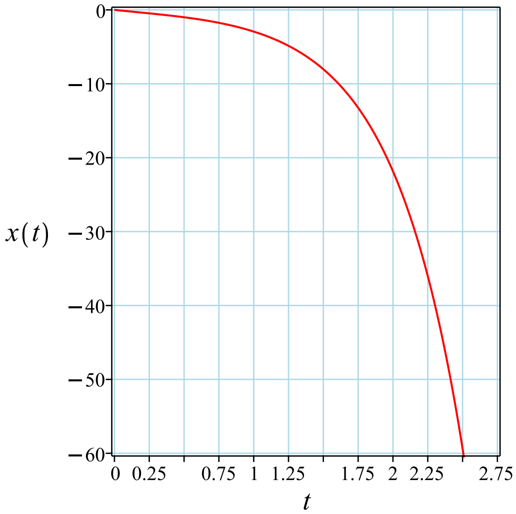
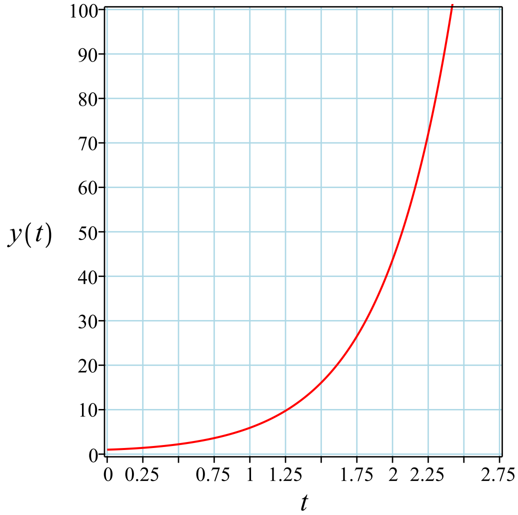


Figure 401: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = -2*x(t)-2*y(t), diff(y(t),t) = -2*x(t)+y(t), x(0) = 0, y(0) = 1], sin
```

$$x(t) = -\frac{2\,\mathrm{e}^{2t}}{5} + \frac{2\,\mathrm{e}^{-3t}}{5}$$
$$y(t) = \frac{4\,\mathrm{e}^{2t}}{5} + \frac{\mathrm{e}^{-3t}}{5}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 42

```
DSolve[{x'[t]==-2*x[t]-2*y[t],y'[t]==-2*x[t]+y[t]},{x[0]==0,y[0]==1},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to -\frac{2}{5}e^{-3t}\left(e^{5t}-1\right)$$
$$y(t) \to \frac{1}{5}e^{-3t}\left(4e^{5t}+1\right)$$

10.13 problem 11 (c)

Internal problem ID [13088]
Internal file name [`OUTPUT/11743_Sunday_December_03_2023_07_16_14_PM_39292626/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 11 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) - 2y \\
y' &= -2x(t) + y
\end{aligned}
$$

With initial conditions

$$[x(0) = 1, y(0) = -2]$$

10.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{5t}+4)e^{-3t}}{5} & -\frac{2(e^{5t}-1)e^{-3t}}{5} \\ -\frac{2(e^{5t}-1)e^{-3t}}{5} & \frac{(4\,e^{5t}+1)e^{-3t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{(e^{5t}+4)e^{-3t}}{5} & -\frac{2(e^{5t}-1)e^{-3t}}{5} \\ -\frac{2(e^{5t}-1)e^{-3t}}{5} & \frac{(4e^{5t}+1)e^{-3t}}{5} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(e^{5t}+4)e^{-3t}}{5} + \frac{4(e^{5t}-1)e^{-3t}}{5} \\ -\frac{2(e^{5t}-1)e^{-3t}}{5} - \frac{2(4e^{5t}+1)e^{-3t}}{5} \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array} \right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -4 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{2t}$$
$$= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{2t}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{2t}}{2} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,e^{-3t} \\ e^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{(c_1 e^{5t} - 4c_2)e^{-3t}}{2} \\ (c_1 e^{5t} + c_2)\,e^{-3t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = -2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{c_1}{2} + 2c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -2 \\ c_2 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}\mathrm{e}^{5t} \\ -2\,\mathrm{e}^{-3t}\mathrm{e}^{5t} \end{bmatrix}$$

The following is the phase plot of the system.

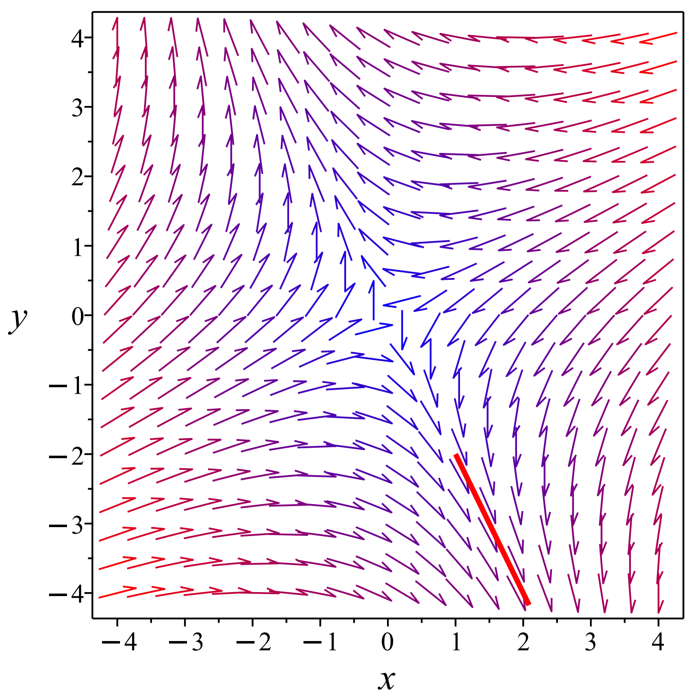
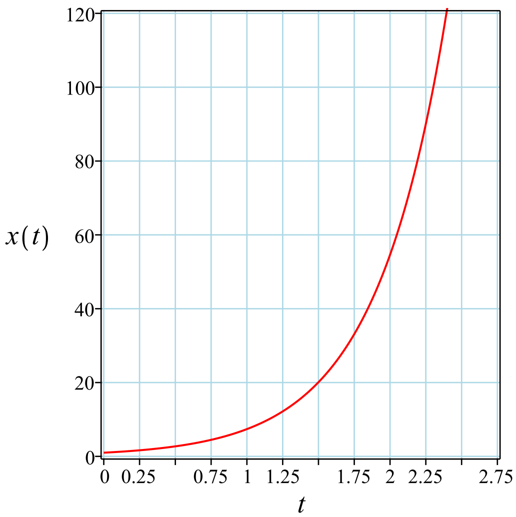
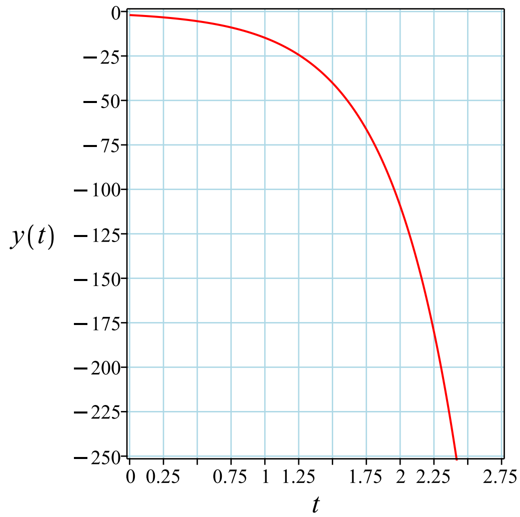


Figure 402: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([diff(x(t),t) = -2*x(t)-2*y(t), diff(y(t),t) = -2*x(t)+y(t), x(0) = 1, y(0) = -2], si
```

$$x(t) = e^{2t}$$
$$y(t) = -2\, e^{2t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 20

```
DSolve[{x'[t]==-2*x[t]-2*y[t],y'[t]==-2*x[t]+y[t]},{x[0]==1,y[0]==-2},{x[t],y[t]},t,IncludeS
```

$$x(t) \to e^{2t}$$
$$y(t) \to -2e^{2t}$$

10.14 problem 12 (a)

Internal problem ID [13089]
Internal file name [`OUTPUT/11744_Sunday_December_03_2023_07_16_15_PM_83843814/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 12 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3x(t)$$
$$y' = x(t) - 2y$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

10.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t} & 0 \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{3t} & 0 \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t} \\ \frac{(e^{5t}-1)e^{-2t}}{5} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 0 \\ 1 & -2-\lambda \end{bmatrix} \right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(3-\lambda)(-2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 5 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1, 1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 5t\}$

Hence the solution is

$$\begin{bmatrix} 5t \\ t \end{bmatrix} = \begin{bmatrix} 5t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 5t \\ t \end{bmatrix} = t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 5t \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-2t}$$
$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{3t}$$
$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{3t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} 5\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 5c_2 \mathrm{e}^{3t} \\ \left(c_2 \mathrm{e}^{5t} + c_1\right) \mathrm{e}^{-2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5c_2 \\ c_2 + c_1 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -\frac{1}{5} \\ c_2 = \frac{1}{5} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} e^{3t} \\ \left(\frac{e^{5t}}{5} - \frac{1}{5}\right) e^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

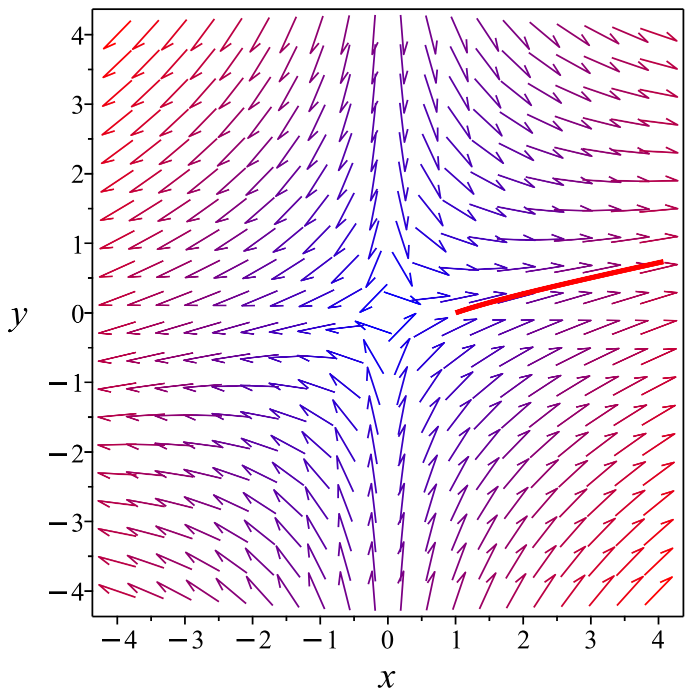
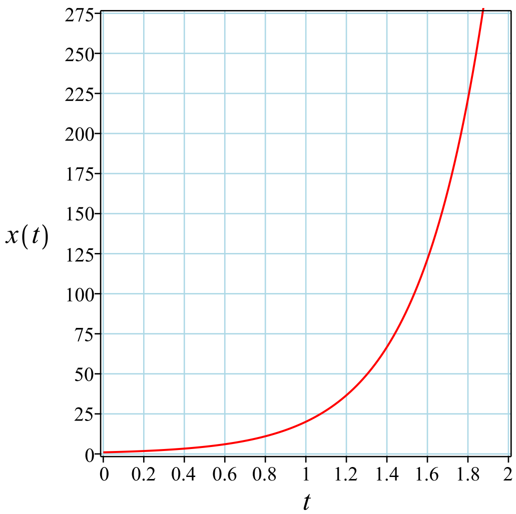
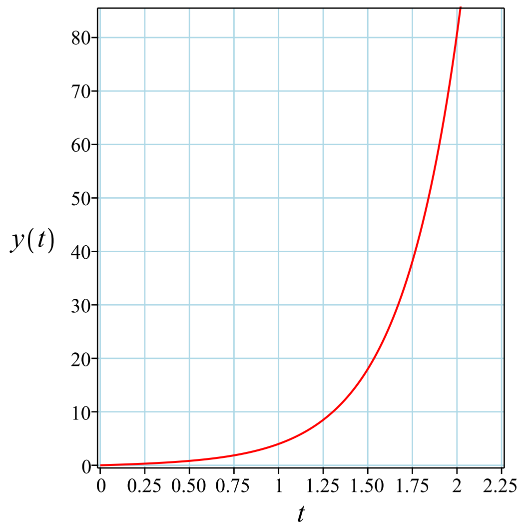


Figure 403: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve([diff(x(t),t) = 3*x(t), diff(y(t),t) = x(t)-2*y(t), x(0) = 1, y(0) = 0], singsol=all)
```

$$x(t) = e^{3t}$$
$$y(t) = \frac{e^{3t}}{5} - \frac{e^{-2t}}{5}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 29

```
DSolve[{x'[t]==3*x[t],y'[t]==x[t]-2*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSingularSol
```

$$x(t) \to e^{3t}$$
$$y(t) \to \frac{1}{5}e^{-2t}\left(e^{5t} - 1\right)$$

10.15 problem 12 (b)

Internal problem ID [13090]
Internal file name [`OUTPUT/11745_Sunday_December_03_2023_07_16_15_PM_83669804/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 12 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3x(t)$$
$$y' = x(t) - 2y$$

With initial conditions

$$[x(0) = 0, y(0) = 1]$$

10.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t} & 0 \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{3t} & 0 \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & 0 \\ 1 & -2-\lambda \end{bmatrix} \right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$
(3-\lambda)(-2-\lambda) = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 5 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1, 1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 5t\}$

Hence the solution is

$$\begin{bmatrix} 5t \\ t \end{bmatrix} = \begin{bmatrix} 5t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 5t \\ t \end{bmatrix} = t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 5t \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 5 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{3t}$$
$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{3t}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-2t}$$
$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 5\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 5c_1\mathrm{e}^{3t} \\ (c_1\mathrm{e}^{5t} + c_2)\,\mathrm{e}^{-2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 0 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

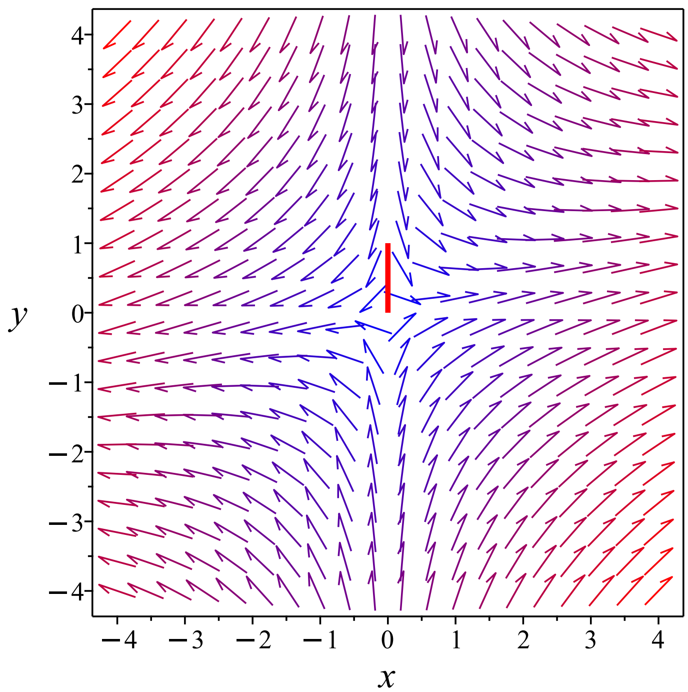
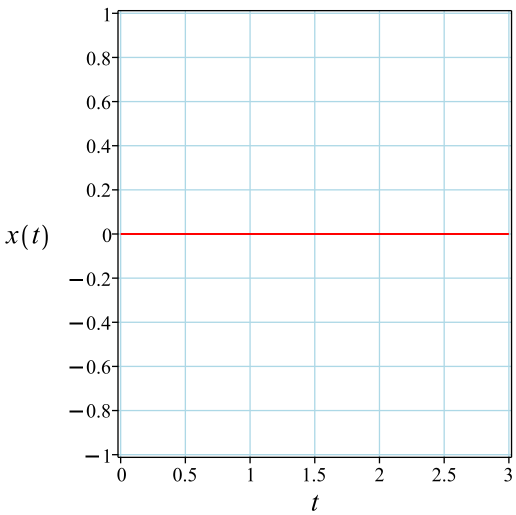
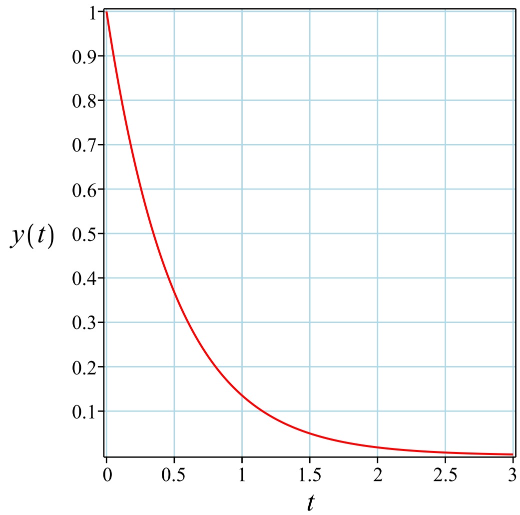


Figure 404: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(x(t),t) = 3*x(t), diff(y(t),t) = x(t)-2*y(t), x(0) = 0, y(0) = 1], singsol=all)
```

$$x(t) = 0$$
$$y(t) = e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 14

```
DSolve[{x'[t]==3*x[t],y'[t]==x[t]-2*y[t]},{x[0]==0,y[0]==1},{x[t],y[t]},t,IncludeSingularSol
```

$$x(t) \to 0$$
$$y(t) \to e^{-2t}$$

10.16 problem 12 (c)

Internal problem ID [13091]
Internal file name [`OUTPUT/11746_Sunday_December_03_2023_07_16_15_PM_18260334/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 12 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3x(t)$$
$$y' = x(t) - 2y$$

With initial conditions

$$[x(0) = 2, y(0) = 2]$$

10.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t} & 0 \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{3t} & 0 \\ \frac{(e^{5t}-1)e^{-2t}}{5} & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2\,e^{3t} \\ \frac{2(e^{5t}-1)e^{-2t}}{5} + 2\,e^{-2t} \end{bmatrix} \\
&= \begin{bmatrix} 2\,e^{3t} \\ \frac{2(e^{5t}+4)e^{-2t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & 0 \\ 1 & -2-\lambda \end{bmatrix} \right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(3-\lambda)(-2-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= 3\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\begin{aligned}\left(\begin{bmatrix}3 & 0\\ 1 & -2\end{bmatrix} - (-2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}5 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}5 & 0 & 0\\ 1 & 0 & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{cc|c}5 & 0 & 0\\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 5t\}$

Hence the solution is

$$\begin{bmatrix} 5t \\ t \end{bmatrix} = \begin{bmatrix} 5t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 5t \\ t \end{bmatrix} = t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 5t \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t} \\ &= \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} 5\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 5c_2 \mathrm{e}^{3t} \\ (c_2 \mathrm{e}^{5t} + c_1)\,\mathrm{e}^{-2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 2 \\ y(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5c_2 \\ c_2 + c_1 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{8}{5} \\ c_2 = \frac{2}{5} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 2\,e^{3t} \\ \left(\frac{2\,e^{5t}}{5} + \frac{8}{5}\right) e^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

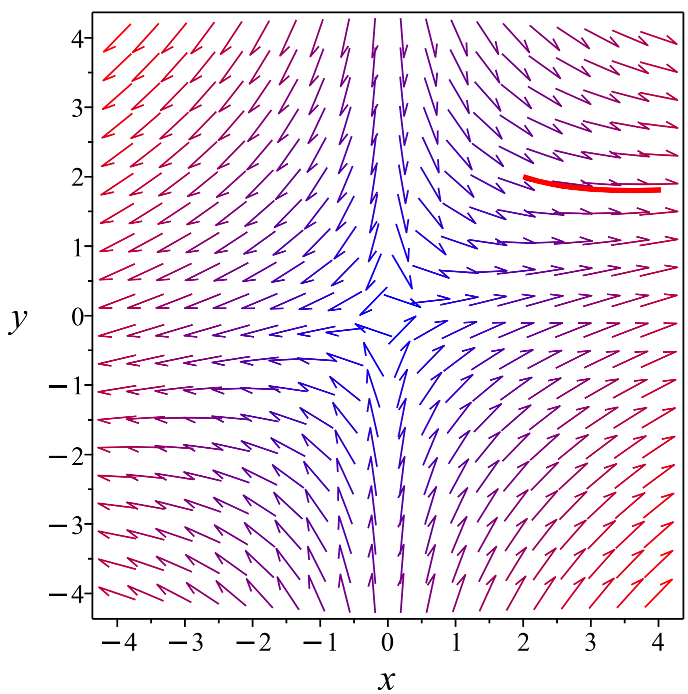
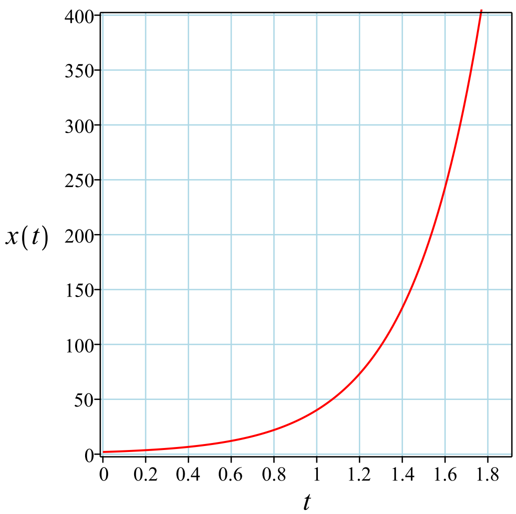
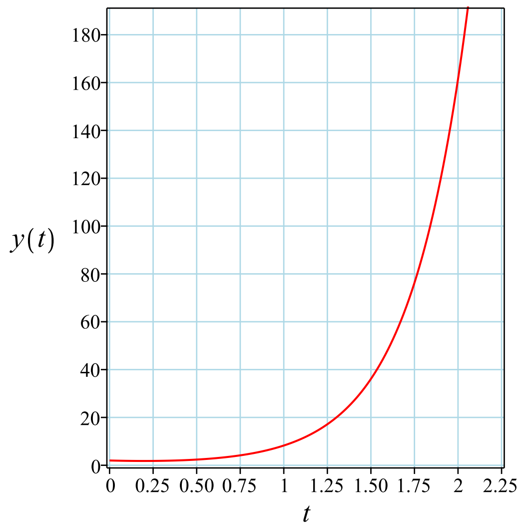


Figure 405: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve([diff(x(t),t) = 3*x(t), diff(y(t),t) = x(t)-2*y(t), x(0) = 2, y(0) = 2], singsol=all)
```

$$x(t) = 2\,e^{3t}$$
$$y(t) = \frac{2\,e^{3t}}{5} + \frac{8\,e^{-2t}}{5}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 31

```
DSolve[{x'[t]==3*x[t],y'[t]==x[t]-2*y[t]},{x[0]==2,y[0]==2},{x[t],y[t]},t,IncludeSingularSol
```

$$x(t) \to 2e^{3t}$$
$$y(t) \to \frac{2}{5}e^{-2t}\left(e^{5t} + 4\right)$$

10.17 problem 13 (a)

Internal problem ID [13092]
Internal file name [OUTPUT/11747_Sunday_December_03_2023_07_16_16_PM_91697910/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 13 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -4x(t) + y \\ y' &= 2x(t) - 3y \end{aligned}$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

10.17.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,e^{-5t}}{3} + \frac{e^{-2t}}{3} & \frac{e^{-2t}}{3} - \frac{e^{-5t}}{3} \\ \frac{2\,e^{-2t}}{3} - \frac{2\,e^{-5t}}{3} & \frac{e^{-5t}}{3} + \frac{2\,e^{-2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{2\,e^{-5t}}{3} + \frac{e^{-2t}}{3} & \frac{e^{-2t}}{3} - \frac{e^{-5t}}{3} \\ \frac{2\,e^{-2t}}{3} - \frac{2\,e^{-5t}}{3} & \frac{e^{-5t}}{3} + \frac{2\,e^{-2t}}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\,e^{-5t}}{3} + \frac{e^{-2t}}{3} \\ \frac{2\,e^{-2t}}{3} - \frac{2\,e^{-5t}}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.17.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -4-\lambda & 1 \\ 2 & -3-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 + 7\lambda + 10 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-5t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{-2t}}{2} \\ e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-5t} \\ e^{-5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-2t}}{2} - c_2 e^{-5t} \\ c_1 e^{-2t} + c_2 e^{-5t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{2} - c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{2}{3} \\ c_2 = -\frac{2}{3} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{2\,e^{-5t}}{3} + \frac{e^{-2t}}{3} \\ \frac{2\,e^{-2t}}{3} - \frac{2\,e^{-5t}}{3} \end{bmatrix}$$

The following is the phase plot of the system.

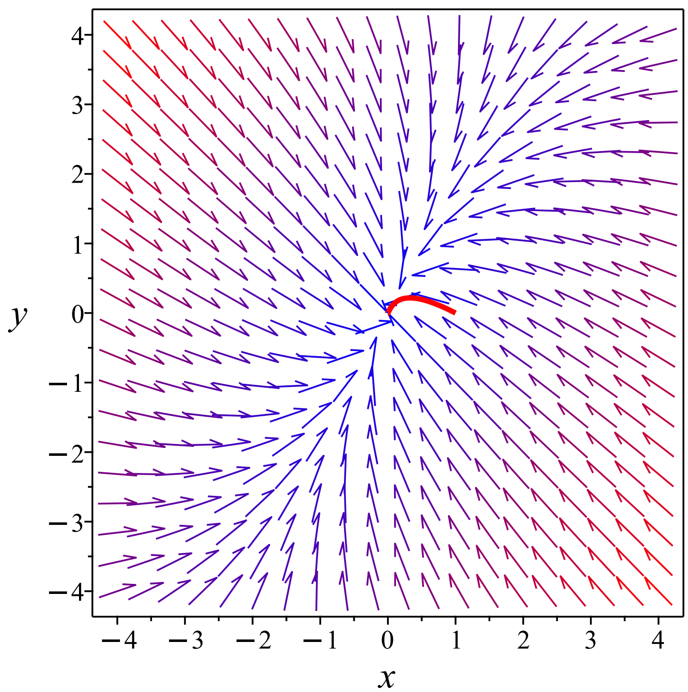
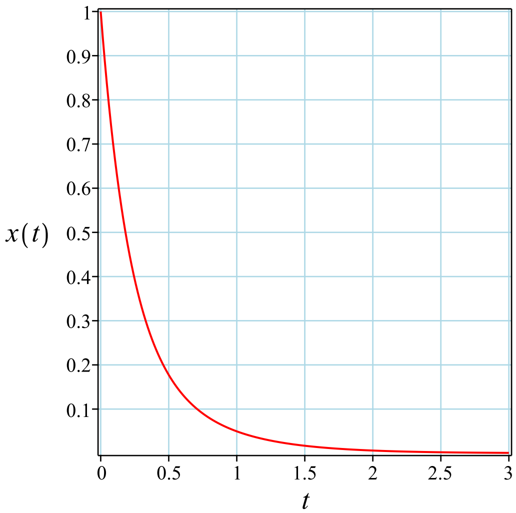
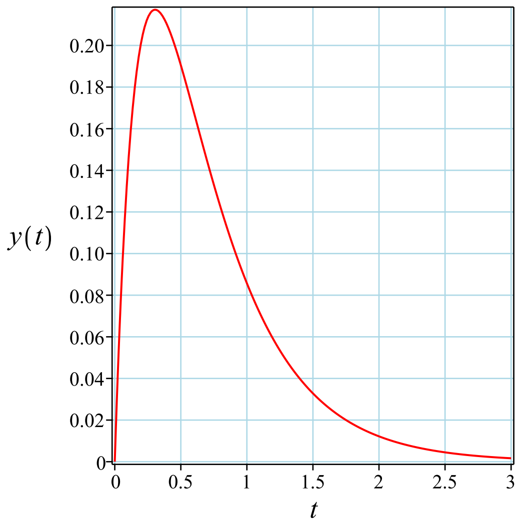


Figure 406: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = -4*x(t)+y(t), diff(y(t),t) = 2*x(t)-3*y(t), x(0) = 1, y(0) = 0], sing
```

$$x(t) = \frac{2\,\mathrm{e}^{-5t}}{3} + \frac{\mathrm{e}^{-2t}}{3}$$
$$y(t) = -\frac{2\,\mathrm{e}^{-5t}}{3} + \frac{2\,\mathrm{e}^{-2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 40

```
DSolve[{x'[t]==-4*x[t]+y[t],y'[t]==2*x[t]-3*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSin
```

$$x(t) \to \frac{1}{3}e^{-5t}\left(e^{3t}+2\right)$$
$$y(t) \to \frac{2}{3}e^{-5t}\left(e^{3t}-1\right)$$

10.18 problem 13 (b)

Internal problem ID [13093]
Internal file name [`OUTPUT/11748_Sunday_December_03_2023_07_16_16_PM_57675736/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 13 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -4x(t) + y \\
y' &= 2x(t) - 3y
\end{aligned}
$$

With initial conditions

$$[x(0) = 2, y(0) = 1]$$

10.18.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,e^{-5t}}{3} + \frac{e^{-2t}}{3} & \frac{e^{-2t}}{3} - \frac{e^{-5t}}{3} \\ \frac{2\,e^{-2t}}{3} - \frac{2\,e^{-5t}}{3} & \frac{e^{-5t}}{3} + \frac{2\,e^{-2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{2\,\mathrm{e}^{-5t}}{3}+\frac{\mathrm{e}^{-2t}}{3} & \frac{\mathrm{e}^{-2t}}{3}-\frac{\mathrm{e}^{-5t}}{3} \\ \frac{2\,\mathrm{e}^{-2t}}{3}-\frac{2\,\mathrm{e}^{-5t}}{3} & \frac{\mathrm{e}^{-5t}}{3}+\frac{2\,\mathrm{e}^{-2t}}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{-5t}+\mathrm{e}^{-2t} \\ 2\,\mathrm{e}^{-2t}-\mathrm{e}^{-5t} \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.18.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & 1 \\ 2 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 7\lambda + 10 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -5$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-5	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -5 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-5t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-2t}$$
$$= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-5t} + \frac{c_2 \mathrm{e}^{-2t}}{2} \\ c_1 \mathrm{e}^{-5t} + c_2 \mathrm{e}^{-2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 2 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1 + \frac{c_2}{2} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = 2 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} e^{-5t} + e^{-2t} \\ 2\,e^{-2t} - e^{-5t} \end{bmatrix}$$

The following is the phase plot of the system.

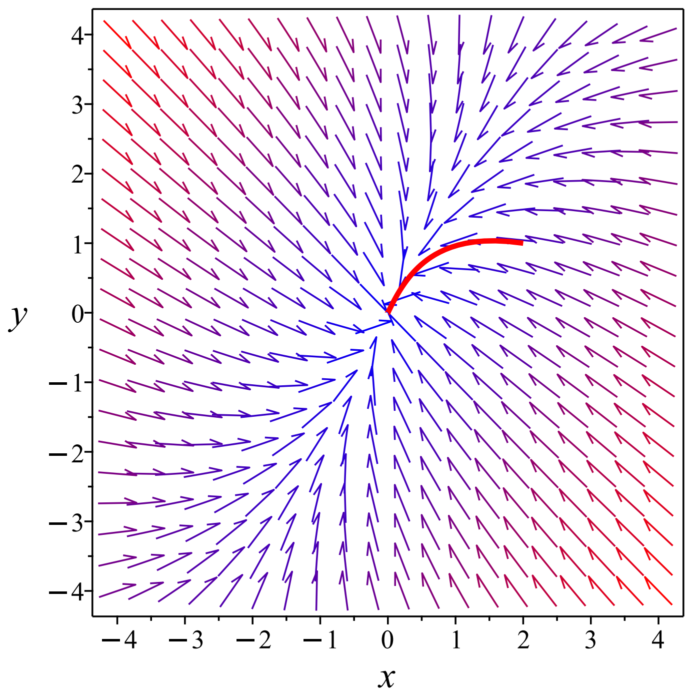
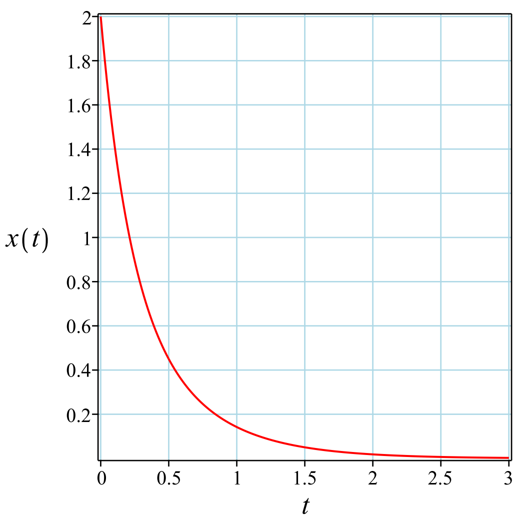
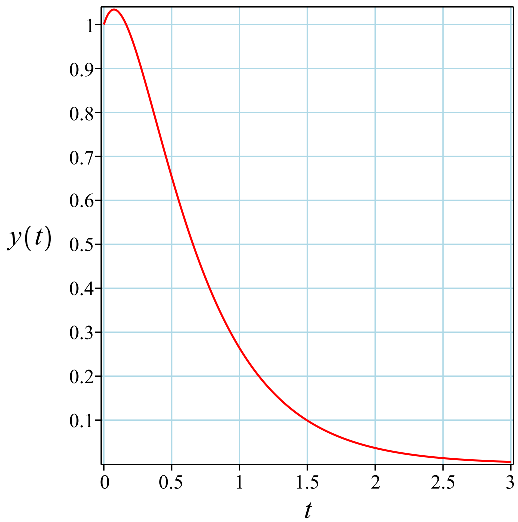


Figure 407: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve([diff(x(t),t) = -4*x(t)+y(t), diff(y(t),t) = 2*x(t)-3*y(t), x(0) = 2, y(0) = 1], sing
```

$$x(t) = e^{-5t} + e^{-2t}$$
$$y(t) = -e^{-5t} + 2e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 34

```
DSolve[{x'[t]==-4*x[t]+y[t],y'[t]==2*x[t]-3*y[t]},{x[0]==2,y[0]==1},{x[t],y[t]},t,IncludeSin
```

$$x(t) \to e^{-5t} + e^{-2t}$$
$$y(t) \to e^{-5t}\left(2e^{3t} - 1\right)$$

10.19 problem 13 (c)

Internal problem ID [13094]
Internal file name [`OUTPUT/11749_Sunday_December_03_2023_07_16_17_PM_40395156/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 13 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -4x(t) + y \\
y' &= 2x(t) - 3y
\end{aligned}
$$

With initial conditions

$$
[x(0) = -1, y(0) = -2]
$$

10.19.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{2\,e^{-5t}}{3} + \frac{e^{-2t}}{3} & \frac{e^{-2t}}{3} - \frac{e^{-5t}}{3} \\ \frac{2\,e^{-2t}}{3} - \frac{2\,e^{-5t}}{3} & \frac{e^{-5t}}{3} + \frac{2\,e^{-2t}}{3} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{2\,e^{-5t}}{3} + \frac{e^{-2t}}{3} & \frac{e^{-2t}}{3} - \frac{e^{-5t}}{3} \\ \frac{2\,e^{-2t}}{3} - \frac{2\,e^{-5t}}{3} & \frac{e^{-5t}}{3} + \frac{2\,e^{-2t}}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} -e^{-2t} \\ -2\,e^{-2t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.19.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & 1 \\ 2 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 7\lambda + 10 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -5$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \implies \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-5	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -5 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-5t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-2t}$$
$$= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -e^{-5t} \\ e^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{-2t}}{2} \\ e^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 e^{-5t} + \frac{c_2 e^{-2t}}{2} \\ c_1 e^{-5t} + c_2 e^{-2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = -1 \\ y(0) = -2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -c_1 + \frac{c_2}{2} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 0 \\ c_2 = -2 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{-2t} \\ -2\,\mathrm{e}^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

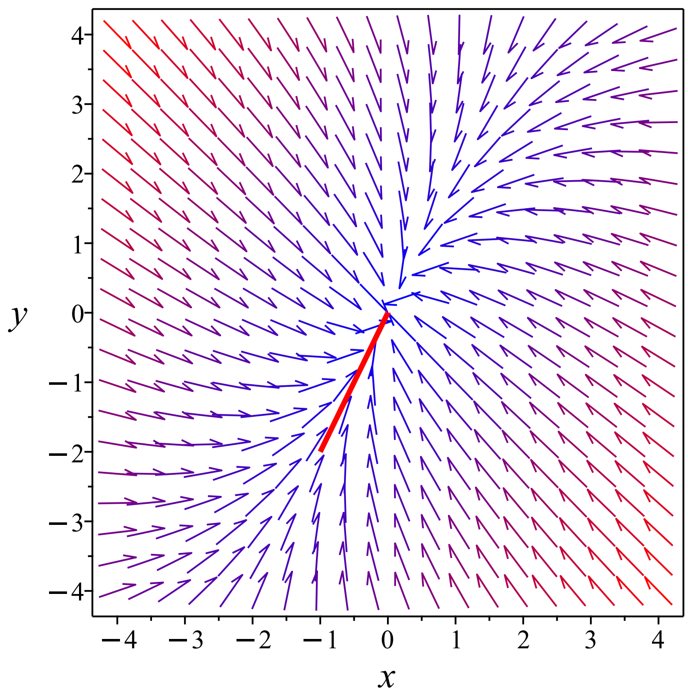
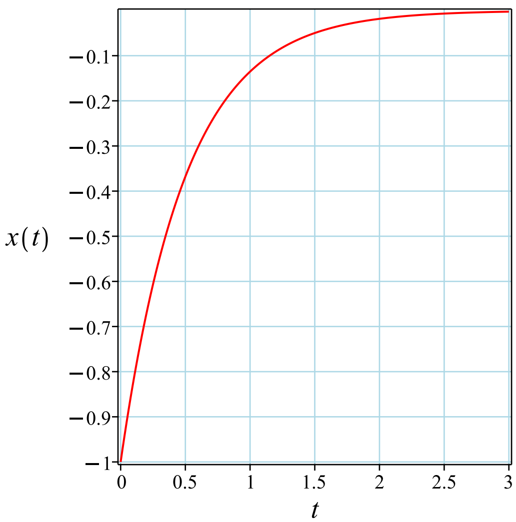
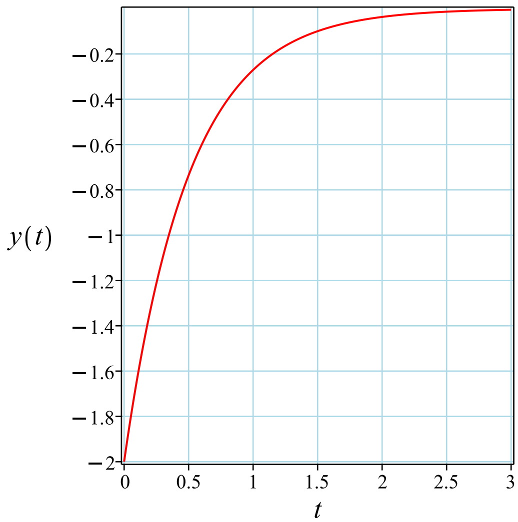


Figure 408: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve([diff(x(t),t) = -4*x(t)+y(t), diff(y(t),t) = 2*x(t)-3*y(t), x(0) = -1, y(0) = -2], si
```

$$x(t) = -e^{-2t}$$
$$y(t) = -2\,e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 22

```
DSolve[{x'[t]==-4*x[t]+y[t],y'[t]==2*x[t]-3*y[t]},{x[0]==-1,y[0]==-2},{x[t],y[t]},t,IncludeS
```

$$x(t) \to -e^{-2t}$$
$$y(t) \to -2e^{-2t}$$

10.20 problem 14 (a)

Internal problem ID [13095]
Internal file name [`OUTPUT/11750_Sunday_December_03_2023_07_16_17_PM_30950620/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 14 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 4x(t) - 2y$$
$$y' = x(t) + y$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

10.20.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -e^{2t} + 2\,e^{3t} & -2\,e^{3t} + 2\,e^{2t} \\ e^{3t} - e^{2t} & 2\,e^{2t} - e^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} -\mathrm{e}^{2t} + 2\,\mathrm{e}^{3t} & -2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{2t} \\ \mathrm{e}^{3t} - \mathrm{e}^{2t} & 2\,\mathrm{e}^{2t} - \mathrm{e}^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{e}^{2t} + 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} - \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.20.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{2t} + 2c_2 \mathrm{e}^{3t} \\ c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{3t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{2t} + 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} - \mathrm{e}^{2t} \end{bmatrix}$$

The following is the phase plot of the system.

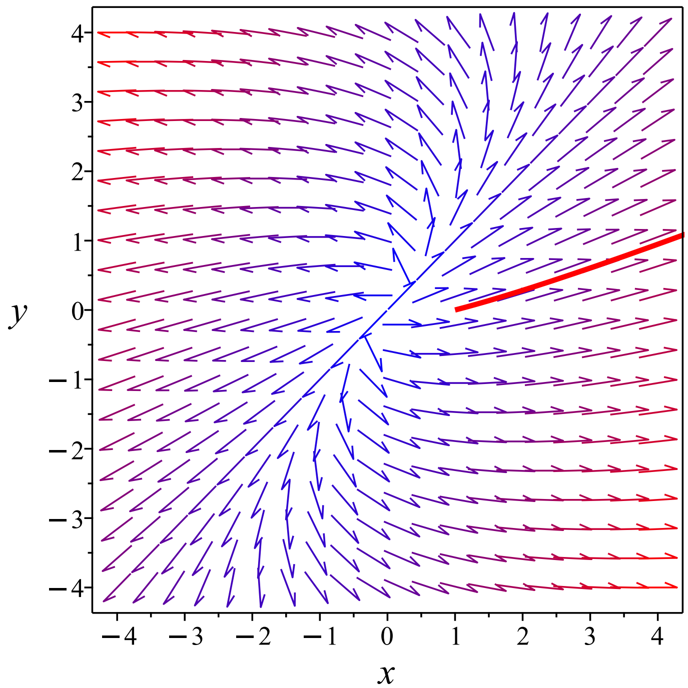
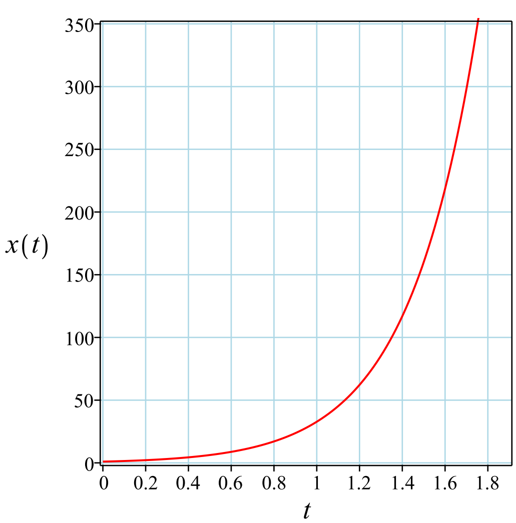
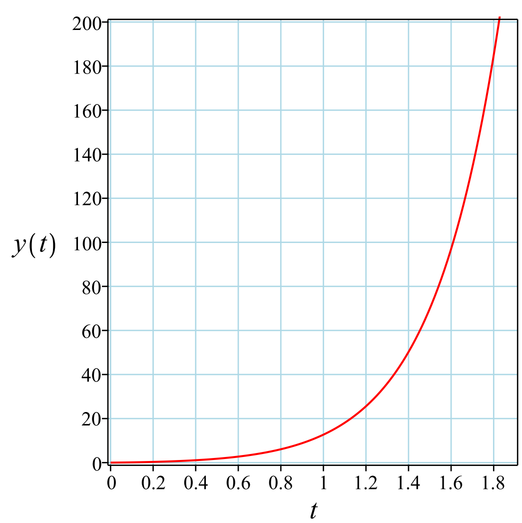


Figure 409: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 32

```
dsolve([diff(x(t),t) = 4*x(t)-2*y(t), diff(y(t),t) = x(t)+y(t), x(0) = 1, y(0) = 0], singsol
```

$$x(t) = 2\,\mathrm{e}^{3t} - \mathrm{e}^{2t}$$
$$y(t) = \mathrm{e}^{3t} - \mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 32

```
DSolve[{x'[t]==4*x[t]-2*y[t],y'[t]==x[t]+y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSingul
```

$$x(t) \to e^{2t}\left(2e^t - 1\right)$$
$$y(t) \to e^{2t}\left(e^t - 1\right)$$

10.21 problem 14 (b)

Internal problem ID [13096]
Internal file name [`OUTPUT/11751_Sunday_December_03_2023_07_16_17_PM_83864725/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 14 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 4x(t) - 2y$$
$$y' = x(t) + y$$

With initial conditions

$$[x(0) = 2, y(0) = 1]$$

10.21.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -e^{2t} + 2\,e^{3t} & -2\,e^{3t} + 2\,e^{2t} \\ e^{3t} - e^{2t} & 2\,e^{2t} - e^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} -e^{2t} + 2\,e^{3t} & -2\,e^{3t} + 2\,e^{2t} \\ e^{3t} - e^{2t} & 2\,e^{2t} - e^{3t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2\,e^{3t} \\ e^{3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.21.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{2t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 2c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 2 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 1 \\ c_2 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$$

The following is the phase plot of the system.

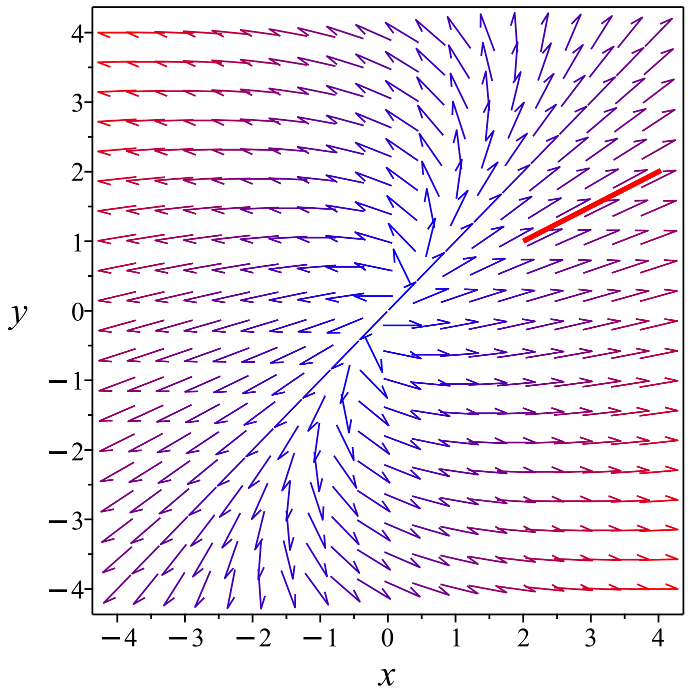
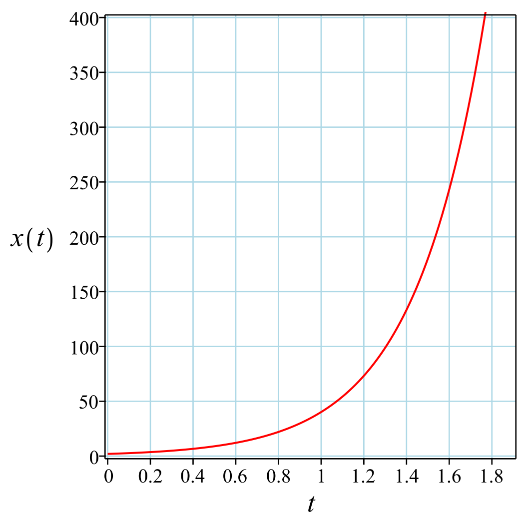
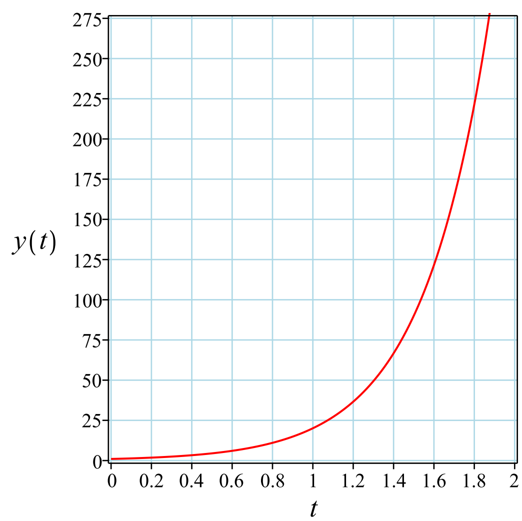


Figure 410: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([diff(x(t),t) = 4*x(t)-2*y(t), diff(y(t),t) = x(t)+y(t), x(0) = 2, y(0) = 1], singsol
```

$$x(t) = 2\,\mathrm{e}^{3t}$$
$$y(t) = \mathrm{e}^{3t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 20

```
DSolve[{x'[t]==4*x[t]-2*y[t],y'[t]==x[t]+y[t]},{x[0]==2,y[0]==1},{x[t],y[t]},t,IncludeSingul
```

$$x(t) \to 2e^{3t}$$
$$y(t) \to e^{3t}$$

10.22 problem 14 (c)

Internal problem ID [13097]
Internal file name [`OUTPUT/11752_Sunday_December_03_2023_07_16_18_PM_8293298/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.2. page 277
Problem number: 14 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 4x(t) - 2y \\
y' &= x(t) + y
\end{aligned}
$$

With initial conditions

$$
[x(0) = -1, y(0) = -2]
$$

10.22.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -e^{2t} + 2\,e^{3t} & -2\,e^{3t} + 2\,e^{2t} \\ e^{3t} - e^{2t} & 2\,e^{2t} - e^{3t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} -\mathrm{e}^{2t} + 2\,\mathrm{e}^{3t} & -2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{2t} \\ \mathrm{e}^{3t} - \mathrm{e}^{2t} & 2\,\mathrm{e}^{2t} - \mathrm{e}^{3t} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} -3\,\mathrm{e}^{2t} + 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} - 3\,\mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

10.22.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 2c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = -1 \\ y(0) = -2 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$
\begin{bmatrix} c_1 = 1 \\ c_2 = -3 \end{bmatrix}
$$

Substituting these constants back in original solution in Eq. (1) gives

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -3\,\mathrm{e}^{2t} + 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} - 3\,\mathrm{e}^{2t} \end{bmatrix}
$$

The following is the phase plot of the system.

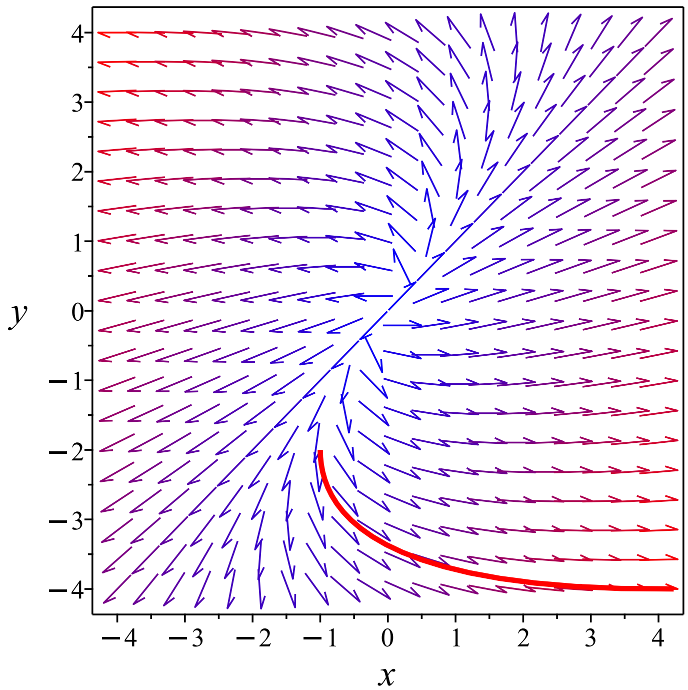
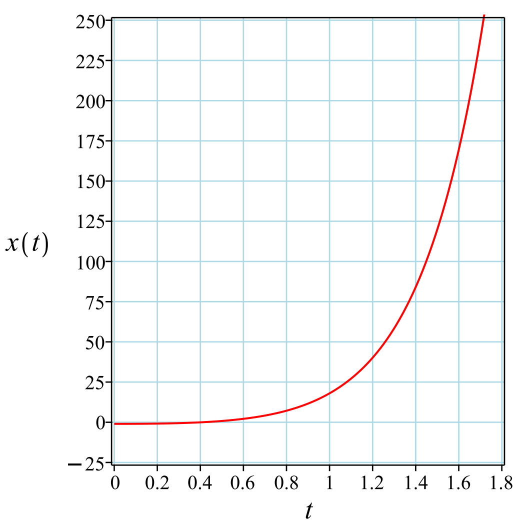
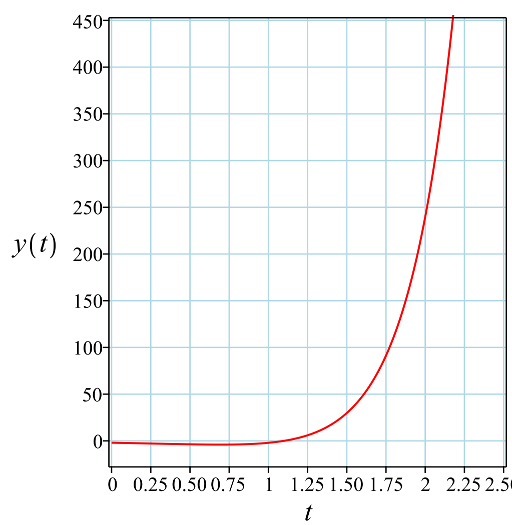


Figure 411: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve([diff(x(t),t) = 4*x(t)-2*y(t), diff(y(t),t) = x(t)+y(t), x(0) = -1, y(0) = -2], sings
```

$$x(t) = 2\,e^{3t} - 3\,e^{2t}$$
$$y(t) = e^{3t} - 3\,e^{2t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 32

```
DSolve[{x'[t]==4*x[t]-2*y[t],y'[t]==x[t]+y[t]},{x[0]==-1,y[0]==-2},{x[t],y[t]},t,IncludeSing
```

$$x(t) \to e^{2t}(2e^t - 3)$$
$$y(t) \to e^{2t}(e^t - 3)$$

11 Chapter 3. Linear Systems. Exercises section 3.4 page 310

11.1 problem 3

Internal problem ID [13098]
Internal file name [`OUTPUT/11753_Sunday_December_03_2023_07_16_18_PM_69937057/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2y \\
y' &= -2x(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = 0]
$$

11.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 + 4 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2i$	1	complex eigenvalue
$-2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - (-2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - (2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -i \mathrm{e}^{2it} \\ \mathrm{e}^{2it} \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{-2it} \\ \mathrm{e}^{-2it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} i(c_2 e^{-2it} - c_1 e^{2it}) \\ c_1 e^{2it} + c_2 e^{-2it} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -i(c_1 - c_2) \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{i}{2} \\ c_2 = -\frac{i}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} i\left(-\frac{ie^{-2it}}{2} - \frac{ie^{2it}}{2}\right) \\ \frac{ie^{2it}}{2} - \frac{ie^{-2it}}{2} \end{bmatrix}$$

The following is the phase plot of the system.

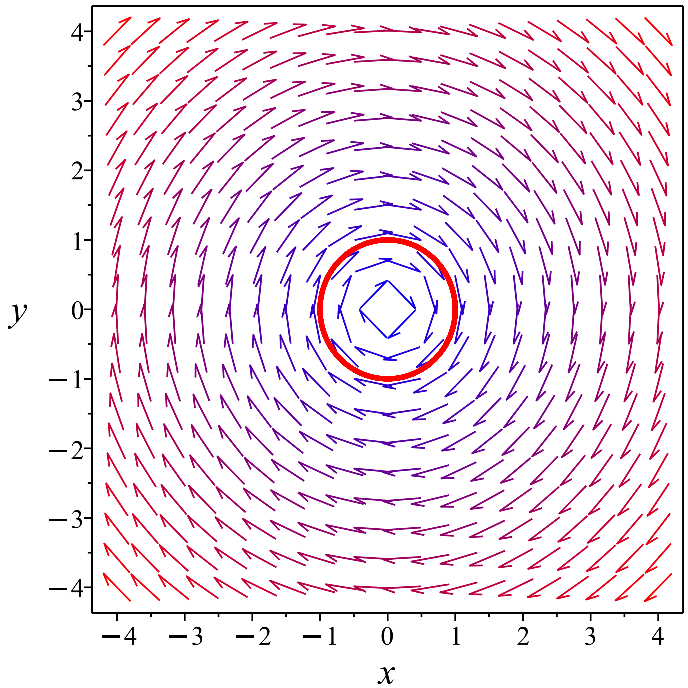


Figure 412: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(x(t),t) = 2*y(t), diff(y(t),t) = -2*x(t), x(0) = 1, y(0) = 0], singsol=all)
```

$$x(t) = \cos(2t)$$
$$y(t) = -\sin(2t)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 18

```
DSolve[{x'[t]==2*y[t],y'[t]==-2*x[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSingularSolutio
```

$$x(t) \to \cos(2t)$$
$$y(t) \to -\sin(2t)$$

11.2 problem 4

Internal problem ID [13099]
Internal file name [`OUTPUT/11754_Sunday_December_03_2023_07_16_19_PM_42285339/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 2x(t) + 2y \\ y' &= -4x(t) + 6y \end{aligned}$$

With initial conditions

$$[x(0) = 1, y(0) = 1]$$

11.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{4t}\cos(2t) - e^{4t}\sin(2t) & e^{4t}\sin(2t) \\ -2\,e^{4t}\sin(2t) & e^{4t}\cos(2t) + e^{4t}\sin(2t) \end{bmatrix} \\ &= \begin{bmatrix} e^{4t}(\cos(2t) - \sin(2t)) & e^{4t}\sin(2t) \\ -2\,e^{4t}\sin(2t) & e^{4t}(\sin(2t) + \cos(2t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{4t}(\cos(2t)-\sin(2t)) & e^{4t}\sin(2t) \\ -2\,e^{4t}\sin(2t) & e^{4t}(\sin(2t)+\cos(2t)) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}(\cos(2t)-\sin(2t)) + e^{4t}\sin(2t) \\ -2\,e^{4t}\sin(2t) + e^{4t}(\sin(2t)+\cos(2t)) \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}\cos(2t) \\ e^{4t}(\cos(2t)-\sin(2t)) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & 2 \\ -4 & 6-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 20 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4 + 2i$$
$$\lambda_2 = 4 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$4 + 2i$	1	complex eigenvalue
$4 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} - (4 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 + 2i & 2 \\ -4 & 2 + 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 + 2i & 2 & 0 \\ -4 & 2 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 + 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 + 2i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} - (4 + 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 - 2i & 2 \\ -4 & 2 - 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 - 2i & 2 & 0 \\ -4 & 2 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 + i) R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 - 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-2i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$4+2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$
$4-2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(4+2i)t} \\ \mathrm{e}^{(4+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(4-2i)t} \\ \mathrm{e}^{(4-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{(4+2i)t} + \left(\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{(4-2i)t} \\ c_1 \mathrm{e}^{(4+2i)t} + c_2 \mathrm{e}^{(4-2i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) c_1 + \left(\frac{1}{2}+\frac{i}{2}\right) c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{1}{2}+\frac{i}{2} \\ c_2 = \frac{1}{2}-\frac{i}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{e^{(4+2i)t}}{2} + \frac{e^{(4-2i)t}}{2} \\ \left(\frac{1}{2} - \frac{i}{2}\right) e^{(4-2i)t} + \left(\frac{1}{2} + \frac{i}{2}\right) e^{(4+2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

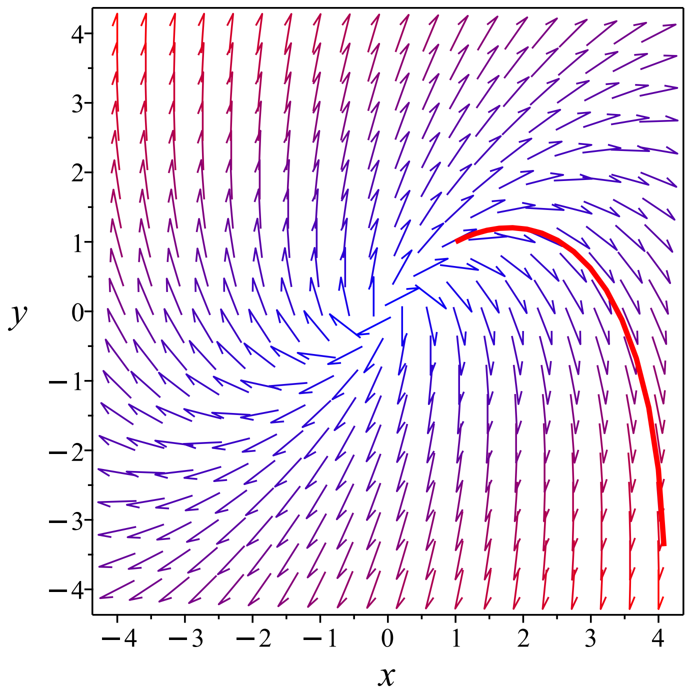


Figure 413: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 33

```
dsolve([diff(x(t),t) = 2*x(t)+2*y(t), diff(y(t),t) = -4*x(t)+6*y(t), x(0) = 1, y(0) = 1], si
```

$$x(t) = e^{4t} \cos(2t)$$
$$y(t) = e^{4t}(\cos(2t) - \sin(2t))$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 35

```
DSolve[{x'[t]==2*x[t]+2*y[t],y'[t]==-4*x[t]+6*y[t]},{x[0]==1,y[0]==1},{x[t],y[t]},t,IncludeS
```

$$x(t) \to e^{4t} \cos(2t)$$
$$y(t) \to e^{4t}(\cos(2t) - \sin(2t))$$

11.3 problem 5

Internal problem ID [13100]
Internal file name [`OUTPUT/11755_Sunday_December_03_2023_07_16_19_PM_13150055/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -3x(t) - 5y$$
$$y' = 3x(t) + y$$

With initial conditions

$$[x(0) = 4, y(0) = 0]$$

11.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos\left(\sqrt{11}\,t\right) - \frac{2\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} & -\frac{5\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \\ \frac{3\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} & e^{-t}\cos\left(\sqrt{11}\,t\right) + \frac{2\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}\left(\cos\left(\sqrt{11}\,t\right) - \frac{2\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11}\right) & -\frac{5\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \\ \frac{3\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} & \frac{e^{-t}\left(2\sin\left(\sqrt{11}\,t\right)\sqrt{11}+11\cos\left(\sqrt{11}\,t\right)\right)}{11} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$

$$= \begin{bmatrix} e^{-t}\left(\cos\left(\sqrt{11}\,t\right) - \frac{2\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11}\right) & -\frac{5\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \\ \frac{3\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} & \frac{e^{-t}\left(2\sin\left(\sqrt{11}\,t\right)\sqrt{11}+11\cos\left(\sqrt{11}\,t\right)\right)}{11} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4\,e^{-t}\left(\cos\left(\sqrt{11}\,t\right) - \frac{2\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11}\right) \\ \frac{12\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-3 & -5\\ 3 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-3-\lambda & -5\\ 3 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 12 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + i\sqrt{11}$$
$$\lambda_2 = -1 - i\sqrt{11}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - i\sqrt{11}$	1	complex eigenvalue
$-1 + i\sqrt{11}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - i\sqrt{11}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-3 & -5\\ 3 & 1\end{bmatrix} - \left(-1 - i\sqrt{11}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-2 + i\sqrt{11} & -5\\ 3 & 2 + i\sqrt{11}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-2 + i\sqrt{11} & -5 & 0\\ 3 & 2 + i\sqrt{11} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-2+i\sqrt{11}} \Longrightarrow \left[\begin{array}{cc|c} -2+i\sqrt{11} & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2+i\sqrt{11} & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{-2+i\sqrt{11}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{-2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{-2+i\sqrt{11}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{-2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{-2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{5t}{-2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{-2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{5t}{-2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{-2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + i\sqrt{11}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix} - \left(-1 + i\sqrt{11}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-i\sqrt{11} & -5 \\ 3 & 2-i\sqrt{11} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2-i\sqrt{11} & -5 & 0 \\ 3 & 2-i\sqrt{11} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-2-i\sqrt{11}} \Longrightarrow \left[\begin{array}{cc|c} -2-i\sqrt{11} & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-i\sqrt{11} & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{5t}{2+i\sqrt{11}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{5t}{2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5t}{2+i\sqrt{11}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{5t}{2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{5t}{2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{5t}{2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$-1+i\sqrt{11}$	1	1	No	$\left[\begin{array}{c}-\frac{5}{2+i\sqrt{11}} \\ 1\end{array}\right]$
$-1-i\sqrt{11}$	1	1	No	$\left[\begin{array}{c}-\frac{5}{2-i\sqrt{11}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = c_1 \left[\begin{array}{c}-\frac{5\,\mathrm{e}^{\left(-1+i\sqrt{11}\right)t}}{2+i\sqrt{11}} \\ \mathrm{e}^{\left(-1+i\sqrt{11}\right)t}\end{array}\right] + c_2 \left[\begin{array}{c}-\frac{5\,\mathrm{e}^{\left(-1-i\sqrt{11}\right)t}}{2-i\sqrt{11}} \\ \mathrm{e}^{\left(-1-i\sqrt{11}\right)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = \left[\begin{array}{c}\frac{2ic_2\left(i-\frac{\sqrt{11}}{2}\right)\mathrm{e}^{-\left(i\sqrt{11}+1\right)t}}{3} + \frac{2i\left(i+\frac{\sqrt{11}}{2}\right)\mathrm{e}^{\left(-1+i\sqrt{11}\right)t}c_1}{3} \\ c_1\mathrm{e}^{\left(-1+i\sqrt{11}\right)t} + c_2\mathrm{e}^{-\left(i\sqrt{11}+1\right)t}\end{array}\right]$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\left[\begin{array}{c}x(0) = 4 \\ y(0) = 0\end{array}\right] \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\left[\begin{array}{c}4 \\ 0\end{array}\right] = \left[\begin{array}{c}\frac{i(c_1-c_2)\sqrt{11}}{3} - \frac{2c_1}{3} - \frac{2c_2}{3} \\ c_1 + c_2\end{array}\right]$$

Solving for the constants of integrations gives

$$\left[\begin{array}{c}c_1 = -\frac{6i\sqrt{11}}{11} \\ c_2 = \frac{6i\sqrt{11}}{11}\end{array}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{4\sqrt{11}\left(i-\frac{\sqrt{11}}{2}\right)e^{-\left(i\sqrt{11}+1\right)t}}{11} + \frac{4\left(i+\frac{\sqrt{11}}{2}\right)e^{\left(-1+i\sqrt{11}\right)t}\sqrt{11}}{11} \\ -\frac{6i\sqrt{11}\,e^{\left(-1+i\sqrt{11}\right)t}}{11} + \frac{6i\sqrt{11}\,e^{-\left(i\sqrt{11}+1\right)t}}{11} \end{bmatrix}$$

The following is the phase plot of the system.

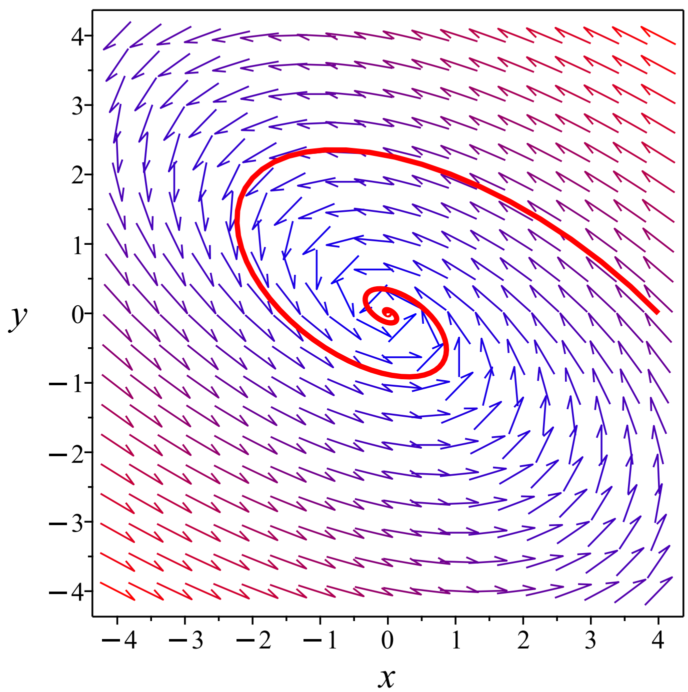


Figure 414: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 48

```
dsolve([diff(x(t),t) = -3*x(t)-5*y(t), diff(y(t),t) = 3*x(t)+y(t), x(0) = 4, y(0) = 0], sing
```

$$x(t) = e^{-t}\left(-\frac{8\sqrt{11}\,\sin\left(\sqrt{11}\,t\right)}{11} + 4\cos\left(\sqrt{11}\,t\right)\right)$$

$$y(t) = \frac{12\,e^{-t}\sqrt{11}\,\sin\left(\sqrt{11}\,t\right)}{11}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 63

```
DSolve[{x'[t]==-3*x[t]-5*y[t],y'[t]==3*x[t]+y[t]},{x[0]==4,y[0]==0},{x[t],y[t]},t,IncludeSin
```

$$x(t) \to \frac{4}{11}e^{-t}\left(11\cos\left(\sqrt{11}t\right) - 2\sqrt{11}\sin\left(\sqrt{11}t\right)\right)$$
$$y(t) \to \frac{12e^{-t}\sin\left(\sqrt{11}t\right)}{\sqrt{11}}$$

11.4 problem 6

Internal problem ID [13101]
Internal file name [`OUTPUT/11756_Sunday_December_03_2023_07_16_20_PM_12272310/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2y$$
$$y' = -2x(t) - y$$

With initial conditions

$$[x(0) = -1, y(0) = 1]$$

11.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} + e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{15}\,t}{2}\right) & \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \\ -\frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} & e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{15}\,t}{2}\right) - \frac{e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)+15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{15} & \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \\ -\frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} & -\frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)-15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{15} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)+15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{15} & \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \\ -\frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} & -\frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)-15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{15} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)+15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{15} + \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \\ \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} - \frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)-15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{15} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)-5\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{5} \\ \frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)+5\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{5} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{15}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{15}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2} + \frac{i\sqrt{15}}{2}$	1	complex eigenvalue
$-\frac{1}{2} - \frac{i\sqrt{15}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2} - \frac{i\sqrt{15}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} - \left(-\frac{1}{2} - \frac{i\sqrt{15}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{2} + \frac{i\sqrt{15}}{2} & 2 \\ -2 & -\frac{1}{2} + \frac{i\sqrt{15}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} + \frac{i\sqrt{15}}{2} & 2 & 0 \\ -2 & -\frac{1}{2} + \frac{i\sqrt{15}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{\frac{1}{2} + \frac{i\sqrt{15}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} + \frac{i\sqrt{15}}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} + \frac{i\sqrt{15}}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{4t}{i\sqrt{15}+1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{4t}{\mathrm{I}\sqrt{15}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{i\sqrt{15}+1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{\mathrm{I}\sqrt{15}+1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{i\sqrt{15}+1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{4t}{\mathrm{I}\sqrt{15}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{i\sqrt{15}+1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{4t}{\mathrm{I}\sqrt{15}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{i\sqrt{15}+1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2} + \frac{i\sqrt{15}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} - \left(-\frac{1}{2} + \frac{i\sqrt{15}}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{15}}{2} & 2 \\ -2 & -\frac{1}{2} - \frac{i\sqrt{15}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{15}}{2} & 2 & 0 \\ -2 & -\frac{1}{2}-\frac{i\sqrt{15}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{\frac{1}{2}-\frac{i\sqrt{15}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{15}}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{i\sqrt{15}}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{-1+i\sqrt{15}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{-1+i\sqrt{15}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{15}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{-1+i\sqrt{15}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{-1+i\sqrt{15}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{-1+i\sqrt{15}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number

of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2}+\frac{i\sqrt{15}}{2}$	1	1	No	$\begin{bmatrix} \frac{2}{-\frac{1}{2}+\frac{i\sqrt{15}}{2}} \\ 1 \end{bmatrix}$
$-\frac{1}{2}-\frac{i\sqrt{15}}{2}$	1	1	No	$\begin{bmatrix} \frac{2}{-\frac{1}{2}-\frac{i\sqrt{15}}{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{15}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{15}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{15}}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{15}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{15}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{15}}{2}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{i\left(-\sqrt{15}+i\right)c_1\mathrm{e}^{\frac{\left(-1+i\sqrt{15}\right)t}{2}}}{4} + \frac{i\left(i+\sqrt{15}\right)\mathrm{e}^{-\frac{\left(i\sqrt{15}+1\right)t}{2}}c_2}{4} \\ c_1\mathrm{e}^{\frac{\left(-1+i\sqrt{15}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(i\sqrt{15}+1\right)t}{2}} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = -1 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t=0$ gives

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{i(-c_1+c_2)\sqrt{15}}{4} - \frac{c_1}{4} - \frac{c_2}{4} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\left[c_1 = -\frac{\left(-\sqrt{15}+3i\right)\sqrt{15}}{30}, c_2 = \frac{\sqrt{15}\left(\sqrt{15}+3i\right)}{30}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{i\left(-\sqrt{15}+i\right)\left(-\sqrt{15}+3i\right)\sqrt{15}\,\mathrm{e}^{\frac{\left(-1+i\sqrt{15}\right)t}{2}}}{120} + \frac{i\left(i+\sqrt{15}\right)\mathrm{e}^{-\frac{\left(i\sqrt{15}+1\right)t}{2}}\sqrt{15}\left(\sqrt{15}+3i\right)}{120} \\ -\frac{\left(-\sqrt{15}+3i\right)\sqrt{15}\,\mathrm{e}^{\frac{\left(-1+i\sqrt{15}\right)t}{2}}}{30} + \frac{\sqrt{15}\left(\sqrt{15}+3i\right)\mathrm{e}^{-\frac{\left(i\sqrt{15}+1\right)t}{2}}}{30} \end{bmatrix}$$

The following is the phase plot of the system.

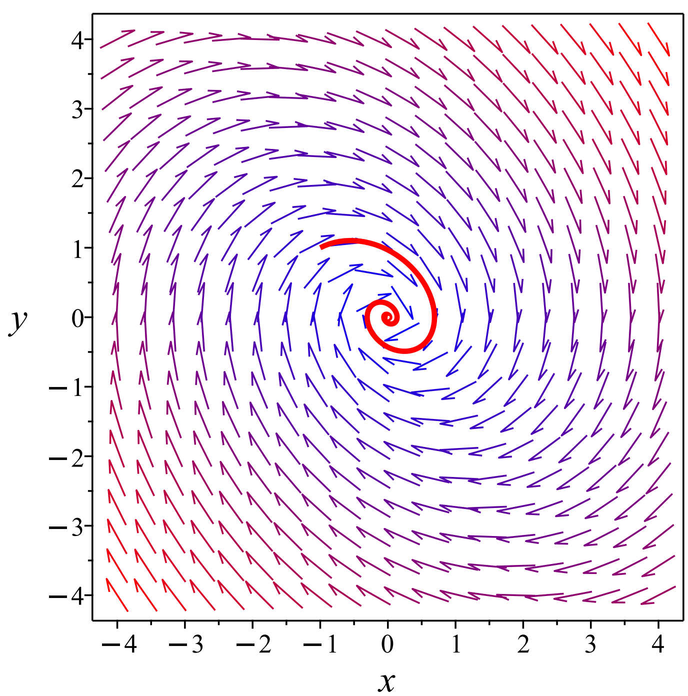


Figure 415: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 63

```
dsolve([diff(x(t),t) = 2*y(t), diff(y(t),t) = -2*x(t)-y(t), x(0) = -1, y(0) = 1], singsol=al
```

$$x(t) = \mathrm{e}^{-\frac{t}{2}}\left(\frac{\sqrt{15}\,\sin\left(\frac{t\sqrt{15}}{2}\right)}{5} - \cos\left(\frac{t\sqrt{15}}{2}\right)\right)$$

$$y(t) = -\frac{\mathrm{e}^{-\frac{t}{2}}\left(-\frac{4\sqrt{15}\,\sin\left(\frac{t\sqrt{15}}{2}\right)}{5} - 4\cos\left(\frac{t\sqrt{15}}{2}\right)\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 92

```
DSolve[{x'[t]==0*x[t]+2*y[t],y'[t]==-2*x[t]-y[t]},{x[0]==-1,y[0]==1},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to \frac{1}{5}e^{-t/2}\left(\sqrt{15}\sin\left(\frac{\sqrt{15}t}{2}\right) - 5\cos\left(\frac{\sqrt{15}t}{2}\right)\right)$$

$$y(t) \to \frac{1}{5}e^{-t/2}\left(\sqrt{15}\sin\left(\frac{\sqrt{15}t}{2}\right) + 5\cos\left(\frac{\sqrt{15}t}{2}\right)\right)$$

11.5 problem 7

Internal problem ID [13102]
Internal file name [`OUTPUT/11757_Sunday_December_03_2023_07_16_20_PM_91629352/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2x(t) - 6y$$
$$y' = 2x(t) + y$$

With initial conditions

$$[x(0) = 2, y(0) = 1]$$

11.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{47}\,t}{2}\right) + \frac{\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & -\frac{12\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ \frac{4\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{47}\,t}{2}\right) - \frac{\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} & -\frac{12\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ \frac{4\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & -\frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)-47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$

$$= \begin{bmatrix} \frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} & -\frac{12\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ \frac{4\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & -\frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)-47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2\,e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} - \frac{12\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ \frac{8\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} - \frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)-47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix}$$

$$= \begin{bmatrix} 2\,e^{\frac{3t}{2}}\left(-\frac{5\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} + \cos\left(\frac{\sqrt{47}\,t}{2}\right)\right) \\ \frac{e^{\frac{3t}{2}}\left(7\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}2 & -6\\ 2 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}2-\lambda & -6\\ 2 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 14 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3}{2} + \frac{i\sqrt{47}}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{i\sqrt{47}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} - \frac{i\sqrt{47}}{2}$	1	complex eigenvalue
$\frac{3}{2} + \frac{i\sqrt{47}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{i\sqrt{47}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}2 & -6\\ 2 & 1\end{bmatrix} - \left(\frac{3}{2} - \frac{i\sqrt{47}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{1}{2} + \frac{i\sqrt{47}}{2} & -6\\ 2 & -\frac{1}{2} + \frac{i\sqrt{47}}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{1}{2} + \frac{i\sqrt{47}}{2} & -6 & 0\\ 2 & -\frac{1}{2} + \frac{i\sqrt{47}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{\frac{1}{2}+\frac{i\sqrt{47}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}+\frac{i\sqrt{47}}{2} & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}+\frac{i\sqrt{47}}{2} & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{12t}{1+i\sqrt{47}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{12t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{12t}{1+i\sqrt{47}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{12t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{12}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{12t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{12}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{12t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{12}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{i\sqrt{47}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix} - \left(\frac{3}{2}+\frac{i\sqrt{47}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}-\frac{i\sqrt{47}}{2} & -6 \\ 2 & -\frac{1}{2}-\frac{i\sqrt{47}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{47}}{2} & -6 & 0 \\ 2 & -\frac{1}{2}-\frac{i\sqrt{47}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{\frac{1}{2}-\frac{i\sqrt{47}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{47}}{2} & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{i\sqrt{47}}{2} & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{12t}{-1+i\sqrt{47}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{12t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12t}{-1+i\sqrt{47}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{12t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{12}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{12t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{12t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{3}{2}+\frac{i\sqrt{47}}{2}$	1	1	No	$\left[\begin{array}{c}-\frac{6}{-\frac{1}{2}+\frac{i\sqrt{47}}{2}} \\ 1\end{array}\right]$
$\frac{3}{2}-\frac{i\sqrt{47}}{2}$	1	1	No	$\left[\begin{array}{c}-\frac{6}{-\frac{1}{2}-\frac{i\sqrt{47}}{2}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = c_1\left[\begin{array}{c}-\frac{6\,\mathrm{e}^{\left(\frac{3}{2}+\frac{i\sqrt{47}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{47}}{2}} \\ \mathrm{e}^{\left(\frac{3}{2}+\frac{i\sqrt{47}}{2}\right)t}\end{array}\right] + c_2\left[\begin{array}{c}-\frac{6\,\mathrm{e}^{\left(\frac{3}{2}-\frac{i\sqrt{47}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{47}}{2}} \\ \mathrm{e}^{\left(\frac{3}{2}-\frac{i\sqrt{47}}{2}\right)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = \left[\begin{array}{c}-\frac{i\left(i-\sqrt{47}\right)c_1\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{4} - \frac{i\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}c_2\left(i+\sqrt{47}\right)}{4} \\ c_1\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}\end{array}\right]$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\left[\begin{array}{c}x(0)=2 \\ y(0)=1\end{array}\right] \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{i(c_1-c_2)\sqrt{47}}{4} + \frac{c_1}{4} + \frac{c_2}{4} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -\frac{\left(-\sqrt{47}+7i\right)\sqrt{47}}{94} \\ c_2 = \frac{\sqrt{47}\left(\sqrt{47}+7i\right)}{94} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{i\left(i-\sqrt{47}\right)\left(-\sqrt{47}+7i\right)\sqrt{47}\,\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{376} - \frac{i\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}\sqrt{47}\left(\sqrt{47}+7i\right)\left(i+\sqrt{47}\right)}{376} \\ -\frac{\left(-\sqrt{47}+7i\right)\sqrt{47}\,\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{94} + \frac{\sqrt{47}\left(\sqrt{47}+7i\right)\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}}{94} \end{bmatrix}$$

The following is the phase plot of the system.

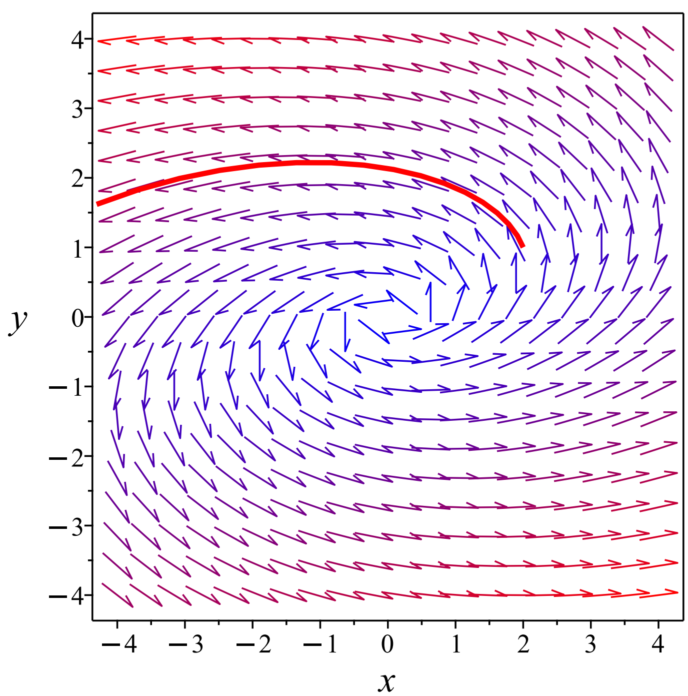


Figure 416: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 63

```
dsolve([diff(x(t),t) = 2*x(t)-6*y(t), diff(y(t),t) = 2*x(t)+y(t), x(0) = 2, y(0) = 1], sings
```

$$x(t) = \mathrm{e}^{\frac{3t}{2}} \left(-\frac{10\sqrt{47}\,\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} + 2\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)$$

$$y(t) = \frac{\mathrm{e}^{\frac{3t}{2}} \left(\frac{84\sqrt{47}\,\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} + 12\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{12}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 94

```
DSolve[{x'[t]==2*x[t]-6*y[t],y'[t]==2*x[t]+y[t]},{x[0]==2,y[0]==1},{x[t],y[t]},t,IncludeSing
```

$$x(t) \to \frac{2}{47} e^{3t/2} \left(47\cos\left(\frac{\sqrt{47}t}{2}\right) - 5\sqrt{47}\sin\left(\frac{\sqrt{47}t}{2}\right)\right)$$

$$y(t) \to \frac{1}{47} e^{3t/2} \left(7\sqrt{47}\sin\left(\frac{\sqrt{47}t}{2}\right) + 47\cos\left(\frac{\sqrt{47}t}{2}\right)\right)$$

11.6 problem 8

Internal problem ID [13103]
Internal file name [OUTPUT/11758_Sunday_December_03_2023_07_16_21_PM_85445911/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + 4y \\
y' &= -3x(t) + 2y
\end{aligned}
$$

With initial conditions

$$[x(0) = 1, y(0) = -1]$$

11.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{47}\, t}{2}\right) - \frac{\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} & \frac{8\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} \\ -\frac{6\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} & e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{47}\, t}{2}\right) + \frac{\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{\frac{3t}{2}} \left(\sqrt{47}\, \sin\left(\frac{\sqrt{47}\, t}{2}\right) - 47 \cos\left(\frac{\sqrt{47}\, t}{2}\right)\right)}{47} & \frac{8\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} \\ -\frac{6\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} & \frac{e^{\frac{3t}{2}} \left(\sqrt{47}\, \sin\left(\frac{\sqrt{47}\, t}{2}\right) + 47 \cos\left(\frac{\sqrt{47}\, t}{2}\right)\right)}{47} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At} \vec{x}_0$$

$$= \begin{bmatrix} -\frac{e^{\frac{3t}{2}} \left(\sqrt{47}\, \sin\left(\frac{\sqrt{47}\, t}{2}\right) - 47 \cos\left(\frac{\sqrt{47}\, t}{2}\right)\right)}{47} & \frac{8\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} \\ -\frac{6\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} & \frac{e^{\frac{3t}{2}} \left(\sqrt{47}\, \sin\left(\frac{\sqrt{47}\, t}{2}\right) + 47 \cos\left(\frac{\sqrt{47}\, t}{2}\right)\right)}{47} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{\frac{3t}{2}} \left(\sqrt{47}\, \sin\left(\frac{\sqrt{47}\, t}{2}\right) - 47 \cos\left(\frac{\sqrt{47}\, t}{2}\right)\right)}{47} - \frac{8\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} \\ -\frac{6\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} - \frac{e^{\frac{3t}{2}} \left(\sqrt{47}\, \sin\left(\frac{\sqrt{47}\, t}{2}\right) + 47 \cos\left(\frac{\sqrt{47}\, t}{2}\right)\right)}{47} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\frac{3t}{2}} \left(-\frac{9\sqrt{47}\, \sin\left(\frac{\sqrt{47}\, t}{2}\right)}{47} + \cos\left(\frac{\sqrt{47}\, t}{2}\right)\right) \\ -\frac{e^{\frac{3t}{2}} \left(7\sqrt{47}\, \sin\left(\frac{\sqrt{47}\, t}{2}\right) + 47 \cos\left(\frac{\sqrt{47}\, t}{2}\right)\right)}{47} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\, \vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}1 & 4\\ -3 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda & 4\\ -3 & 2-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 14 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3}{2} + \frac{i\sqrt{47}}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{i\sqrt{47}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} - \frac{i\sqrt{47}}{2}$	1	complex eigenvalue
$\frac{3}{2} + \frac{i\sqrt{47}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{i\sqrt{47}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}1 & 4\\ -3 & 2\end{bmatrix} - \left(\frac{3}{2} - \frac{i\sqrt{47}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-\frac{1}{2} + \frac{i\sqrt{47}}{2} & 4\\ -3 & \frac{1}{2} + \frac{i\sqrt{47}}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-\frac{1}{2} + \frac{i\sqrt{47}}{2} & 4 & 0\\ -3 & \frac{1}{2} + \frac{i\sqrt{47}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{-\frac{1}{2} + \frac{i\sqrt{47}}{2}} \implies \left[\begin{array}{cc|c} -\frac{1}{2} + \frac{i\sqrt{47}}{2} & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2} + \frac{i\sqrt{47}}{2} & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{8t}{-1+i\sqrt{47}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{8t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{8t}{-1+i\sqrt{47}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{8t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{8}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{8t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{8}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{8t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{8}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{i\sqrt{47}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} - \left(\frac{3}{2} + \frac{i\sqrt{47}}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{47}}{2} & 4 \\ -3 & \frac{1}{2} - \frac{i\sqrt{47}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{1}{2}-\frac{i\sqrt{47}}{2} & 4 & 0 \\ -3 & \frac{1}{2}-\frac{i\sqrt{47}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{-\frac{1}{2}-\frac{i\sqrt{47}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{1}{2}-\frac{i\sqrt{47}}{2} & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2}-\frac{i\sqrt{47}}{2} & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{8t}{1+i\sqrt{47}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{8t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{8t}{1+i\sqrt{47}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{8t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{8}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{8t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{8}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{8t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{8}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number

of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{3}{2}+\frac{i\sqrt{47}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{4}{\frac{1}{2}+\frac{i\sqrt{47}}{2}} \\ 1\end{array}\right]$
$\frac{3}{2}-\frac{i\sqrt{47}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{4}{\frac{1}{2}-\frac{i\sqrt{47}}{2}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = c_1\left[\begin{array}{c}\frac{4\,\mathrm{e}^{\left(\frac{3}{2}+\frac{i\sqrt{47}}{2}\right)t}}{\frac{1}{2}+\frac{i\sqrt{47}}{2}} \\ \mathrm{e}^{\left(\frac{3}{2}+\frac{i\sqrt{47}}{2}\right)t}\end{array}\right] + c_2\left[\begin{array}{c}\frac{4\,\mathrm{e}^{\left(\frac{3}{2}-\frac{i\sqrt{47}}{2}\right)t}}{\frac{1}{2}-\frac{i\sqrt{47}}{2}} \\ \mathrm{e}^{\left(\frac{3}{2}-\frac{i\sqrt{47}}{2}\right)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = \left[\begin{array}{c}-\frac{i\left(i+\sqrt{47}\right)c_1\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{6} - \frac{i\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}c_2\left(i-\sqrt{47}\right)}{6} \\ c_1\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}\end{array}\right]$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\left[\begin{array}{c}x(0) = 1 \\ y(0) = -1\end{array}\right] \tag{1}$$

Substituting initial conditions into the above solution at $t=0$ gives

$$\left[\begin{array}{c}1 \\ -1\end{array}\right] = \left[\begin{array}{c}\frac{i(-c_1+c_2)\sqrt{47}}{6} + \frac{c_1}{6} + \frac{c_2}{6} \\ c_1 + c_2\end{array}\right]$$

Solving for the constants of integrations gives

$$\left[c_1 = -\frac{\left(\sqrt{47}-7i\right)\sqrt{47}}{94}, c_2 = -\frac{\sqrt{47}\left(\sqrt{47}+7i\right)}{94}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{i\left(i+\sqrt{47}\right)\left(\sqrt{47}-7i\right)\sqrt{47}\,\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{564} + \frac{i\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}\sqrt{47}\left(\sqrt{47}+7i\right)\left(i-\sqrt{47}\right)}{564} \\ -\frac{\left(\sqrt{47}-7i\right)\sqrt{47}\,\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{94} - \frac{\sqrt{47}\left(\sqrt{47}+7i\right)\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}}{94} \end{bmatrix}$$

The following is the phase plot of the system.

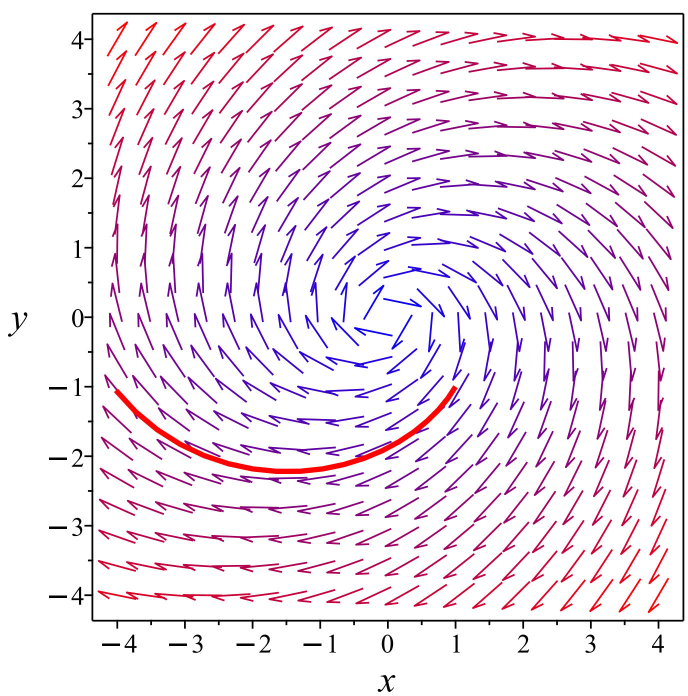


Figure 417: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 61

```
dsolve([diff(x(t),t) = x(t)+4*y(t), diff(y(t),t) = -3*x(t)+2*y(t), x(0) = 1, y(0) = -1], sin
```

$$x(t) = e^{\frac{3t}{2}} \left(-\frac{9\sqrt{47}\,\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} + \cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)$$

$$y(t) = -\frac{e^{\frac{3t}{2}} \left(\frac{56\sqrt{47}\,\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} + 8\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{8}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 94

```
DSolve[{x'[t]==1*x[t]+4*y[t],y'[t]==-3*x[t]+2*y[t]},{x[0]==1,y[0]==-1},{x[t],y[t]},t,Include
```

$$x(t) \to \frac{1}{47} e^{3t/2} \left(47\cos\left(\frac{\sqrt{47}t}{2}\right) - 9\sqrt{47}\sin\left(\frac{\sqrt{47}t}{2}\right)\right)$$

$$y(t) \to -\frac{1}{47} e^{3t/2} \left(7\sqrt{47}\sin\left(\frac{\sqrt{47}t}{2}\right) + 47\cos\left(\frac{\sqrt{47}t}{2}\right)\right)$$

11.7 problem 9

Internal problem ID [13104]
Internal file name [`OUTPUT/11759_Sunday_December_03_2023_07_16_22_PM_2645872/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2y$$
$$y' = -2x(t)$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

11.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2i$	1	complex eigenvalue
$-2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - (-2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - (2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{2it} \\ \mathrm{e}^{2it} \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{-2it} \\ \mathrm{e}^{-2it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -i(c_1 e^{2it} - c_2 e^{-2it}) \\ c_1 e^{2it} + c_2 e^{-2it} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -i(c_1 - c_2) \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{i}{2} \\ c_2 = -\frac{i}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -i\left(\frac{ie^{2it}}{2} + \frac{ie^{-2it}}{2}\right) \\ \frac{ie^{2it}}{2} - \frac{ie^{-2it}}{2} \end{bmatrix}$$

The following is the phase plot of the system.

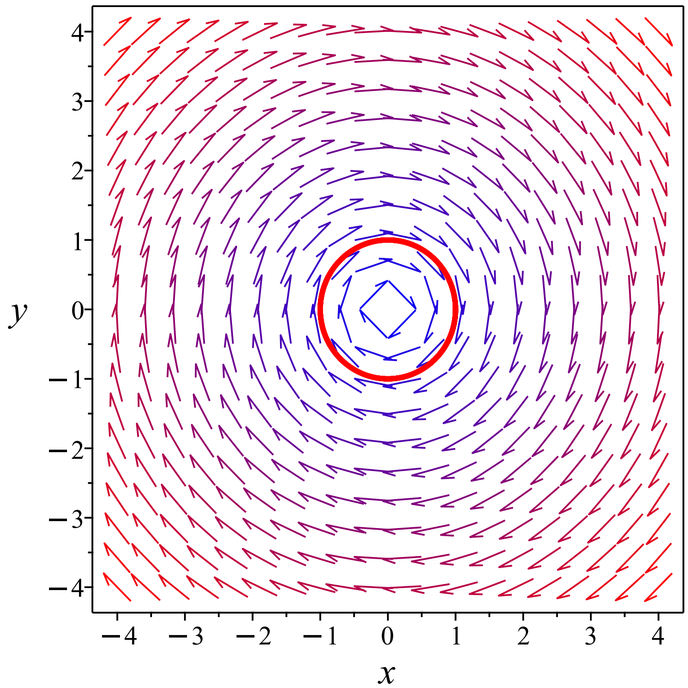


Figure 418: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([diff(x(t),t) = 2*y(t), diff(y(t),t) = -2*x(t), x(0) = 1, y(0) = 0], singsol=all)
```

$$x(t) = \cos(2t)$$
$$y(t) = -\sin(2t)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 18

```
DSolve[{x'[t]==0*x[t]+2*y[t],y'[t]==-2*x[t]+0*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeS
```

$$x(t) \to \cos(2t)$$
$$y(t) \to -\sin(2t)$$

11.8 problem 10

Internal problem ID [13105]
Internal file name [OUTPUT/11760_Sunday_December_03_2023_07_16_22_PM_70722247/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + 2y \\
y' &= -4x(t) + 6y
\end{aligned}
$$

With initial conditions

$$[x(0) = 1, y(0) = 1]$$

11.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{4t}\cos(2t) - e^{4t}\sin(2t) & e^{4t}\sin(2t) \\ -2\,e^{4t}\sin(2t) & e^{4t}\cos(2t) + e^{4t}\sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}(\cos(2t) - \sin(2t)) & e^{4t}\sin(2t) \\ -2\,e^{4t}\sin(2t) & e^{4t}(\sin(2t) + \cos(2t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{4t}(\cos(2t)-\sin(2t)) & e^{4t}\sin(2t) \\ -2\,e^{4t}\sin(2t) & e^{4t}(\sin(2t)+\cos(2t)) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}(\cos(2t)-\sin(2t)) + e^{4t}\sin(2t) \\ -2\,e^{4t}\sin(2t) + e^{4t}(\sin(2t)+\cos(2t)) \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}\cos(2t) \\ e^{4t}(\cos(2t)-\sin(2t)) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & 2 \\ -4 & 6-\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 20 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4 + 2i$$
$$\lambda_2 = 4 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$4 + 2i$	1	complex eigenvalue
$4 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} - (4 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2+2i & 2 \\ -4 & 2+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2+2i & 2 & 0 \\ -4 & 2+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} -2+2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2+2i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix} - (4 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 - 2i & 2 \\ -4 & 2 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 - 2i & 2 & 0 \\ -4 & 2 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 + i) R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 - 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-2i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$4+2i$	1	1	No	$\left[\begin{array}{c}\frac{1}{2}-\frac{i}{2} \\ 1\end{array}\right]$
$4-2i$	1	1	No	$\left[\begin{array}{c}\frac{1}{2}+\frac{i}{2} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = c_1\left[\begin{array}{c}\left(\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(4+2i)t} \\ \mathrm{e}^{(4+2i)t}\end{array}\right] + c_2\left[\begin{array}{c}\left(\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(4-2i)t} \\ \mathrm{e}^{(4-2i)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = \left[\begin{array}{c}\left(\frac{1}{2}-\frac{i}{2}\right)c_1\mathrm{e}^{(4+2i)t} + \left(\frac{1}{2}+\frac{i}{2}\right)c_2\mathrm{e}^{(4-2i)t} \\ c_1\mathrm{e}^{(4+2i)t} + c_2\mathrm{e}^{(4-2i)t}\end{array}\right]$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\left[\begin{array}{c}x(0)=1 \\ y(0)=1\end{array}\right] \tag{1}$$

Substituting initial conditions into the above solution at $t=0$ gives

$$\left[\begin{array}{c}1 \\ 1\end{array}\right] = \left[\begin{array}{c}\left(\frac{1}{2}-\frac{i}{2}\right)c_1 + \left(\frac{1}{2}+\frac{i}{2}\right)c_2 \\ c_1+c_2\end{array}\right]$$

Solving for the constants of integrations gives

$$\left[\begin{array}{c}c_1=\frac{1}{2}+\frac{i}{2} \\ c_2=\frac{1}{2}-\frac{i}{2}\end{array}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{e^{(4+2i)t}}{2} + \frac{e^{(4-2i)t}}{2} \\ \left(\frac{1}{2} - \frac{i}{2}\right) e^{(4-2i)t} + \left(\frac{1}{2} + \frac{i}{2}\right) e^{(4+2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

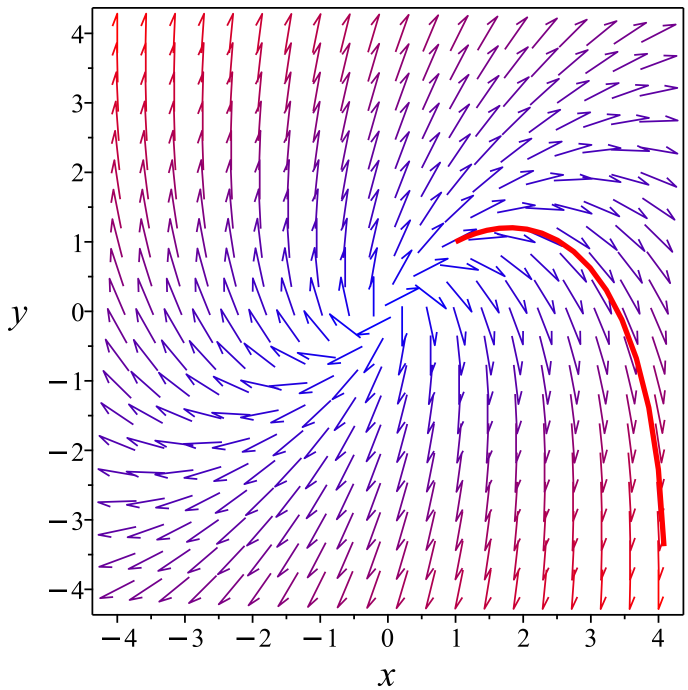


Figure 419: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve([diff(x(t),t) = 2*x(t)+2*y(t), diff(y(t),t) = -4*x(t)+6*y(t), x(0) = 1, y(0) = 1], si
```

$$x(t) = e^{4t} \cos(2t)$$
$$y(t) = e^{4t}(\cos(2t) - \sin(2t))$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 35

```
DSolve[{x'[t]==2*x[t]+2*y[t],y'[t]==-4*x[t]+6*y[t]},{x[0]==1,y[0]==1},{x[t],y[t]},t,IncludeS
```

$$x(t) \to e^{4t} \cos(2t)$$
$$y(t) \to e^{4t}(\cos(2t) - \sin(2t))$$

11.9 problem 11

Internal problem ID [13106]
Internal file name [`OUTPUT/11761_Sunday_December_03_2023_07_16_22_PM_58303665/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -3x(t) - 5y$$
$$y' = 3x(t) + y$$

With initial conditions

$$[x(0) = 4, y(0) = 0]$$

11.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos\left(\sqrt{11}\,t\right) - \frac{2\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} & -\frac{5\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \\ \frac{3\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} & e^{-t}\cos\left(\sqrt{11}\,t\right) + \frac{2\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}\left(\cos\left(\sqrt{11}\,t\right) - \frac{2\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11}\right) & -\frac{5\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \\ \frac{3\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} & \frac{e^{-t}\left(2\sin\left(\sqrt{11}\,t\right)\sqrt{11}+11\cos\left(\sqrt{11}\,t\right)\right)}{11} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$

$$= \begin{bmatrix} e^{-t}\left(\cos\left(\sqrt{11}\,t\right) - \frac{2\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11}\right) & -\frac{5\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \\ \frac{3\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} & \frac{e^{-t}\left(2\sin\left(\sqrt{11}\,t\right)\sqrt{11}+11\cos\left(\sqrt{11}\,t\right)\right)}{11} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4\,e^{-t}\left(\cos\left(\sqrt{11}\,t\right) - \frac{2\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11}\right) \\ \frac{12\,e^{-t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-3 & -5\\ 3 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-3-\lambda & -5\\ 3 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 12 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + i\sqrt{11}$$
$$\lambda_2 = -1 - i\sqrt{11}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - i\sqrt{11}$	1	complex eigenvalue
$-1 + i\sqrt{11}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - i\sqrt{11}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-3 & -5\\ 3 & 1\end{bmatrix} - \left(-1 - i\sqrt{11}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-2 + i\sqrt{11} & -5\\ 3 & 2 + i\sqrt{11}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-2 + i\sqrt{11} & -5 & 0\\ 3 & 2 + i\sqrt{11} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-2+i\sqrt{11}} \Longrightarrow \left[\begin{array}{cc|c} -2+i\sqrt{11} & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2+i\sqrt{11} & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{5t}{-2+i\sqrt{11}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{5t}{-2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5t}{-2+i\sqrt{11}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{5t}{-2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{-2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{5t}{-2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{-2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{5t}{-2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{-2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + i\sqrt{11}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix} - \left(-1+i\sqrt{11}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-i\sqrt{11} & -5 \\ 3 & 2-i\sqrt{11} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2-i\sqrt{11} & -5 & 0 \\ 3 & 2-i\sqrt{11} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-2-i\sqrt{11}} \Longrightarrow \left[\begin{array}{cc|c} -2-i\sqrt{11} & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-i\sqrt{11} & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{5t}{2+i\sqrt{11}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{5t}{2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5t}{2+i\sqrt{11}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{5t}{2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{5t}{2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{5t}{2+\mathrm{I}\sqrt{11}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{2+i\sqrt{11}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-1+i\sqrt{11}$	1	1	No	$\begin{bmatrix} -\frac{5}{2+i\sqrt{11}} \\ 1 \end{bmatrix}$
$-1-i\sqrt{11}$	1	1	No	$\begin{bmatrix} -\frac{5}{2-i\sqrt{11}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{5\,\mathrm{e}^{\left(-1+i\sqrt{11}\right)t}}{2+i\sqrt{11}} \\ \mathrm{e}^{\left(-1+i\sqrt{11}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{5\,\mathrm{e}^{\left(-1-i\sqrt{11}\right)t}}{2-i\sqrt{11}} \\ \mathrm{e}^{\left(-1-i\sqrt{11}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{2ic_2\left(i-\frac{\sqrt{11}}{2}\right)\mathrm{e}^{-\left(i\sqrt{11}+1\right)t}}{3} + \frac{2i\left(i+\frac{\sqrt{11}}{2}\right)\mathrm{e}^{\left(-1+i\sqrt{11}\right)t}c_1}{3} \\ c_1\mathrm{e}^{\left(-1+i\sqrt{11}\right)t} + c_2\mathrm{e}^{-\left(i\sqrt{11}+1\right)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 4 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{i(c_1-c_2)\sqrt{11}}{3} - \frac{2c_1}{3} - \frac{2c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -\frac{6i\sqrt{11}}{11} \\ c_2 = \frac{6i\sqrt{11}}{11} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{4\sqrt{11}\left(i-\frac{\sqrt{11}}{2}\right)\mathrm{e}^{-\left(i\sqrt{11}+1\right)t}}{11} + \frac{4\left(i+\frac{\sqrt{11}}{2}\right)\mathrm{e}^{\left(-1+i\sqrt{11}\right)t}\sqrt{11}}{11} \\ -\frac{6i\sqrt{11}\,\mathrm{e}^{\left(-1+i\sqrt{11}\right)t}}{11} + \frac{6i\sqrt{11}\,\mathrm{e}^{-\left(i\sqrt{11}+1\right)t}}{11} \end{bmatrix}$$

The following is the phase plot of the system.

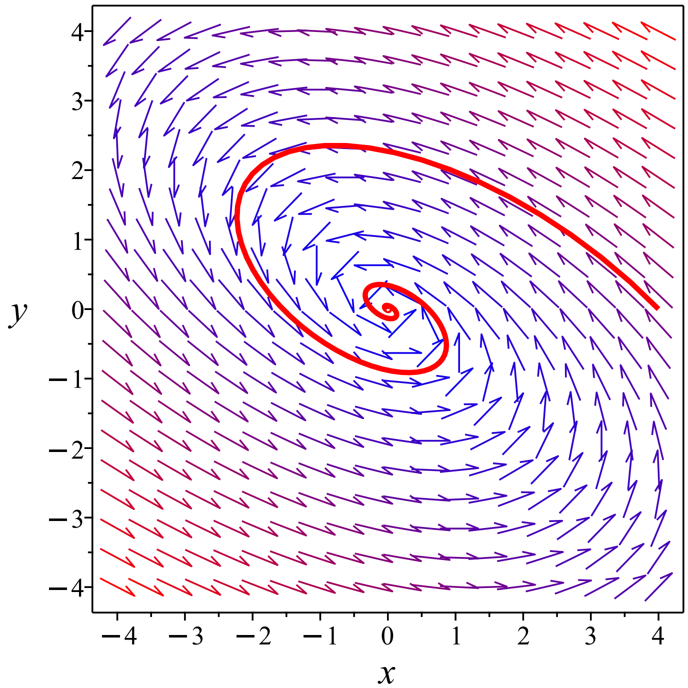


Figure 420: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 48

```
dsolve([diff(x(t),t) = -3*x(t)-5*y(t), diff(y(t),t) = 3*x(t)+y(t), x(0) = 4, y(0) = 0], sing
```

$$x(t) = \mathrm{e}^{-t}\left(-\frac{8\sqrt{11}\,\sin\left(\sqrt{11}\,t\right)}{11} + 4\cos\left(\sqrt{11}\,t\right)\right)$$

$$y(t) = \frac{12\,\mathrm{e}^{-t}\sqrt{11}\,\sin\left(\sqrt{11}\,t\right)}{11}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 63

```
DSolve[{x'[t]==-3*x[t]-5*y[t],y'[t]==3*x[t]+1*y[t]},{x[0]==4,y[0]==0},{x[t],y[t]},t,IncludeS
```

$$x(t) \to \frac{4}{11}e^{-t}\left(11\cos\left(\sqrt{11}t\right) - 2\sqrt{11}\sin\left(\sqrt{11}t\right)\right)$$
$$y(t) \to \frac{12e^{-t}\sin\left(\sqrt{11}t\right)}{\sqrt{11}}$$

11.10 problem 12

Internal problem ID [13107]
Internal file name [OUTPUT/11762_Sunday_December_03_2023_07_16_23_PM_20812460/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2y \\
y' &= -2x(t) - y
\end{aligned}
$$

With initial conditions

$$
[x(0) = -1, y(0) = 1]
$$

11.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} + e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{15}\,t}{2}\right) & \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \\ -\frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} & e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{15}\,t}{2}\right) - \frac{e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)+15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{15} & \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \\ -\frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} & -\frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)-15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{15} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)+15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{15} & \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \\ -\frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} & -\frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)-15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{15} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)+15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{15} + \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} \\ \frac{4\,e^{-\frac{t}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)}{15} - \frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)-15\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{15} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)-5\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{5} \\ \frac{e^{-\frac{t}{2}}\left(\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,t}{2}\right)+5\cos\left(\frac{\sqrt{15}\,t}{2}\right)\right)}{5} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{15}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{15}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2} + \frac{i\sqrt{15}}{2}$	1	complex eigenvalue
$-\frac{1}{2} - \frac{i\sqrt{15}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2} - \frac{i\sqrt{15}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} - \left(-\frac{1}{2} - \frac{i\sqrt{15}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{2} + \frac{i\sqrt{15}}{2} & 2 \\ -2 & -\frac{1}{2} + \frac{i\sqrt{15}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} + \frac{i\sqrt{15}}{2} & 2 & 0 \\ -2 & -\frac{1}{2} + \frac{i\sqrt{15}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{\frac{1}{2} + \frac{i\sqrt{15}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} + \frac{i\sqrt{15}}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} + \frac{i\sqrt{15}}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{4t}{i\sqrt{15}+1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{4t}{\mathrm{I}\sqrt{15}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{i\sqrt{15}+1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{\mathrm{I}\sqrt{15}+1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{i\sqrt{15}+1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{4t}{\mathrm{I}\sqrt{15}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{i\sqrt{15}+1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{4t}{\mathrm{I}\sqrt{15}+1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{i\sqrt{15}+1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2} + \frac{i\sqrt{15}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix} - \left(-\frac{1}{2} + \frac{i\sqrt{15}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{15}}{2} & 2 \\ -2 & -\frac{1}{2} - \frac{i\sqrt{15}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{15}}{2} & 2 & 0 \\ -2 & -\frac{1}{2}-\frac{i\sqrt{15}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{\frac{1}{2}-\frac{i\sqrt{15}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{15}}{2} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{i\sqrt{15}}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{-1+i\sqrt{15}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{-1+i\sqrt{15}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{15}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{-1+i\sqrt{15}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{-1+i\sqrt{15}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{4t}{-1+\mathrm{I}\sqrt{15}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{-1+i\sqrt{15}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number

of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2}+\frac{i\sqrt{15}}{2}$	1	1	No	$\begin{bmatrix} \frac{2}{-\frac{1}{2}+\frac{i\sqrt{15}}{2}} \\ 1 \end{bmatrix}$
$-\frac{1}{2}-\frac{i\sqrt{15}}{2}$	1	1	No	$\begin{bmatrix} \frac{2}{-\frac{1}{2}-\frac{i\sqrt{15}}{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{15}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{15}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{15}}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{15}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{15}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{15}}{2}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{i\left(-\sqrt{15}+i\right)c_1 \mathrm{e}^{\frac{\left(-1+i\sqrt{15}\right)t}{2}}}{4} + \frac{i\left(i+\sqrt{15}\right)\mathrm{e}^{-\frac{\left(i\sqrt{15}+1\right)t}{2}} c_2}{4} \\ c_1 \mathrm{e}^{\frac{\left(-1+i\sqrt{15}\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(i\sqrt{15}+1\right)t}{2}} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = -1 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{i(-c_1+c_2)\sqrt{15}}{4} - \frac{c_1}{4} - \frac{c_2}{4} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\left[c_1 = -\frac{\left(-\sqrt{15}+3i\right)\sqrt{15}}{30}, c_2 = \frac{\sqrt{15}\left(\sqrt{15}+3i\right)}{30}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{i\left(-\sqrt{15}+i\right)\left(-\sqrt{15}+3i\right)\sqrt{15}\,e^{\frac{\left(-1+i\sqrt{15}\right)t}{2}}}{120} + \frac{i\left(i+\sqrt{15}\right)e^{-\frac{\left(i\sqrt{15}+1\right)t}{2}}\sqrt{15}\left(\sqrt{15}+3i\right)}{120} \\ -\frac{\left(-\sqrt{15}+3i\right)\sqrt{15}\,e^{\frac{\left(-1+i\sqrt{15}\right)t}{2}}}{30} + \frac{\sqrt{15}\left(\sqrt{15}+3i\right)e^{-\frac{\left(i\sqrt{15}+1\right)t}{2}}}{30} \end{bmatrix}$$

The following is the phase plot of the system.

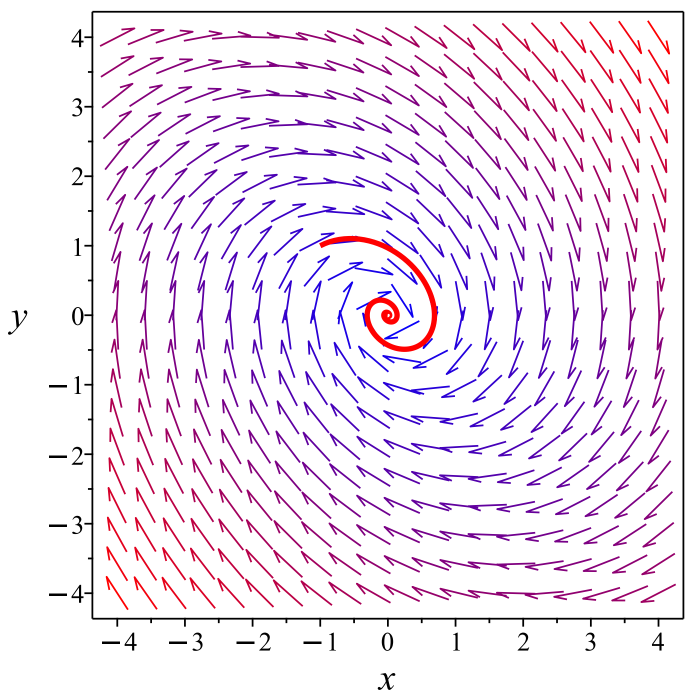


Figure 421: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 63

```
dsolve([diff(x(t),t) = 2*y(t), diff(y(t),t) = -2*x(t)-y(t), x(0) = -1, y(0) = 1], singsol=al
```

$$x(t) = e^{-\frac{t}{2}} \left(\frac{\sqrt{15}\, \sin\left(\frac{t\sqrt{15}}{2}\right)}{5} - \cos\left(\frac{t\sqrt{15}}{2}\right)\right)$$

$$y(t) = -\frac{e^{-\frac{t}{2}} \left(-\frac{4\sqrt{15}\, \sin\left(\frac{t\sqrt{15}}{2}\right)}{5} - 4\cos\left(\frac{t\sqrt{15}}{2}\right)\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 92

```
DSolve[{x'[t]==2*y[t],y'[t]==-2*x[t]-1*y[t]},{x[0]==-1,y[0]==1},{x[t],y[t]},t,IncludeSingula
```

$$x(t) \to \frac{1}{5} e^{-t/2} \left(\sqrt{15} \sin\left(\frac{\sqrt{15}t}{2}\right) - 5\cos\left(\frac{\sqrt{15}t}{2}\right)\right)$$

$$y(t) \to \frac{1}{5} e^{-t/2} \left(\sqrt{15} \sin\left(\frac{\sqrt{15}t}{2}\right) + 5\cos\left(\frac{\sqrt{15}t}{2}\right)\right)$$

11.11 problem 13

Internal problem ID [13108]
Internal file name [`OUTPUT/11763_Sunday_December_03_2023_07_16_23_PM_57207460/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 2x(t) - 6y \\ y' &= 2x(t) + y \end{aligned}$$

With initial conditions

$$[x(0) = 2, y(0) = 1]$$

11.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{47}\,t}{2}\right) + \frac{\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & -\frac{12\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ \frac{4\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{47}\,t}{2}\right) - \frac{\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} & -\frac{12\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ \frac{4\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & -\frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)-47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$

$$= \begin{bmatrix} \frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} & -\frac{12\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ \frac{4\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & -\frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)-47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2\, e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} - \frac{12\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ \frac{8\sqrt{47}\, e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} - \frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)-47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix}$$

$$= \begin{bmatrix} 2\, e^{\frac{3t}{2}}\left(-\frac{5\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} + \cos\left(\frac{\sqrt{47}\,t}{2}\right)\right) \\ \frac{e^{\frac{3t}{2}}\left(7\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -6 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 14 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3}{2} + \frac{i\sqrt{47}}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{i\sqrt{47}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} - \frac{i\sqrt{47}}{2}$	1	complex eigenvalue
$\frac{3}{2} + \frac{i\sqrt{47}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{i\sqrt{47}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix} - \left(\frac{3}{2} - \frac{i\sqrt{47}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{2} + \frac{i\sqrt{47}}{2} & -6 \\ 2 & -\frac{1}{2} + \frac{i\sqrt{47}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} + \frac{i\sqrt{47}}{2} & -6 & 0 \\ 2 & -\frac{1}{2} + \frac{i\sqrt{47}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{\frac{1}{2} + \frac{i\sqrt{47}}{2}} \implies \left[\begin{array}{cc|c} \frac{1}{2} + \frac{i\sqrt{47}}{2} & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} + \frac{i\sqrt{47}}{2} & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{12t}{1+i\sqrt{47}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{12t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{12t}{1+i\sqrt{47}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{12t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{12}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{12t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{12}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{12t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{12}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{i\sqrt{47}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix} - \left(\frac{3}{2} + \frac{i\sqrt{47}}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{47}}{2} & -6 \\ 2 & -\frac{1}{2} - \frac{i\sqrt{47}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{47}}{2} & -6 & 0 \\ 2 & -\frac{1}{2}-\frac{i\sqrt{47}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{\frac{1}{2}-\frac{i\sqrt{47}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2}-\frac{i\sqrt{47}}{2} & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2}-\frac{i\sqrt{47}}{2} & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{12t}{-1+i\sqrt{47}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{12t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12t}{-1+i\sqrt{47}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{12t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{12}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{12t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{12t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{3}{2}+\frac{i\sqrt{47}}{2}$	1	1	No	$\left[\begin{array}{c}-\frac{6}{-\frac{1}{2}+\frac{i\sqrt{47}}{2}} \\ 1\end{array}\right]$
$\frac{3}{2}-\frac{i\sqrt{47}}{2}$	1	1	No	$\left[\begin{array}{c}-\frac{6}{-\frac{1}{2}-\frac{i\sqrt{47}}{2}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = c_1\left[\begin{array}{c}-\frac{6\,\mathrm{e}^{\left(\frac{3}{2}+\frac{i\sqrt{47}}{2}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{47}}{2}} \\ \mathrm{e}^{\left(\frac{3}{2}+\frac{i\sqrt{47}}{2}\right)t}\end{array}\right] + c_2\left[\begin{array}{c}-\frac{6\,\mathrm{e}^{\left(\frac{3}{2}-\frac{i\sqrt{47}}{2}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{47}}{2}} \\ \mathrm{e}^{\left(\frac{3}{2}-\frac{i\sqrt{47}}{2}\right)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = \left[\begin{array}{c}-\frac{i\left(i-\sqrt{47}\right)c_1\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{4} - \frac{i\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}c_2\left(i+\sqrt{47}\right)}{4} \\ c_1\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}\end{array}\right]$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\left[\begin{array}{c}x(0) = 2 \\ y(0) = 1\end{array}\right] \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\left[\begin{array}{c} 2 \\ 1 \end{array}\right] = \left[\begin{array}{c} \frac{i(c_1-c_2)\sqrt{47}}{4} + \frac{c_1}{4} + \frac{c_2}{4} \\ c_1 + c_2 \end{array}\right]$$

Solving for the constants of integrations gives

$$\left[\begin{array}{c} c_1 = -\frac{\left(-\sqrt{47}+7i\right)\sqrt{47}}{94} \\ c_2 = \frac{\sqrt{47}\left(\sqrt{47}+7i\right)}{94} \end{array}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{i\left(i-\sqrt{47}\right)\left(-\sqrt{47}+7i\right)\sqrt{47}\,\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{376} - \frac{i\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}\sqrt{47}\left(\sqrt{47}+7i\right)\left(i+\sqrt{47}\right)}{376} \\ -\frac{\left(-\sqrt{47}+7i\right)\sqrt{47}\,\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{94} + \frac{\sqrt{47}\left(\sqrt{47}+7i\right)\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}}{94} \end{array}\right]$$

The following is the phase plot of the system.

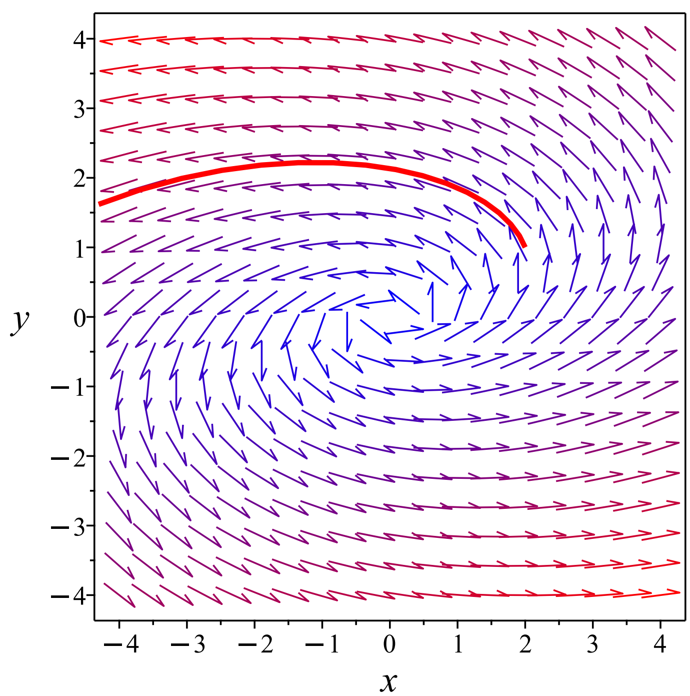


Figure 422: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
dsolve([diff(x(t),t) = 2*x(t)-6*y(t), diff(y(t),t) = 2*x(t)+y(t), x(0) = 2, y(0) = 1], sings
```

$$x(t) = e^{\frac{3t}{2}}\left(-\frac{10\sqrt{47}\,\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47}+2\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)$$

$$y(t) = \frac{e^{\frac{3t}{2}}\left(\frac{84\sqrt{47}\,\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47}+12\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{12}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 94

```
DSolve[{x'[t]==2*x[t]-6*y[t],y'[t]==2*x[t]+1*y[t]},{x[0]==2,y[0]==1},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to \frac{2}{47}e^{3t/2}\left(47\cos\left(\frac{\sqrt{47}t}{2}\right)-5\sqrt{47}\sin\left(\frac{\sqrt{47}t}{2}\right)\right)$$

$$y(t) \to \frac{1}{47}e^{3t/2}\left(7\sqrt{47}\sin\left(\frac{\sqrt{47}t}{2}\right)+47\cos\left(\frac{\sqrt{47}t}{2}\right)\right)$$

11.12 problem 14

Internal problem ID [13109]
Internal file name [`OUTPUT/11764_Sunday_December_03_2023_07_16_24_PM_94633801/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + 4y \\
y' &= -3x(t) + 2y
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = -1]
$$

11.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{\frac{3t}{2}}\cos\left(\frac{\sqrt{47}\,t}{2}\right) - \frac{\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & \frac{8\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ -\frac{6\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & e^{\frac{3t}{2}}\cos\left(\frac{\sqrt{47}\,t}{2}\right) + \frac{\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)-47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} & \frac{8\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ -\frac{6\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & \frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$

$$= \begin{bmatrix} -\frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)-47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} & \frac{8\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ -\frac{6\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} & \frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)-47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} - \frac{8\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} \\ -\frac{6\sqrt{47}\,e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} - \frac{e^{\frac{3t}{2}}\left(\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\frac{3t}{2}}\left(-\frac{9\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} + \cos\left(\frac{\sqrt{47}\,t}{2}\right)\right) \\ -\frac{e^{\frac{3t}{2}}\left(7\sqrt{47}\sin\left(\frac{\sqrt{47}\,t}{2}\right)+47\cos\left(\frac{\sqrt{47}\,t}{2}\right)\right)}{47} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}1 & 4\\ -3 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}1-\lambda & 4\\ -3 & 2-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 14 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3}{2} + \frac{i\sqrt{47}}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{i\sqrt{47}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} - \frac{i\sqrt{47}}{2}$	1	complex eigenvalue
$\frac{3}{2} + \frac{i\sqrt{47}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{i\sqrt{47}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}1 & 4\\ -3 & 2\end{bmatrix} - \left(\frac{3}{2} - \frac{i\sqrt{47}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-\frac{1}{2} + \frac{i\sqrt{47}}{2} & 4\\ -3 & \frac{1}{2} + \frac{i\sqrt{47}}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-\frac{1}{2} + \frac{i\sqrt{47}}{2} & 4 & 0\\ -3 & \frac{1}{2} + \frac{i\sqrt{47}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{-\frac{1}{2} + \frac{i\sqrt{47}}{2}} \implies \left[\begin{array}{cc|c} -\frac{1}{2} + \frac{i\sqrt{47}}{2} & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2} + \frac{i\sqrt{47}}{2} & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{8t}{-1+i\sqrt{47}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{8t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{8t}{-1+i\sqrt{47}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{8t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{8}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{8t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{8}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{8t}{-1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{8}{-1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{i\sqrt{47}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} - \left(\frac{3}{2} + \frac{i\sqrt{47}}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{47}}{2} & 4 \\ -3 & \frac{1}{2} - \frac{i\sqrt{47}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{1}{2}-\frac{i\sqrt{47}}{2} & 4 & 0 \\ -3 & \frac{1}{2}-\frac{i\sqrt{47}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{-\frac{1}{2}-\frac{i\sqrt{47}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{1}{2}-\frac{i\sqrt{47}}{2} & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2}-\frac{i\sqrt{47}}{2} & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{8t}{1+i\sqrt{47}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{8t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{8t}{1+i\sqrt{47}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{8t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{8}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{8t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{8}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{8t}{1+\mathrm{I}\sqrt{47}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{8}{1+i\sqrt{47}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number

of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{3}{2}+\frac{i\sqrt{47}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{4}{\frac{1}{2}+\frac{i\sqrt{47}}{2}} \\ 1\end{array}\right]$
$\frac{3}{2}-\frac{i\sqrt{47}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{4}{\frac{1}{2}-\frac{i\sqrt{47}}{2}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right]=c_1\left[\begin{array}{c}\frac{4\,\mathrm{e}^{\left(\frac{3}{2}+\frac{i\sqrt{47}}{2}\right)t}}{\frac{1}{2}+\frac{i\sqrt{47}}{2}} \\ \mathrm{e}^{\left(\frac{3}{2}+\frac{i\sqrt{47}}{2}\right)t}\end{array}\right]+c_2\left[\begin{array}{c}\frac{4\,\mathrm{e}^{\left(\frac{3}{2}-\frac{i\sqrt{47}}{2}\right)t}}{\frac{1}{2}-\frac{i\sqrt{47}}{2}} \\ \mathrm{e}^{\left(\frac{3}{2}-\frac{i\sqrt{47}}{2}\right)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right]=\left[\begin{array}{c}-\frac{i\left(i+\sqrt{47}\right)c_1\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{6}-\frac{i\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}c_2\left(i-\sqrt{47}\right)}{6} \\ c_1\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}+c_2\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}\end{array}\right]$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\left[\begin{array}{c}x(0)=1 \\ y(0)=-1\end{array}\right] \tag{1}$$

Substituting initial conditions into the above solution at $t=0$ gives

$$\left[\begin{array}{c}1 \\ -1\end{array}\right]=\left[\begin{array}{c}\frac{i(-c_1+c_2)\sqrt{47}}{6}+\frac{c_1}{6}+\frac{c_2}{6} \\ c_1+c_2\end{array}\right]$$

Solving for the constants of integrations gives

$$\left[c_1 = -\frac{\left(\sqrt{47}-7i\right)\sqrt{47}}{94}, c_2 = -\frac{\sqrt{47}\left(\sqrt{47}+7i\right)}{94}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{i\left(i+\sqrt{47}\right)\left(\sqrt{47}-7i\right)\sqrt{47}\,\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{564} + \frac{i\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}\sqrt{47}\left(\sqrt{47}+7i\right)\left(i-\sqrt{47}\right)}{564} \\ -\frac{\left(\sqrt{47}-7i\right)\sqrt{47}\,\mathrm{e}^{\frac{\left(3+i\sqrt{47}\right)t}{2}}}{94} - \frac{\sqrt{47}\left(\sqrt{47}+7i\right)\mathrm{e}^{-\frac{\left(i\sqrt{47}-3\right)t}{2}}}{94} \end{bmatrix}$$

The following is the phase plot of the system.

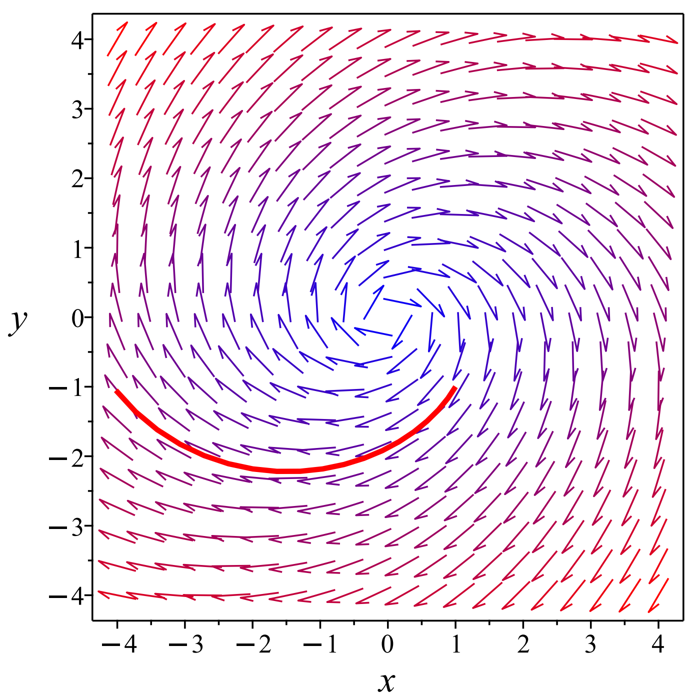


Figure 423: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
dsolve([diff(x(t),t) = x(t)+4*y(t), diff(y(t),t) = -3*x(t)+2*y(t), x(0) = 1, y(0) = -1], sin
```

$$x(t) = e^{\frac{3t}{2}} \left(-\frac{9\sqrt{47}\,\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} + \cos\left(\frac{\sqrt{47}\,t}{2}\right) \right)$$

$$y(t) = -\frac{e^{\frac{3t}{2}} \left(\frac{56\sqrt{47}\,\sin\left(\frac{\sqrt{47}\,t}{2}\right)}{47} + 8\cos\left(\frac{\sqrt{47}\,t}{2}\right) \right)}{8}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 94

```
DSolve[{x'[t]==1*x[t]+4*y[t],y'[t]==-3*x[t]+2*y[t]},{x[0]==1,y[0]==-1},{x[t],y[t]},t,Include
```

$$x(t) \to \frac{1}{47} e^{3t/2} \left(47\cos\left(\frac{\sqrt{47}t}{2}\right) - 9\sqrt{47}\sin\left(\frac{\sqrt{47}t}{2}\right) \right)$$

$$y(t) \to -\frac{1}{47} e^{3t/2} \left(7\sqrt{47}\sin\left(\frac{\sqrt{47}t}{2}\right) + 47\cos\left(\frac{\sqrt{47}t}{2}\right) \right)$$

11.13 problem 24

Internal problem ID [13110]
Internal file name [`OUTPUT/11765_Sunday_December_03_2023_07_16_24_PM_63702795/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -\frac{9x(t)}{10} - 2y$$
$$y' = x(t) + \frac{11y}{10}$$

With initial conditions

$$[x(0) = 1, y(0) = 1]$$

11.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{9}{10} & -2 \\ 1 & \frac{11}{10} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{\frac{t}{10}} \cos(t) - e^{\frac{t}{10}} \sin(t) & -2\, e^{\frac{t}{10}} \sin(t) \\ e^{\frac{t}{10}} \sin(t) & e^{\frac{t}{10}} \cos(t) + e^{\frac{t}{10}} \sin(t) \end{bmatrix} \\ &= \begin{bmatrix} e^{\frac{t}{10}} (\cos(t) - \sin(t)) & -2\, e^{\frac{t}{10}} \sin(t) \\ e^{\frac{t}{10}} \sin(t) & e^{\frac{t}{10}} (\cos(t) + \sin(t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At} \vec{x}_0 \\ &= \begin{bmatrix} e^{\frac{t}{10}} (\cos(t) - \sin(t)) & -2\, e^{\frac{t}{10}} \sin(t) \\ e^{\frac{t}{10}} \sin(t) & e^{\frac{t}{10}} (\cos(t) + \sin(t)) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{\frac{t}{10}} (\cos(t) - \sin(t)) - 2\, e^{\frac{t}{10}} \sin(t) \\ e^{\frac{t}{10}} \sin(t) + e^{\frac{t}{10}} (\cos(t) + \sin(t)) \end{bmatrix} \\ &= \begin{bmatrix} e^{\frac{t}{10}} (\cos(t) - 3\sin(t)) \\ e^{\frac{t}{10}} (\cos(t) + 2\sin(t)) \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\, \vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{9}{10} & -2 \\ 1 & \frac{11}{10} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -\frac{9}{10} & -2 \\ 1 & \frac{11}{10} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\frac{9}{10}-\lambda & -2 \\ 1 & \frac{11}{10}-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \frac{1}{5}\lambda + \frac{101}{100} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{10} + i$$
$$\lambda_2 = \frac{1}{10} - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{10} - i$	1	complex eigenvalue
$\frac{1}{10} + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{10} - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{9}{10} & -2 \\ 1 & \frac{11}{10} \end{bmatrix} - \left(\frac{1}{10} - i\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+i & -2 & 0 \\ 1 & 1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} + \frac{i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-1-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (-1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (-1-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-1-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{10} + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{9}{10} & -2 \\ 1 & \frac{11}{10} \end{bmatrix} - \left(\frac{1}{10} + i\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-i & -2 & 0 \\ 1 & 1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1-i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-1+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (-1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (-1+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-1+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\frac{1}{10}+i$	1	1	No	$\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$
$\frac{1}{10}-i$	1	1	No	$\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} (-1+i)\,\mathrm{e}^{\left(\frac{1}{10}+i\right)t} \\ \mathrm{e}^{\left(\frac{1}{10}+i\right)t} \end{bmatrix} + c_2 \begin{bmatrix} (-1-i)\,\mathrm{e}^{\left(\frac{1}{10}-i\right)t} \\ \mathrm{e}^{\left(\frac{1}{10}-i\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} (-1+i)\,c_1\mathrm{e}^{\left(\frac{1}{10}+i\right)t} + (-1-i)\,c_2\mathrm{e}^{\left(\frac{1}{10}-i\right)t} \\ c_1\mathrm{e}^{\left(\frac{1}{10}+i\right)t} + c_2\mathrm{e}^{\left(\frac{1}{10}-i\right)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1+i)\,c_1 + (-1-i)\,c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{1}{2} - i \\ c_2 = \frac{1}{2} + i \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{3i}{2}\right)\mathrm{e}^{\left(\frac{1}{10}+i\right)t} + \left(\frac{1}{2} - \frac{3i}{2}\right)\mathrm{e}^{\left(\frac{1}{10}-i\right)t} \\ \left(\frac{1}{2} - i\right)\mathrm{e}^{\left(\frac{1}{10}+i\right)t} + \left(\frac{1}{2} + i\right)\mathrm{e}^{\left(\frac{1}{10}-i\right)t} \end{bmatrix}$$

The following is the phase plot of the system.

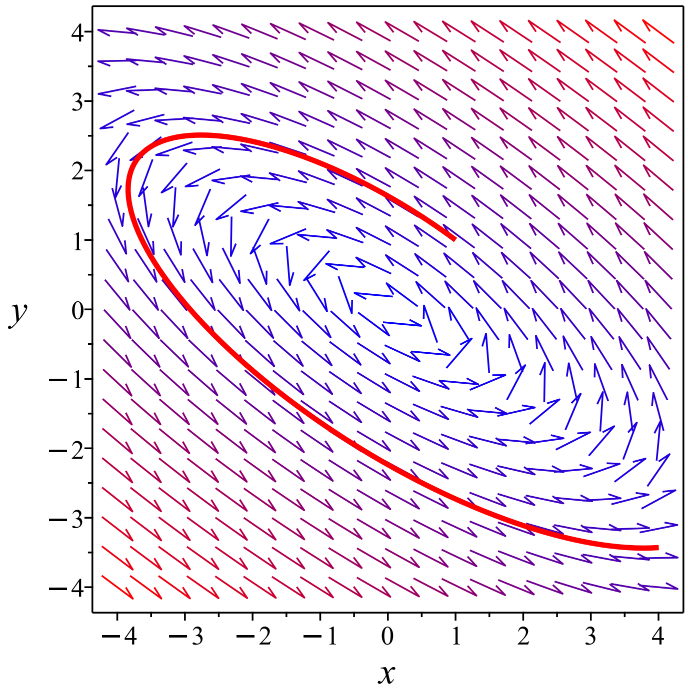


Figure 424: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve([diff(x(t),t) = -9/10*x(t)-2*y(t), diff(y(t),t) = x(t)+11/10*y(t), x(0) = 1, y(0) = 1
```

$$x(t) = e^{\frac{t}{10}}(-3\sin(t) + \cos(t))$$
$$y(t) = -\frac{e^{\frac{t}{10}}(-4\sin(t) - 2\cos(t))}{2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 38

```
DSolve[{x'[t]==-9/10*x[t]-2*y[t],y'[t]==x[t]+11/10*y[t]},{x[0]==1,y[0]==1},{x[t],y[t]},t,Inc
```

$$x(t) \to e^{t/10}(\cos(t) - 3\sin(t))$$
$$y(t) \to e^{t/10}(2\sin(t) + \cos(t))$$

11.14 problem 26

Internal problem ID [13111]
Internal file name [`OUTPUT/11766_Sunday_December_03_2023_07_16_25_PM_55203300/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.4 page 310
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -3x(t) + 10y \\
y' &= -x(t) + 3y
\end{aligned}
$$

11.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 10 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \cos(t) - 3\sin(t) & 10\sin(t) \\ -\sin(t) & \cos(t) + 3\sin(t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t) - 3\sin(t) & 10\sin(t) \\ -\sin(t) & \cos(t) + 3\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(t) - 3\sin(t))\, c_1 + 10\sin(t)\, c_2 \\ -\sin(t)\, c_1 + (\cos(t) + 3\sin(t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-3c_1 + 10c_2)\sin(t) + c_1\cos(t) \\ (-c_1 + 3c_2)\sin(t) + c_2\cos(t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

11.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 10 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 10 \\ -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 10 \\ -1 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 10 \\ -1 & 3 \end{bmatrix} - (-i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3+i & 10 \\ -1 & 3+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3+i & 10 & 0 \\ -1 & 3+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{10} - \frac{i}{10}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -3+i & 10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3+i & 10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (3+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (3+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (3+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (3+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 3+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (3+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 3+i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 10 \\ -1 & 3 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-i & 10 \\ -1 & 3-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3-i & 10 & 0 \\ -1 & 3-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{10} + \frac{i}{10}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -3-i & 10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3-i & 10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (3 - i)\, t\}$

Hence the solution is

$$\begin{bmatrix} (3 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} (3 - i)\, t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (3 - \mathrm{I})\, t \\ t \end{bmatrix} = t \begin{bmatrix} 3 - i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (3 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} 3 - i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} 3 - i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} 3 + i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} (3-i)\,\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} (3+i)\,\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} (3-i)\,c_1\mathrm{e}^{it} + (3+i)\,c_2\mathrm{e}^{-it} \\ c_1\mathrm{e}^{it} + c_2\mathrm{e}^{-it} \end{bmatrix}$$

The following is the phase plot of the system.

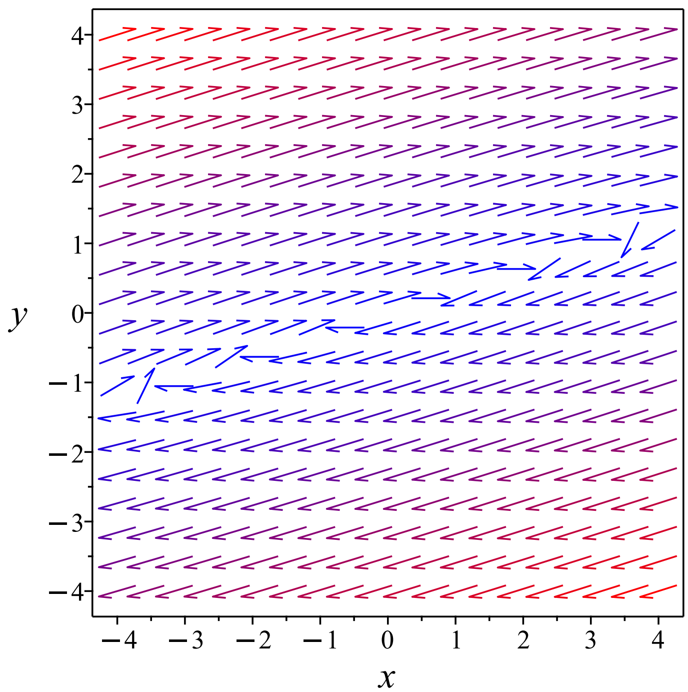


Figure 425: Phase plot

11.14.3 Maple step by step solution

Let's solve

$[x'(t) = -3x(t) + 10y, y' = -x(t) + 3y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -3 & 10 \\ -1 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -3 & 10 \\ -1 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 10 \\ -1 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \begin{bmatrix} 3+\mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} 3-\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} 3+\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} 3+\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} 3+\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\left[\begin{array}{c}(3+\mathrm{I})(\cos(t)-\mathrm{I}\sin(t))\\ \cos(t)-\mathrm{I}\sin(t)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t)=\left[\begin{array}{c}3\cos(t)+\sin(t)\\ \cos(t)\end{array}\right], \vec{x}_2(t)=\left[\begin{array}{c}\cos(t)-3\sin(t)\\ -\sin(t)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{x}=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x}=\left[\begin{array}{c}c_2(\cos(t)-3\sin(t))+c_1(3\cos(t)+\sin(t))\\ c_1\cos(t)-c_2\sin(t)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t)\\ y\end{array}\right]=\left[\begin{array}{c}(3c_1+c_2)\cos(t)+\sin(t)(c_1-3c_2)\\ c_1\cos(t)-c_2\sin(t)\end{array}\right]$$

- Solution to the system of ODEs

$$\{x(t)=(3c_1+c_2)\cos(t)+\sin(t)(c_1-3c_2), y=c_1\cos(t)-c_2\sin(t)\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve([diff(x(t),t)=-3*x(t)+10*y(t),diff(y(t),t)=-x(t)+3*y(t)],singsol=all)
```

$$x(t)=c_1\sin(t)+c_2\cos(t)$$
$$y(t)=\frac{c_1\cos(t)}{10}-\frac{c_2\sin(t)}{10}+\frac{3c_1\sin(t)}{10}+\frac{3c_2\cos(t)}{10}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 42

```
DSolve[{x'[t]==-3*x[t]+10*y[t],y'[t]==-x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t)\to 10c_2\sin(t)+c_1(\cos(t)-3\sin(t))$$
$$y(t)\to c_2(3\sin(t)+\cos(t))-c_1\sin(t)$$

12 Chapter 3. Linear Systems. Exercises section 3.5 page 327

12.1 problem 1

Internal problem ID [13112]
Internal file name [`OUTPUT/11767_Sunday_December_03_2023_07_16_25_PM_83371828/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -3x(t) \\
y' &= x(t) - 3y
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = 0]
$$

12.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{-3t} & 0 \\ t\,e^{-3t} & e^{-3t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \mathrm{e}^{-3t} & 0 \\ t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-3t} \\ t\,\mathrm{e}^{-3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -3-\lambda & 0 \\ 1 & -3-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$
(-3-\lambda)(-3-\lambda) = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

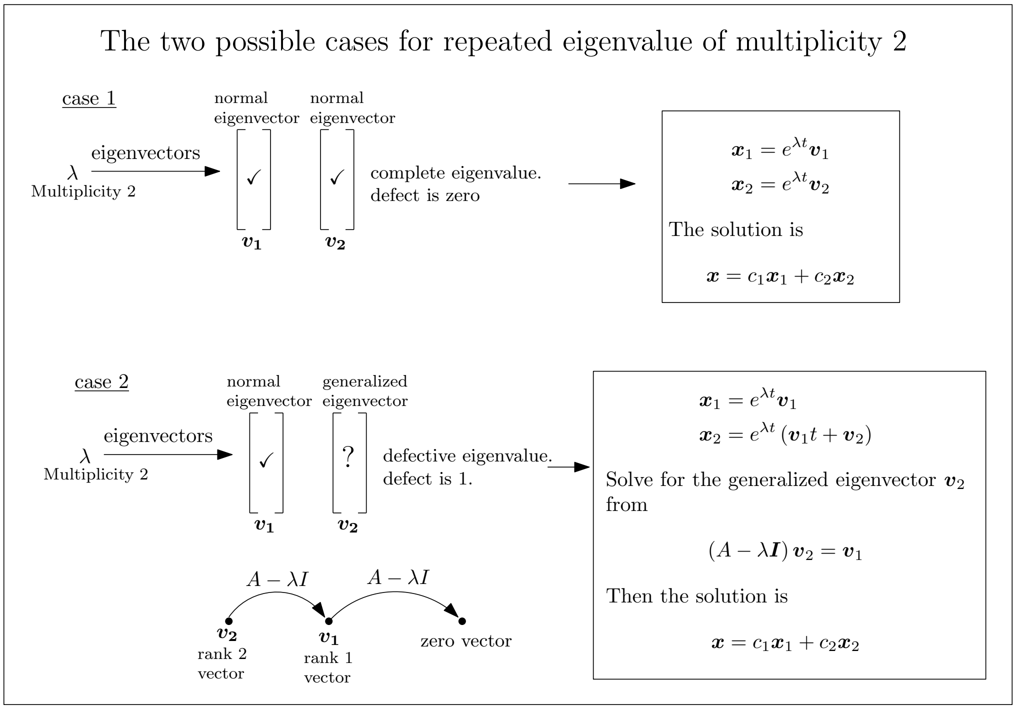


Figure 426: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\ &= \begin{bmatrix} 0 \\ \mathrm{e}^{-3t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \mathrm{e}^{-3t} \\ &= \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(1+t) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} e^{-3t} \\ t\,e^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

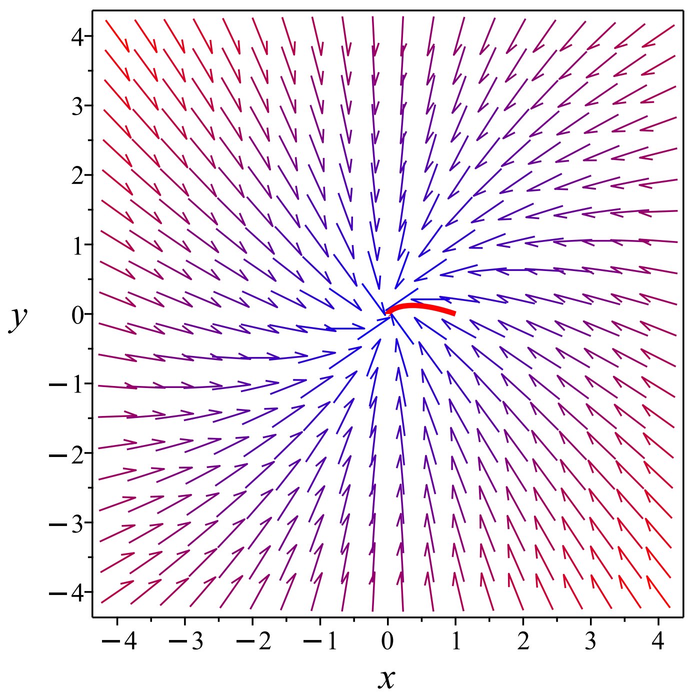
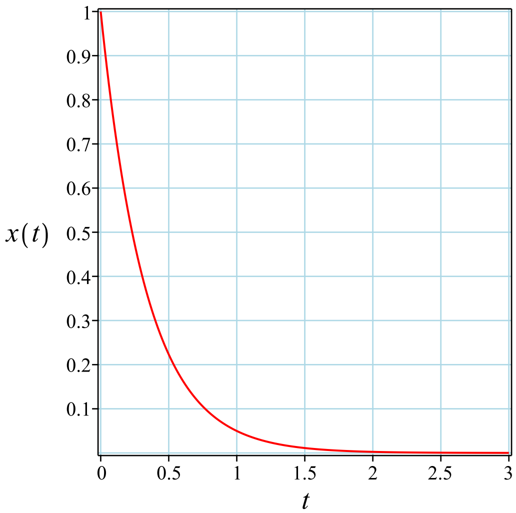
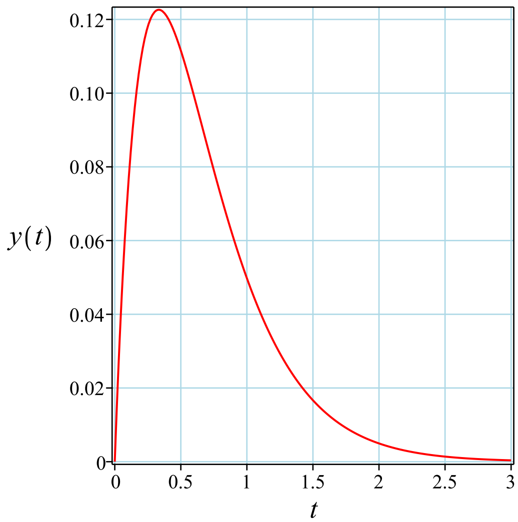


Figure 427: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 18

```
dsolve([diff(x(t),t) = -3*x(t), diff(y(t),t) = x(t)-3*y(t), x(0) = 1, y(0) = 0], singsol=all
```

$$x(t) = \mathrm{e}^{-3t}$$
$$y(t) = t\,\mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 20

```
DSolve[{x'[t]==-3*x[t],y'[t]==x[t]-3*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSingularSo
```

$$x(t) \to e^{-3t}$$
$$y(t) \to e^{-3t}t$$

12.2 problem 2

Internal problem ID [13113]
Internal file name [OUTPUT/11768_Sunday_December_03_2023_07_16_26_PM_94654993/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + y \\
y' &= -x(t) - 2y
\end{aligned}
$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

12.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(3-2\sqrt{3})\,e^{-\sqrt{3}\,t}}{6} + \frac{e^{\sqrt{3}\,t}(3+2\sqrt{3})}{6} & \frac{(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t})\sqrt{3}}{6} \\ -\frac{(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t})\sqrt{3}}{6} & \frac{(3+2\sqrt{3})\,e^{-\sqrt{3}\,t}}{6} + \frac{(3-2\sqrt{3})\,e^{\sqrt{3}\,t}}{6} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{(3-2\sqrt{3})\,e^{-\sqrt{3}\,t}}{6} + \frac{e^{\sqrt{3}\,t}(3+2\sqrt{3})}{6} & \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} \\ -\frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} & \frac{(3+2\sqrt{3})\,e^{-\sqrt{3}\,t}}{6} + \frac{(3-2\sqrt{3})\,e^{\sqrt{3}\,t}}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(3-2\sqrt{3})\,e^{-\sqrt{3}\,t}}{6} + \frac{e^{\sqrt{3}\,t}(3+2\sqrt{3})}{6} \\ -\frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 3 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{3}$$
$$\lambda_2 = -\sqrt{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\sqrt{3}$	1	real eigenvalue
$-\sqrt{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} - \left(\sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2-\sqrt{3} & 1 \\ -1 & -2-\sqrt{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-\sqrt{3} & 1 & 0 \\ -1 & -2-\sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2-\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} 2-\sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-\sqrt{3} & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{\sqrt{3}-2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{\sqrt{3}-2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{\sqrt{3}-2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{\sqrt{3}-2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{\sqrt{3}-2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{\sqrt{3}-2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}-2} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} - \left(-\sqrt{3}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & \sqrt{3}-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+\sqrt{3} & 1 & 0 \\ -1 & \sqrt{3}-2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2+\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} 2+\sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+\sqrt{3} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2+\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2+\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2+\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2+\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2+\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2+\sqrt{3}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\sqrt{3}$	1	1	No	$\begin{bmatrix} \frac{1}{\sqrt{3}-2} \\ 1 \end{bmatrix}$
$-\sqrt{3}$	1	1	No	$\begin{bmatrix} \frac{1}{-2-\sqrt{3}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\sqrt{3}\,t} \\
&= \left[\begin{array}{c} \frac{1}{\sqrt{3}-2} \\ 1 \end{array}\right] e^{\sqrt{3}\,t}
\end{aligned}
$$

Since eigenvalue $-\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-\sqrt{3}\,t} \\
&= \left[\begin{array}{c} \frac{1}{-2-\sqrt{3}} \\ 1 \end{array}\right] e^{-\sqrt{3}\,t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \frac{\mathrm{e}^{\sqrt{3}\,t}}{\sqrt{3}-2} \\ \mathrm{e}^{\sqrt{3}\,t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{\mathrm{e}^{-\sqrt{3}\,t}}{-2-\sqrt{3}} \\ \mathrm{e}^{-\sqrt{3}\,t} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} c_2\left(\sqrt{3}-2\right) \mathrm{e}^{-\sqrt{3}\,t} - c_1 \mathrm{e}^{\sqrt{3}\,t}\left(2+\sqrt{3}\right) \\ c_1 \mathrm{e}^{\sqrt{3}\,t} + c_2 \mathrm{e}^{-\sqrt{3}\,t} \end{array}\right]
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\left[\begin{array}{c} x(0) = 1 \\ y(0) = 0 \end{array}\right] \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} (-c_1 + c_2)\sqrt{3} - 2c_1 - 2c_2 \\ c_1 + c_2 \end{array}\right]
$$

Solving for the constants of integrations gives

$$\left[\begin{array}{c} c_1 = -\frac{\sqrt{3}}{6} \\ c_2 = \frac{\sqrt{3}}{6} \end{array}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{\sqrt{3}\left(\sqrt{3}-2\right)e^{-\sqrt{3}\,t}}{6} + \frac{\sqrt{3}\,e^{\sqrt{3}\,t}\left(2+\sqrt{3}\right)}{6} \\ -\frac{\sqrt{3}\,e^{\sqrt{3}\,t}}{6} + \frac{\sqrt{3}\,e^{-\sqrt{3}\,t}}{6} \end{array}\right]$$

The following is the phase plot of the system.

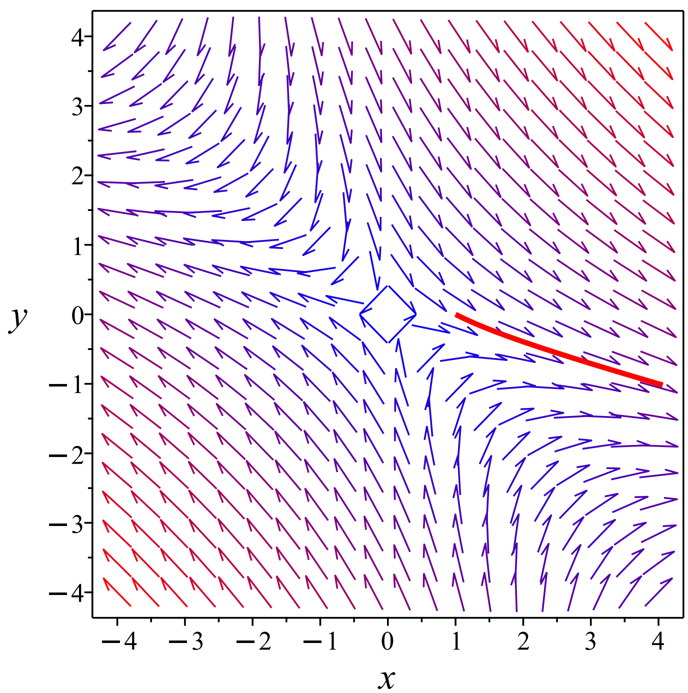
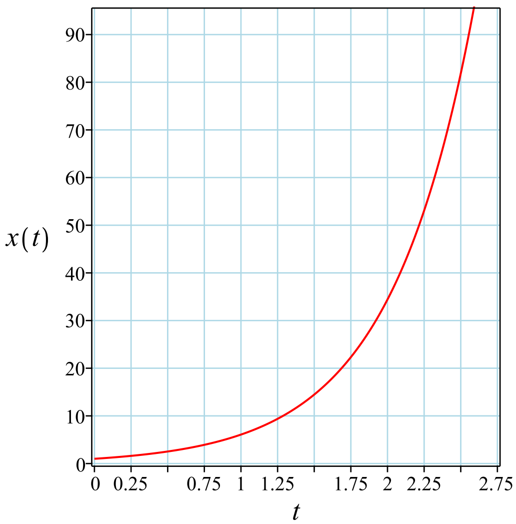
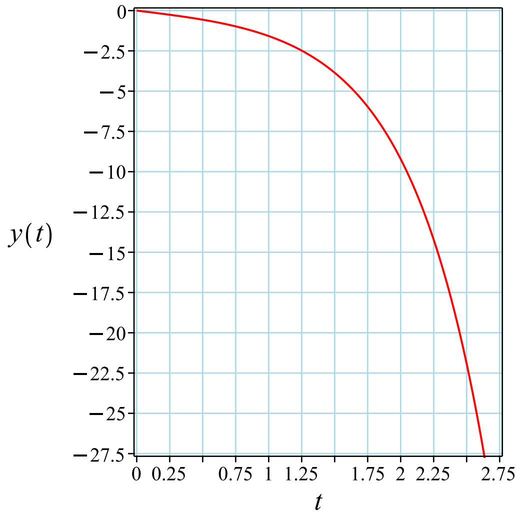


Figure 428: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 106

```
dsolve([diff(x(t),t) = 2*x(t)+y(t), diff(y(t),t) = -x(t)-2*y(t), x(0) = 1, y(0) = 0], singso
```

$$
\begin{aligned}
x(t) &= \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right) \mathrm{e}^{\sqrt{3}\,t}+\left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right) \mathrm{e}^{-\sqrt{3}\,t} \\
y(t) &= \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right) \sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t}-\left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right) \sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t} \\
&\quad -2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right) \mathrm{e}^{\sqrt{3}\,t}-2\left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right) \mathrm{e}^{-\sqrt{3}\,t}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 82

```
DSolve[{x'[t]==2*x[t]+1*y[t],y'[t]==-1*x[t]-2*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeS
```

$$
\begin{aligned}
x(t) &\to \frac{1}{6}e^{-\sqrt{3}t}\left(\left(3+2\sqrt{3}\right)e^{2\sqrt{3}t}+3-2\sqrt{3}\right) \\
y(t) &\to -\frac{e^{-\sqrt{3}t}\left(e^{2\sqrt{3}t}-1\right)}{2\sqrt{3}}
\end{aligned}
$$

12.3 problem 3

Internal problem ID [13114]
Internal file name [`OUTPUT/11769_Sunday_December_03_2023_07_16_26_PM_4516032/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -2x(t) - y$$
$$y' = x(t) - 4y$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

12.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-3t}(1+t) & -t\,e^{-3t} \\ t\,e^{-3t} & e^{-3t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-3t}(1+t) & -t\,e^{-3t} \\ t\,e^{-3t} & e^{-3t}(1-t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}(1+t) \\ t\,e^{-3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

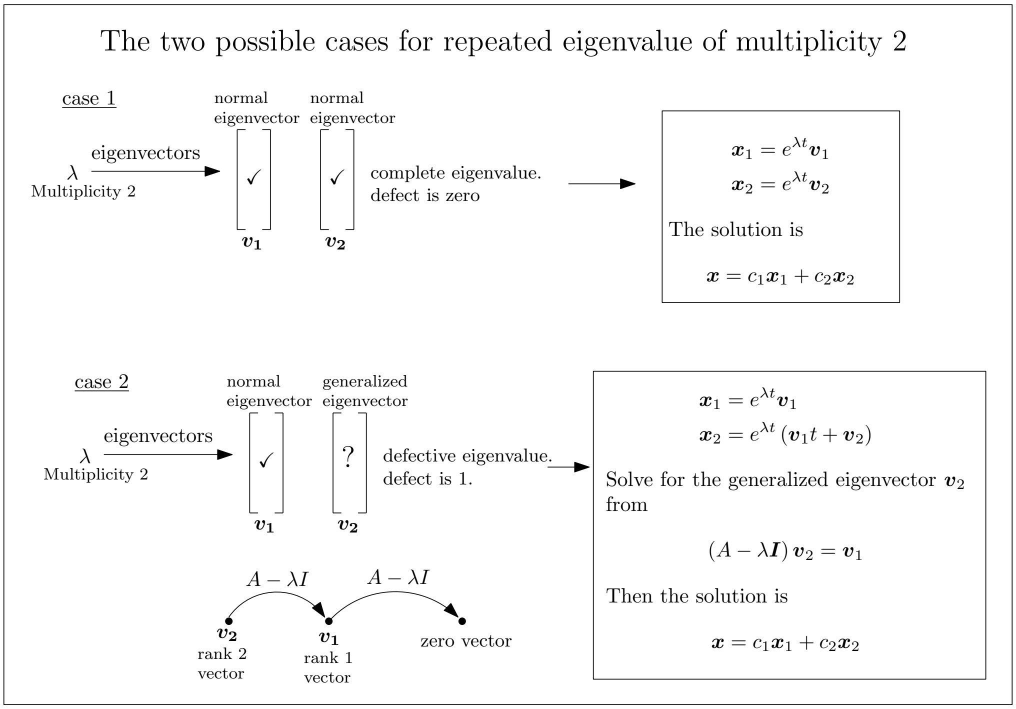


Figure 429: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \mathrm{e}^{-3t}(t+2) \\ \mathrm{e}^{-3t}(1+t) \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}(t+2) \\ \mathrm{e}^{-3t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} ((t+2)\, c_2 + c_1)\, \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_2 + c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}(1 + t) \\ t\,\mathrm{e}^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

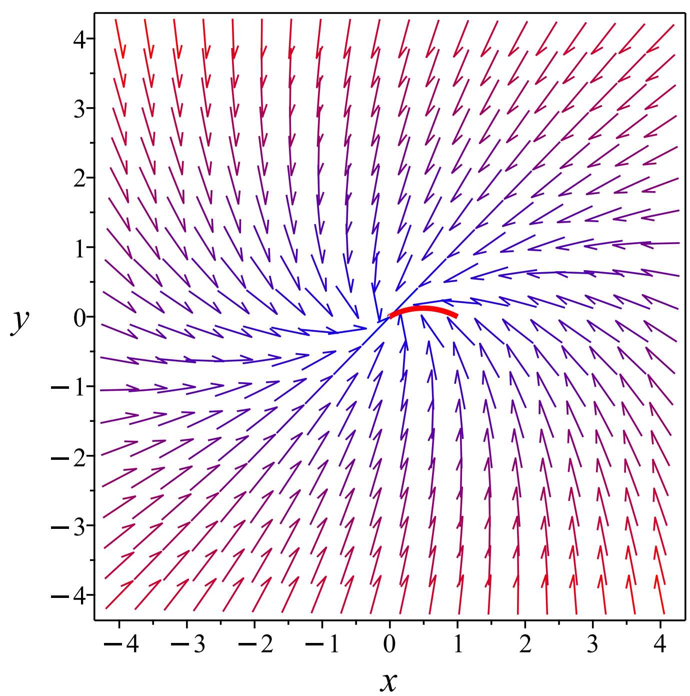
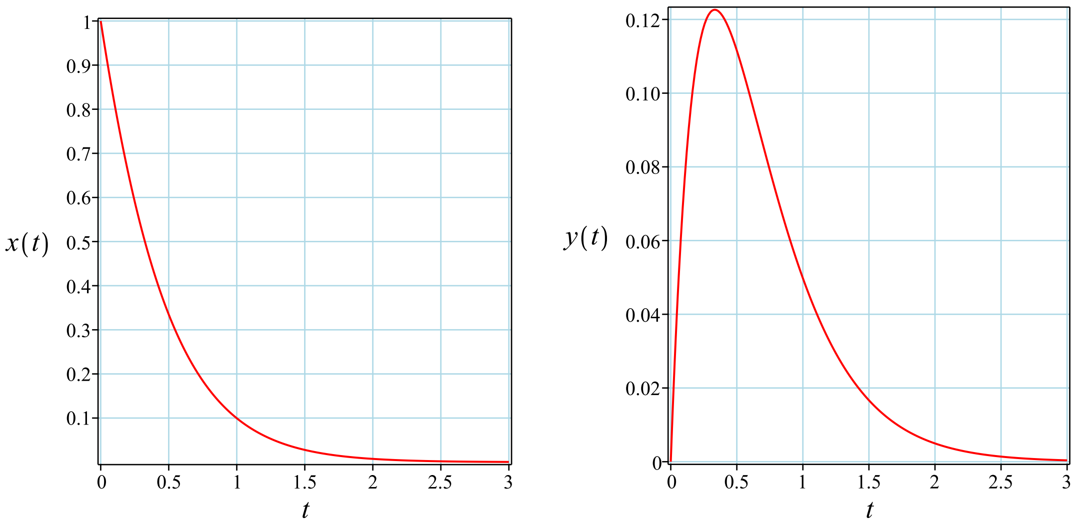


Figure 430: Phase plot

The following are plots of each solution.



✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 22

```
dsolve([diff(x(t),t) = -2*x(t)-y(t), diff(y(t),t) = x(t)-4*y(t), x(0) = 1, y(0) = 0], singso
```

$$x(t) = (t+1)\,\mathrm{e}^{-3t}$$
$$y(t) = t\,\mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 24

```
DSolve[{x'[t]==-2*x[t]-1*y[t],y'[t]==1*x[t]-4*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeS
```

$$x(t) \to e^{-3t}(t+1)$$
$$y(t) \to e^{-3t}t$$

12.4 problem 4

Internal problem ID [13115]
Internal file name [`OUTPUT/11770_Sunday_December_03_2023_07_16_27_PM_63489899/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= y \\
y' &= -x(t) - 2y
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = 0]
$$

12.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} (1+t)\,\mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1-t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} (1+t)\,\mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1-t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (1+t)\,\mathrm{e}^{-t} \\ -t\,\mathrm{e}^{-t} \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

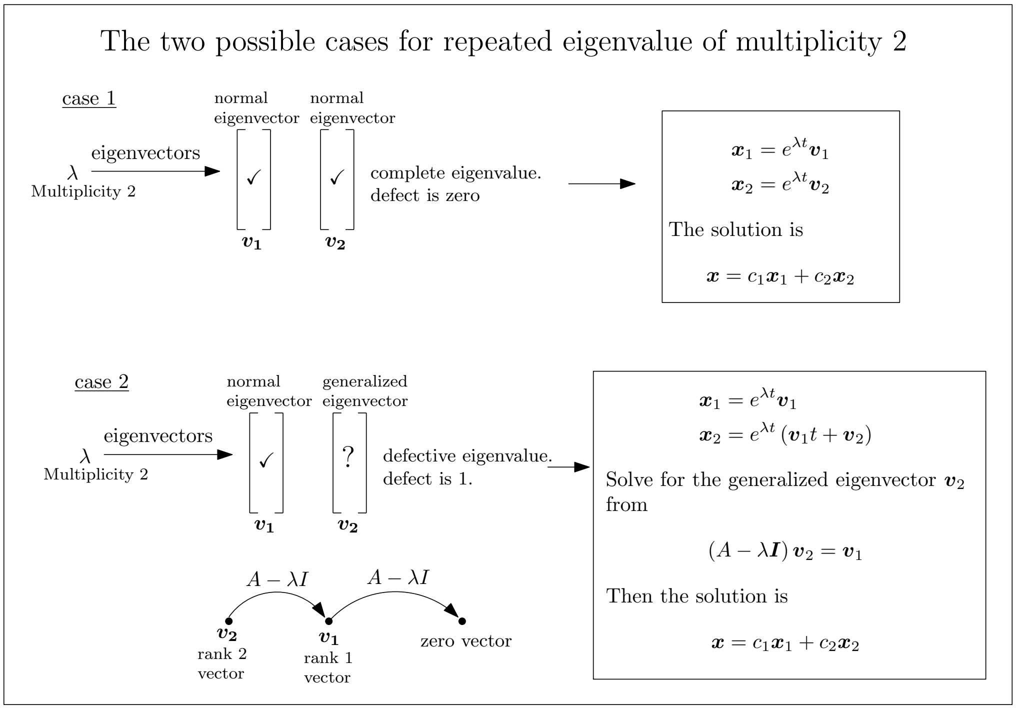


Figure 431: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I) \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} -(t+2)\,\mathrm{e}^{-t} \\ (1+t)\,\mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}(-t-2) \\ (1+t)\,\mathrm{e}^{-t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -((t+2)\,c_2 + c_1)\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2c_2 - c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 1 \\ c_2 = -1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -(-1-t)\, e^{-t} \\ -t\, e^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

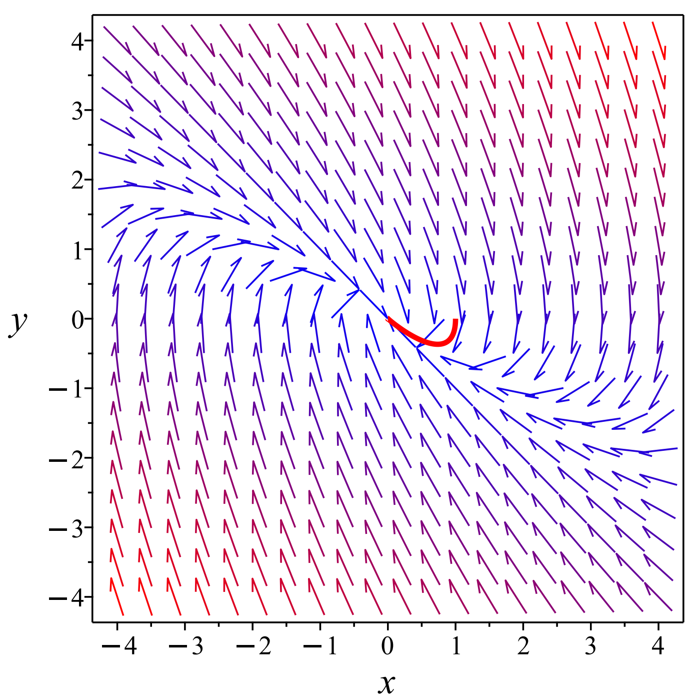
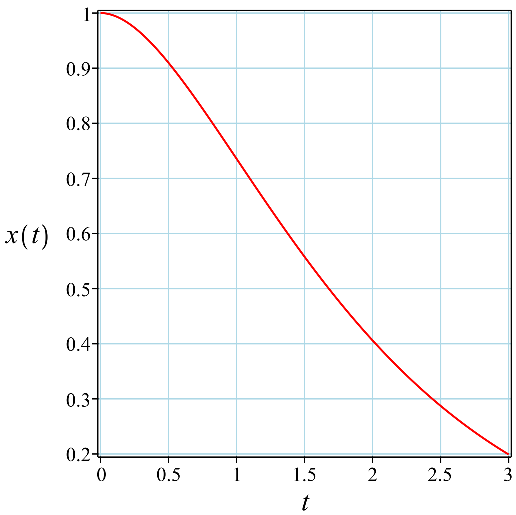
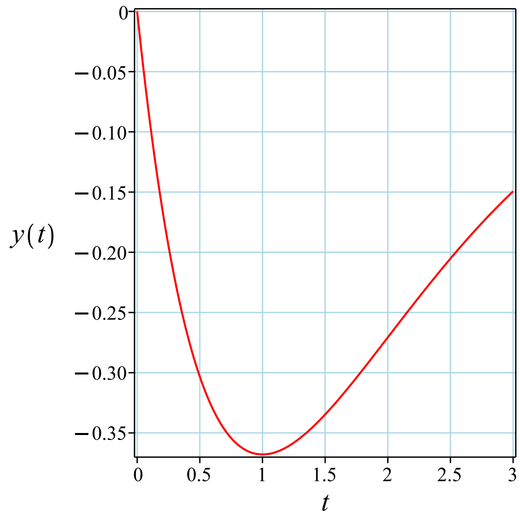


Figure 432: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(x(t),t) = y(t), diff(y(t),t) = -x(t)-2*y(t), x(0) = 1, y(0) = 0], singsol=all)
```

$$x(t) = e^{-t}(t+1)$$
$$y(t) = -t\,e^{-t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 25

```
DSolve[{x'[t]==1*y[t],y'[t]==-1*x[t]-2*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSingular
```

$$x(t) \to e^{-t}(t+1)$$
$$y(t) \to -e^{-t}t$$

12.5 problem 5

Internal problem ID [13116]
Internal file name [`OUTPUT/11771_Sunday_December_03_2023_07_16_27_PM_19636284/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -3x(t)$$
$$y' = x(t) - 3y$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

12.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-3t} & 0 \\ t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \mathrm{e}^{-3t} & 0 \\ t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-3t} \\ t\,\mathrm{e}^{-3t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 0 \\ 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-3-\lambda)(-3-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

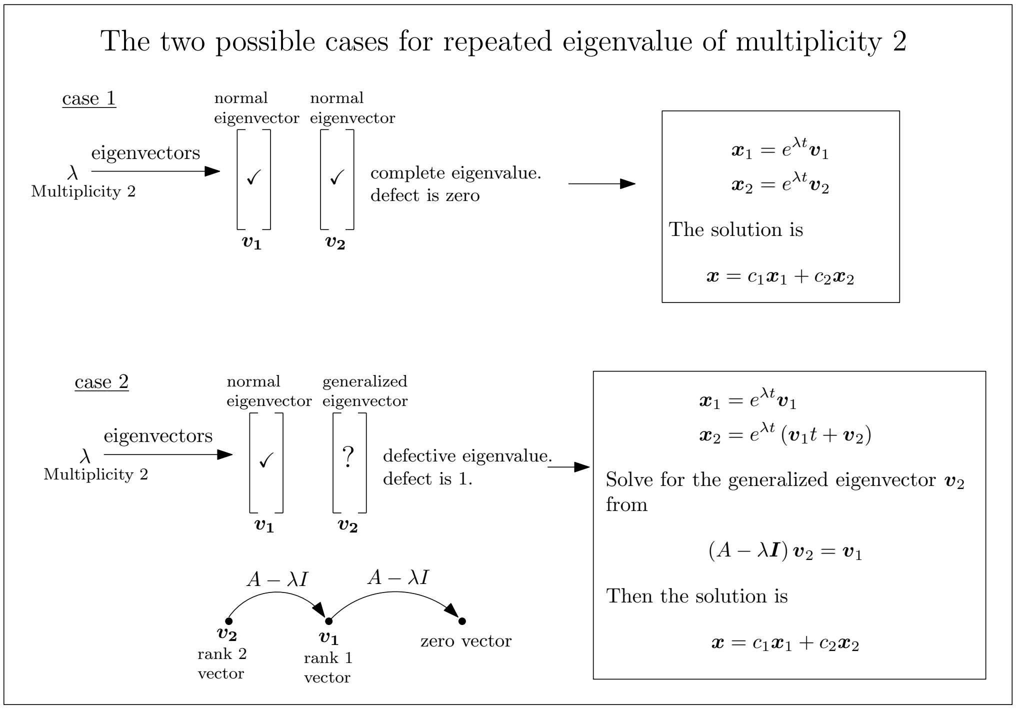


Figure 433: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\
&= \begin{bmatrix} 0 \\ \mathrm{e}^{-3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-3t} \\
&= \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(1+t) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(1+t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} e^{-3t} \\ t\,e^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

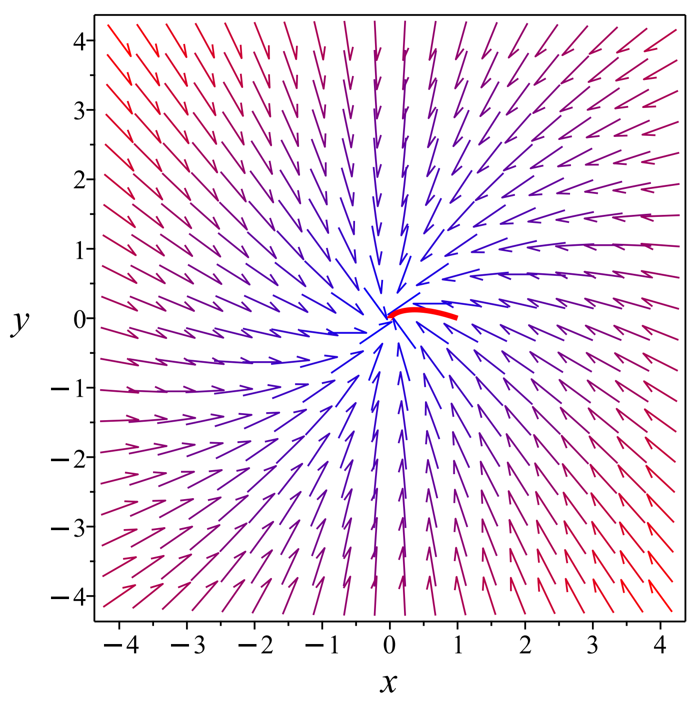
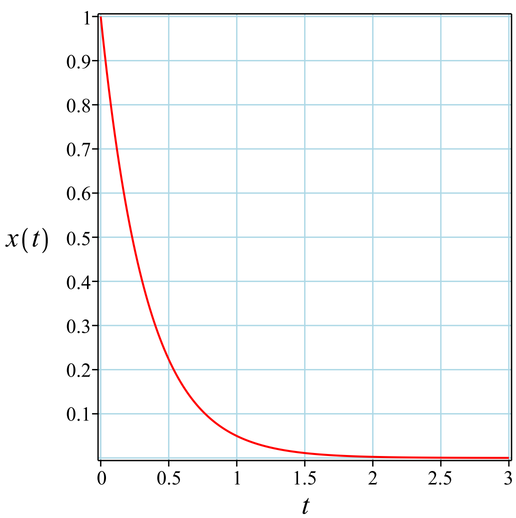
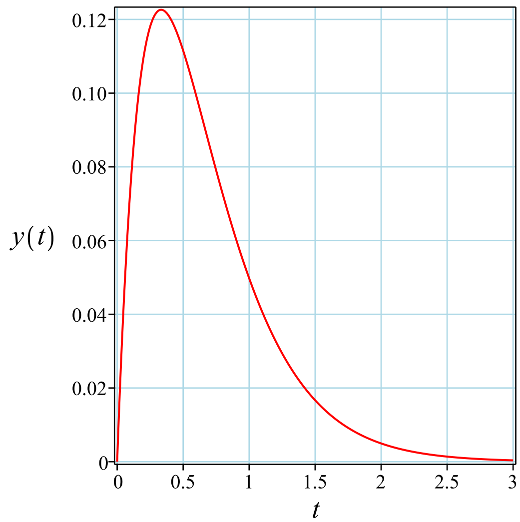


Figure 434: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve([diff(x(t),t) = -3*x(t), diff(y(t),t) = x(t)-3*y(t), x(0) = 1, y(0) = 0], singsol=all
```

$$x(t) = e^{-3t}$$
$$y(t) = t\, e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 20

```
DSolve[{x'[t]==-3*x[t]+0*y[t],y'[t]==1*x[t]-3*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeS
```

$$x(t) \to e^{-3t}$$
$$y(t) \to e^{-3t} t$$

12.6 problem 6

Internal problem ID [13117]
Internal file name [OUTPUT/11772_Sunday_December_03_2023_07_16_27_PM_72485441/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2x(t) + y$$
$$y' = -x(t) + 4y$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

12.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{3t}(1-t) & t\,\mathrm{e}^{3t} \\ -t\,\mathrm{e}^{3t} & \mathrm{e}^{3t}(1+t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{3t}(1-t) & t\,e^{3t} \\ -t\,e^{3t} & e^{3t}(1+t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(1-t) \\ -t\,e^{3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

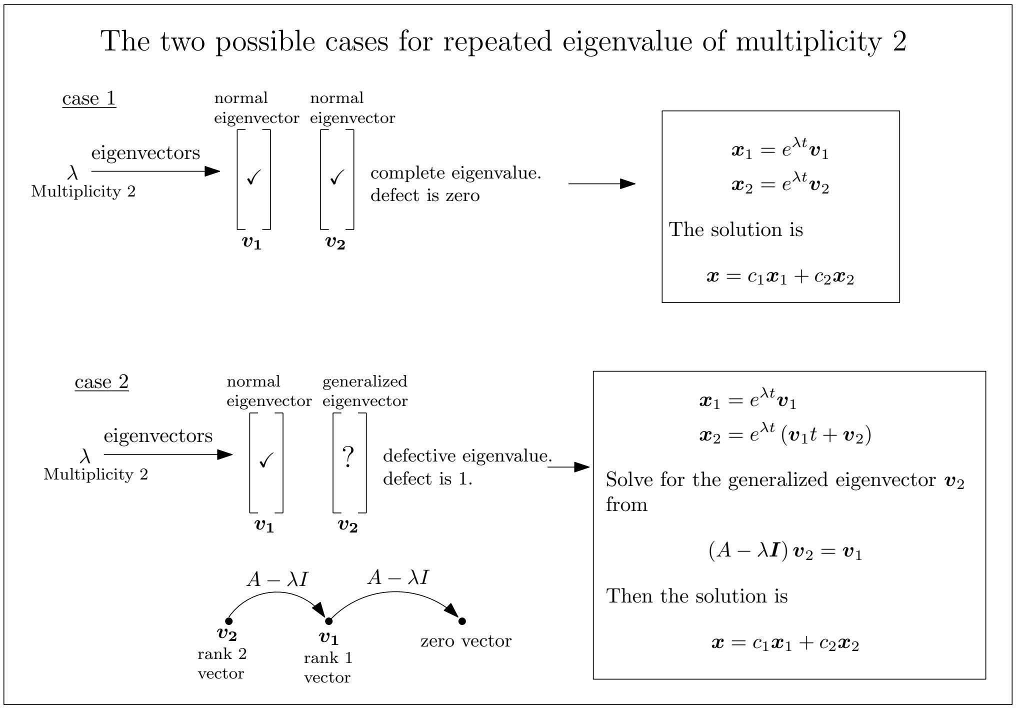


Figure 435: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{3t} \\ &= \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \mathrm{e}^{3t} \\ &= \begin{bmatrix} t\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} t\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{3t}(t c_2 + c_1) \\ \mathrm{e}^{3t}(t c_2 + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 1 \\ c_2 = -1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} e^{3t}(1 - t) \\ -t\, e^{3t} \end{bmatrix}$$

The following is the phase plot of the system.

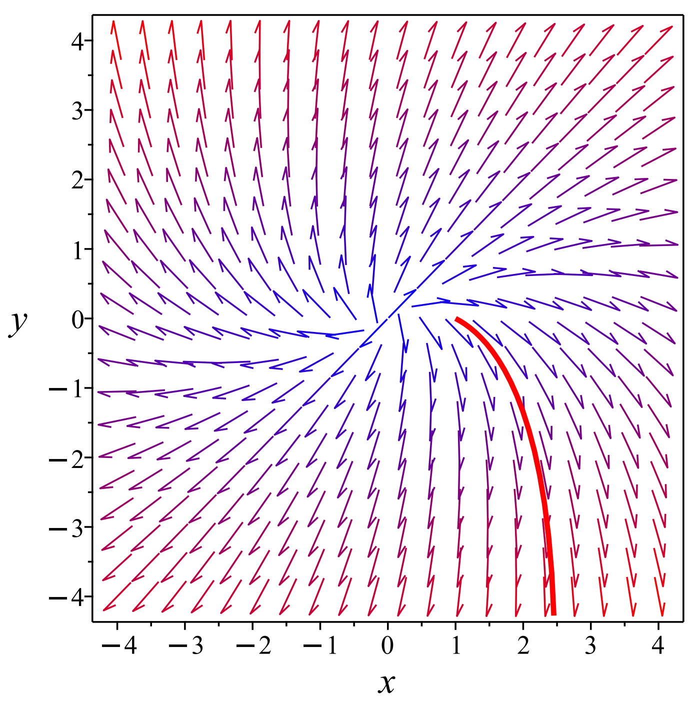
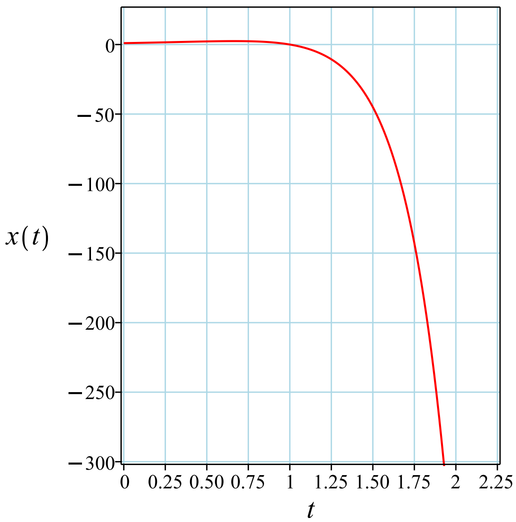
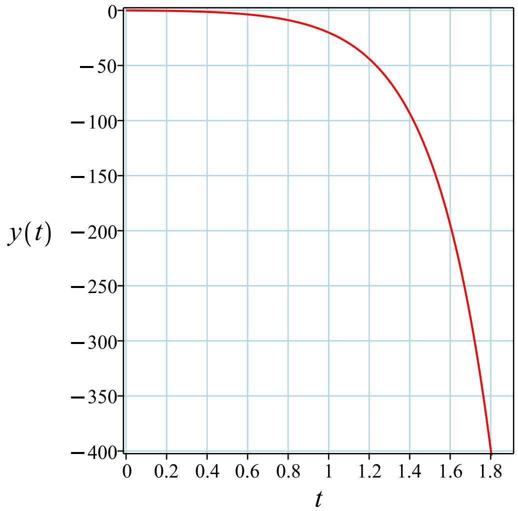


Figure 436: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve([diff(x(t),t) = 2*x(t)+y(t), diff(y(t),t) = -x(t)+4*y(t), x(0) = 1, y(0) = 0], singso
```

$$x(t) = e^{3t}(-t+1)$$
$$y(t) = -e^{3t}t$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 26

```
DSolve[{x'[t]==2*x[t]+1*y[t],y'[t]==-1*x[t]+4*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeS
```

$$x(t) \to -e^{3t}(t-1)$$
$$y(t) \to -e^{3t}t$$

12.7 problem 7

Internal problem ID [13118]
Internal file name [`OUTPUT/11773_Sunday_December_03_2023_07_16_28_PM_78819708/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -2x(t) - y$$
$$y' = x(t) - 4y$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

12.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-3t}(1+t) & -t\,\mathrm{e}^{-3t} \\ t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-3t}(1+t) & -t\,e^{-3t} \\ t\,e^{-3t} & e^{-3t}(1-t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}(1+t) \\ t\,e^{-3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

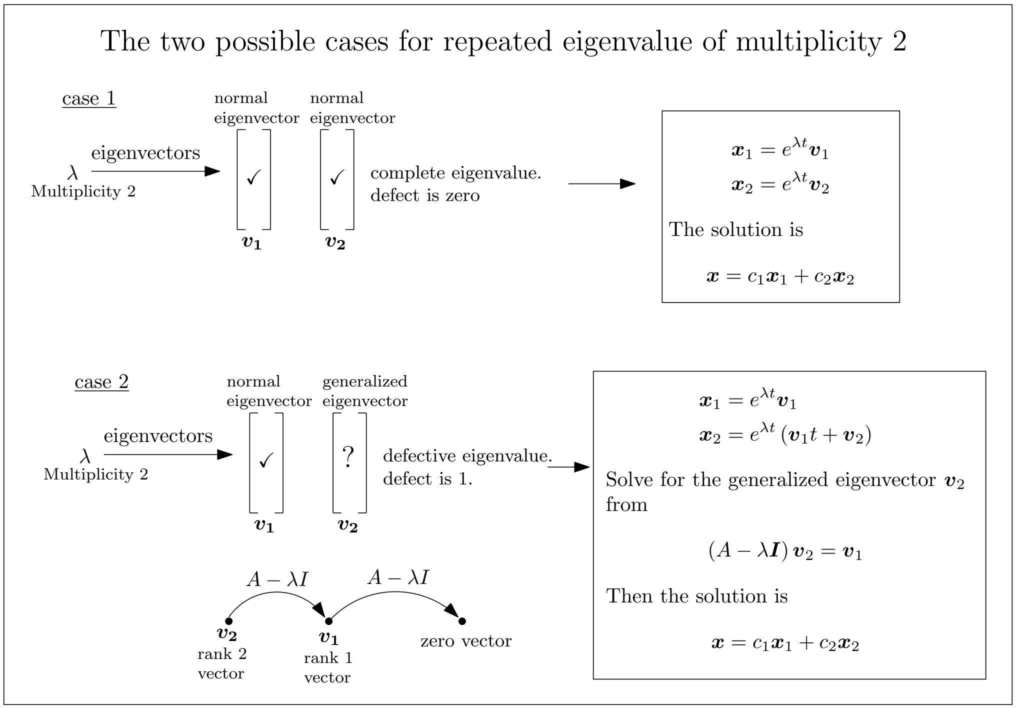


Figure 437: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\ &= \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \mathrm{e}^{-3t} \\ &= \begin{bmatrix} \mathrm{e}^{-3t}(t+2) \\ \mathrm{e}^{-3t}(1+t) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t}(t+2) \\ \mathrm{e}^{-3t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} ((t+2)\, c_2 + c_1)\, \mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_2 + c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-3t}(1+t) \\ t\,\mathrm{e}^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

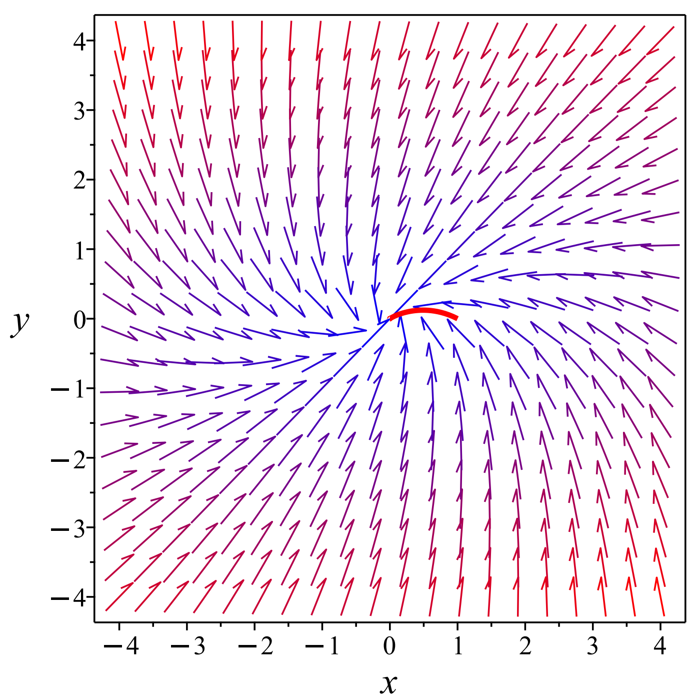
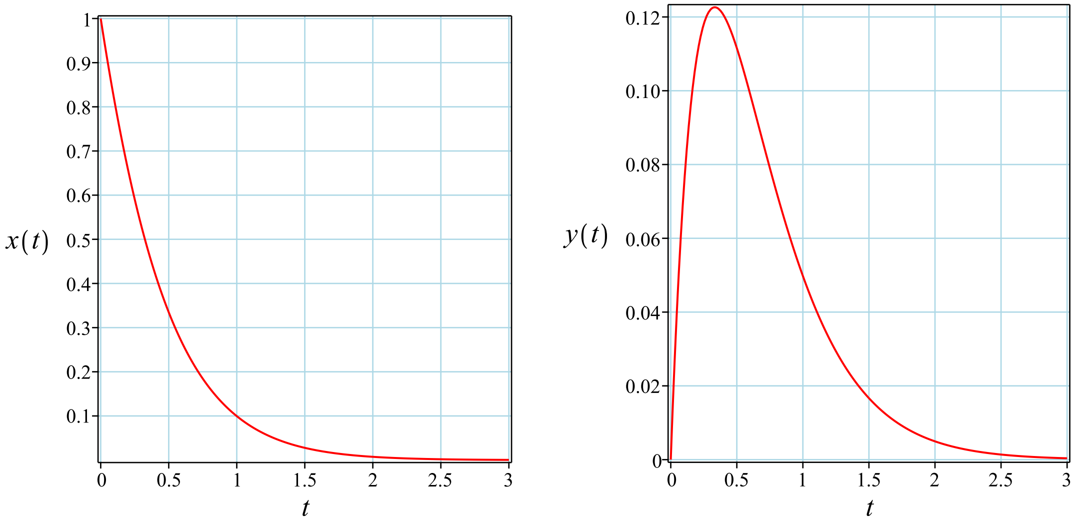


Figure 438: Phase plot

The following are plots of each solution.



✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve([diff(x(t),t) = -2*x(t)-y(t), diff(y(t),t) = x(t)-4*y(t), x(0) = 1, y(0) = 0], singso
```

$$x(t) = (t+1)\,\mathrm{e}^{-3t}$$
$$y(t) = t\,\mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 24

```
DSolve[{x'[t]==-2*x[t]-1*y[t],y'[t]==1*x[t]-4*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeS
```

$$x(t) \to e^{-3t}(t+1)$$
$$y(t) \to e^{-3t}t$$

12.8 problem 8

Internal problem ID [13119]
Internal file name [`OUTPUT/11774_Sunday_December_03_2023_07_16_28_PM_718308/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= y \\
y' &= -x(t) - 2y
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = 0]
$$

12.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} (1+t)\,\mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1-t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} (1+t)\,\mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1-t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (1+t)\,\mathrm{e}^{-t} \\ -t\,\mathrm{e}^{-t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix} \right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

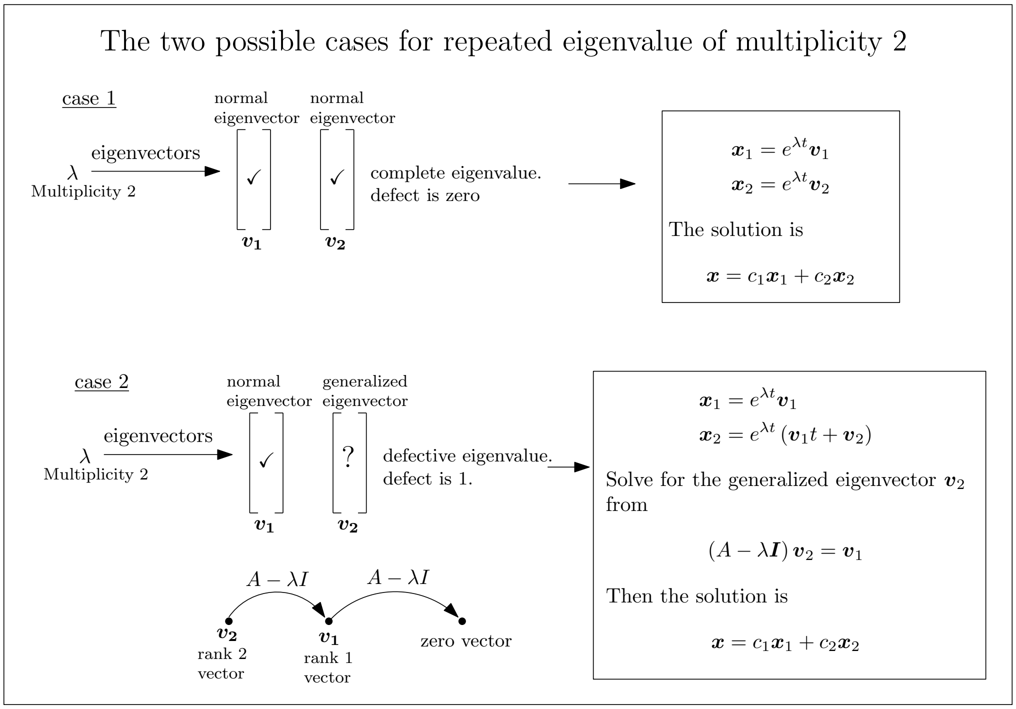


Figure 439: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$
(A-\lambda I)\,\vec{v}_2=\vec{v}_1
$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$
\left(\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}-(-1)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}
$$

$$
\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}
$$

Solving for $\vec{v}_2$ gives

$$
\vec{v}_2=\begin{bmatrix}-2\\ 1\end{bmatrix}
$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} -(t+2)\,\mathrm{e}^{-t} \\ (1+t)\,\mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}(-t-2) \\ (1+t)\,\mathrm{e}^{-t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -((t+2)\,c_2 + c_1)\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2c_2 - c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 1 \\ c_2 = -1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -(-1-t)\,\mathrm{e}^{-t} \\ -t\,\mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

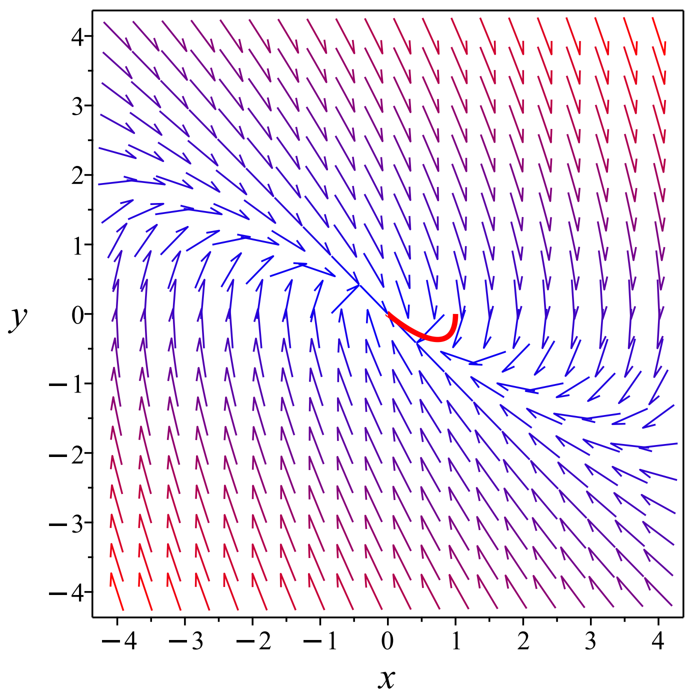
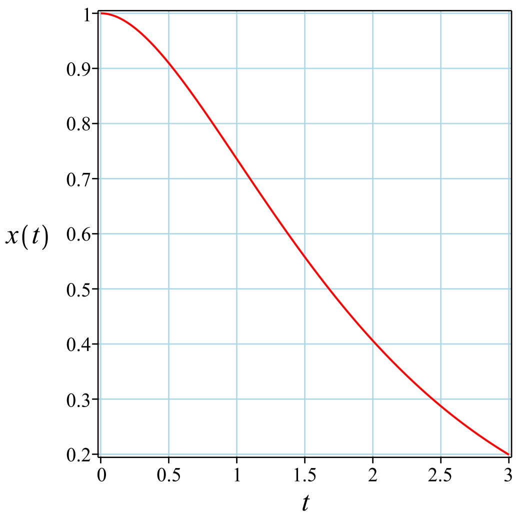
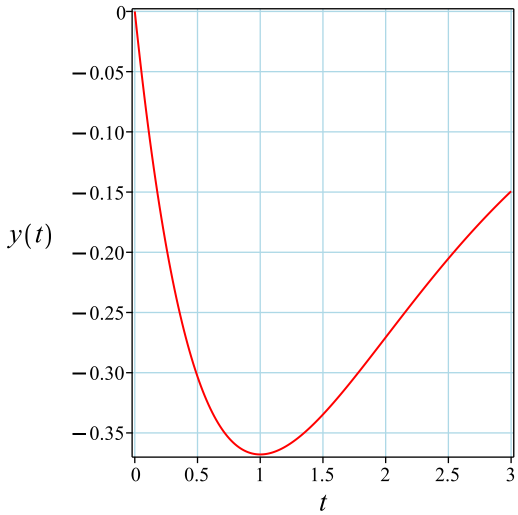


Figure 440: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(x(t),t) = y(t), diff(y(t),t) = -x(t)-2*y(t), x(0) = 1, y(0) = 0], singsol=all)
```

$$x(t) = e^{-t}(t+1)$$
$$y(t) = -t\,e^{-t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 25

```
DSolve[{x'[t]==1*y[t],y'[t]==-1*x[t]-2*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSingular
```

$$x(t) \to e^{-t}(t+1)$$
$$y(t) \to -e^{-t}t$$

12.9 problem 17

Internal problem ID [13120]
Internal file name [`OUTPUT/11775_Sunday_December_03_2023_07_16_28_PM_39181784/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2y$$
$$y' = -y$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

12.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & 2-2\,\mathrm{e}^{-t} \\ 0 & \mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} 1 & 2-2\,\mathrm{e}^{-t} \\ 0 & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 \\ 0 & -1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(-1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^0 \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^0
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -2c_1 \mathrm{e}^{-t} + c_2 \\ c_1 \mathrm{e}^{-t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2c_1 + c_2 \\ c_1 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$
\begin{bmatrix} c_1 = 0 \\ c_2 = 1 \end{bmatrix}
$$

Substituting these constants back in original solution in Eq. (1) gives

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

The following is the phase plot of the system.

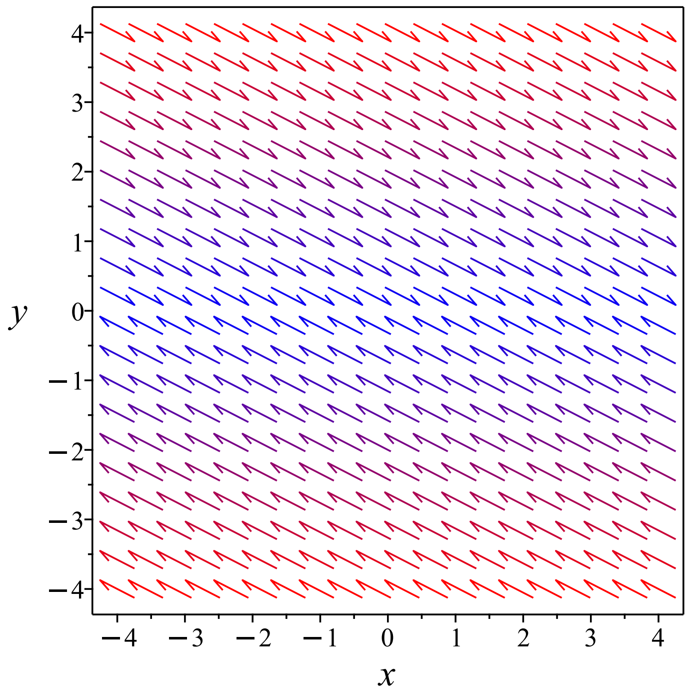
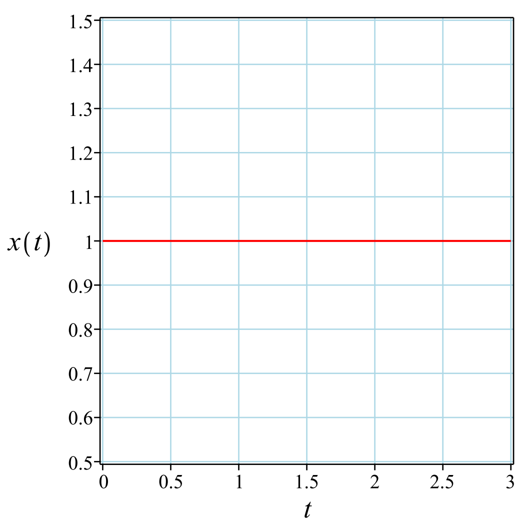
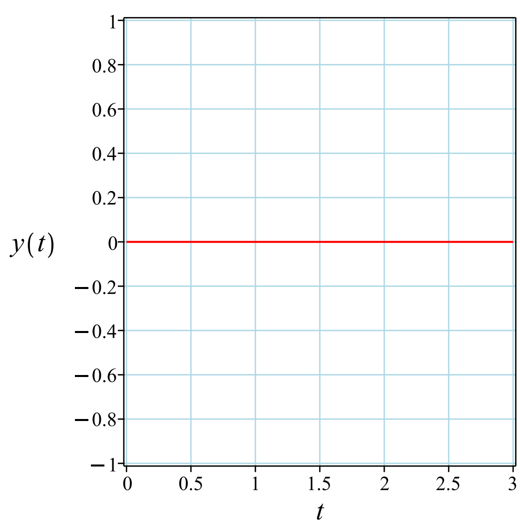


Figure 441: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(x(t),t) = 2*y(t), diff(y(t),t) = -y(t), x(0) = 1, y(0) = 0], singsol=all)
```

$$x(t) = 1$$
$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 10

```
DSolve[{x'[t]==2*y[t],y'[t]==0*x[t]-1*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSingularS
```

$$x(t) \to 1$$
$$y(t) \to 0$$

12.10 problem 18

Internal problem ID [13121]
Internal file name [`OUTPUT/11776_Sunday_December_03_2023_07_16_29_PM_17148776/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + 4y \\
y' &= 3x(t) + 6y
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = 0]
$$

12.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{3}{4} + \frac{e^{8t}}{4} & \frac{e^{8t}}{2} - \frac{1}{2} \\ \frac{3\,e^{8t}}{8} - \frac{3}{8} & \frac{1}{4} + \frac{3\,e^{8t}}{4} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{3}{4}+\frac{e^{8t}}{4} & \frac{e^{8t}}{2}-\frac{1}{2} \\ \frac{3\,e^{8t}}{8}-\frac{3}{8} & \frac{1}{4}+\frac{3\,e^{8t}}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{4}+\frac{e^{8t}}{4} \\ \frac{3\,e^{8t}}{8}-\frac{3}{8} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & 4 \\ 3 & 6-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 8\lambda = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 8$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
8	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 3 & 6 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 8$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} - (8) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -6 & 4 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -6 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
8	1	1	No	$\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the

corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{0} \\
&= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{0}
\end{aligned}
$$

Since eigenvalue 8 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{8t} \\
&= \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} e^{8t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{8t}}{3} \\ \mathrm{e}^{8t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -2c_1 + \frac{2c_2 \mathrm{e}^{8t}}{3} \\ c_1 + c_2 \mathrm{e}^{8t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2c_1 + \frac{2c_2}{3} \\ c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$
\begin{bmatrix} c_1 = -\frac{3}{8} \\ c_2 = \frac{3}{8} \end{bmatrix}
$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{4} + \frac{e^{8t}}{4} \\ \frac{3\,e^{8t}}{8} - \frac{3}{8} \end{bmatrix}$$

The following is the phase plot of the system.

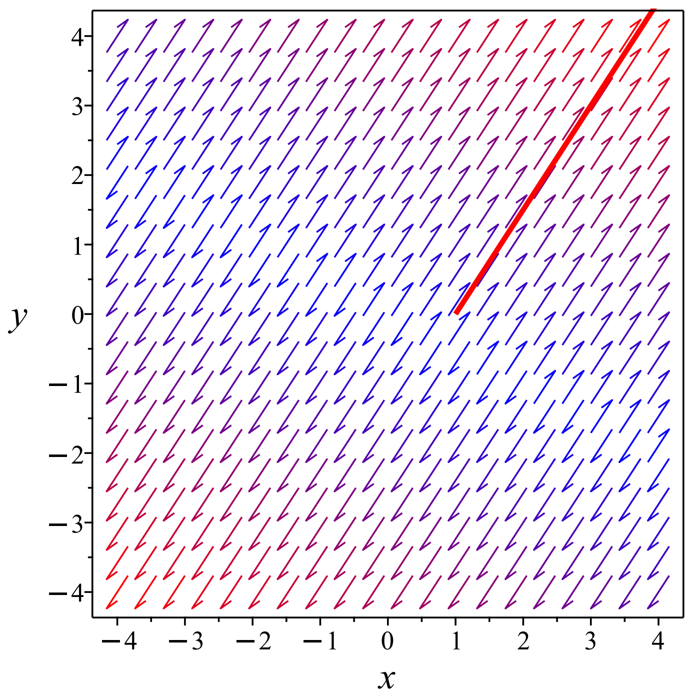
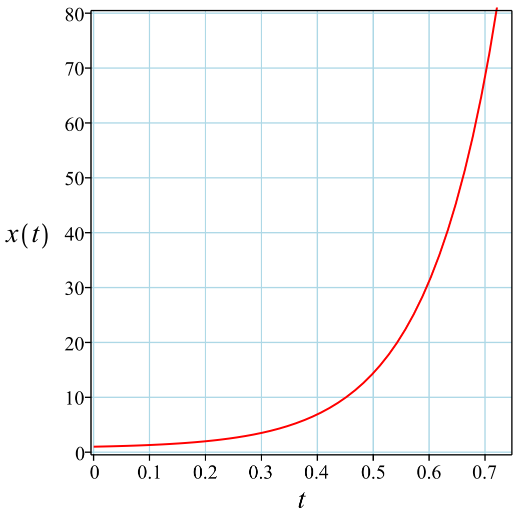
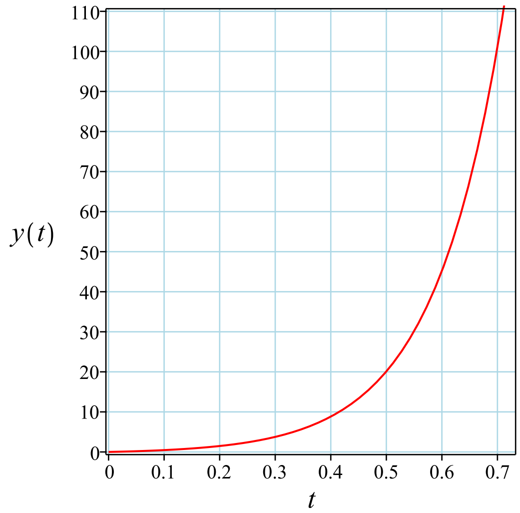


Figure 442: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve([diff(x(t),t) = 2*x(t)+4*y(t), diff(y(t),t) = 3*x(t)+6*y(t), x(0) = 1, y(0) = 0], sin
```

$$x(t) = \frac{3}{4} + \frac{e^{8t}}{4}$$
$$y(t) = \frac{3\,e^{8t}}{8} - \frac{3}{8}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 30

```
DSolve[{x'[t]==2*x[t]+4*y[t],y'[t]==3*x[t]+6*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to \frac{1}{4}\left(e^{8t} + 3\right)$$
$$y(t) \to \frac{3}{8}\left(e^{8t} - 1\right)$$

12.11 problem 19

Internal problem ID [13122]
Internal file name [OUTPUT/11777_Sunday_December_03_2023_07_16_29_PM_46630471/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 4x(t) + 2y$$
$$y' = 2x(t) + y$$

With initial conditions

$$[x(0) = 1, y(0) = 0]$$

12.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{1}{5} + \frac{4\,e^{5t}}{5} & \frac{2\,e^{5t}}{5} - \frac{2}{5} \\ \frac{2\,e^{5t}}{5} - \frac{2}{5} & \frac{4}{5} + \frac{e^{5t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{1}{5}+\frac{4\,e^{5t}}{5} & \frac{2\,e^{5t}}{5}-\frac{2}{5} \\ \frac{2\,e^{5t}}{5}-\frac{2}{5} & \frac{4}{5}+\frac{e^{5t}}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{5}+\frac{4\,e^{5t}}{5} \\ \frac{2\,e^{5t}}{5}-\frac{2}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 5\lambda = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 5 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{5t}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{0}$$
$$= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{0}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 2c_1 \mathrm{e}^{5t} - \frac{c_2}{2} \\ c_1 \mathrm{e}^{5t} + c_2 \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c_1 - \frac{c_2}{2} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{2}{5} \\ c_2 = -\frac{2}{5} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{5} + \frac{4\,e^{5t}}{5} \\ \frac{2\,e^{5t}}{5} - \frac{2}{5} \end{bmatrix}$$

The following is the phase plot of the system.

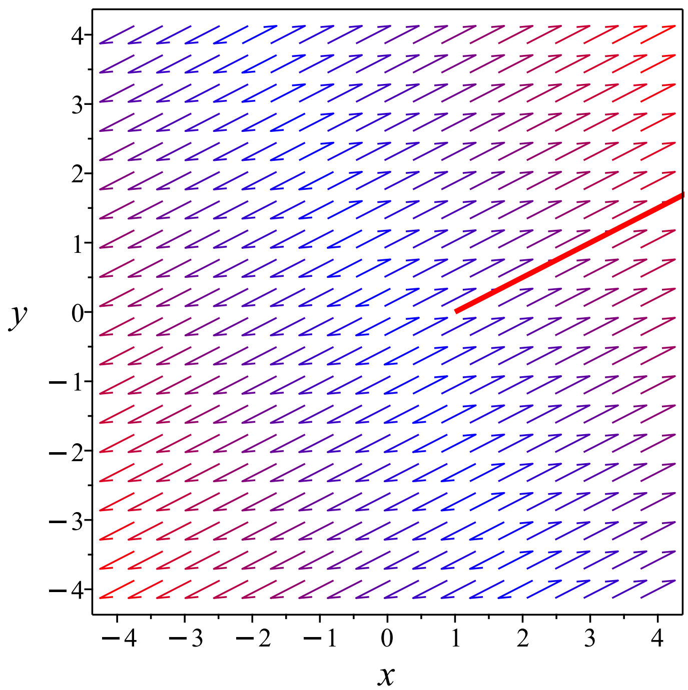
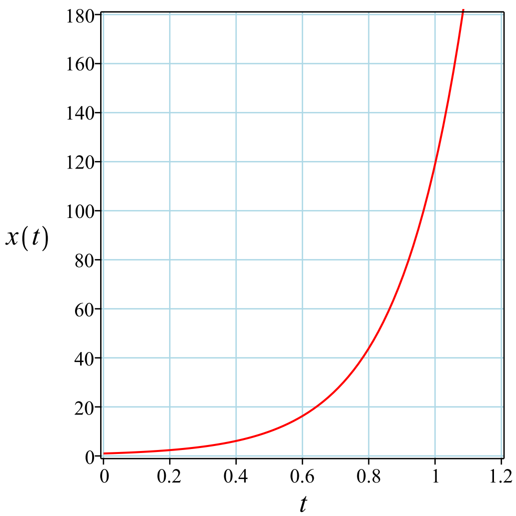
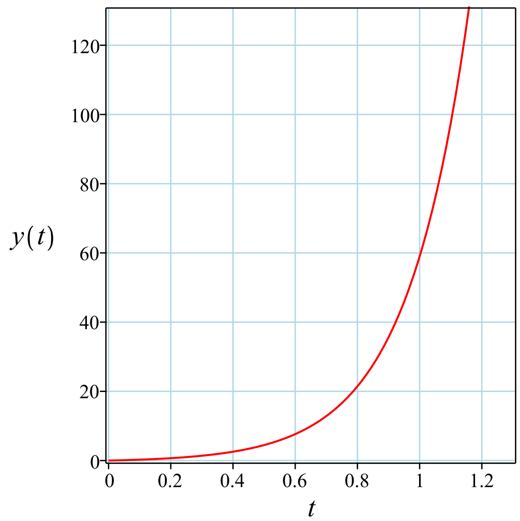


Figure 443: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(x(t),t) = 4*x(t)+2*y(t), diff(y(t),t) = 2*x(t)+y(t), x(0) = 1, y(0) = 0], sings
```

$$x(t) = \frac{1}{5} + \frac{4\,\mathrm{e}^{5t}}{5}$$
$$y(t) = \frac{2\,\mathrm{e}^{5t}}{5} - \frac{2}{5}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 32

```
DSolve[{x'[t]==4*x[t]+2*y[t],y'[t]==2*x[t]+1*y[t]},{x[0]==1,y[0]==0},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to \frac{1}{5}\left(4e^{5t} + 1\right)$$
$$y(t) \to \frac{2}{5}\left(e^{5t} - 1\right)$$

12.12 problem 21(a)

Internal problem ID [13123]
Internal file name [`OUTPUT/11778_Sunday_December_03_2023_07_16_30_PM_55848813/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 21(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2y$$
$$y' = 0$$

12.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} 2tc_2 + c_1 \\ c_2 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 \\ 0 & -\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

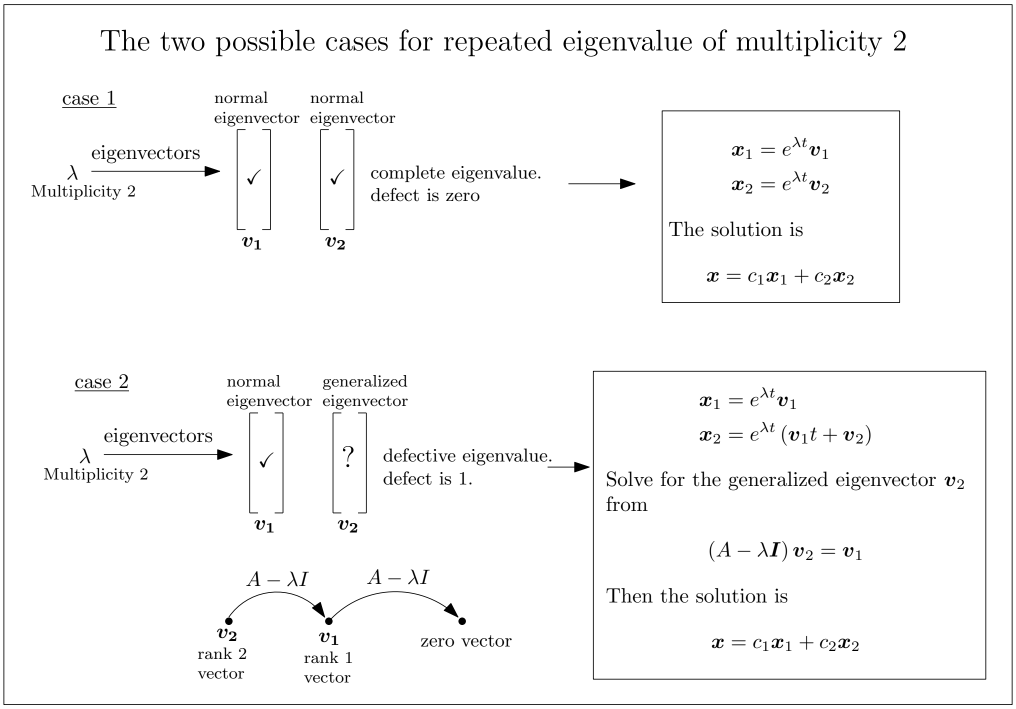


Figure 444: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \right) 1 \\ &= \begin{bmatrix} 1+t \\ \frac{1}{2} \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1+t \\ \frac{1}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 t + c_1 + c_2 \\ \frac{c_2}{2} \end{bmatrix}$$

The following is the phase plot of the system.

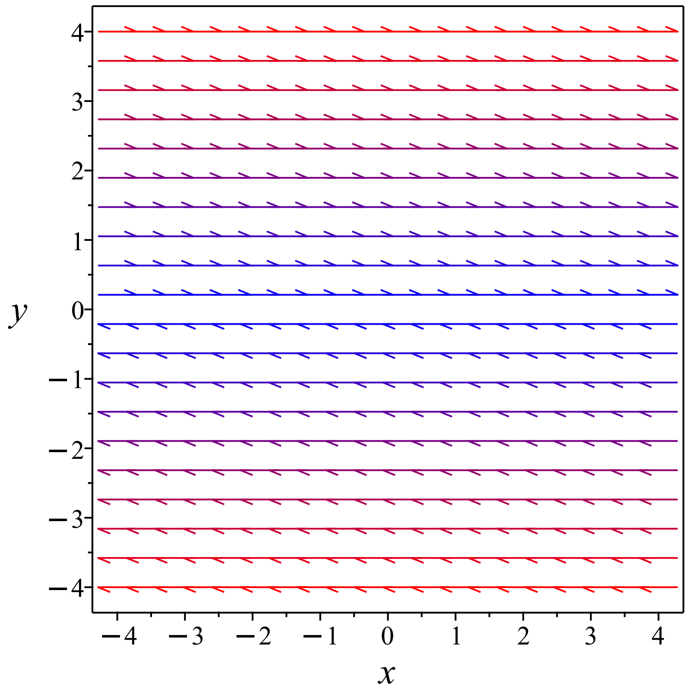


Figure 445: Phase plot

12.12.3 Maple step by step solution

Let's solve

$[x'(t) = 2y, y' = 0]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = c_1, y = 0\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(x(t),t)=2*y(t),diff(y(t),t)=0],singsol=all)
```

$$x(t) = 2c_2t + c_1$$
$$y(t) = c_2$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 18

```
DSolve[{x'[t]==2*y[t],y'[t]==0*x[t]+0*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to 2c_2t + c_1$$
$$y(t) \to c_2$$

12.13 problem 21(b)

Internal problem ID [13124]
Internal file name [`OUTPUT/11779_Sunday_December_03_2023_07_16_30_PM_27040092/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 21(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -2y$$
$$y' = 0$$

12.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & -2t \\ 0 & 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} 1 & -2t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} -2tc_2 + c_1 \\ c_2 \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -2 \\ 0 & -\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

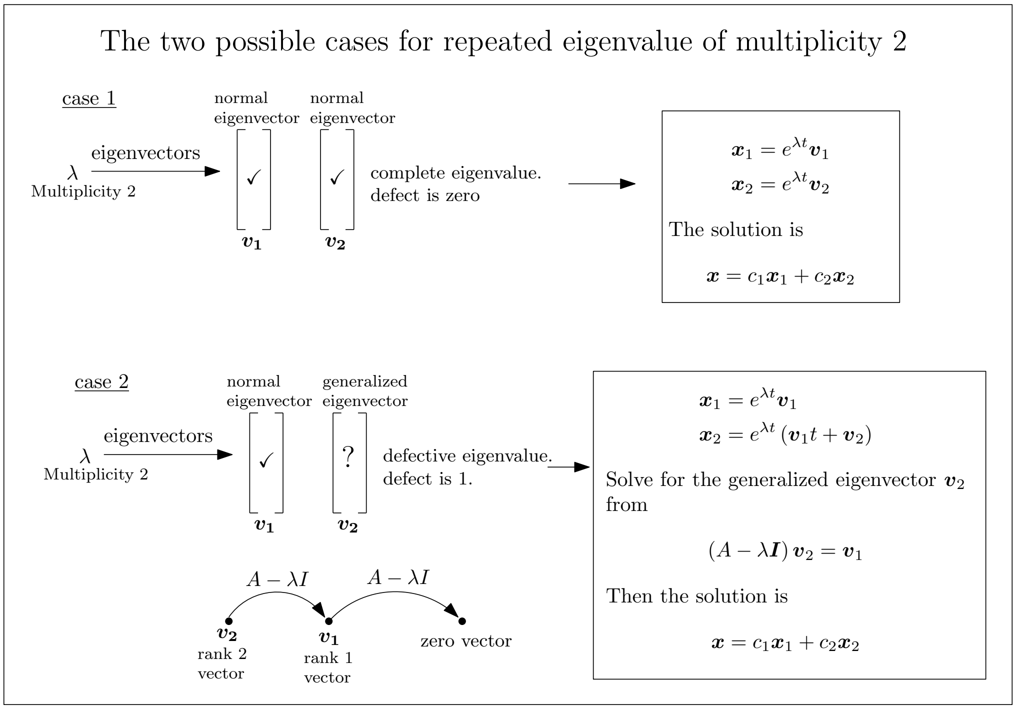


Figure 446: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \right) 1 \\
&= \begin{bmatrix} 1+t \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1+t \\ -\frac{1}{2} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 t + c_1 + c_2 \\ -\frac{c_2}{2} \end{bmatrix}
$$

The following is the phase plot of the system.

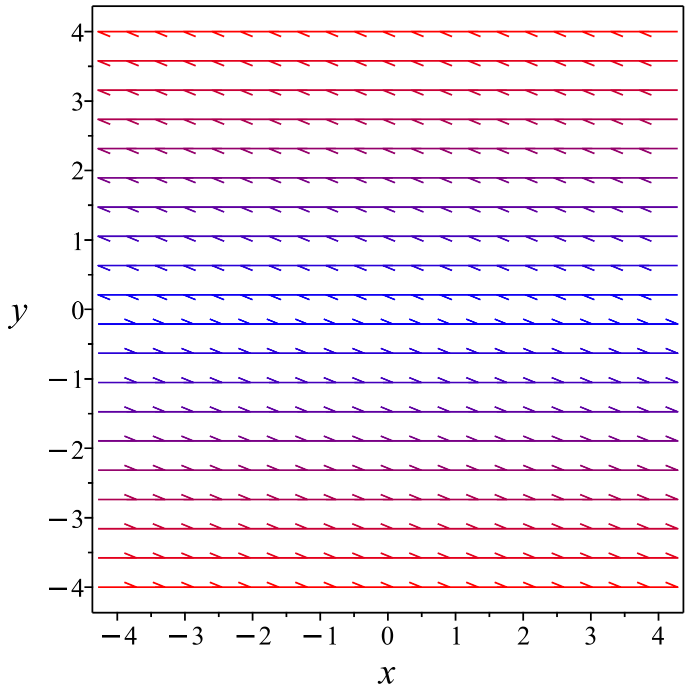


Figure 447: Phase plot

12.13.3 Maple step by step solution

Let's solve

$[x'(t) = -2y, y' = 0]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$
- Consider eigenpair

  $$\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair

  $$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
- Consider eigenpair

  $$\left[0, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair

  $$\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
- General solution to the system of ODEs

  $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$
- Substitute solutions into the general solution

  $$\vec{x} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$
- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = c_1, y = 0\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(x(t),t)=-2*y(t),diff(y(t),t)=0],singsol=all)
```

$$x(t) = -2c_2t + c_1$$
$$y(t) = c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 18

```
DSolve[{x'[t]==-2*y[t],y'[t]==0*x[t]+0*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 - 2c_2t$$
$$y(t) \to c_2$$

12.14 problem 24

Internal problem ID [13125]
Internal file name [`OUTPUT/11780_Sunday_December_03_2023_07_16_30_PM_90758387/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.5 page 327
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -3x(t) - y$$
$$y' = 4x(t) + y$$

With initial conditions

$$[x(0) = -1, y(0) = 2]$$

12.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}(1-2t) & -t\,\mathrm{e}^{-t} \\ 4t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(2t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-t}(1-2t) & -t\,e^{-t} \\ 4t\,e^{-t} & e^{-t}(2t+1) \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -e^{-t}(1-2t) - 2t\,e^{-t} \\ -4t\,e^{-t} + 2\,e^{-t}(2t+1) \end{bmatrix} \\
&= \begin{bmatrix} -e^{-t} \\ 2\,e^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

12.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & -1 \\ 4 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & -1 & 0 \\ 4 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

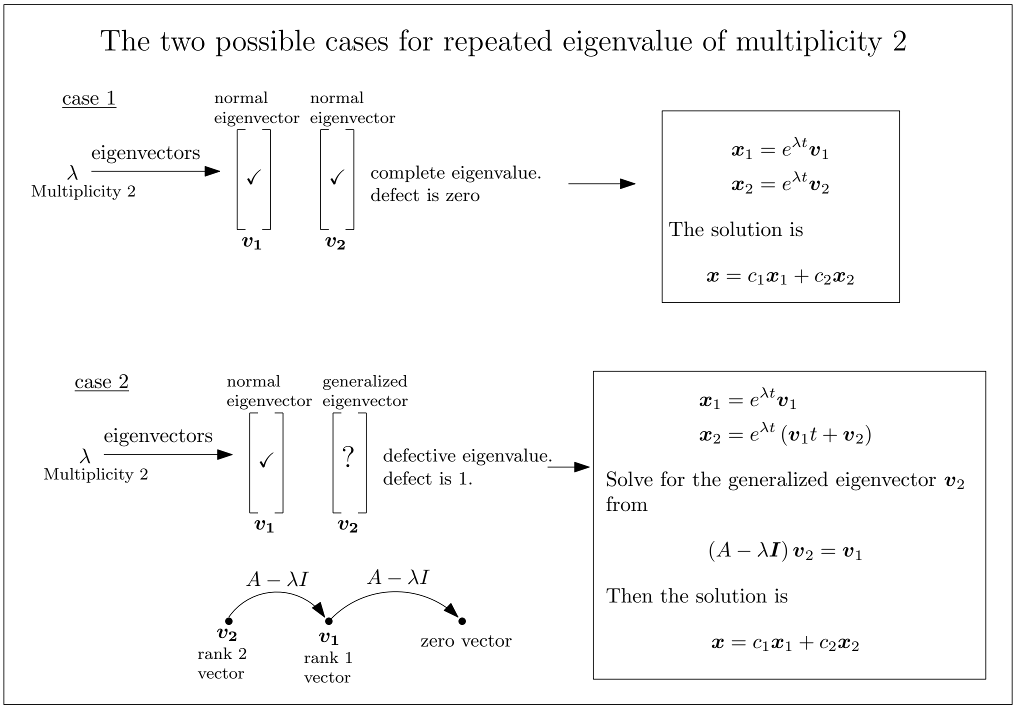


Figure 448: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{-t}(-2+t)}{2} \\ \frac{\mathrm{e}^{-t}(2t-3)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}\left(-\frac{t}{2} + 1\right) \\ \mathrm{e}^{-t}\left(t - \frac{3}{2}\right) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{((-2+t)c_2 + c_1)\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t}\left(c_1 + c_2 t - \frac{3}{2} c_2\right) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = -1 \\ y(0) = 2 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_2 - \frac{c_1}{2} \\ c_1 - \frac{3c_2}{2} \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 2 \\ c_2 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{-t} \\ 2\,\mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

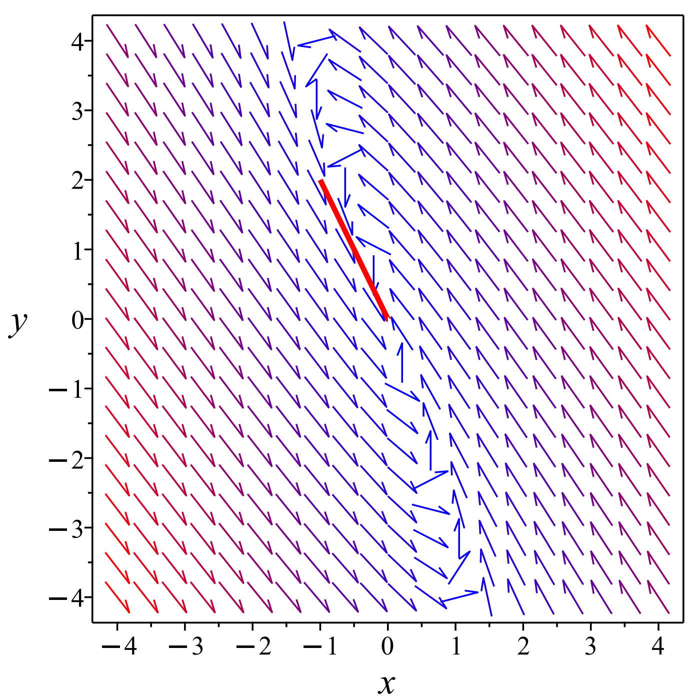
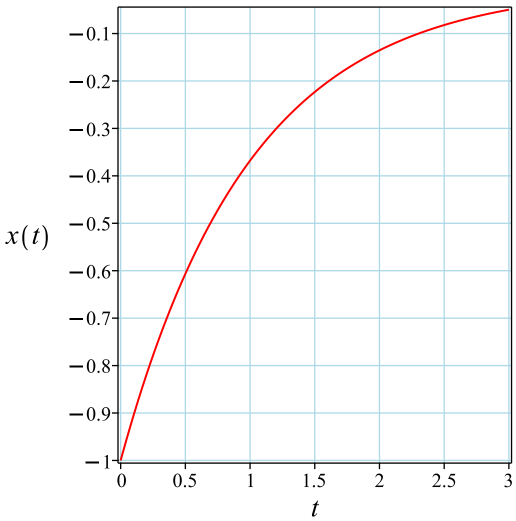
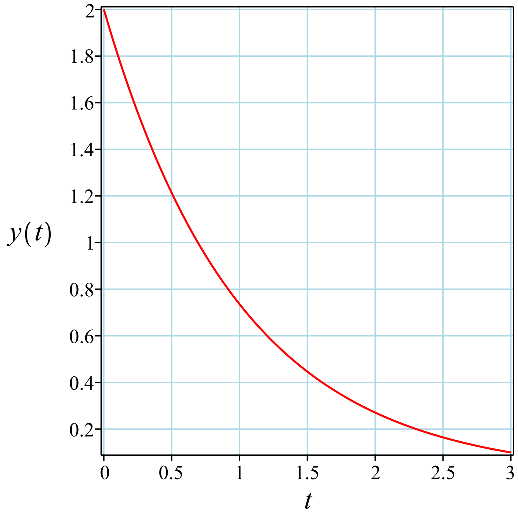


Figure 449: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([diff(x(t),t) = -3*x(t)-y(t), diff(y(t),t) = 4*x(t)+y(t), x(0) = -1, y(0) = 2], sings
```

$$x(t) = -e^{-t}$$
$$y(t) = 2\,e^{-t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 22

```
DSolve[{x'[t]==-3*x[t]-y[t],y'[t]==4*x[t]+y[t]},{x[0]==-1,y[0]==2},{x[t],y[t]},t,IncludeSing
```

$$x(t) \to -e^{-t}$$
$$y(t) \to 2e^{-t}$$

13 Chapter 3. Linear Systems. Exercises section 3.6 page 342

13.1 problem 1

Internal problem ID [13126]
Internal file name [`OUTPUT/11781_Sunday_December_03_2023_07_16_31_PM_41270165/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.6 page 342
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 6y' - 7y = 0}$$

13.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -6, C = -7$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 6\lambda e^{\lambda t} - 7 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 6\lambda - 7 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = -7$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-6^2 - (4)(1)(-7)} \\ &= 3 \pm 4\end{aligned}$$

Hence

$$\lambda_1 = 3 + 4$$
$$\lambda_2 = 3 - 4$$

Which simplifies to

$$\lambda_1 = 7$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(7)t} + c_2 e^{(-1)t}$$

Or

$$y = c_1 \mathrm{e}^{7t} + c_2 \mathrm{e}^{-t}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{7t} + c_2 \mathrm{e}^{-t} \tag{1}$$

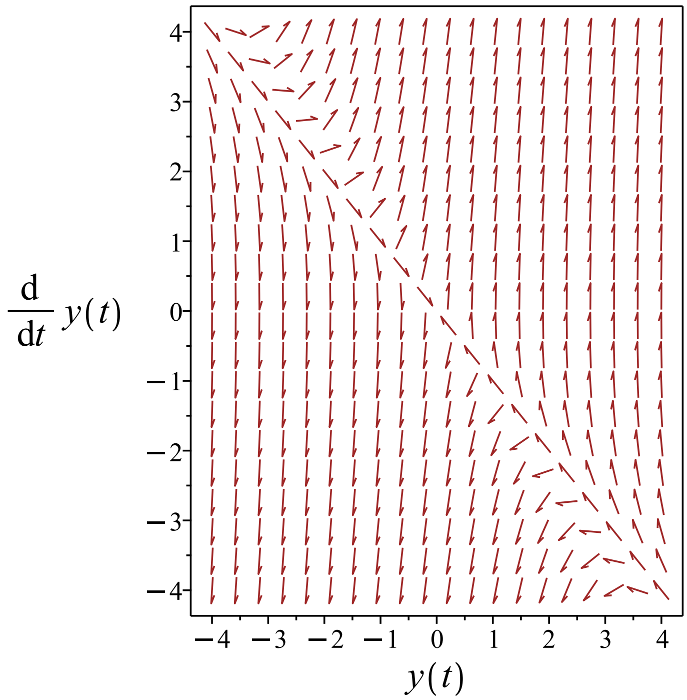


Figure 450: Slope field plot

Verification of solutions

$$y = c_1 e^{7t} + c_2 e^{-t}$$

Verified OK.

13.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' - 7y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= -7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 16z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 378: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 16$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-4t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dt} \\ &= z_1 e^{3t} \\ &= z_1\left(\mathrm{e}^{3t}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{6t}}{(y_1)^2}\,dt \\ &= y_1\left(\frac{\mathrm{e}^{8t}}{8}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t} \left(\frac{\mathrm{e}^{8t}}{8}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{7t}}{8} \tag{1}
$$

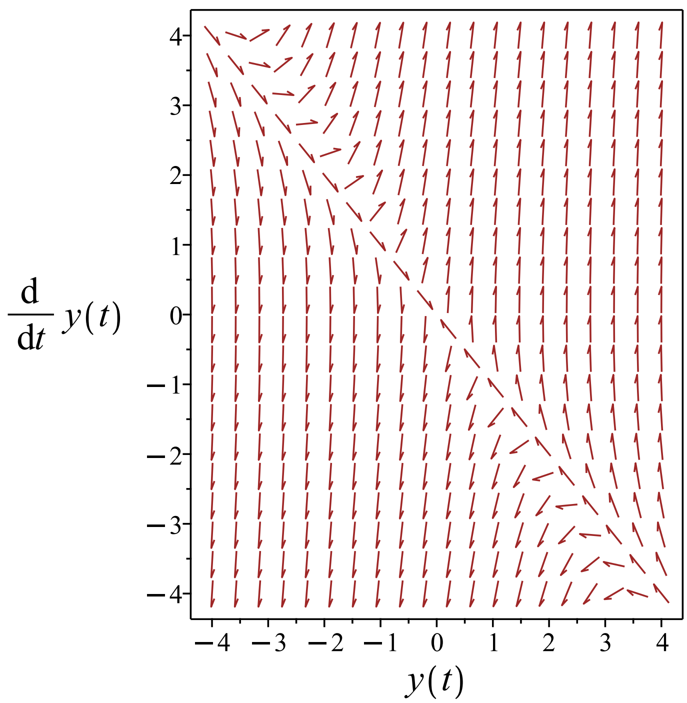


Figure 451: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{7t}}{8}
$$

Verified OK.

13.1.3 Maple step by step solution

Let's solve

$y'' - 6y' - 7y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 6r - 7 = 0$

- Factor the characteristic polynomial

  $(r + 1)(r - 7) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 7)$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$

- 2nd solution of the ODE

  $y_2(t) = e^{7t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-t} + c_2 e^{7t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(t),t$2)-6*diff(y(t),t)-7*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 e^{7t} + c_2 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 22

```
DSolve[y''[t]-6*y'[t]-7*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}\left(c_2 e^{8t} + c_1\right)$$

13.2 problem 2

Internal problem ID [13127]
Internal file name [`OUTPUT/11782_Sunday_December_03_2023_07_16_32_PM_88545654/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.6 page 342
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 12y = 0}$$

13.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -1, C = -12$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - 12 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - \lambda - 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -12$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-1^2 - (4)(1)(-12)} \\
&= \frac{1}{2} \pm \frac{7}{2}
\end{aligned}
$$

Hence

$$
\lambda_1 = \frac{1}{2} + \frac{7}{2}
$$

$$
\lambda_2 = \frac{1}{2} - \frac{7}{2}
$$

Which simplifies to

$$
\lambda_1 = 4
$$

$$
\lambda_2 = -3
$$

Since roots are real and distinct, then the solution is

$$
y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}
$$

$$
y = c_1 e^{(4)t} + c_2 e^{(-3)t}
$$

Or

$$
y = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-3t}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-3t} \tag{1}
$$

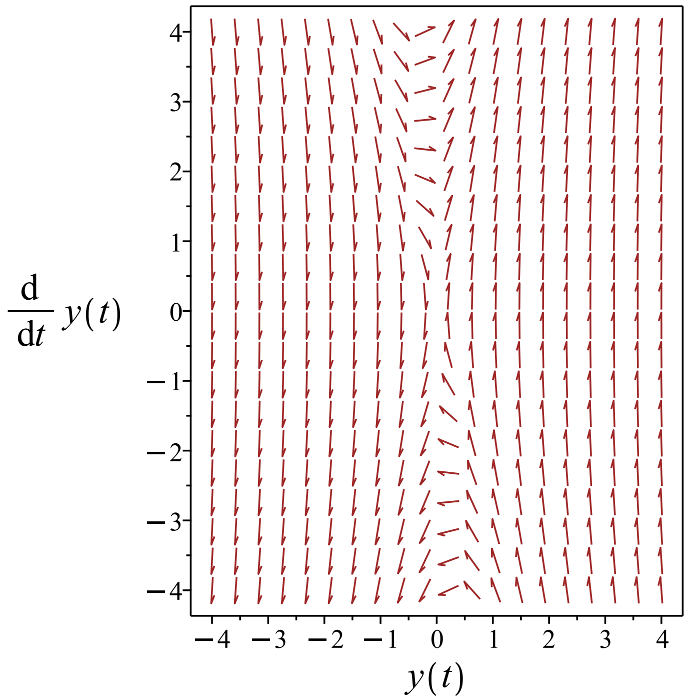


Figure 452: Slope field plot

Verification of solutions

$$y = c_1 e^{4t} + c_2 e^{-3t}$$

Verified OK.

13.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 12y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 49$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{49z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 380: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{49}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{7t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dt}\\ &= z_1 e^{\frac{t}{2}}\\ &= z_1\left(\mathrm{e}^{\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dt}}{(y_1)^2}\,dt\\ &= y_1 \int \frac{e^{t}}{(y_1)^2}\,dt\\ &= y_1\left(\frac{\mathrm{e}^{7t}}{7}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-3t}\right) + c_2\left(e^{-3t}\left(\frac{e^{7t}}{7}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-3t}c_1 + \frac{c_2 e^{4t}}{7} \tag{1}$$

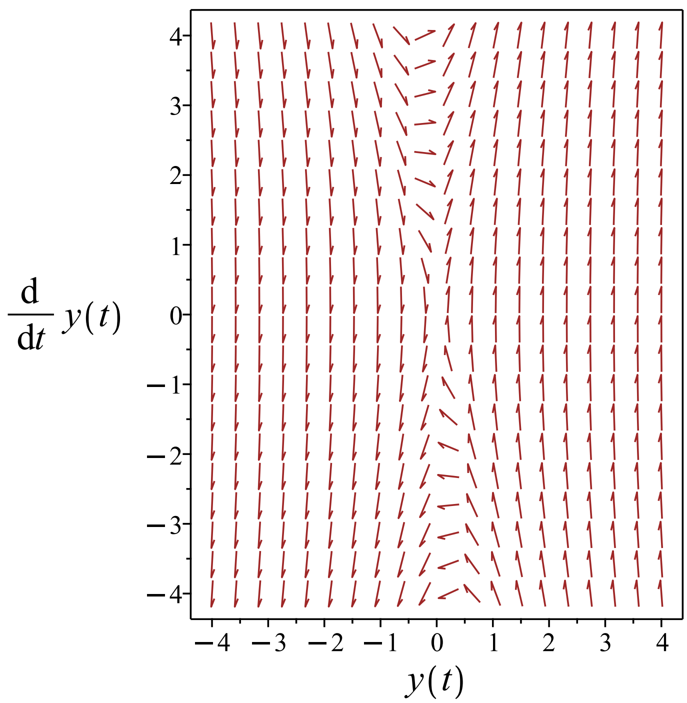


Figure 453: Slope field plot

Verification of solutions

$$y = e^{-3t}c_1 + \frac{c_2 e^{4t}}{7}$$

Verified OK.

13.2.3 Maple step by step solution

Let's solve

$y'' - y' - 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 12 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 4)$

- 1st solution of the ODE

  $y_1(t) = e^{-3t}$

- 2nd solution of the ODE

  $y_2(t) = e^{4t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = e^{-3t} c_1 + c_2 e^{4t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(t),t$2)-diff(y(t),t)-12*y(t)=0,y(t), singsol=all)
```

$$y(t) = \left(e^{7t}c_2 + c_1\right)e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 22

```
DSolve[y''[t]-y'[t]-12*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-3t}\left(c_2e^{7t} + c_1\right)$$

14 Chapter 3. Linear Systems. Exercises section 3.8 page 371

14.1 problem 1

Internal problem ID [13128]
Internal file name [`OUTPUT/11783_Sunday_December_03_2023_07_16_34_PM_72024113/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{y}{10} \\
y' &= \frac{z(t)}{5} \\
z'(t) &= \frac{2x(t)}{5}
\end{aligned}
$$

14.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \left[\begin{array}{ccc}
\frac{e^{\frac{t}{5}}}{3}+\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3} & -\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{6}+\frac{\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{6}+\frac{e^{\frac{t}{5}}}{6} & -\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{6} \cdots \\
-\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}-\frac{2\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\,e^{\frac{t}{5}}}{3} & \frac{e^{\frac{t}{5}}}{3}+\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3} & -\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3} \cdots \\
-\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\,e^{\frac{t}{5}}}{3} & -\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}-\frac{\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{e^{\frac{t}{5}}}{3} & \frac{e^{\frac{t}{5}}}{3}+ \cdots
\end{array}\right.
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{ccc}
\frac{e^{\frac{t}{5}}}{3}+\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3} & -\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{6}+\frac{\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{6}+\frac{e^{\frac{t}{5}}}{6} & -\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{6} \cdots \\
-\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}-\frac{2\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\,e^{\frac{t}{5}}}{3} & \frac{e^{\frac{t}{5}}}{3}+\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3} & -\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3} \cdots \\
-\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\,e^{\frac{t}{5}}}{3} & -\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}-\frac{\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{e^{\frac{t}{5}}}{3} & \frac{e^{\frac{t}{5}}}{3}+ \cdots
\end{array}\right. \\
&= \left[\begin{array}{c}
\left(\frac{e^{\frac{t}{5}}}{3}+\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}\right)c_1+\left(-\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{6}+\frac{\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{6}+\frac{e^{\frac{t}{5}}}{6}\right)c_2+\left(-\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{6}- \cdots\right. \\
\left(-\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}-\frac{2\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\,e^{\frac{t}{5}}}{3}\right)c_1+\left(\frac{e^{\frac{t}{5}}}{3}+\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}\right)c_2+\left(-\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+ \cdots\right. \\
\left(-\frac{2\,e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\,e^{\frac{t}{5}}}{3}\right)c_1+\left(-\frac{e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}-\frac{\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{e^{\frac{t}{5}}}{3}\right)c_2+ \cdots
\end{array}\right. \\
&= \left[\begin{array}{c}
\frac{2\left(c_1-\frac{c_2}{4}-\frac{c_3}{4}\right)e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{\sqrt{3}\,e^{-\frac{t}{10}}\left(c_2-c_3\right)\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{6}+\frac{e^{\frac{t}{5}}\left(c_1+\frac{c_2}{2}+\frac{c_3}{2}\right)}{3} \\
-\frac{2\left(c_1-c_2+\frac{c_3}{2}\right)e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}-\frac{2\left(c_1-\frac{c_3}{2}\right)\sqrt{3}\,e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\,e^{\frac{t}{5}}\left(c_1+\frac{c_2}{2}+\frac{c_3}{2}\right)}{3} \\
-\frac{2\,e^{-\frac{t}{10}}\left(c_1+\frac{c_2}{2}-c_3\right)\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\sqrt{3}\,e^{-\frac{t}{10}}\left(c_1-\frac{c_2}{2}\right)\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3}+\frac{2\,e^{\frac{t}{5}}\left(c_1+\frac{c_2}{2}+\frac{c_3}{2}\right)}{3}
\end{array}\right]
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\lambda & \frac{1}{10} & 0 \\ 0 & -\lambda & \frac{1}{5} \\ \frac{2}{5} & 0 & -\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$
\lambda^3 - \frac{1}{125} = 0
$$

The roots of the above are the eigenvalues.

$$
\begin{aligned}
\lambda_1 &= \frac{1}{5} \\
\lambda_2 &= -\frac{1}{10} + \frac{i\sqrt{3}}{10} \\
\lambda_3 &= -\frac{1}{10} - \frac{i\sqrt{3}}{10}
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{10} - \frac{i\sqrt{3}}{10}$	1	complex eigenvalue
$\frac{1}{5}$	1	real eigenvalue
$-\frac{1}{10} + \frac{i\sqrt{3}}{10}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{5}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & 0 \end{bmatrix} - \left(\frac{1}{5}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{5} & \frac{1}{10} & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & 0 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\frac{1}{5} & \frac{1}{10} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & 0 & -\frac{1}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -\frac{1}{5} & \frac{1}{10} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -\frac{1}{5} & \frac{1}{10} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{5} & \frac{1}{10} & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = t\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{10} - \frac{i\sqrt{3}}{10}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & 0 \end{bmatrix} - \left(-\frac{1}{10} - \frac{i\sqrt{3}}{10} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{10} + \frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 \\ 0 & \frac{1}{10} + \frac{i\sqrt{3}}{10} & \frac{1}{5} \\ \frac{2}{5} & 0 & \frac{1}{10} + \frac{i\sqrt{3}}{10} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{10}+\frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10}+\frac{i\sqrt{3}}{10} & \frac{1}{5} & 0 \\ \frac{2}{5} & 0 & \frac{1}{10}+\frac{i\sqrt{3}}{10} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{5\left(\frac{1}{10}+\frac{i\sqrt{3}}{10}\right)} \implies \left[\begin{array}{ccc|c} \frac{1}{10}+\frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10}+\frac{i\sqrt{3}}{10} & \frac{1}{5} & 0 \\ 0 & -\frac{2}{5i\sqrt{3}+5} & \frac{1}{10}+\frac{i\sqrt{3}}{10} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_2}{\left(5i\sqrt{3}+5\right)\left(\frac{1}{10}+\frac{i\sqrt{3}}{10}\right)} \implies \left[\begin{array}{ccc|c} \frac{1}{10}+\frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10}+\frac{i\sqrt{3}}{10} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{10}+\frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 \\ 0 & \frac{1}{10}+\frac{i\sqrt{3}}{10} & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{i\sqrt{3}-1}, v_2 = -\frac{2t}{1+i\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{\mathrm{I}\sqrt{3}-1} \\ -\frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{i\sqrt{3}-1} \\ -\frac{2t}{1+i\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{\mathrm{I}\sqrt{3}-1} \\ -\frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{i\sqrt{3}-1} \\ -\frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{\mathrm{I}\sqrt{3}-1} \\ -\frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{i\sqrt{3}-1} \\ -\frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{\mathrm{I}\sqrt{3}-1} \\ -\frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{i\sqrt{3}-1} \\ -\frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -\frac{1}{10} + \frac{i\sqrt{3}}{10}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & 0 \end{bmatrix} - \left(-\frac{1}{10} + \frac{i\sqrt{3}}{10} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{10} - \frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 \\ 0 & \frac{1}{10} - \frac{i\sqrt{3}}{10} & \frac{1}{5} \\ \frac{2}{5} & 0 & \frac{1}{10} - \frac{i\sqrt{3}}{10} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{10} - \frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} - \frac{i\sqrt{3}}{10} & \frac{1}{5} & 0 \\ \frac{2}{5} & 0 & \frac{1}{10} - \frac{i\sqrt{3}}{10} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_1}{5\left(\frac{1}{10} - \frac{i\sqrt{3}}{10}\right)} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{10} - \frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} - \frac{i\sqrt{3}}{10} & \frac{1}{5} & 0 \\ 0 & \frac{2}{-5+5i\sqrt{3}} & \frac{1}{10} - \frac{i\sqrt{3}}{10} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_2}{\left(-5+5i\sqrt{3}\right)\left(\frac{1}{10} - \frac{i\sqrt{3}}{10}\right)} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{10} - \frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} - \frac{i\sqrt{3}}{10} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{10}-\frac{i\sqrt{3}}{10} & \frac{1}{10} & 0 \\ 0 & \frac{1}{10}-\frac{i\sqrt{3}}{10} & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{-i\sqrt{3}-1}, v_2 = \frac{2t}{i\sqrt{3}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{-\mathrm{I}\sqrt{3}-1} \\ \frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{-i\sqrt{3}-1} \\ \frac{2t}{i\sqrt{3}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{-\mathrm{I}\sqrt{3}-1} \\ \frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{-i\sqrt{3}-1} \\ \frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{-\mathrm{I}\sqrt{3}-1} \\ \frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{-i\sqrt{3}-1} \\ \frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{-\mathrm{I}\sqrt{3}-1} \\ \frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{-i\sqrt{3}-1} \\ \frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\frac{1}{5}$	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$
$-\frac{1}{10}+\frac{i\sqrt{3}}{10}$	1	1	No	$\begin{bmatrix} \frac{1}{50\left(-\frac{1}{10}+\frac{i\sqrt{3}}{10}\right)^2} \\ \frac{1}{-\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ 1 \end{bmatrix}$
$-\frac{1}{10}-\frac{i\sqrt{3}}{10}$	1	1	No	$\begin{bmatrix} \frac{1}{50\left(-\frac{1}{10}-\frac{i\sqrt{3}}{10}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{i\sqrt{3}}{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{1}{5}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\frac{t}{5}} \\
&= \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} e^{\frac{t}{5}}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{\frac{t}{5}}}{2} \\ e^{\frac{t}{5}} \\ e^{\frac{t}{5}} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{\left(-\frac{1}{10}+\frac{i\sqrt{3}}{10}\right)t}}{50\left(-\frac{1}{10}+\frac{i\sqrt{3}}{10}\right)^2} \\ \frac{e^{\left(-\frac{1}{10}+\frac{i\sqrt{3}}{10}\right)t}}{-\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ e^{\left(-\frac{1}{10}+\frac{i\sqrt{3}}{10}\right)t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{e^{\left(-\frac{1}{10}-\frac{i\sqrt{3}}{10}\right)t}}{50\left(-\frac{1}{10}-\frac{i\sqrt{3}}{10}\right)^2} \\ \frac{e^{\left(-\frac{1}{10}-\frac{i\sqrt{3}}{10}\right)t}}{-\frac{1}{2}-\frac{i\sqrt{3}}{2}} \\ e^{\left(-\frac{1}{10}-\frac{i\sqrt{3}}{10}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{c_3\left(-i\sqrt{3}-1\right)e^{-\frac{\left(1+i\sqrt{3}\right)t}{10}}}{4} + \frac{c_2\left(i\sqrt{3}-1\right)e^{\frac{\left(i\sqrt{3}-1\right)t}{10}}}{4} + \frac{c_1 e^{\frac{t}{5}}}{2} \\ \frac{c_3\left(i\sqrt{3}-1\right)e^{-\frac{\left(1+i\sqrt{3}\right)t}{10}}}{2} + \frac{c_2\left(-i\sqrt{3}-1\right)e^{\frac{\left(i\sqrt{3}-1\right)t}{10}}}{2} + c_1 e^{\frac{t}{5}} \\ c_1 e^{\frac{t}{5}} + c_2 e^{\frac{\left(i\sqrt{3}-1\right)t}{10}} + c_3 e^{-\frac{\left(1+i\sqrt{3}\right)t}{10}} \end{bmatrix}$$

14.1.3 Maple step by step solution

Let's solve

$\left[x'(t) = \frac{y}{10}, y' = \frac{z(t)}{5}, z'(t) = \frac{2x(t)}{5}\right]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{5}, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{10} - \frac{I\sqrt{3}}{10}, \begin{bmatrix} \frac{1}{50\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)^2} \\ \frac{1}{5\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{10} + \frac{I\sqrt{3}}{10}, \begin{bmatrix} \frac{1}{50\left(-\frac{1}{10}+\frac{I\sqrt{3}}{10}\right)^2} \\ \frac{1}{5\left(-\frac{1}{10}+\frac{I\sqrt{3}}{10}\right)} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{1}{5}, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = e^{\frac{t}{5}} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{10} - \frac{I\sqrt{3}}{10}, \begin{bmatrix} \frac{1}{50\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)^2} \\ \frac{1}{5\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)t} \cdot \begin{bmatrix} \frac{1}{50\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)^2} \\ \frac{1}{5\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-\frac{t}{10}} \cdot \left(\cos\left(\frac{\sqrt{3}\,t}{10}\right) - I\sin\left(\frac{\sqrt{3}\,t}{10}\right)\right) \cdot \begin{bmatrix} \frac{1}{50\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)^2} \\ \frac{1}{5\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-\frac{t}{10}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)-I\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{50\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)^2} \\ \frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)-I\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{5\left(-\frac{1}{10}-\frac{I\sqrt{3}}{10}\right)} \\ \cos\left(\frac{\sqrt{3}\,t}{10}\right) - I\sin\left(\frac{\sqrt{3}\,t}{10}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = e^{-\frac{t}{10}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{4} - \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{4} \\ -\frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{2} + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{2} \\ \cos\left(\frac{\sqrt{3}\,t}{10}\right) \end{bmatrix}, \vec{x}_3(t) = e^{-\frac{t}{10}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)\sqrt{3}}{4} + \frac{\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{4} \\ \frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,t}{10}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{\frac{t}{5}} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-\frac{t}{10}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{4} - \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{4} \\ -\frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{2} + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{2} \\ \cos\left(\frac{\sqrt{3}\,t}{10}\right) \end{bmatrix} + c_3 e^{-\frac{t}{10}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)\sqrt{3}}{4} + \frac{\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{4} \\ \frac{\cos\left(\frac{\sqrt{3}\,t}{10}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,t}{10}\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{e^{-\frac{t}{10}}\left(c_3\sqrt{3}+c_2\right)\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{4} - \frac{e^{-\frac{t}{10}}\left(c_2\sqrt{3}-c_3\right)\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{4} + \frac{c_1 e^{\frac{t}{5}}}{2} \\ -\frac{e^{-\frac{t}{10}}\left(-c_3\sqrt{3}+c_2\right)\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{2} + \frac{e^{-\frac{t}{10}}\left(c_2\sqrt{3}+c_3\right)\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{2} + c_1 e^{\frac{t}{5}} \\ c_1 e^{\frac{t}{5}} + c_2 e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right) - c_3 e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{e^{-\frac{t}{10}}\left(c_3\sqrt{3}+c_2\right)\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{4} - \frac{e^{-\frac{t}{10}}\left(c_2\sqrt{3}-c_3\right)\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{4} + \frac{c_1 e^{\frac{t}{5}}}{2}, y = -\frac{e^{-\frac{t}{10}}\left(-c_3\sqrt{3}+c_2\right)\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{2} + \frac{e^{-\frac{t}{1}}}{}\right.$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 183

```
dsolve([diff(x(t),t)=0*x(t)+1/10*y(t)+0*z(t),diff(y(t),t)=0*x(t)+0*y(t)+2/10*z(t),diff(z(t),
```

$$x(t) = \frac{e^{\frac{t}{5}}c_1}{2} - \frac{c_2 e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{4} + \frac{c_2 e^{-\frac{t}{10}}\sqrt{3}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{4} - \frac{c_3 e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{4} - \frac{c_3 e^{-\frac{t}{10}}\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{4}$$

$$y(t) = e^{\frac{t}{5}}c_1 - \frac{c_2 e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{2} - \frac{c_2 e^{-\frac{t}{10}}\sqrt{3}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{2} - \frac{c_3 e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)}{2} + \frac{c_3 e^{-\frac{t}{10}}\sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{2}$$

$$z(t) = e^{\frac{t}{5}}c_1 + c_2 e^{-\frac{t}{10}}\sin\left(\frac{\sqrt{3}\,t}{10}\right) + c_3 e^{-\frac{t}{10}}\cos\left(\frac{\sqrt{3}\,t}{10}\right)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 269

```
DSolve[{x'[t]==0*x[t]+1/10*y[t]+0*z[t],y'[t]==0*x[t]+0*y[t]+2/10*z[t],z'[t]==4/10*x[t]+0*y[t
```

$$x(t) \to \frac{1}{6}e^{-t/10}\left((2c_1 + c_2 + c_3)e^{t/10}\sqrt[5]{e^t} + (4c_1 - c_2 - c_3)\cos\left(\frac{\sqrt{3}t}{10}\right) + \sqrt{3}(c_2 - c_3)\sin\left(\frac{\sqrt{3}t}{10}\right)\right)$$

$$y(t) \to \frac{1}{3}e^{-t/10}\left((2c_1 + c_2 + c_3)e^{t/10}\sqrt[5]{e^t} - (2c_1 - 2c_2 + c_3)\cos\left(\frac{\sqrt{3}t}{10}\right) - \sqrt{3}(2c_1 - c_3)\sin\left(\frac{\sqrt{3}t}{10}\right)\right)$$

$$z(t) \to \frac{1}{3}e^{-t/10}\left((2c_1 + c_2 + c_3)e^{t/10}\sqrt[5]{e^t} - (2c_1 + c_2 - 2c_3)\cos\left(\frac{\sqrt{3}t}{10}\right) + \sqrt{3}(2c_1 - c_2)\sin\left(\frac{\sqrt{3}t}{10}\right)\right)$$

14.2 problem 4

Internal problem ID [13129]
Internal file name [`OUTPUT/11784_Sunday_December_03_2023_07_16_35_PM_23060983/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= y \\
y' &= -x(t) \\
z'(t) &= 2z(t)
\end{aligned}
$$

14.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & \mathrm{e}^{2t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \cos(t)\, c_1 + \sin(t)\, c_2 \\ -\sin(t)\, c_1 + \cos(t)\, c_2 \\ e^{2t} c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 + \lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (-i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & 2+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} i & 1 & 0 & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 2+i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2+i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} i & 1 & 0 & 0 \\ 0 & 0 & 2+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 & 0 \\ 0 & 0 & 2+i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} it \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & 2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 2-i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2-i & 0 \end{array}\right]$$

Since the current pivot $A(2, 3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ 0 & 0 & 2-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 & 0 \\ 0 & 0 & 2-i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -it \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
i	1	1	No	$\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{it} \\ \mathrm{e}^{it} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} i(c_2 e^{-it} - c_1 e^{it}) \\ c_1 e^{it} + c_2 e^{-it} \\ c_3 e^{2t} \end{bmatrix}$$

14.2.3 Maple step by step solution

Let's solve

$[x'(t) = y, y' = -x(t), z'(t) = 2z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \left[\begin{array}{c}0\\0\\1\end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c}\mathrm{I}\\1\\0\end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c}-\mathrm{I}\\1\\0\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c}0\\0\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{2t} \cdot \left[\begin{array}{c}0\\0\\1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c}\mathrm{I}\\1\\0\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \left[\begin{array}{c}\mathrm{I}\\1\\0\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c}\mathrm{I}\\1\\0\end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c}\mathrm{I}(\cos(t) - \mathrm{I}\sin(t))\\ \cos(t) - \mathrm{I}\sin(t)\\ 0\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ 0 \end{bmatrix}, \vec{x}_3(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{2t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} c_3 \cos(t) + c_2 \sin(t) \\ c_2 \cos(t) - c_3 \sin(t) \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_3 \cos(t) + c_2 \sin(t) \\ c_2 \cos(t) - c_3 \sin(t) \\ c_1 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = c_3 \cos(t) + c_2 \sin(t), y = c_2 \cos(t) - c_3 \sin(t), z(t) = c_1 e^{2t}\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=0*x(t)+1*y(t)+0*z(t),diff(y(t),t)=-1*x(t)+0*y(t)+0*z(t),diff(z(t),t)=0*
```

$$x(t) = c_1 \sin(t) + c_2 \cos(t)$$
$$y(t) = c_1 \cos(t) - c_2 \sin(t)$$
$$z(t) = c_3 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 76

```
DSolve[{x'[t]==0*x[t]+1*y[t]+0*z[t],y'[t]==-1*x[t]+0*y[t]+0*z[t],z'[t]==0*x[t]+0*y[t]+2*z[t]
```

$$
\begin{aligned}
x(t) &\to c_1 \cos(t) + c_2 \sin(t) \\
y(t) &\to c_2 \cos(t) - c_1 \sin(t) \\
z(t) &\to c_3 e^{2t} \\
x(t) &\to c_1 \cos(t) + c_2 \sin(t) \\
y(t) &\to c_2 \cos(t) - c_1 \sin(t) \\
z(t) &\to 0
\end{aligned}
$$

14.3 problem 5

Internal problem ID [13130]
Internal file name [`OUTPUT/11785_Sunday_December_03_2023_07_16_35_PM_48437138/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) + 3y \\
y' &= 3x(t) - 2y \\
z'(t) &= -z(t)
\end{aligned}
$$

14.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{6t}+1)e^{-5t}}{2} & \frac{(e^{6t}-1)e^{-5t}}{2} & 0 \\ \frac{(e^{6t}-1)e^{-5t}}{2} & \frac{(e^{6t}+1)e^{-5t}}{2} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(e^{6t}+1)e^{-5t}}{2} & \frac{(e^{6t}-1)e^{-5t}}{2} & 0 \\ \frac{(e^{6t}-1)e^{-5t}}{2} & \frac{(e^{6t}+1)e^{-5t}}{2} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(e^{6t}+1)e^{-5t}c_1}{2} + \frac{(e^{6t}-1)e^{-5t}c_2}{2} \\ \frac{(e^{6t}-1)e^{-5t}c_1}{2} + \frac{(e^{6t}+1)e^{-5t}c_2}{2} \\ e^{-t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{((c_1+c_2)e^{6t}-c_2+c_1)e^{-5t}}{2} \\ \frac{e^{-5t}((c_1+c_2)e^{6t}+c_2-c_1)}{2} \\ e^{-t}c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 3 & 0 \\ 3 & -2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 5\lambda^2 - \lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -5$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \implies \left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 3 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 3 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-5	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-5t} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{t}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{-t}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\left(-c_2\mathrm{e}^{6t} + c_1\right)\mathrm{e}^{-5t} \\ \left(c_2\mathrm{e}^{6t} + c_1\right)\mathrm{e}^{-5t} \\ c_3\mathrm{e}^{-t} \end{bmatrix}$$

14.3.3 Maple step by step solution

Let's solve

$[x'(t) = -2x(t) + 3y, y' = 3x(t) - 2y, z'(t) = -z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-5t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}0 \\ 0 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{2} = \mathrm{e}^{-t} \cdot \left[\begin{array}{c}0 \\ 0 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 0\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{3} = \mathrm{e}^{t} \cdot \left[\begin{array}{c}1 \\ 1 \\ 0\end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_{1} + c_2 \overrightarrow{x}_{2} + c_3 \overrightarrow{x}_{3}$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{-5t} \cdot \left[\begin{array}{c}-1 \\ 1 \\ 0\end{array}\right] + c_2 \mathrm{e}^{-t} \cdot \left[\begin{array}{c}0 \\ 0 \\ 1\end{array}\right] + c_3 \mathrm{e}^{t} \cdot \left[\begin{array}{c}1 \\ 1 \\ 0\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t) \\ y \\ z(t)\end{array}\right] = \left[\begin{array}{c}-\left(-c_3 \mathrm{e}^{6t} + c_1\right) \mathrm{e}^{-5t} \\ \left(c_3 \mathrm{e}^{6t} + c_1\right) \mathrm{e}^{-5t} \\ c_2 \mathrm{e}^{-t}\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\left(-c_3 \mathrm{e}^{6t} + c_1\right) \mathrm{e}^{-5t}, y = \left(c_3 \mathrm{e}^{6t} + c_1\right) \mathrm{e}^{-5t}, z(t) = c_2 \mathrm{e}^{-t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 40

```
dsolve([diff(x(t),t)=-2*x(t)+3*y(t)+0*z(t),diff(y(t),t)=3*x(t)-2*y(t)+0*z(t),diff(z(t),t)=0*
```

$$x(t) = c_1 e^{-5t} + c_2 e^{t}$$
$$y(t) = -c_1 e^{-5t} + c_2 e^{t}$$
$$z(t) = c_3 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 150

```
DSolve[{x'[t]==-2*x[t]+3*y[t]+0*z[t],y'[t]==3*x[t]-2*y[t]+0*z[t],z'[t]==0*x[t]+0*y[t]-1*z[t]
```

$$x(t) \to \frac{1}{2}e^{-5t}\left(c_1\left(e^{6t}+1\right)+c_2\left(e^{6t}-1\right)\right)$$
$$y(t) \to \frac{1}{2}e^{-5t}\left(c_1\left(e^{6t}-1\right)+c_2\left(e^{6t}+1\right)\right)$$
$$z(t) \to c_3 e^{-t}$$
$$x(t) \to \frac{1}{2}e^{-5t}\left(c_1\left(e^{6t}+1\right)+c_2\left(e^{6t}-1\right)\right)$$
$$y(t) \to \frac{1}{2}e^{-5t}\left(c_1\left(e^{6t}-1\right)+c_2\left(e^{6t}+1\right)\right)$$
$$z(t) \to 0$$

14.4 problem 6

Internal problem ID [13131]
Internal file name [`OUTPUT/11786_Sunday_December_03_2023_07_16_36_PM_56995902/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}x'(t) &= x(t) + 3z(t)\\ y' &= -y\\ z'(t) &= -3x(t) + z(t)\end{aligned}$$

14.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t}\cos(3t) & 0 & \mathrm{e}^{t}\sin(3t) \\ 0 & \mathrm{e}^{-t} & 0 \\ -\mathrm{e}^{t}\sin(3t) & 0 & \mathrm{e}^{t}\cos(3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t \cos(3t) & 0 & e^t \sin(3t) \\ 0 & e^{-t} & 0 \\ -e^t \sin(3t) & 0 & e^t \cos(3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t \cos(3t) c_1 + e^t \sin(3t) c_3 \\ e^{-t} c_2 \\ -e^t \sin(3t) c_1 + e^t \cos(3t) c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t (\cos(3t) c_1 + \sin(3t) c_3) \\ e^{-t} c_2 \\ -e^t (\sin(3t) c_1 - \cos(3t) c_3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 3 \\ 0 & -1-\lambda & 0 \\ -3 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda^2 + 8\lambda + 10 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= 1+3i \\ \lambda_3 &= 1-3i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
$1+3i$	1	complex eigenvalue
$1-3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 2 & 0 \end{array}\right]$$

$$
R_3 = R_3 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{13}{2} & 0 \end{array}\right]
$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$
\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & 0 & \frac{13}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & \frac{13}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$
\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

Considering the eigenvalue $\lambda_2 = 1 - 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} - (1 - 3i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3i & 0 & 3 \\ 0 & -2+3i & 0 \\ -3 & 0 & 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3i & 0 & 3 & 0 \\ 0 & -2+3i & 0 & 0 \\ -3 & 0 & 3i & 0 \end{array}\right]$$

$$R_3 = -iR_1 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} 3i & 0 & 3 & 0 \\ 0 & -2+3i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3i & 0 & 3 \\ 0 & -2+3i & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} it \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}\,t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}\,t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1 + 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} - (1+3i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3i & 0 & 3 \\ 0 & -2-3i & 0 \\ -3 & 0 & -3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3i & 0 & 3 & 0 \\ 0 & -2-3i & 0 & 0 \\ -3 & 0 & -3i & 0 \end{array}\right]$$

$$R_3 = iR_1 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} -3i & 0 & 3 & 0 \\ 0 & -2-3i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3i & 0 & 3 \\ 0 & -2-3i & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -it \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
$1+3i$	1	1	No	$\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$
$1-3i$	1	1	No	$\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{(1+3i)t} \\ 0 \\ \mathrm{e}^{(1+3i)t} \end{bmatrix} + c_3 \begin{bmatrix} i\mathrm{e}^{(1-3i)t} \\ 0 \\ \mathrm{e}^{(1-3i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} i\left(c_3 e^{(1-3i)t} - c_2 e^{(1+3i)t}\right) \\ c_1 e^{-t} \\ c_2 e^{(1+3i)t} + c_3 e^{(1-3i)t} \end{bmatrix}$$

14.4.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + 3z(t), y' = -y, z'(t) = -3x(t) + z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right], \left[1-3\,\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 0 \\ 1 \end{bmatrix}\right], \left[1+3\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-3\,\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-3\,\mathrm{I})t} \cdot \begin{bmatrix} \mathrm{I} \\ 0 \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{t} \cdot (\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} \mathrm{I} \\ 0 \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{t} \cdot \begin{bmatrix} \mathrm{I}(\cos(3t) - \mathrm{I}\sin(3t)) \\ 0 \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} \sin(3t) \\ 0 \\ \cos(3t) \end{bmatrix}, \vec{x}_3(t) = \mathrm{e}^{t} \cdot \begin{bmatrix} \cos(3t) \\ 0 \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1\mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2\mathrm{e}^{t} \cdot \begin{bmatrix} \sin(3t) \\ 0 \\ \cos(3t) \end{bmatrix} + c_3\mathrm{e}^{t} \cdot \begin{bmatrix} \cos(3t) \\ 0 \\ -\sin(3t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{t}(c_2\sin(3t) + c_3\cos(3t)) \\ c_1\mathrm{e}^{-t} \\ \mathrm{e}^{t}(c_2\cos(3t) - c_3\sin(3t)) \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = \mathrm{e}^{t}(c_2\sin(3t) + c_3\cos(3t)), y = c_1\mathrm{e}^{-t}, z(t) = \mathrm{e}^{t}(c_2\cos(3t) - c_3\sin(3t))\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve([diff(x(t),t)=1*x(t)+0*y(t)+3*z(t),diff(y(t),t)=0*x(t)-1*y(t)+0*z(t),diff(z(t),t)=-3*
```

$$x(t) = \mathrm{e}^{t}(c_1\sin(3t) + c_2\cos(3t))$$
$$y(t) = c_3\mathrm{e}^{-t}$$
$$z(t) = \mathrm{e}^{t}(c_1\cos(3t) - c_2\sin(3t))$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 108

```
DSolve[{x'[t]==1*x[t]+0*y[t]+3*z[t],y'[t]==0*x[t]-1*y[t]+0*z[t],z'[t]==-3*x[t]+0*y[t]+1*z[t]
```

$$x(t) \to e^t(c_1 \cos(3t) + c_2 \sin(3t))$$
$$z(t) \to e^t(c_2 \cos(3t) - c_1 \sin(3t))$$
$$y(t) \to c_3 e^{-t}$$
$$x(t) \to e^t(c_1 \cos(3t) + c_2 \sin(3t))$$
$$z(t) \to e^t(c_2 \cos(3t) - c_1 \sin(3t))$$
$$y(t) \to 0$$

14.5 problem 7

Internal problem ID [13132]
Internal file name [`OUTPUT/11787_Sunday_December_03_2023_07_16_37_PM_85458287/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) \\
y' &= 2y - z(t) \\
z'(t) &= -y + 2z(t)
\end{aligned}
$$

14.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ 0 & \frac{\mathrm{e}^{t}}{2} + \frac{\mathrm{e}^{3t}}{2} & -\frac{\mathrm{e}^{3t}}{2} + \frac{\mathrm{e}^{t}}{2} \\ 0 & -\frac{\mathrm{e}^{3t}}{2} + \frac{\mathrm{e}^{t}}{2} & \frac{\mathrm{e}^{t}}{2} + \frac{\mathrm{e}^{3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^t & 0 & 0 \\ 0 & \frac{\mathrm{e}^t}{2}+\frac{\mathrm{e}^{3t}}{2} & -\frac{\mathrm{e}^{3t}}{2}+\frac{\mathrm{e}^t}{2} \\ 0 & -\frac{\mathrm{e}^{3t}}{2}+\frac{\mathrm{e}^t}{2} & \frac{\mathrm{e}^t}{2}+\frac{\mathrm{e}^{3t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^t c_1 \\ \left(\frac{\mathrm{e}^t}{2}+\frac{\mathrm{e}^{3t}}{2}\right) c_2 + \left(-\frac{\mathrm{e}^{3t}}{2}+\frac{\mathrm{e}^t}{2}\right) c_3 \\ \left(-\frac{\mathrm{e}^{3t}}{2}+\frac{\mathrm{e}^t}{2}\right) c_2 + \left(\frac{\mathrm{e}^t}{2}+\frac{\mathrm{e}^{3t}}{2}\right) c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^t c_1 \\ \frac{(-c_3+c_2)\mathrm{e}^{3t}}{2} + \frac{\mathrm{e}^t(c_2+c_3)}{2} \\ \frac{(c_3-c_2)\mathrm{e}^{3t}}{2} + \frac{\mathrm{e}^t(c_2+c_3)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_3\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = s\}$

Hence the solution is

$$\begin{bmatrix} t \\ s \\ s \end{bmatrix} = \begin{bmatrix} t \\ s \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t \\ s \\ s \end{bmatrix} &= \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ s \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	2	2	No	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

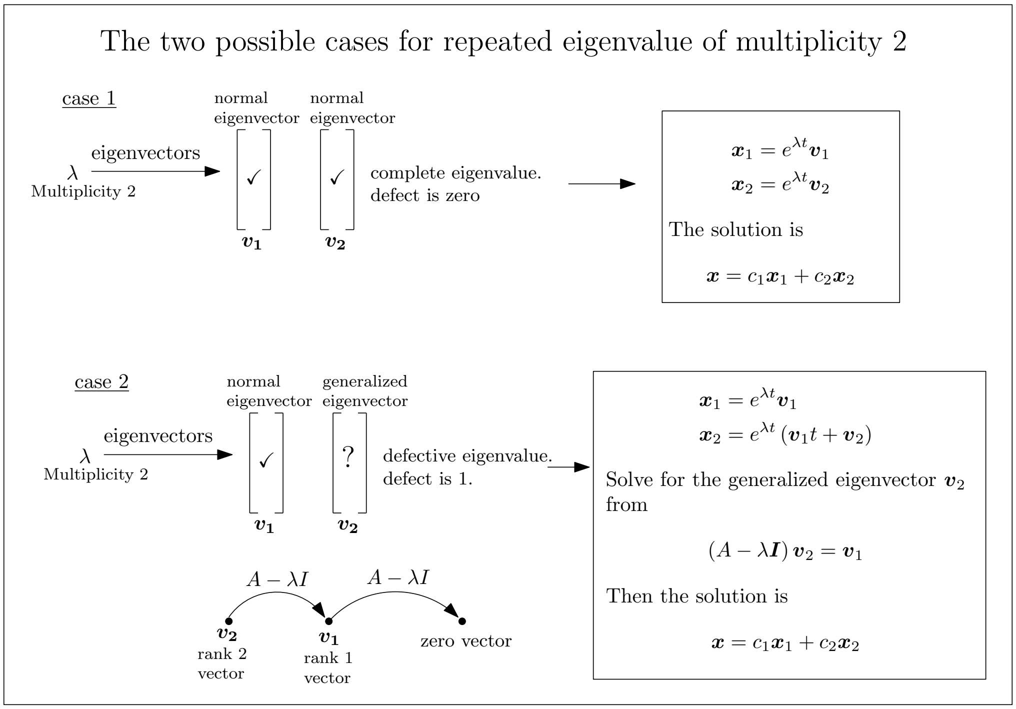


Figure 454: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

$$\vec{x}_2(t) = \vec{v}_2 e^{t}$$
$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{3t}$$
$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{3t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{t} \\ c_1 \mathrm{e}^{t} - c_3 \mathrm{e}^{3t} \\ c_1 \mathrm{e}^{t} + c_3 \mathrm{e}^{3t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 38

```
dsolve([diff(x(t),t)=1*x(t)+0*y(t)+0*z(t),diff(y(t),t)=0*x(t)+2*y(t)-1*z(t),diff(z(t),t)=0*x
```

$$x(t) = c_3 \mathrm{e}^{t}$$
$$y(t) = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{3t}$$
$$z(t) = c_1 \mathrm{e}^{t} - c_2 \mathrm{e}^{3t}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 144

```
DSolve[{x'[t]==1*x[t]+0*y[t]+0*z[t],y'[t]==0*x[t]+2*y[t]-1*z[t],z'[t]==0*x[t]-1*y[t]+2*z[t]}
```

$$x(t) \to c_1 e^t$$
$$y(t) \to \frac{1}{2}e^t\left(c_2 e^{2t} - c_3 e^{2t} + c_2 + c_3\right)$$
$$z(t) \to \frac{1}{2}e^t\left(c_2\left(-e^{2t}\right) + c_3 e^{2t} + c_2 + c_3\right)$$
$$x(t) \to 0$$
$$y(t) \to \frac{1}{2}e^t\left(c_2 e^{2t} - c_3 e^{2t} + c_2 + c_3\right)$$
$$z(t) \to \frac{1}{2}e^t\left(c_2\left(-e^{2t}\right) + c_3 e^{2t} + c_2 + c_3\right)$$

14.6 problem 10

Internal problem ID [13133]
Internal file name [`OUTPUT/11788_Sunday_December_03_2023_07_16_37_PM_79513523/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) + y \\
y' &= -2y \\
z'(t) &= -z(t)
\end{aligned}
$$

14.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} & 0 \\ 0 & \mathrm{e}^{-2t} & 0 \\ 0 & 0 & \mathrm{e}^{-t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} & 0 \\ 0 & \mathrm{e}^{-2t} & 0 \\ 0 & 0 & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-2t}c_1 + t\,\mathrm{e}^{-2t}c_2 \\ \mathrm{e}^{-2t}c_2 \\ \mathrm{e}^{-t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-2t}(c_2 t + c_1) \\ \mathrm{e}^{-2t}c_2 \\ \mathrm{e}^{-t}c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix}-2-\lambda & 1 & 0\\ 0 & -2-\lambda & 0\\ 0 & 0 & -1-\lambda\end{bmatrix}\right)=0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-2-\lambda)(-2-\lambda)(-1-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-2$$
$$\lambda_2=-1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-2$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 0\\ 0 & 0 & -1\end{bmatrix}-(-2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

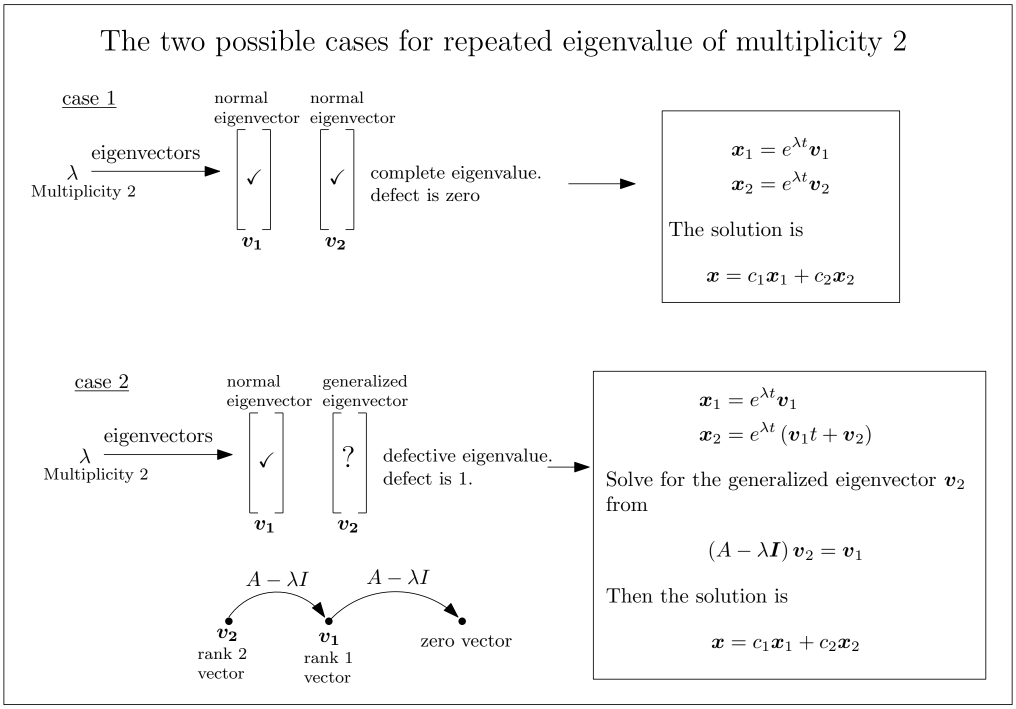


Figure 455: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{-2t} \\ &= \begin{bmatrix} \mathrm{e}^{-2t} \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \mathrm{e}^{-2t} \\ &= \begin{bmatrix} \mathrm{e}^{-2t}(1+t) \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_3(t) &= \vec{v}_3 e^{-t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t}(1+t) \\ e^{-2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ e^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} e^{-2t}(c_2 t + c_1 + c_2) \\ c_2 e^{-2t} \\ c_3 e^{-t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 33

```
dsolve([diff(x(t),t)=-2*x(t)+1*y(t)+0*z(t),diff(y(t),t)=0*x(t)-2*y(t)+0*z(t),diff(z(t),t)=0*
```

$$x(t) = (c_2 t + c_1)\, e^{-2t}$$
$$y(t) = c_2 e^{-2t}$$
$$z(t) = c_3 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 72

```
DSolve[{x'[t]==-2*x[t]+1*y[t]+0*z[t],y'[t]==0*x[t]-2*y[t]+0*z[t],z'[t]==0*x[t]+0*y[t]-1*z[t]
```

$$x(t) \to e^{-2t}(c_2 t + c_1)$$
$$y(t) \to c_2 e^{-2t}$$
$$z(t) \to c_3 e^{-t}$$
$$x(t) \to e^{-2t}(c_2 t + c_1)$$
$$y(t) \to c_2 e^{-2t}$$
$$z(t) \to 0$$

14.7 problem 11

Internal problem ID [13134]
Internal file name [OUTPUT/11789_Sunday_December_03_2023_07_16_38_PM_61681793/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}x'(t) &= -2x(t) + y \\ y' &= -2y \\ z'(t) &= z(t)\end{aligned}$$

14.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} & 0 \\ 0 & \mathrm{e}^{-2t} & 0 \\ 0 & 0 & \mathrm{e}^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} & 0 \\ 0 & \mathrm{e}^{-2t} & 0 \\ 0 & 0 & \mathrm{e}^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-2t}c_1 + t\,\mathrm{e}^{-2t}c_2 \\ \mathrm{e}^{-2t}c_2 \\ \mathrm{e}^{t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-2t}(c_2 t + c_1) \\ \mathrm{e}^{-2t}c_2 \\ \mathrm{e}^{t}c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 1 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-2-\lambda)(-2-\lambda)(1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

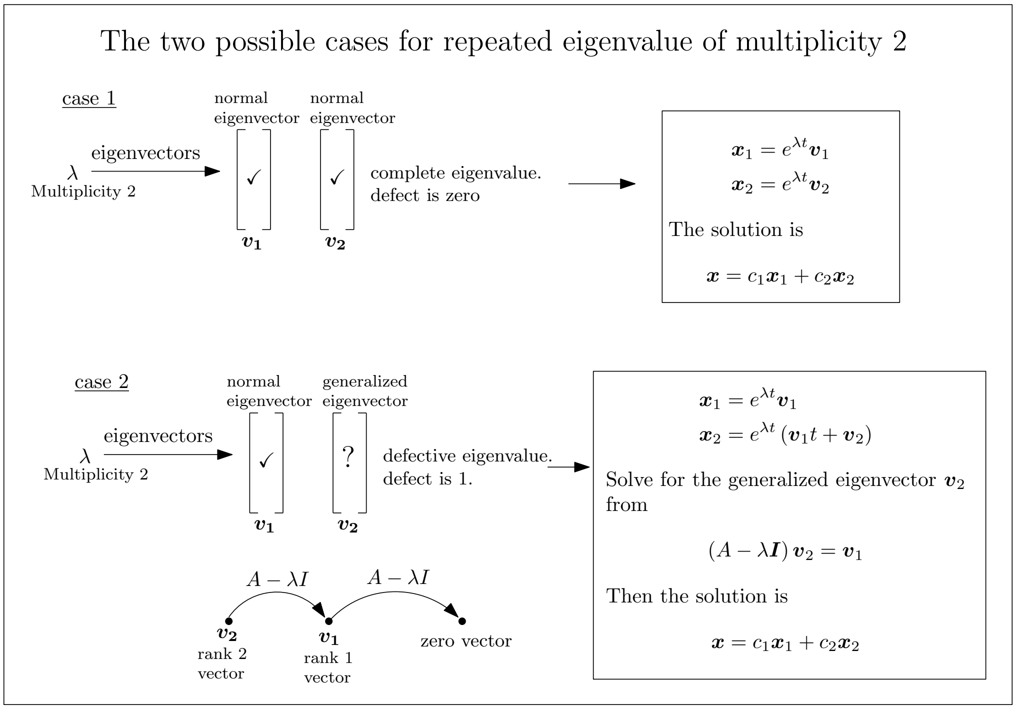


Figure 456: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{-2t} \\ &= \begin{bmatrix} \mathrm{e}^{-2t} \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \mathrm{e}^{-2t} \\ &= \begin{bmatrix} \mathrm{e}^{-2t}(1+t) \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_3(t) &= \vec{v}_3 e^{t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t}(1+t) \\ e^{-2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ e^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} e^{-2t}(c_2 t + c_1 + c_2) \\ c_2 e^{-2t} \\ c_3 e^{t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)=-2*x(t)+1*y(t)+0*z(t),diff(y(t),t)=0*x(t)-2*y(t)+0*z(t),diff(z(t),t)=0*
```

$$\begin{aligned} x(t) &= (c_2 t + c_1)\, e^{-2t} \\ y(t) &= c_2 e^{-2t} \\ z(t) &= c_3 e^{t} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 70

```
DSolve[{x'[t]==-2*x[t]+1*y[t]+0*z[t],y'[t]==0*x[t]-2*y[t]+0*z[t],z'[t]==0*x[t]+0*y[t]+1*z[t]
```

$$\begin{aligned} x(t) &\to e^{-2t}(c_2 t + c_1) \\ y(t) &\to c_2 e^{-2t} \\ z(t) &\to c_3 e^{t} \\ x(t) &\to e^{-2t}(c_2 t + c_1) \\ y(t) &\to c_2 e^{-2t} \\ z(t) &\to 0 \end{aligned}$$

14.8 problem 12

Internal problem ID [13135]
Internal file name [`OUTPUT/11790_Sunday_December_03_2023_07_16_38_PM_74909972/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) + 2y \\
y' &= 2x(t) - 4y \\
z'(t) &= -z(t)
\end{aligned}
$$

14.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{4}{5} + \frac{e^{-5t}}{5} & \frac{2}{5} - \frac{2\,e^{-5t}}{5} & 0 \\ \frac{2}{5} - \frac{2\,e^{-5t}}{5} & \frac{4\,e^{-5t}}{5} + \frac{1}{5} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{4}{5}+\frac{e^{-5t}}{5} & \frac{2}{5}-\frac{2\,e^{-5t}}{5} & 0 \\ \frac{2}{5}-\frac{2\,e^{-5t}}{5} & \frac{4\,e^{-5t}}{5}+\frac{1}{5} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{4}{5}+\frac{e^{-5t}}{5}\right)c_1 + \left(\frac{2}{5}-\frac{2\,e^{-5t}}{5}\right)c_2 \\ \left(\frac{2}{5}-\frac{2\,e^{-5t}}{5}\right)c_1 + \left(\frac{4\,e^{-5t}}{5}+\frac{1}{5}\right)c_2 \\ e^{-t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-2c_2)e^{-5t}}{5} + \frac{4c_1}{5} + \frac{2c_2}{5} \\ \frac{(-2c_1+4c_2)e^{-5t}}{5} + \frac{2c_1}{5} + \frac{c_2}{5} \\ e^{-t}c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-1-\lambda & 2 & 0\\ 2 & -4-\lambda & 0\\ 0 & 0 & -1-\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3+6\lambda^2+5\lambda=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-1$$
$$\lambda_2=-5$$
$$\lambda_3=0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
0	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-5$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-1 & 2 & 0\\ 2 & -4 & 0\\ 0 & 0 & -1\end{bmatrix}-(-5)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}4 & 2 & 0\\ 2 & 1 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}4 & 2 & 0 & 0\\ 2 & 1 & 0 & 0\\ 0 & 0 & 4 & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 4 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_3 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 2 & -4 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-5t} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} e^{-5t} \end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^0 \\ &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} e^0\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{\mathrm{e}^{-5t}}{2} \\ \mathrm{e}^{-5t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_2 \mathrm{e}^{-5t}}{2} + 2c_3 \\ c_2 \mathrm{e}^{-5t} + c_3 \\ c_1 \mathrm{e}^{-t} \end{bmatrix}$$

14.8.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + 2y, y' = 2x(t) - 4y, z'(t) = -z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-5t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 2 \\ 1 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \left[\begin{array}{c} 2 \\ 1 \\ 0 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-5t} \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \\ 0 \end{array}\right] + c_2 \mathrm{e}^{-t} \cdot \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] + \left[\begin{array}{c} 2c_3 \\ c_3 \\ 0 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} -\frac{c_1 \mathrm{e}^{-5t}}{2} + 2c_3 \\ c_1 \mathrm{e}^{-5t} + c_3 \\ c_2 \mathrm{e}^{-t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{c_1 \mathrm{e}^{-5t}}{2} + 2c_3, y = c_1 \mathrm{e}^{-5t} + c_3, z(t) = c_2 \mathrm{e}^{-t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=-1*x(t)+2*y(t)+0*z(t),diff(y(t),t)=2*x(t)-4*y(t)+0*z(t),diff(z(t),t)=0*
```

$$x(t) = c_1 + c_2 e^{-5t}$$
$$y(t) = -2c_2 e^{-5t} + \frac{c_1}{2}$$
$$z(t) = c_3 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 158

```
DSolve[{x'[t]==-1*x[t]+2*y[t]+0*z[t],y'[t]==2*x[t]-4*y[t]+0*z[t],z'[t]==0*x[t]+0*y[t]-1*z[t]
```

$$x(t) \to \frac{1}{5}e^{-5t}\left(c_1\left(4e^{5t}+1\right)+2c_2\left(e^{5t}-1\right)\right)$$
$$y(t) \to \frac{1}{5}e^{-5t}\left(2c_1\left(e^{5t}-1\right)+c_2\left(e^{5t}+4\right)\right)$$
$$z(t) \to c_3 e^{-t}$$
$$x(t) \to \frac{1}{5}e^{-5t}\left(c_1\left(4e^{5t}+1\right)+2c_2\left(e^{5t}-1\right)\right)$$
$$y(t) \to \frac{1}{5}e^{-5t}\left(2c_1\left(e^{5t}-1\right)+c_2\left(e^{5t}+4\right)\right)$$
$$z(t) \to 0$$

14.9 problem 13

Internal problem ID [13136]
Internal file name [`OUTPUT/11791_Sunday_December_03_2023_07_16_39_PM_16858110/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) + 2y \\
y' &= 2x(t) - 4y \\
z'(t) &= 0
\end{aligned}
$$

14.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{4}{5} + \frac{e^{-5t}}{5} & \frac{2}{5} - \frac{2\,e^{-5t}}{5} & 0 \\ \frac{2}{5} - \frac{2\,e^{-5t}}{5} & \frac{4\,e^{-5t}}{5} + \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{4}{5}+\frac{e^{-5t}}{5} & \frac{2}{5}-\frac{2\,e^{-5t}}{5} & 0 \\ \frac{2}{5}-\frac{2\,e^{-5t}}{5} & \frac{4\,e^{-5t}}{5}+\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{4}{5}+\frac{e^{-5t}}{5}\right)c_1 + \left(\frac{2}{5}-\frac{2\,e^{-5t}}{5}\right)c_2 \\ \left(\frac{2}{5}-\frac{2\,e^{-5t}}{5}\right)c_1 + \left(\frac{4\,e^{-5t}}{5}+\frac{1}{5}\right)c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-2c_2)e^{-5t}}{5}+\frac{4c_1}{5}+\frac{2c_2}{5} \\ \frac{(-2c_1+4c_2)e^{-5t}}{5}+\frac{2c_1}{5}+\frac{c_2}{5} \\ c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 2 & 0 \\ 2 & -4-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 5\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$$
R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]
$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$
\left[\begin{array}{ccc|c} 4 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 4 & 2 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_3 = 0\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}
$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} 2t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} 2t \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	2	2	No	$\begin{bmatrix} 0 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

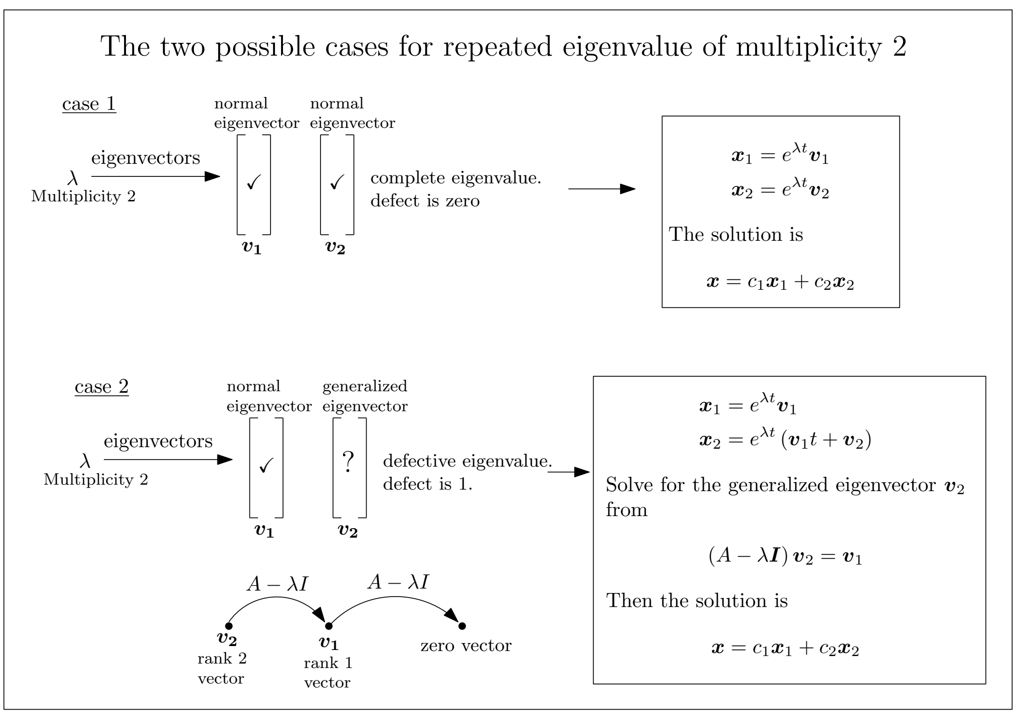


Figure 457: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{0} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{0} \\ &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} e^{0}\end{aligned}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-5t} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} e^{-5t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -\frac{\mathrm{e}^{-5t}}{2} \\ \mathrm{e}^{-5t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} 2c_2 - \frac{c_3 \mathrm{e}^{-5t}}{2} \\ c_2 + c_3 \mathrm{e}^{-5t} \\ c_1 \end{bmatrix}$$

14.9.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + 2y, y' = 2x(t) - 4y, z'(t) = 0]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \left[\begin{array}{c} -\frac{1}{2} \\ 1 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-5t} \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 2 \\ 1 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \left[\begin{array}{c} 2 \\ 1 \\ 0 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-5t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2c_3 \\ c_3 \\ c_2 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 \mathrm{e}^{-5t}}{2} + 2c_3 \\ c_1 \mathrm{e}^{-5t} + c_3 \\ c_2 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{c_1 \mathrm{e}^{-5t}}{2} + 2c_3, y = c_1 \mathrm{e}^{-5t} + c_3, z(t) = c_2\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)=-1*x(t)+2*y(t)+0*z(t),diff(y(t),t)=2*x(t)-4*y(t)+0*z(t),diff(z(t),t)=0*
```

$$x(t) = c_1 + c_2 \mathrm{e}^{-5t}$$
$$y(t) = -2c_2 \mathrm{e}^{-5t} + \frac{c_1}{2}$$
$$z(t) = c_3$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 77

```
DSolve[{x'[t]==-1*x[t]+2*y[t]+0*z[t],y'[t]==2*x[t]-4*y[t]+0*z[t],z'[t]==0*x[t]+0*y[t]+0*z[t]
```

$$x(t) \to \frac{1}{5} e^{-5t} \left(c_1 \left(4e^{5t} + 1\right) + 2c_2 \left(e^{5t} - 1\right)\right)$$
$$y(t) \to \frac{1}{5} e^{-5t} \left(2c_1 \left(e^{5t} - 1\right) + c_2 \left(e^{5t} + 4\right)\right)$$
$$z(t) \to c_3$$

14.10 problem 14

Internal problem ID [13137]
Internal file name [OUTPUT/11792_Sunday_December_03_2023_07_16_39_PM_6009185/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -2x(t) + y \\ y' &= -2y + z(t) \\ z'(t) &= -2z(t) \end{aligned}$$

14.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-2t} & t\,e^{-2t} & \frac{e^{-2t}t^2}{2} \\ 0 & e^{-2t} & t\,e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} & \frac{\mathrm{e}^{-2t}t^2}{2} \\ 0 & \mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} \\ 0 & 0 & \mathrm{e}^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-2t}c_1 + t\,\mathrm{e}^{-2t}c_2 + \frac{\mathrm{e}^{-2t}t^2c_3}{2} \\ \mathrm{e}^{-2t}c_2 + t\,\mathrm{e}^{-2t}c_3 \\ \mathrm{e}^{-2t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-2t}\left(c_1 + c_2t + \frac{1}{2}c_3t^2\right) \\ \mathrm{e}^{-2t}(c_3t + c_2) \\ \mathrm{e}^{-2t}c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 1 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & -2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-2-\lambda)(-2-\lambda)(-2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	3	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

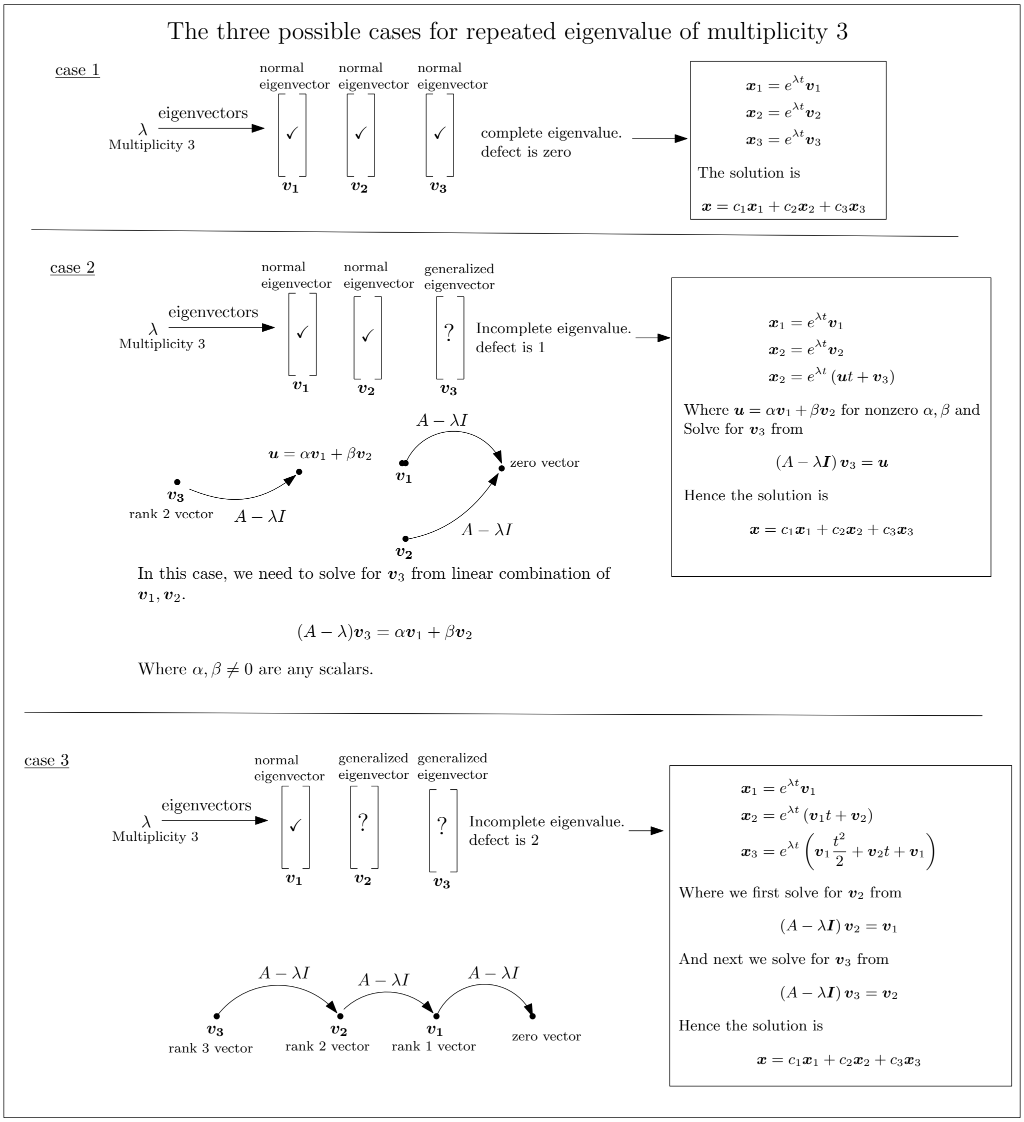


Figure 458: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector

$\vec{v}_3$ of rank $3.\vec{v}_2$ is found by solving

$$(A-\lambda I)\,\vec{v}_2=\vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 1\\ 0 & 0 & -2\end{bmatrix}-(-2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A-\lambda I)\,\vec{v}_3=\vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 1\\ 0 & 0 & -2\end{bmatrix}-(-2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$
$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue -2. Therefore the three basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{-2t} \\
&= \begin{bmatrix} \mathrm{e}^{-2t} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{-2t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} \mathrm{e}^{-2t}(1+t) \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3 \right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-2t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-2t}(t^2+2t+2)}{2} \\ \mathrm{e}^{-2t}(1+t) \\ \mathrm{e}^{-2t} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t}(1+t) \\ e^{-2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{-2t}\left(t+\frac{1}{2}t^2+1\right) \\ e^{-2t}(1+t) \\ e^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{-2t}((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)}{2} \\ e^{-2t}(c_3t+c_2+c_3) \\ c_3e^{-2t} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 46

```
dsolve([diff(x(t),t)=-2*x(t)+1*y(t)+0*z(t),diff(y(t),t)=0*x(t)-2*y(t)+1*z(t),diff(z(t),t)=0*
```

$$x(t) = \frac{(c_3t^2+2c_2t+2c_1)\,e^{-2t}}{2}$$
$$y(t) = (c_3t+c_2)\,e^{-2t}$$
$$z(t) = c_3e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 57

```
DSolve[{x'[t]==-2*x[t]+1*y[t]+0*z[t],y'[t]==0*x[t]-2*y[t]+1*z[t],z'[t]==0*x[t]+0*y[t]-2*z[t]
```

$$x(t) \to \frac{1}{2}e^{-2t}(t(c_3t+2c_2)+2c_1)$$
$$y(t) \to e^{-2t}(c_3t+c_2)$$
$$z(t) \to c_3e^{-2t}$$

14.11 problem 15

Internal problem ID [13138]
Internal file name [`OUTPUT/11793_Sunday_December_03_2023_07_16_40_PM_54813/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= y \\
y' &= z(t) \\
z'(t) &= 0
\end{aligned}
$$

14.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} c_1 + tc_2 + \frac{1}{2}t^2 c_3 \\ tc_3 + c_2 \\ c_3 \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(-\lambda)(-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	3	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

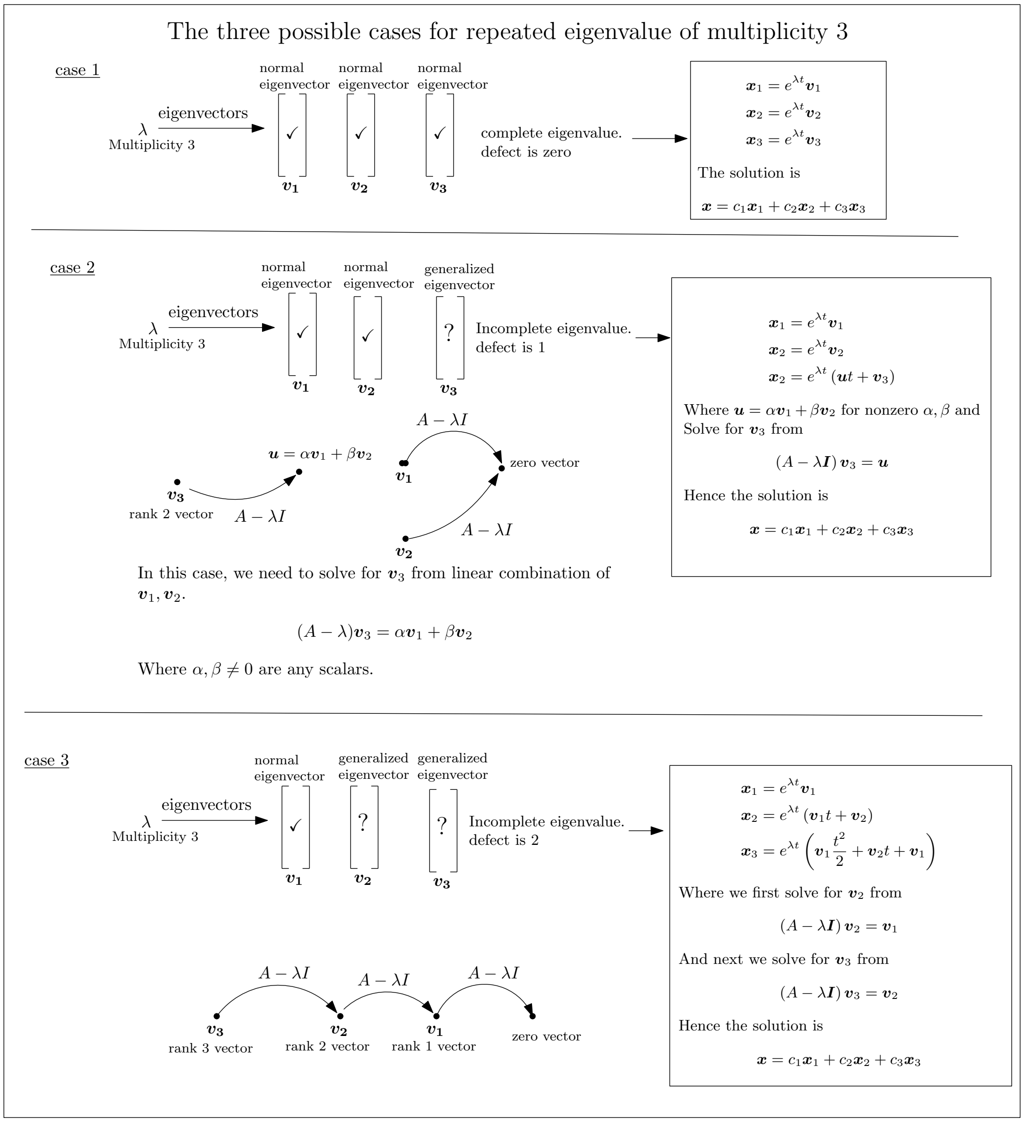


Figure 459: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}-(0)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A-\lambda I)\,\vec{v}_3=\vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}-(0)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$
$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 0. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix}1\\0\\0\end{bmatrix} 1\\ &= \begin{bmatrix}1\\0\\0\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2)\\ &= 1\left(\begin{bmatrix}1\\0\\0\end{bmatrix} t + \begin{bmatrix}1\\1\\0\end{bmatrix}\right)\\ &= \begin{bmatrix}1+t\\1\\0\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t}\\ &= \left(\begin{bmatrix}1\\0\\0\end{bmatrix} \frac{t^2}{2} + \begin{bmatrix}1\\1\\0\end{bmatrix} t + \begin{bmatrix}1\\1\\1\end{bmatrix}\right) 1\\ &= \begin{bmatrix}t + \frac{1}{2}t^2 + 1\\ 1+t\\ 1\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1+t \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} t+\frac{1}{2}t^2+1 \\ 1+t \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{(t^2+2t+2)c_3}{2} + c_2 t + c_1 + c_2 \\ c_3 t + c_2 + c_3 \\ c_3 \end{bmatrix}$$

14.11.3 Maple step by step solution

Let's solve

$[x'(t) = y, y' = z(t), z'(t) = 0]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = c_1, y = 0, z(t) = 0\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve([diff(x(t),t)=0*x(t)+1*y(t)+0*z(t),diff(y(t),t)=0*x(t)+0*y(t)+1*z(t),diff(z(t),t)=0*x
```

$$x(t) = \frac{1}{2}c_3t^2 + c_2t + c_1$$
$$y(t) = c_3t + c_2$$
$$z(t) = c_3$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 36

```
DSolve[{x'[t]==0*x[t]+1*y[t]+0*z[t],y'[t]==0*x[t]+0*y[t]+1*z[t],z'[t]==0*x[t]+0*y[t]+0*z[t]}
```

$$x(t) \to \frac{c_3 t^2}{2} + c_2 t + c_1$$
$$y(t) \to c_3 t + c_2$$
$$z(t) \to c_3$$

14.12 problem 16

Internal problem ID [13139]
Internal file name [`OUTPUT/11794_Sunday_December_03_2023_07_16_40_PM_17857581/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) - y \\
y' &= -2y + 3z(t) \\
z'(t) &= -x(t) + 3y - z(t)
\end{aligned}
$$

14.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,e^t}{8} + \frac{(5+3\sqrt{3})e^{(-1+2\sqrt{3})t}}{16} + \frac{(-3\sqrt{3}+5)e^{(-1-2\sqrt{3})t}}{16} & \frac{(-1-\sqrt{3})e^{(-1+2\sqrt{3})t}}{8} + \frac{(\sqrt{3}-1)e^{(-1-2\sqrt{3})t}}{8} + \frac{e^t}{4} & \frac{(-\sqrt{3}-\cdots}{\cdots} \\ \frac{(-\sqrt{3}-3)e^{(-1+2\sqrt{3})t}}{16} + \frac{(\sqrt{3}-3)e^{(-1-2\sqrt{3})t}}{16} + \frac{3\,e^t}{8} & \frac{(-\sqrt{3}+3)e^{(-1+2\sqrt{3})t}}{8} + \frac{(\sqrt{3}+3)e^{(-1-2\sqrt{3})t}}{8} + \frac{e^t}{4} & \frac{(3\sqrt{3}-3)\cdots}{\cdots} \\ \frac{(-7\sqrt{3}-9)e^{(-1+2\sqrt{3})t}}{48} + \frac{(7\sqrt{3}-9)e^{(-1-2\sqrt{3})t}}{48} + \frac{3\,e^t}{8} & \frac{(5\sqrt{3}-3)e^{(-1+2\sqrt{3})t}}{24} + \frac{(-5\sqrt{3}-3)e^{(-1-2\sqrt{3})t}}{24} + \frac{e^t}{4} & \frac{(-\sqrt{3}+\cdots}{\cdots} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{3\,e^t}{8} + \frac{(5+3\sqrt{3})e^{(-1+2\sqrt{3})t}}{16} + \frac{(-3\sqrt{3}+5)e^{(-1-2\sqrt{3})t}}{16} & \frac{(-1-\sqrt{3})e^{(-1+2\sqrt{3})t}}{8} + \frac{(\sqrt{3}-1)e^{(-1-2\sqrt{3})t}}{8} + \frac{e^t}{4} & \frac{(-\sqrt{\cdots}}{\cdots} \\ \frac{(-\sqrt{3}-3)e^{(-1+2\sqrt{3})t}}{16} + \frac{(\sqrt{3}-3)e^{(-1-2\sqrt{3})t}}{16} + \frac{3\,e^t}{8} & \frac{(-\sqrt{3}+3)e^{(-1+2\sqrt{3})t}}{8} + \frac{(\sqrt{3}+3)e^{(-1-2\sqrt{3})t}}{8} + \frac{e^t}{4} & \frac{(3\sqrt{3}-\cdots}{\cdots} \\ \frac{(-7\sqrt{3}-9)e^{(-1+2\sqrt{3})t}}{48} + \frac{(7\sqrt{3}-9)e^{(-1-2\sqrt{3})t}}{48} + \frac{3\,e^t}{8} & \frac{(5\sqrt{3}-3)e^{(-1+2\sqrt{3})t}}{24} + \frac{(-5\sqrt{3}-3)e^{(-1-2\sqrt{3})t}}{24} + \frac{e^t}{4} & \frac{(-\sqrt{\cdots}}{\cdots} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{3\,e^t}{8} + \frac{(5+3\sqrt{3})e^{(-1+2\sqrt{3})t}}{16} + \frac{(-3\sqrt{3}+5)e^{(-1-2\sqrt{3})t}}{16}\right)c_1 + \left(\frac{(-1-\sqrt{3})e^{(-1+2\sqrt{3})t}}{8} + \frac{(\sqrt{3}-1)e^{(-1-2\sqrt{3})t}}{8} + \frac{e^t}{4}\cdots\right. \\ \left(\frac{(-\sqrt{3}-3)e^{(-1+2\sqrt{3})t}}{16} + \frac{(\sqrt{3}-3)e^{(-1-2\sqrt{3})t}}{16} + \frac{3\,e^t}{8}\right)c_1 + \left(\frac{(-\sqrt{3}+3)e^{(-1+2\sqrt{3})t}}{8} + \frac{(\sqrt{3}+3)e^{(-1-2\sqrt{3})t}}{8} + \frac{e^t}{4}\right)\cdots \\ \left(\frac{(-7\sqrt{3}-9)e^{(-1+2\sqrt{3})t}}{48} + \frac{(7\sqrt{3}-9)e^{(-1-2\sqrt{3})t}}{48} + \frac{3\,e^t}{8}\right)c_1 + \left(\frac{(5\sqrt{3}-3)e^{(-1+2\sqrt{3})t}}{24} + \frac{(-5\sqrt{3}-3)e^{(-1-2\sqrt{3})t}}{24} + \cdots\right. \end{bmatrix}$$

$$= \begin{bmatrix} \frac{((3c_1-2c_2-c_3)\sqrt{3}+5c_1-2c_2-3c_3)e^{(-1+2\sqrt{3})t}}{16} + \frac{((-3c_1+2c_2+c_3)\sqrt{3}+5c_1-2c_2-3c_3)e^{(-1-2\sqrt{3})t}}{16} + \frac{3\,e^t\left(c_1+\frac{2c_2}{3}+c_3\right)}{8} \\ \frac{((-c_1-2c_2+3c_3)\sqrt{3}-3c_1+6c_2-3c_3)e^{(-1+2\sqrt{3})t}}{16} + \frac{((c_1+2c_2-3c_3)\sqrt{3}-3c_1+6c_2-3c_3)e^{(-1-2\sqrt{3})t}}{16} + \frac{3\,e^t\left(c_1+\frac{2c_2}{3}+c_3\right)}{8} \\ \frac{((-7c_1+10c_2-3c_3)\sqrt{3}-9c_1-6c_2+15c_3)e^{(-1+2\sqrt{3})t}}{48} + \frac{((7c_1-10c_2+3c_3)\sqrt{3}-9c_1-6c_2+15c_3)e^{(-1-2\sqrt{3})t}}{48} + \frac{3\,e^t\left(c_1+\frac{2c_2}{3}+c_3\cdots\right.}{8} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -1 & 0 \\ 0 & -2-\lambda & 3 \\ -1 & 3 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \lambda^2 - 13\lambda + 11 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -1 + 2\sqrt{3} \\ \lambda_2 &= -1 - 2\sqrt{3} \\ \lambda_3 &= 1 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
$-1-2\sqrt{3}$	1	real eigenvalue
$-1+2\sqrt{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 3 \\ -1 & 3 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ -1 & 3 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 - 2\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix} - \left(-1 - 2\sqrt{3}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 + 2\sqrt{3} & -1 & 0 \\ 0 & -1 + 2\sqrt{3} & 3 \\ -1 & 3 & 2\sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 + 2\sqrt{3} & -1 & 0 & 0 \\ 0 & -1 + 2\sqrt{3} & 3 & 0 \\ -1 & 3 & 2\sqrt{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{3+2\sqrt{3}} \Longrightarrow \left[\begin{array}{ccc|c} 3+2\sqrt{3} & -1 & 0 & 0 \\ 0 & -1+2\sqrt{3} & 3 & 0 \\ 0 & \frac{8+6\sqrt{3}}{3+2\sqrt{3}} & 2\sqrt{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{\left(8+6\sqrt{3}\right) R_2}{\left(3+2\sqrt{3}\right)\left(-1+2\sqrt{3}\right)} \Longrightarrow \left[\begin{array}{ccc|c} 3+2\sqrt{3} & -1 & 0 & 0 \\ 0 & -1+2\sqrt{3} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3+2\sqrt{3} & -1 & 0 \\ 0 & -1+2\sqrt{3} & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{\left(4\sqrt{3}-9\right)t}{11}, v_2 = -\frac{3\left(1+2\sqrt{3}\right)t}{11}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\left(4\sqrt{3}-9\right)t}{11} \\ -\frac{3\left(1+2\sqrt{3}\right)t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{\left(4\sqrt{3}-9\right)t}{11} \\ -\frac{3\left(1+2\sqrt{3}\right)t}{11} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\left(4\sqrt{3}-9\right)t}{11} \\ -\frac{3\left(1+2\sqrt{3}\right)t}{11} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4\sqrt{3}}{11} - \frac{9}{11} \\ -\frac{3}{11} - \frac{6\sqrt{3}}{11} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{\left(4\sqrt{3}-9\right)t}{11} \\ -\frac{3\left(1+2\sqrt{3}\right)t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4\sqrt{3}}{11} - \frac{9}{11} \\ -\frac{3}{11} - \frac{6\sqrt{3}}{11} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{\left(4\sqrt{3}-9\right)t}{11} \\ -\frac{3\left(1+2\sqrt{3}\right)t}{11} \\ t \end{bmatrix} = \begin{bmatrix} 4\sqrt{3}-9 \\ -3-6\sqrt{3} \\ 11 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -1 + 2\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix} - \left(-1+2\sqrt{3}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-2\sqrt{3} & -1 & 0 \\ 0 & -1-2\sqrt{3} & 3 \\ -1 & 3 & -2\sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3-2\sqrt{3} & -1 & 0 & 0 \\ 0 & -1-2\sqrt{3} & 3 & 0 \\ -1 & 3 & -2\sqrt{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{3-2\sqrt{3}} \implies \left[\begin{array}{ccc|c} 3-2\sqrt{3} & -1 & 0 & 0 \\ 0 & -1-2\sqrt{3} & 3 & 0 \\ 0 & \frac{-8+6\sqrt{3}}{-3+2\sqrt{3}} & -2\sqrt{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{\left(-8+6\sqrt{3}\right) R_2}{\left(-3+2\sqrt{3}\right)\left(-1-2\sqrt{3}\right)} \implies \left[\begin{array}{ccc|c} 3-2\sqrt{3} & -1 & 0 & 0 \\ 0 & -1-2\sqrt{3} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-2\sqrt{3} & -1 & 0 \\ 0 & -1-2\sqrt{3} & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{(9+4\sqrt{3})t}{11}, v_2 = \frac{3(-1+2\sqrt{3})t}{11}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{(9+4\sqrt{3})t}{11} \\ \frac{3(-1+2\sqrt{3})t}{11} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{(9+4\sqrt{3})t}{11} \\ \frac{3(-1+2\sqrt{3})t}{11} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{(9+4\sqrt{3})t}{11} \\ \frac{3(-1+2\sqrt{3})t}{11} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{9}{11} - \frac{4\sqrt{3}}{11} \\ \frac{6\sqrt{3}}{11} - \frac{3}{11} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{(9+4\sqrt{3})t}{11} \\ \frac{3(-1+2\sqrt{3})t}{11} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{9}{11} - \frac{4\sqrt{3}}{11} \\ \frac{6\sqrt{3}}{11} - \frac{3}{11} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{(9+4\sqrt{3})t}{11} \\ \frac{3(-1+2\sqrt{3})t}{11} \\ t \end{bmatrix} = \begin{bmatrix} -9 - 4\sqrt{3} \\ 6\sqrt{3} - 3 \\ 11 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-1+2\sqrt{3}$	1	1	No	$\left[\begin{array}{c} -\frac{3}{\left(1+2\sqrt{3}\right)\left(-3+2\sqrt{3}\right)} \\ \frac{3}{1+2\sqrt{3}} \\ 1 \end{array}\right]$
$-1-2\sqrt{3}$	1	1	No	$\left[\begin{array}{c} -\frac{3}{\left(1-2\sqrt{3}\right)\left(-3-2\sqrt{3}\right)} \\ \frac{3}{1-2\sqrt{3}} \\ 1 \end{array}\right]$
1	1	1	No	$\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-1+2\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-1+2\sqrt{3}\right)t} \\
&= \left[\begin{array}{c} -\frac{3}{\left(1+2\sqrt{3}\right)\left(-3+2\sqrt{3}\right)} \\ \frac{3}{1+2\sqrt{3}} \\ 1 \end{array}\right] e^{\left(-1+2\sqrt{3}\right)t}
\end{aligned}
$$

Since eigenvalue $-1-2\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-1-2\sqrt{3}\right)t} \\
&= \left[\begin{array}{c} -\frac{3}{\left(1-2\sqrt{3}\right)\left(-3-2\sqrt{3}\right)} \\ \frac{3}{1-2\sqrt{3}} \\ 1 \end{array}\right] e^{\left(-1-2\sqrt{3}\right)t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{3\,\mathrm{e}^{\left(-1+2\sqrt{3}\right)t}}{\left(1+2\sqrt{3}\right)\left(-3+2\sqrt{3}\right)} \\ \frac{3\,\mathrm{e}^{\left(-1+2\sqrt{3}\right)t}}{1+2\sqrt{3}} \\ \mathrm{e}^{\left(-1+2\sqrt{3}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{3\,\mathrm{e}^{\left(-1-2\sqrt{3}\right)t}}{\left(1-2\sqrt{3}\right)\left(-3-2\sqrt{3}\right)} \\ \frac{3\,\mathrm{e}^{\left(-1-2\sqrt{3}\right)t}}{1-2\sqrt{3}} \\ \mathrm{e}^{\left(-1-2\sqrt{3}\right)t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1\left(-9-4\sqrt{3}\right)\mathrm{e}^{\left(-1+2\sqrt{3}\right)t}}{11} + \frac{c_2\left(4\sqrt{3}-9\right)\mathrm{e}^{\left(-1-2\sqrt{3}\right)t}}{11} + c_3\mathrm{e}^{t} \\ \frac{3c_1\left(-1+2\sqrt{3}\right)\mathrm{e}^{\left(-1+2\sqrt{3}\right)t}}{11} + \frac{3c_2\left(-1-2\sqrt{3}\right)\mathrm{e}^{\left(-1-2\sqrt{3}\right)t}}{11} + c_3\mathrm{e}^{t} \\ c_1\mathrm{e}^{\left(-1+2\sqrt{3}\right)t} + c_2\mathrm{e}^{\left(-1-2\sqrt{3}\right)t} + c_3\mathrm{e}^{t} \end{bmatrix}
$$

14.12.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) - y, y' = -2y + 3z(t), z'(t) = -x(t) + 3y - z(t)]$

- Define vector

$$
\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix} \cdot \overrightarrow{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix}$$

- Rewrite the system as

$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-1-2\sqrt{3}, \begin{bmatrix} -\frac{3}{(1-2\sqrt{3})(-3-2\sqrt{3})} \\ \frac{3}{1-2\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[-1+2\sqrt{3}, \begin{bmatrix} -\frac{3}{(1+2\sqrt{3})(-3+2\sqrt{3})} \\ \frac{3}{1+2\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1-2\sqrt{3}, \left[\begin{array}{c} -\frac{3}{\left(1-2\sqrt{3}\right)\left(-3-2\sqrt{3}\right)} \\ \frac{3}{1-2\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{2} = \mathrm{e}^{\left(-1-2\sqrt{3}\right)t} \cdot \left[\begin{array}{c} -\frac{3}{\left(1-2\sqrt{3}\right)\left(-3-2\sqrt{3}\right)} \\ \frac{3}{1-2\sqrt{3}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1+2\sqrt{3}, \left[\begin{array}{c} -\frac{3}{\left(1+2\sqrt{3}\right)\left(-3+2\sqrt{3}\right)} \\ \frac{3}{1+2\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{3} = \mathrm{e}^{\left(-1+2\sqrt{3}\right)t} \cdot \left[\begin{array}{c} -\frac{3}{\left(1+2\sqrt{3}\right)\left(-3+2\sqrt{3}\right)} \\ \frac{3}{1+2\sqrt{3}} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1}\overrightarrow{x}_{1} + c_{2}\overrightarrow{x}_{2} + c_{3}\overrightarrow{x}_{3}$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_{1}\mathrm{e}^{t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] + c_{2}\mathrm{e}^{\left(-1-2\sqrt{3}\right)t} \cdot \left[\begin{array}{c} -\frac{3}{\left(1-2\sqrt{3}\right)\left(-3-2\sqrt{3}\right)} \\ \frac{3}{1-2\sqrt{3}} \\ 1 \end{array}\right] + c_{3}\mathrm{e}^{\left(-1+2\sqrt{3}\right)t} \cdot \left[\begin{array}{c} -\frac{3}{\left(1+2\sqrt{3}\right)\left(-3+2\sqrt{3}\right)} \\ \frac{3}{1+2\sqrt{3}} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{c_3(-9-4\sqrt{3})e^{(-1+2\sqrt{3})t}}{11} + \frac{c_2(4\sqrt{3}-9)e^{(-1-2\sqrt{3})t}}{11} + c_1e^t \\ \frac{3c_3(-1+2\sqrt{3})e^{(-1+2\sqrt{3})t}}{11} + \frac{3c_2(-1-2\sqrt{3})e^{(-1-2\sqrt{3})t}}{11} + c_1e^t \\ c_1e^t + c_2e^{(-1-2\sqrt{3})t} + c_3e^{(-1+2\sqrt{3})t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{c_3(-9-4\sqrt{3})e^{(-1+2\sqrt{3})t}}{11} + \frac{c_2(4\sqrt{3}-9)e^{(-1-2\sqrt{3})t}}{11} + c_1e^t, y = \frac{3c_3(-1+2\sqrt{3})e^{(-1+2\sqrt{3})t}}{11} + \frac{3c_2(-1-2\sqrt{3})e^{(-}}{11}\right.$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 171

```
dsolve([diff(x(t),t)=2*x(t)-1*y(t)+0*z(t),diff(y(t),t)=0*x(t)-2*y(t)+3*z(t),diff(z(t),t)=-1*
```

$$x(t) = -c_2e^{(-1+2\sqrt{3})t} - c_3e^{-(1+2\sqrt{3})t} - \frac{2c_2e^{(-1+2\sqrt{3})t}\sqrt{3}}{3} + \frac{2c_3e^{-(1+2\sqrt{3})t}\sqrt{3}}{3} + c_1e^t$$

$$y(t) = c_1e^t + c_2e^{(-1+2\sqrt{3})t} + c_3e^{-(1+2\sqrt{3})t}$$

$$z(t) = \frac{2c_2e^{(-1+2\sqrt{3})t}\sqrt{3}}{3} - \frac{2c_3e^{-(1+2\sqrt{3})t}\sqrt{3}}{3} + \frac{c_2e^{(-1+2\sqrt{3})t}}{3} + \frac{c_3e^{-(1+2\sqrt{3})t}}{3} + c_1e^t$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 474

```
DSolve[{x'[t]==2*x[t]-1*y[t]+0*z[t],y'[t]==0*x[t]-2*y[t]+3*z[t],z'[t]==-1*x[t]+3*y[t]-1*z[t]
```

$$x(t) \to \frac{1}{16}e^{-\left(\left(1+2\sqrt{3}\right)t\right)}\left(c_1\left(\left(5+3\sqrt{3}\right)e^{4\sqrt{3}t}+6e^{2\left(1+\sqrt{3}\right)t}+5-3\sqrt{3}\right)\right.$$
$$-2c_2\left(\left(1+\sqrt{3}\right)e^{4\sqrt{3}t}-2e^{2\left(1+\sqrt{3}\right)t}+1-\sqrt{3}\right)$$
$$\left.-c_3\left(\left(3+\sqrt{3}\right)e^{4\sqrt{3}t}-6e^{2\left(1+\sqrt{3}\right)t}+3-\sqrt{3}\right)\right)$$

$$y(t) \to \frac{1}{16}e^{-\left(\left(1+2\sqrt{3}\right)t\right)}\left(c_1\left(-\left(3+\sqrt{3}\right)e^{4\sqrt{3}t}+6e^{2\left(1+\sqrt{3}\right)t}-3+\sqrt{3}\right)\right.$$
$$+2c_2\left(-\left(\sqrt{3}-3\right)e^{4\sqrt{3}t}+2e^{2\left(1+\sqrt{3}\right)t}+3+\sqrt{3}\right)$$
$$\left.+3c_3\left(\left(\sqrt{3}-1\right)e^{4\sqrt{3}t}+2e^{2\left(1+\sqrt{3}\right)t}-1-\sqrt{3}\right)\right)$$

$$z(t) \to -\frac{1}{48}e^{-\left(\left(1+2\sqrt{3}\right)t\right)}\left(c_1\left(\left(9+7\sqrt{3}\right)e^{4\sqrt{3}t}-18e^{2\left(1+\sqrt{3}\right)t}+9-7\sqrt{3}\right)\right.$$
$$-2c_2\left(\left(5\sqrt{3}-3\right)e^{4\sqrt{3}t}+6e^{2\left(1+\sqrt{3}\right)t}-3-5\sqrt{3}\right)$$
$$\left.+3c_3\left(\left(\sqrt{3}-5\right)e^{4\sqrt{3}t}-6e^{2\left(1+\sqrt{3}\right)t}-5-\sqrt{3}\right)\right)$$

14.13 problem 17

Internal problem ID [13140]
Internal file name [`OUTPUT/11795_Sunday_December_03_2023_07_16_42_PM_89067/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -4x(t) + 3y \\
y' &= -y + z(t) \\
z'(t) &= 5x(t) - 5y
\end{aligned}
$$

14.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{5\,\mathrm{e}^{-t}}{2}-\frac{3\cos(t)\mathrm{e}^{-2t}}{2}-\frac{9\sin(t)\mathrm{e}^{-2t}}{2} & \frac{3\cos(t)\mathrm{e}^{-2t}}{2}+\frac{9\sin(t)\mathrm{e}^{-2t}}{2}-\frac{3\,\mathrm{e}^{-t}}{2} & -\frac{3\cos(t)\mathrm{e}^{-2t}}{2}-\frac{3\sin(t)\mathrm{e}^{-2t}}{2}+\frac{3\,\mathrm{e}^{-t}}{2} \\ -\frac{5\cos(t)\mathrm{e}^{-2t}}{2}-\frac{5\sin(t)\mathrm{e}^{-2t}}{2}+\frac{5\,\mathrm{e}^{-t}}{2} & -\frac{3\,\mathrm{e}^{-t}}{2}+\frac{5\cos(t)\mathrm{e}^{-2t}}{2}+\frac{5\sin(t)\mathrm{e}^{-2t}}{2} & -\frac{3\cos(t)\mathrm{e}^{-2t}}{2}-\frac{\sin(t)\mathrm{e}^{-2t}}{2}+\frac{3\,\mathrm{e}^{-t}}{2} \\ 5\sin(t)\,\mathrm{e}^{-2t} & -5\sin(t)\,\mathrm{e}^{-2t} & \cos(t)\,\mathrm{e}^{-2t}+2\sin(t)\,\mathrm{e}^{-2t} \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{(-3\cos(t)-9\sin(t))\mathrm{e}^{-2t}}{2}+\frac{5\,\mathrm{e}^{-t}}{2} & \frac{(3\cos(t)+9\sin(t))\mathrm{e}^{-2t}}{2}-\frac{3\,\mathrm{e}^{-t}}{2} & \frac{(-3\cos(t)-3\sin(t))\mathrm{e}^{-2t}}{2}+\frac{3\,\mathrm{e}^{-t}}{2} \\ \frac{(-5\cos(t)-5\sin(t))\mathrm{e}^{-2t}}{2}+\frac{5\,\mathrm{e}^{-t}}{2} & \frac{(5\cos(t)+5\sin(t))\mathrm{e}^{-2t}}{2}-\frac{3\,\mathrm{e}^{-t}}{2} & \frac{(-3\cos(t)-\sin(t))\mathrm{e}^{-2t}}{2}+\frac{3\,\mathrm{e}^{-t}}{2} \\ 5\sin(t)\,\mathrm{e}^{-2t} & -5\sin(t)\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}(\cos(t)+2\sin(t)) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = e^{At}\vec{c}
$$

$$
= \begin{bmatrix} \frac{(-3\cos(t)-9\sin(t))\mathrm{e}^{-2t}}{2}+\frac{5\,\mathrm{e}^{-t}}{2} & \frac{(3\cos(t)+9\sin(t))\mathrm{e}^{-2t}}{2}-\frac{3\,\mathrm{e}^{-t}}{2} & \frac{(-3\cos(t)-3\sin(t))\mathrm{e}^{-2t}}{2}+\frac{3\,\mathrm{e}^{-t}}{2} \\ \frac{(-5\cos(t)-5\sin(t))\mathrm{e}^{-2t}}{2}+\frac{5\,\mathrm{e}^{-t}}{2} & \frac{(5\cos(t)+5\sin(t))\mathrm{e}^{-2t}}{2}-\frac{3\,\mathrm{e}^{-t}}{2} & \frac{(-3\cos(t)-\sin(t))\mathrm{e}^{-2t}}{2}+\frac{3\,\mathrm{e}^{-t}}{2} \\ 5\sin(t)\,\mathrm{e}^{-2t} & -5\sin(t)\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}(\cos(t)+2\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} \left(\frac{(-3\cos(t)-9\sin(t))\mathrm{e}^{-2t}}{2}+\frac{5\,\mathrm{e}^{-t}}{2}\right)c_1+\left(\frac{(3\cos(t)+9\sin(t))\mathrm{e}^{-2t}}{2}-\frac{3\,\mathrm{e}^{-t}}{2}\right)c_2+\left(\frac{(-3\cos(t)-3\sin(t))\mathrm{e}^{-2t}}{2}+\frac{3\,\mathrm{e}^{-t}}{2}\right)c_3 \\ \left(\frac{(-5\cos(t)-5\sin(t))\mathrm{e}^{-2t}}{2}+\frac{5\,\mathrm{e}^{-t}}{2}\right)c_1+\left(\frac{(5\cos(t)+5\sin(t))\mathrm{e}^{-2t}}{2}-\frac{3\,\mathrm{e}^{-t}}{2}\right)c_2+\left(\frac{(-3\cos(t)-\sin(t))\mathrm{e}^{-2t}}{2}+\frac{3\,\mathrm{e}^{-t}}{2}\right)c_3 \\ 5\sin(t)\,\mathrm{e}^{-2t}c_1-5\sin(t)\,\mathrm{e}^{-2t}c_2+\mathrm{e}^{-2t}(\cos(t)+2\sin(t))\,c_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{((-3c_1+3c_2-3c_3)\cos(t)-9\sin(t)\left(c_1-c_2+\frac{c_3}{3}\right))\mathrm{e}^{-2t}}{2}+\frac{5\left(c_1-\frac{3c_2}{5}+\frac{3c_3}{5}\right)\mathrm{e}^{-t}}{2} \\ \frac{((-5c_1+5c_2-3c_3)\cos(t)-5\sin(t)\left(c_1-c_2+\frac{c_3}{5}\right))\mathrm{e}^{-2t}}{2}+\frac{5\left(c_1-\frac{3c_2}{5}+\frac{3c_3}{5}\right)\mathrm{e}^{-t}}{2} \\ 5\left(\left(c_1-c_2+\frac{2c_3}{5}\right)\sin(t)+\frac{c_3\cos(t)}{5}\right)\mathrm{e}^{-2t} \end{bmatrix}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & 3 & 0 \\ 0 & -1-\lambda & 1 \\ 5 & -5 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 5\lambda^2 + 9\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= -2 + i \\ \lambda_2 &= -2 - i \\ \lambda_3 &= -1\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
$-2 - i$	1	complex eigenvalue
$-2 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 & 0 \\ 0 & 0 & 1 \\ 5 & -5 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & -5 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{bmatrix} - (-2 - i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 + i & 3 & 0 \\ 0 & 1 + i & 1 \\ 5 & -5 & 2 + i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 + i & 3 & 0 & 0 \\ 0 & 1 + i & 1 & 0 \\ 5 & -5 & 2 + i & 0 \end{array}\right]$$

$$R_3 = R_3 + (2 + i) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 + i & 3 & 0 & 0 \\ 0 & 1 + i & 1 & 0 \\ 0 & 1 + 3i & 2 + i & 0 \end{array}\right]$$

$$R_3 = R_3 + (-2 - i) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 + i & 3 & 0 & 0 \\ 0 & 1 + i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2+i & 3 & 0 \\ 0 & 1+i & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{9}{10} + \frac{3i}{10}\right) t, v_2 = \left(-\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{9}{10} + \frac{3\,\mathrm{I}}{10}\right) t \\ \left(-\frac{1}{2} + \frac{\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{9}{10} + \frac{3i}{10}\right) t \\ \left(-\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{9}{10} + \frac{3\,\mathrm{I}}{10}\right) t \\ \left(-\frac{1}{2} + \frac{\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{9}{10} + \frac{3i}{10} \\ -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{9}{10} + \frac{3\,\mathrm{I}}{10}\right) t \\ \left(-\frac{1}{2} + \frac{\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{9}{10} + \frac{3i}{10} \\ -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{9}{10} + \frac{3\,\mathrm{I}}{10}\right) t \\ \left(-\frac{1}{2} + \frac{\mathrm{I}}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -9+3i \\ -5+5i \\ 10 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -2 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{bmatrix} - (-2+i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-i & 3 & 0 \\ 0 & 1-i & 1 \\ 5 & -5 & 2-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2-i & 3 & 0 & 0 \\ 0 & 1-i & 1 & 0 \\ 5 & -5 & 2-i & 0 \end{array}\right]$$

$$R_3 = R_3 + (2-i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2-i & 3 & 0 & 0 \\ 0 & 1-i & 1 & 0 \\ 0 & 1-3i & 2-i & 0 \end{array}\right]$$

$$R_3 = R_3 + (-2+i)\,R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2-i & 3 & 0 & 0 \\ 0 & 1-i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-i & 3 & 0 \\ 0 & 1-i & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{9}{10} - \frac{3i}{10}\right)t, v_2 = \left(-\frac{1}{2} - \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{9}{10} - \frac{3\,I}{10}\right)t \\ \left(-\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{9}{10} - \frac{3i}{10}\right)t \\ \left(-\frac{1}{2} - \frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{9}{10}-\frac{3\,\mathrm{I}}{10}\right)t \\ \left(-\frac{1}{2}-\frac{\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{9}{10}-\frac{3i}{10} \\ -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{9}{10}-\frac{3\,\mathrm{I}}{10}\right)t \\ \left(-\frac{1}{2}-\frac{\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{9}{10}-\frac{3i}{10} \\ -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{9}{10}-\frac{3\,\mathrm{I}}{10}\right)t \\ \left(-\frac{1}{2}-\frac{\mathrm{I}}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -9-3i \\ -5-5i \\ 10 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-2+i$	1	1	No	$\begin{bmatrix} -\frac{9}{10}-\frac{3i}{10} \\ -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$
$-2-i$	1	1	No	$\begin{bmatrix} -\frac{9}{10}+\frac{3i}{10} \\ -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{9}{10}-\frac{3i}{10}\right) \mathrm{e}^{(-2+i)t} \\ \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(-2+i)t} \\ \mathrm{e}^{(-2+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{9}{10}+\frac{3i}{10}\right) \mathrm{e}^{(-2-i)t} \\ \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(-2-i)t} \\ \mathrm{e}^{(-2-i)t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{9}{10}-\frac{3i}{10}\right) c_1 e^{(-2+i)t} + \left(-\frac{9}{10}+\frac{3i}{10}\right) c_2 e^{(-2-i)t} + c_3 e^{-t} \\ \left(-\frac{1}{2}-\frac{i}{2}\right) c_1 e^{(-2+i)t} + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(-2-i)t} + c_3 e^{-t} \\ c_1 e^{(-2+i)t} + c_2 e^{(-2-i)t} \end{bmatrix}$$

14.13.3 Maple step by step solution

Let's solve

$[x'(t) = -4x(t) + 3y, y' = -y + z(t), z'(t) = 5x(t) - 5y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -4 & 3 & 0 \\ 0 & -1 & 1 \\ 5 & -5 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c}1\\1\\0\end{array}\right]\right], \left[-2-\mathrm{I}, \left[\begin{array}{c}-\frac{9}{10}+\frac{3\,\mathrm{I}}{10}\\-\frac{1}{2}+\frac{\mathrm{I}}{2}\\1\end{array}\right]\right], \left[-2+\mathrm{I}, \left[\begin{array}{c}-\frac{9}{10}-\frac{3\,\mathrm{I}}{10}\\-\frac{1}{2}-\frac{\mathrm{I}}{2}\\1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1\\1\\0\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-t}\cdot\left[\begin{array}{c}1\\1\\0\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2-\mathrm{I}, \left[\begin{array}{c}-\frac{9}{10}+\frac{3\,\mathrm{I}}{10}\\-\frac{1}{2}+\frac{\mathrm{I}}{2}\\1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-2-\mathrm{I})t}\cdot\left[\begin{array}{c}-\frac{9}{10}+\frac{3\,\mathrm{I}}{10}\\-\frac{1}{2}+\frac{\mathrm{I}}{2}\\1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{-2t}\cdot(\cos(t)-\mathrm{I}\sin(t))\cdot\left[\begin{array}{c}-\frac{9}{10}+\frac{3\,\mathrm{I}}{10}\\-\frac{1}{2}+\frac{\mathrm{I}}{2}\\1\end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-2t}\cdot\left[\begin{array}{c}\left(-\frac{9}{10}+\frac{3\,\mathrm{I}}{10}\right)(\cos(t)-\mathrm{I}\sin(t))\\\left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(t)-\mathrm{I}\sin(t))\\\cos(t)-\mathrm{I}\sin(t)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = e^{-2t} \cdot \begin{bmatrix} -\frac{9\cos(t)}{10} + \frac{3\sin(t)}{10} \\ \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \\ \cos(t) \end{bmatrix}, \vec{x}_3(t) = e^{-2t} \cdot \begin{bmatrix} \frac{9\sin(t)}{10} + \frac{3\cos(t)}{10} \\ \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \cdot \begin{bmatrix} -\frac{9\cos(t)}{10} + \frac{3\sin(t)}{10} \\ \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \\ \cos(t) \end{bmatrix} + c_3 e^{-2t} \cdot \begin{bmatrix} \frac{9\sin(t)}{10} + \frac{3\cos(t)}{10} \\ \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ -\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{((-9c_2+3c_3)\cos(t)+3\sin(t)(c_2+3c_3))e^{-2t}}{10} + c_1 e^{-t} \\ \frac{((-c_2+c_3)\cos(t)+\sin(t)(c_2+c_3))e^{-2t}}{2} + c_1 e^{-t} \\ e^{-2t}(c_2\cos(t) - c_3\sin(t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{((-9c_2+3c_3)\cos(t)+3\sin(t)(c_2+3c_3))e^{-2t}}{10} + c_1 e^{-t}, y = \frac{((-c_2+c_3)\cos(t)+\sin(t)(c_2+c_3))e^{-2t}}{2} + c_1 e^{-t}, z(t) = e\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 101

```
dsolve([diff(x(t),t)=-4*x(t)+3*y(t)+0*z(t),diff(y(t),t)=0*x(t)-1*y(t)+1*z(t),diff(z(t),t)=5*
```

$$x(t) = e^{-t}c_1 + \frac{6c_2 e^{-2t}\sin(t)}{5} - \frac{3c_2 e^{-2t}\cos(t)}{5} + \frac{6\,e^{-2t}\cos(t)\,c_3}{5} + \frac{3\,e^{-2t}\sin(t)\,c_3}{5}$$
$$y(t) = e^{-t}c_1 + c_2 e^{-2t}\sin(t) + e^{-2t}\cos(t)\,c_3$$
$$z(t) = -e^{-2t}(c_2\sin(t) + c_3\sin(t) - c_2\cos(t) + c_3\cos(t))$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 152

```
DSolve[{x'[t]==-4*x[t]+3*y[t]+0*z[t],y'[t]==0*x[t]-1*y[t]+1*z[t],z'[t]==5*x[t]-5*y[t]+0*z[t]
```

$$x(t) \to \frac{1}{2}e^{-2t}\left((5c_1 - 3c_2 + 3c_3)e^t - 3(c_1 - c_2 + c_3)\cos(t) - 3(3c_1 - 3c_2 + c_3)\sin(t)\right)$$
$$y(t) \to \frac{1}{2}e^{-2t}\left((5c_1 - 3c_2 + 3c_3)e^t + (-5c_1 + 5c_2 - 3c_3)\cos(t) - (5c_1 - 5c_2 + c_3)\sin(t)\right)$$
$$z(t) \to e^{-2t}(c_3\cos(t) + (5c_1 - 5c_2 + 2c_3)\sin(t))$$

14.14 problem 18

Internal problem ID [13141]
Internal file name [`OUTPUT/11796_Sunday_December_03_2023_07_16_43_PM_66458785/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -10x(t) + 10y \\
y' &= 28x(t) - y \\
z'(t) &= -\frac{8z(t)}{3}
\end{aligned}
$$

14.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{cc} \frac{\left(-9\sqrt{1201}+1201\right)\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{2402}+\frac{\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\left(9\sqrt{1201}+1201\right)}{2402} & -\frac{10\left(-\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}+\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\right)\sqrt{}}{1201} \\ -\frac{28\left(-\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}+\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\right)\sqrt{1201}}{1201} & \frac{\left(9\sqrt{1201}+1201\right)\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{2402}+\frac{\left(-9\sqrt{1201}+1201\right)\mathrm{e}}{2402} \\ 0 & 0 \end{array}\right.$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \left[\begin{array}{cc} \frac{\left(-9\sqrt{1201}+1201\right)\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{2402}+\frac{\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\left(9\sqrt{1201}+1201\right)}{2402} & -\frac{10\left(-\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}+\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\right)}{1201} \\ -\frac{28\left(-\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}+\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\right)\sqrt{1201}}{1201} & \frac{\left(9\sqrt{1201}+1201\right)\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{2402}+\frac{\left(-9\sqrt{1201}+1201\right)}{24} \\ 0 & 0 \end{array}\right.$$

$$= \left[\begin{array}{c} \left(\frac{\left(-9\sqrt{1201}+1201\right)\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{2402}+\frac{\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\left(9\sqrt{1201}+1201\right)}{2402}\right)c_1-\frac{10\left(-\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}+\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\right)\sqrt{}}{1201} \\ -\frac{28\left(-\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}+\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\right)\sqrt{1201}\,c_1}{1201}+\left(\frac{\left(9\sqrt{1201}+1201\right)\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{2402}+\frac{\left(-9\sqrt{1201}+1201\right)\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)}{2}}}{2402}\right. \\ \mathrm{e}^{-\frac{8t}{3}}c_3 \end{array}\right.$$

$$= \left[\begin{array}{c} \frac{\left((-9c_1+20c_2)\sqrt{1201}+1201c_1\right)\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{2402}+\frac{9\,\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\left(\left(c_1-\frac{20c_2}{9}\right)\sqrt{1201}+\frac{1201c_1}{9}\right)}{2402} \\ \frac{\left((56c_1+9c_2)\sqrt{1201}+1201c_2\right)\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{2402}-\frac{28\,\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}\left(\left(c_1+\frac{9c_2}{56}\right)\sqrt{1201}-\frac{1201c_2}{56}\right)}{1201} \\ \mathrm{e}^{-\frac{8t}{3}}c_3 \end{array}\right]$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -10-\lambda & 10 & 0 \\ 28 & -1-\lambda & 0 \\ 0 & 0 & -\frac{8}{3}-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \frac{41}{3}\lambda^2 - \frac{722}{3}\lambda - 720 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{11}{2} + \frac{\sqrt{1201}}{2}$$
$$\lambda_2 = -\frac{11}{2} - \frac{\sqrt{1201}}{2}$$
$$\lambda_3 = -\frac{8}{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{11}{2}+\frac{\sqrt{1201}}{2}$	1	real eigenvalue
$-\frac{11}{2}-\frac{\sqrt{1201}}{2}$	1	real eigenvalue
$-\frac{8}{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{8}{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} - \left(-\frac{8}{3}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{22}{3} & 10 & 0 \\ 28 & \frac{5}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\frac{22}{3} & 10 & 0 & 0 \\ 28 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{42R_1}{11} \Longrightarrow \left[\begin{array}{ccc|c} -\frac{22}{3} & 10 & 0 & 0 \\ 0 & \frac{1315}{33} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{22}{3} & 10 & 0 \\ 0 & \frac{1315}{33} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{11}{2} - \frac{\sqrt{1201}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} - \left(-\frac{11}{2} - \frac{\sqrt{1201}}{2} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{9}{2} + \frac{\sqrt{1201}}{2} & 10 & 0 \\ 28 & \frac{9}{2} + \frac{\sqrt{1201}}{2} & 0 \\ 0 & 0 & \frac{17}{6} + \frac{\sqrt{1201}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\frac{9}{2} + \frac{\sqrt{1201}}{2} & 10 & 0 & 0 \\ 28 & \frac{9}{2} + \frac{\sqrt{1201}}{2} & 0 & 0 \\ 0 & 0 & \frac{17}{6} + \frac{\sqrt{1201}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{28R_1}{-\frac{9}{2} + \frac{\sqrt{1201}}{2}} \Longrightarrow \left[\begin{array}{ccc|c} -\frac{9}{2} + \frac{\sqrt{1201}}{2} & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{17}{6} + \frac{\sqrt{1201}}{2} & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -\frac{9}{2}+\frac{\sqrt{1201}}{2} & 10 & 0 & 0 \\ 0 & 0 & \frac{17}{6}+\frac{\sqrt{1201}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{9}{2}+\frac{\sqrt{1201}}{2} & 10 & 0 \\ 0 & 0 & \frac{17}{6}+\frac{\sqrt{1201}}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{\left(9+\sqrt{1201}\right)t}{56}, v_3 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{\left(9+\sqrt{1201}\right)t}{56} \\ t \\ 0 \end{bmatrix}=\begin{bmatrix} -\frac{\left(9+\sqrt{1201}\right)t}{56} \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{\left(9+\sqrt{1201}\right)t}{56} \\ t \\ 0 \end{bmatrix}=t\begin{bmatrix} -\frac{9}{56}-\frac{\sqrt{1201}}{56} \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{\left(9+\sqrt{1201}\right)t}{56} \\ t \\ 0 \end{bmatrix}=\begin{bmatrix} -\frac{9}{56}-\frac{\sqrt{1201}}{56} \\ 1 \\ 0 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{(9+\sqrt{1201})t}{56} \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -9-\sqrt{1201} \\ 56 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -\frac{11}{2} + \frac{\sqrt{1201}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} - \left(-\frac{11}{2} + \frac{\sqrt{1201}}{2} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{9}{2} - \frac{\sqrt{1201}}{2} & 10 & 0 \\ 28 & \frac{9}{2} - \frac{\sqrt{1201}}{2} & 0 \\ 0 & 0 & \frac{17}{6} - \frac{\sqrt{1201}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\frac{9}{2} - \frac{\sqrt{1201}}{2} & 10 & 0 & 0 \\ 28 & \frac{9}{2} - \frac{\sqrt{1201}}{2} & 0 & 0 \\ 0 & 0 & \frac{17}{6} - \frac{\sqrt{1201}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{28R_1}{-\frac{9}{2} - \frac{\sqrt{1201}}{2}} \Longrightarrow \left[\begin{array}{ccc|c} -\frac{9}{2} - \frac{\sqrt{1201}}{2} & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{17}{6} - \frac{\sqrt{1201}}{2} & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -\frac{9}{2} - \frac{\sqrt{1201}}{2} & 10 & 0 & 0 \\ 0 & 0 & \frac{17}{6} - \frac{\sqrt{1201}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{9}{2}-\frac{\sqrt{1201}}{2} & 10 & 0 \\ 0 & 0 & \frac{17}{6}-\frac{\sqrt{1201}}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{\left(-9+\sqrt{1201}\right)t}{56}, v_3 = 0\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\left(-9+\sqrt{1201}\right)t}{56} \\ t \\ 0 \end{bmatrix}=\begin{bmatrix} \frac{\left(-9+\sqrt{1201}\right)t}{56} \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\left(-9+\sqrt{1201}\right)t}{56} \\ t \\ 0 \end{bmatrix}=t\begin{bmatrix} \frac{\sqrt{1201}}{56}-\frac{9}{56} \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{\left(-9+\sqrt{1201}\right)t}{56} \\ t \\ 0 \end{bmatrix}=\begin{bmatrix} \frac{\sqrt{1201}}{56}-\frac{9}{56} \\ 1 \\ 0 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{\left(-9+\sqrt{1201}\right)t}{56} \\ t \\ 0 \end{bmatrix}=\begin{bmatrix} -9+\sqrt{1201} \\ 56 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-\frac{11}{2}+\frac{\sqrt{1201}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{10}{\frac{9}{2}+\frac{\sqrt{1201}}{2}} \\ 1 \\ 0\end{array}\right]$
$-\frac{11}{2}-\frac{\sqrt{1201}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{10}{\frac{9}{2}-\frac{\sqrt{1201}}{2}} \\ 1 \\ 0\end{array}\right]$
$-\frac{8}{3}$	1	1	No	$\left[\begin{array}{c}0 \\ 0 \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{11}{2}+\frac{\sqrt{1201}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{11}{2}+\frac{\sqrt{1201}}{2}\right)t} \\
&= \left[\begin{array}{c}\frac{10}{\frac{9}{2}+\frac{\sqrt{1201}}{2}} \\ 1 \\ 0\end{array}\right] e^{\left(-\frac{11}{2}+\frac{\sqrt{1201}}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $-\frac{11}{2}-\frac{\sqrt{1201}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\left(-\frac{11}{2}-\frac{\sqrt{1201}}{2}\right)t} \\ &= \left[\begin{array}{c} \frac{10}{\frac{9}{2}-\frac{\sqrt{1201}}{2}} \\ 1 \\ 0 \end{array}\right] e^{\left(-\frac{11}{2}-\frac{\sqrt{1201}}{2}\right)t}\end{aligned}$$

Since eigenvalue $-\frac{8}{3}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-\frac{8t}{3}} \\ &= \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] e^{-\frac{8t}{3}}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = c_1 \left[\begin{array}{c} \frac{10\,\mathrm{e}^{\left(-\frac{11}{2}+\frac{\sqrt{1201}}{2}\right)t}}{\frac{9}{2}+\frac{\sqrt{1201}}{2}} \\ \mathrm{e}^{\left(-\frac{11}{2}+\frac{\sqrt{1201}}{2}\right)t} \\ 0 \end{array}\right] + c_2 \left[\begin{array}{c} \frac{10\,\mathrm{e}^{\left(-\frac{11}{2}-\frac{\sqrt{1201}}{2}\right)t}}{\frac{9}{2}-\frac{\sqrt{1201}}{2}} \\ \mathrm{e}^{\left(-\frac{11}{2}-\frac{\sqrt{1201}}{2}\right)t} \\ 0 \end{array}\right] + c_3 \left[\begin{array}{c} 0 \\ 0 \\ \mathrm{e}^{-\frac{8t}{3}} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} \frac{c_1\left(-9+\sqrt{1201}\right)\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{56} - \frac{\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}} c_2\left(9+\sqrt{1201}\right)}{56} \\ c_1\mathrm{e}^{\frac{\left(-11+\sqrt{1201}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(11+\sqrt{1201}\right)t}{2}} \\ c_3\mathrm{e}^{-\frac{8t}{3}} \end{array}\right]$$

14.14.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -10x(t) + 10y, y' = 28x(t) - y, z'(t) = -\frac{8z(t)}{3}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{8}{3}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[-\frac{11}{2} - \frac{\sqrt{1201}}{2}, \begin{bmatrix} \frac{10}{\frac{9}{2} - \frac{\sqrt{1201}}{2}} \\ 1 \\ 0 \end{bmatrix}\right], \left[-\frac{11}{2} + \frac{\sqrt{1201}}{2}, \begin{bmatrix} \frac{10}{\frac{9}{2} + \frac{\sqrt{1201}}{2}} \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{8}{3}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-\frac{8t}{3}} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{11}{2} - \frac{\sqrt{1201}}{2}, \begin{bmatrix} \frac{10}{\frac{9}{2} - \frac{\sqrt{1201}}{2}} \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(-\frac{11}{2} - \frac{\sqrt{1201}}{2}\right)t} \cdot \begin{bmatrix} \frac{10}{\frac{9}{2} - \frac{\sqrt{1201}}{2}} \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{11}{2} + \frac{\sqrt{1201}}{2}, \begin{bmatrix} \frac{10}{\frac{9}{2} + \frac{\sqrt{1201}}{2}} \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \mathrm{e}^{\left(-\frac{11}{2} + \frac{\sqrt{1201}}{2}\right)t} \cdot \begin{bmatrix} \frac{10}{\frac{9}{2} + \frac{\sqrt{1201}}{2}} \\ 1 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-\frac{8t}{3}} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{\left(-\frac{11}{2}-\frac{\sqrt{1201}}{2}\right)t} \cdot \begin{bmatrix} \frac{10}{\frac{9}{2}-\frac{\sqrt{1201}}{2}} \\ 1 \\ 0 \end{bmatrix} + c_3 e^{\left(-\frac{11}{2}+\frac{\sqrt{1201}}{2}\right)t} \cdot \begin{bmatrix} \frac{10}{\frac{9}{2}+\frac{\sqrt{1201}}{2}} \\ 1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{c_3\left(-9+\sqrt{1201}\right) e^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{56} - \frac{e^{-\frac{\left(11+\sqrt{1201}\right)t}{2}} c_2 \left(9+\sqrt{1201}\right)}{56} \\ c_2 e^{-\frac{\left(11+\sqrt{1201}\right)t}{2}} + c_3 e^{\frac{\left(-11+\sqrt{1201}\right)t}{2}} \\ c_1 e^{-\frac{8t}{3}} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{c_3\left(-9+\sqrt{1201}\right) e^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{56} - \frac{e^{-\frac{\left(11+\sqrt{1201}\right)t}{2}} c_2 \left(9+\sqrt{1201}\right)}{56}, y = c_2 e^{-\frac{\left(11+\sqrt{1201}\right)t}{2}} + c_3 e^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}, z(t\right.$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 95

```
dsolve([diff(x(t),t)=-10*x(t)+10*y(t)+0*z(t),diff(y(t),t)=28*x(t)-1*y(t)+0*z(t),diff(z(t),t)
```

$$x(t) = c_1 e^{\frac{\left(-11+\sqrt{1201}\right)t}{2}} + c_2 e^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}$$

$$y(t) = \frac{c_1 e^{\frac{\left(-11+\sqrt{1201}\right)t}{2}} \sqrt{1201}}{20} - \frac{c_2 e^{-\frac{\left(11+\sqrt{1201}\right)t}{2}} \sqrt{1201}}{20} + \frac{9 c_1 e^{\frac{\left(-11+\sqrt{1201}\right)t}{2}}}{20} + \frac{9 c_2 e^{-\frac{\left(11+\sqrt{1201}\right)t}{2}}}{20}$$

$$z(t) = c_3 e^{-\frac{8t}{3}}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 312

```
DSolve[{x'[t]==-10*x[t]+10*y[t]+0*z[t],y'[t]==28*x[t]-1*y[t]+0*z[t],z'[t]==0*x[t]+0*y[t]-8/3
```

$$x(t) \to \frac{e^{-\frac{1}{2}\left(11+\sqrt{1201}\right)t}\left(c_1\left(\left(1201-9\sqrt{1201}\right)e^{\sqrt{1201}t}+1201+9\sqrt{1201}\right)+20\sqrt{1201}c_2\left(e^{\sqrt{1201}t}-1\right)\right)}{2402}$$

$$y(t) \to \frac{e^{-\frac{1}{2}\left(11+\sqrt{1201}\right)t}\left(56\sqrt{1201}c_1\left(e^{\sqrt{1201}t}-1\right)+c_2\left(\left(1201+9\sqrt{1201}\right)e^{\sqrt{1201}t}+1201-9\sqrt{1201}\right)\right)}{2402}$$

$$z(t) \to c_3e^{-8t/3}$$

$$x(t) \to \frac{e^{-\frac{1}{2}\left(11+\sqrt{1201}\right)t}\left(c_1\left(\left(1201-9\sqrt{1201}\right)e^{\sqrt{1201}t}+1201+9\sqrt{1201}\right)+20\sqrt{1201}c_2\left(e^{\sqrt{1201}t}-1\right)\right)}{2402}$$

$$y(t) \to \frac{e^{-\frac{1}{2}\left(11+\sqrt{1201}\right)t}\left(56\sqrt{1201}c_1\left(e^{\sqrt{1201}t}-1\right)+c_2\left(\left(1201+9\sqrt{1201}\right)e^{\sqrt{1201}t}+1201-9\sqrt{1201}\right)\right)}{2402}$$

$$z(t) \to 0$$

14.15 problem 20

Internal problem ID [13142]
Internal file name [`OUTPUT/11797_Sunday_December_03_2023_07_16_43_PM_32031629/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Exercises section 3.8 page 371
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -y + z(t) \\ y' &= -x(t) + z(t) \\ z'(t) &= z(t) \end{aligned}$$

14.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ -\frac{e^{t}}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^{t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ 0 & 0 & e^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-t}}{2}+\frac{e^{t}}{2} & -\frac{e^{t}}{2}+\frac{e^{-t}}{2} & \frac{e^{t}}{2}-\frac{e^{-t}}{2} \\ -\frac{e^{t}}{2}+\frac{e^{-t}}{2} & \frac{e^{-t}}{2}+\frac{e^{t}}{2} & \frac{e^{t}}{2}-\frac{e^{-t}}{2} \\ 0 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-t}}{2}+\frac{e^{t}}{2}\right)c_1+\left(-\frac{e^{t}}{2}+\frac{e^{-t}}{2}\right)c_2+\left(\frac{e^{t}}{2}-\frac{e^{-t}}{2}\right)c_3 \\ \left(-\frac{e^{t}}{2}+\frac{e^{-t}}{2}\right)c_1+\left(\frac{e^{-t}}{2}+\frac{e^{t}}{2}\right)c_2+\left(\frac{e^{t}}{2}-\frac{e^{-t}}{2}\right)c_3 \\ e^{t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1+c_2-c_3)e^{-t}}{2}+\frac{e^{t}(c_1-c_2+c_3)}{2} \\ \frac{(c_1+c_2-c_3)e^{-t}}{2}-\frac{e^{t}(c_1-c_2-c_3)}{2} \\ e^{t}c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

14.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A-\lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 & 1 \\ -1 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t + s\}$

Hence the solution is

$$\begin{bmatrix} -t + s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t + s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
1	2	2	No	$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t}\end{aligned}$$

eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

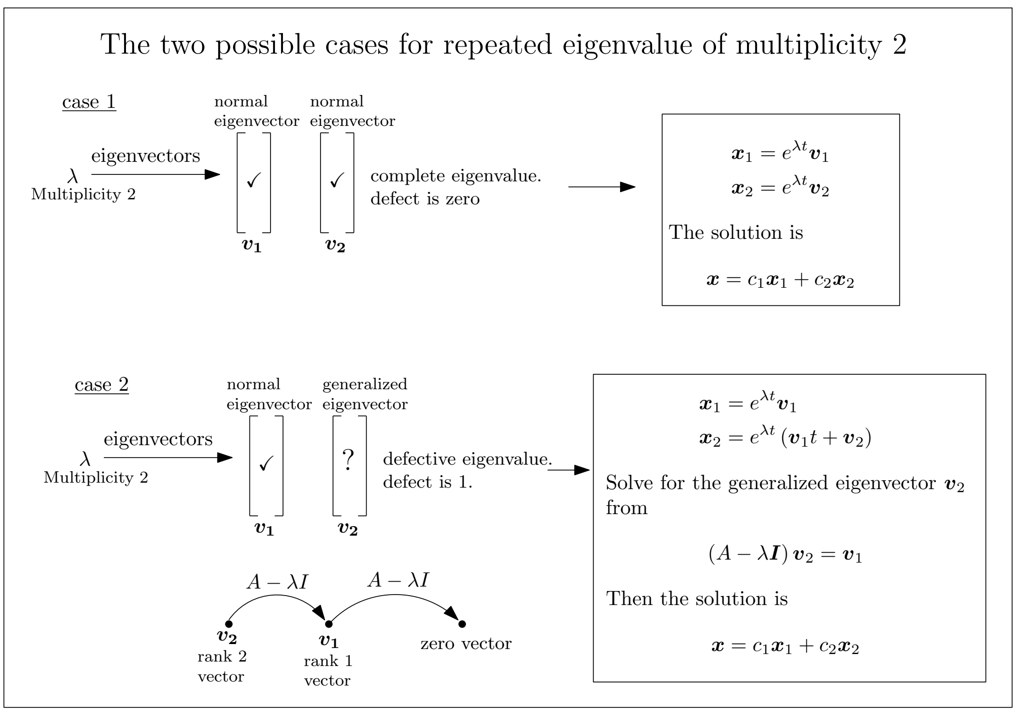


Figure 460: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above.

Hence the corresponding eigenvector basis are

$$\vec{x}_2(t) = \vec{v}_2 e^{t} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{t}$$

$$\vec{x}_3(t) = \vec{v}_3 e^{t} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{t} \\ \mathrm{e}^{t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1\mathrm{e}^{-t} + \mathrm{e}^{t}(c_2 - c_3) \\ c_1\mathrm{e}^{-t} + c_3\mathrm{e}^{t} \\ c_2\mathrm{e}^{t} \end{bmatrix}$$

14.15.3 Maple step by step solution

Let's solve

$[x'(t) = -y + z(t)\,, y' = -x(t) + z(t)\,, z'(t) = z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{x}_2(t) = e^{t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

$$\vec{x}_3(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_3(t)$ into the homogeneous system

$$\lambda\, e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\, e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_3(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{x}_3(t) = \mathrm{e}^t \cdot \left(t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \mathrm{e}^t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^t \cdot \left(t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-t} + \mathrm{e}^t ((t-1) c_3 + c_2) \\ c_1 \mathrm{e}^{-t} \\ \mathrm{e}^t (c_3 t + c_2) \end{bmatrix}$$

- Solution to the system of ODEs

$\{x(t) = c_1 \mathrm{e}^{-t} + \mathrm{e}^t ((t-1) c_3 + c_2), y = c_1 \mathrm{e}^{-t}, z(t) = \mathrm{e}^t (c_3 t + c_2)\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 42

```
dsolve([diff(x(t),t)=-y(t)+z(t),diff(y(t),t)=-x(t)+z(t),diff(z(t),t)=z(t)],singsol=all)
```

$$x(t) = c_1 e^t + c_2 e^{-t}$$
$$y(t) = -c_1 e^t + c_2 e^{-t} + c_3 e^t$$
$$z(t) = c_3 e^t$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 94

```
DSolve[{x'[t]==-y[t]+z[t],y'[t]==-x[t]+z[t],z'[t]==z[t]},{x[t],y[t],z[t]},t,IncludeSingularS
```

$$x(t) \to \frac{1}{2} e^{-t} \left(c_1 \left(e^{2t} + 1 \right) - (c_2 - c_3) \left(e^{2t} - 1 \right) \right)$$
$$y(t) \to \frac{1}{2} e^{-t} \left(-\left(c_1 \left(e^{2t} - 1 \right) \right) + c_2 \left(e^{2t} + 1 \right) + c_3 \left(e^{2t} - 1 \right) \right)$$
$$z(t) \to c_3 e^t$$

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15.1 problem 1

Internal problem ID [13143]
Internal file name [`OUTPUT/11798_Sunday_December_03_2023_07_16_44_PM_86889940/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = 0$$
$$(-1+\lambda)(-2+\lambda) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]
$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$
\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$
\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
2	1	2	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

15.2 problem 2

Internal problem ID [13144]
Internal file name [OUTPUT/11799_Sunday_December_03_2023_07_16_44_PM_33469809/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} -\lambda & 1 \\ 2 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = \sqrt{2}$$
$$\lambda_2 = -\sqrt{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\sqrt{2}$	1	real eigenvalue
$-\sqrt{2}$	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = \sqrt{2}$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \left(\sqrt{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -\sqrt{2} & 1 \\ 2 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} -\sqrt{2} & 1 & 0 \\ 2 & -\sqrt{2} & 0 \end{array} \right]
$$

$$
R_2 = R_2 + \sqrt{2}\, R_1 \Longrightarrow \left[\begin{array}{cc|c} -\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -\sqrt{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{\sqrt{2}\,t}{2}\right\}$

Hence the solution is

$$
\begin{bmatrix} \frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$$

Considering $\lambda = -\sqrt{2}$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \left(-\sqrt{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} -\sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} \sqrt{2} & 1 & 0 \\ 2 & \sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \sqrt{2}\, R_1 \Longrightarrow \left[\begin{array}{cc|c} \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \sqrt{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{\sqrt{2}\,t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{\sqrt{2}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$\sqrt{2}$	1	2	No	$\begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$
$-\sqrt{2}$	1	2	No	$\begin{bmatrix} -\sqrt{2} \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$$
$$P = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 2 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 2 & 2 \end{bmatrix}^{-1}$$

15.3 problem 3

Internal problem ID [13145]
Internal file name [`OUTPUT/11800_Sunday_December_03_2023_07_16_45_PM_2968719/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3x(t)$$
$$y' = -2y$$

15.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{3t} & 0 \\ 0 & \mathrm{e}^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{3t}c_1 \\ e^{-2t}c_2 \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 0 \\ 0 & -2-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(3-\lambda)(-2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 0 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{3t} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 e^{3t} \\ c_1 e^{-2t} \end{bmatrix}
$$

The following is the phase plot of the system.

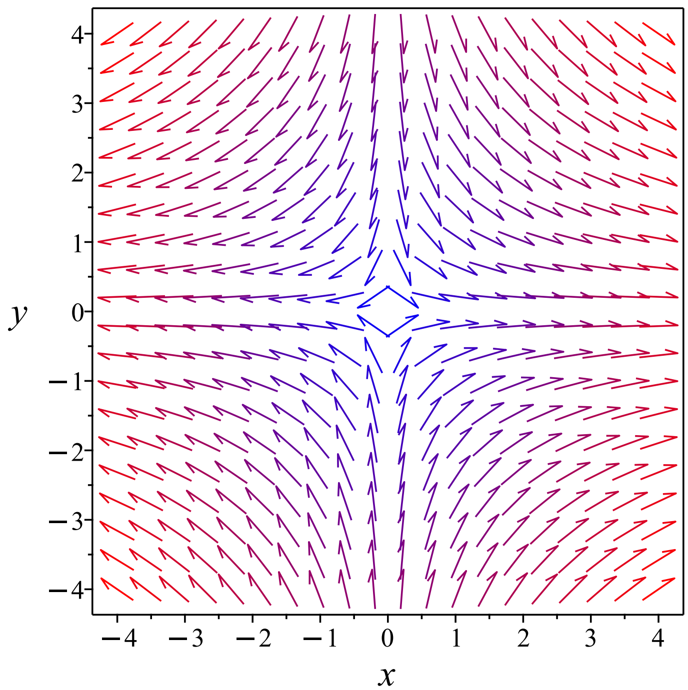


Figure 461: Phase plot

15.3.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t), y' = -2y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{3t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-2t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 e^{3t} \\ c_1 e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = c_2 e^{3t}, y = c_1 e^{-2t}\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([diff(x(t),t)=3*x(t)+0*y(t),diff(y(t),t)=0*x(t)-2*y(t)],singsol=all)
```

$$x(t) = c_2 e^{3t}$$
$$y(t) = c_1 e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 65

```
DSolve[{x'[t]==3*x[t]+0*y[t],y'[t]==0*x[t]-2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$
\begin{aligned}
x(t) &\to c_1 e^{3t} \\
y(t) &\to c_2 e^{-2t} \\
x(t) &\to c_1 e^{3t} \\
y(t) &\to 0 \\
x(t) &\to 0 \\
y(t) &\to c_2 e^{-2t} \\
x(t) &\to 0 \\
y(t) &\to 0
\end{aligned}
$$

15.4 problem 4

Internal problem ID [13146]
Internal file name [OUTPUT/11801_Sunday_December_03_2023_07_16_45_PM_69760170/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{bmatrix} = 0$$
$$(-1+\lambda)(-3+\lambda) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	2	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
3	1	2	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

15.5 problem 6

Internal problem ID [13147]
Internal file name [`OUTPUT/11802_Sunday_December_03_2023_07_16_45_PM_86155900/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 0$$
$$y' = x(t) - y$$

15.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 1 - e^{-t} & e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1 & 0 \\ 1-\mathrm{e}^{-t} & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} c_1 \\ (1-\mathrm{e}^{-t})\,c_1 + \mathrm{e}^{-t}c_2 \end{bmatrix} \\
&= \begin{bmatrix} c_1 \\ (-c_1+c_2)\,\mathrm{e}^{-t} + c_1 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(-1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{0} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \mathrm{e}^{-t} + c_2 \end{bmatrix}$$

The following is the phase plot of the system.

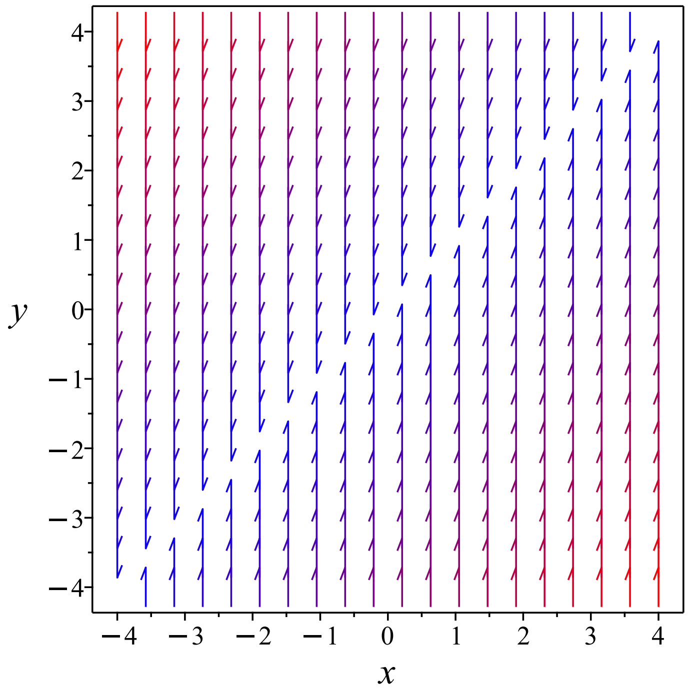


Figure 462: Phase plot

15.5.3 Maple step by step solution

Let's solve

$[x'(t) = 0, y' = x(t) - y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 e^{-t} + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = c_2, y = c_1 e^{-t} + c_2\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(x(t),t)=0*x(t)+0*y(t),diff(y(t),t)=1*x(t)-1*y(t)],singsol=all)
```

$$x(t) = c_2$$
$$y(t) = c_2 + e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 27

```
DSolve[{x'[t]==0*x[t]+0*y[t],y'[t]==1*x[t]-1*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to c_1$$
$$y(t) \to e^{-t}(c_1(e^t - 1) + c_2)$$

15.6 problem 7

Internal problem ID [13148]
Internal file name [`OUTPUT/11803_Sunday_December_03_2023_07_16_46_PM_40371899/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = \pi^2 x(t) + \frac{187y}{5}$$
$$y' = \sqrt{555}\, x(t) + \frac{400617y}{5000}$$

With initial conditions

$$[x(0) = 0, y(0) = 0]$$

15.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \pi^2 & \frac{187}{5} \\ \sqrt{555} & \frac{400617}{5000} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{c} \frac{\left(-5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}+400617\right)e^{\frac{\left(5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+374000\cdots}\right.}{10000}}}{2\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}} \\ -\frac{5000\sqrt{555}\left(-e^{\frac{\left(5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+37\cdots}\right.}{10000}}\right.}{\sqrt{\cdots}} \end{array}\right.$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$

$$= \left[\begin{array}{c} \frac{\left(-5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}+400617\right)e^{\frac{\left(5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+3740\cdots}\right.}{10000}}}{2\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}} \\ -\frac{5000\sqrt{555}\left(-e^{\frac{\left(5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+\cdots}\right.}{10\cdots}}\right.}{\cdots} \end{array}\right.$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \pi^2 & \frac{187}{5} \\ \sqrt{555} & \frac{400617}{5000} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} \pi^2 & \frac{187}{5} \\ \sqrt{555} & \frac{400617}{5000} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \pi^2-\lambda & \frac{187}{5} \\ \sqrt{555} & \frac{400617}{5000}-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \left(-\frac{400617}{5000} - \pi^2\right)\lambda - \frac{187\sqrt{555}}{5} + \frac{400617\pi^2}{5000} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{\pi^2}{2} + \frac{400617}{10000} + \frac{\sqrt{25000000\pi^4 - 4006170000\pi^2 + 3740000000\sqrt{555} + 160493980689}}{10000}$$
$$\lambda_2 = \frac{\pi^2}{2} + \frac{400617}{10000} - \frac{\sqrt{25000000\pi^4 - 4006170000\pi^2 + 3740000000\sqrt{555} + 160493980689}}{10000}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenval
$\frac{\pi^2}{2} + \frac{400617}{10000} + \frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}$	1	real eigenvalue
$\frac{\pi^2}{2} + \frac{400617}{10000} - \frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{\pi^2}{2} + \frac{400617}{10000} - \frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \pi^2 & \frac{187}{5} \\ \sqrt{555} & \frac{400617}{5000} \end{bmatrix} - \left(\frac{\pi^2}{2} + \frac{400617}{10000} - \frac{\sqrt{25000000\pi^4 - 4006170000\pi^2 + 3740000000\sqrt{555} + 160}}{10000}\right.\right.$$

$$\begin{bmatrix} \frac{\pi^2}{2} - \frac{400617}{10000} + \frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000} & \frac{187}{5} \\ \sqrt{555} & \frac{400617}{10000} - \frac{\pi^2}{2} + \frac{\sqrt{25000000\pi^4-4006170000\pi^2+374}}{1000} \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} \frac{\pi^2}{2} - \frac{400617}{10000} + \frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000} & \frac{187}{5} \\ \sqrt{555} & \frac{400617}{10000} - \frac{\pi^2}{2} + \frac{\sqrt{25000000\pi^4-4006170000\pi^2+374}}{10000} \end{bmatrix}$$

$$R_2 = R_2 - \frac{\sqrt{555}\, R_1}{\frac{\pi^2}{2} - \frac{400617}{10000} + \frac{\sqrt{25000000\pi^4 - 4006170000\pi^2 + 3740000000\sqrt{555} + 160493980689}}{10000}} \Longrightarrow \left[\frac{\pi^2}{2} - \frac{400617}{10000} + \frac{\sqrt{25000000\pi^4}}{}\right.$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{\pi^2}{2} - \frac{400617}{10000} + \frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000} & \frac{187}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{374000t}{5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{374000t}{5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617} \\ t \end{bmatrix} = \left[\begin{matrix} -\frac{374}{5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+}} \\ t \end{matrix}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{374000t}{5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617} \\ t \end{bmatrix} = t\left[\begin{matrix} -\frac{37}{5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2}} \\ \end{matrix}\right.$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{374000t}{5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617} \\ t \end{bmatrix} = \left[\begin{matrix} -\frac{374}{5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+}} \\ 1 \end{matrix}\right.$$

Which is normalized to

$$\begin{bmatrix} -\frac{374000t}{5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617} \\ t \end{bmatrix} = \left[\begin{matrix} -\frac{374}{5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+}} \\ 1 \end{matrix}\right.$$

Considering the eigenvalue $\lambda_2 = \frac{\pi^2}{2} + \frac{400617}{10000} + \frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \pi^2 & \frac{187}{5} \\ \sqrt{555} & \frac{400617}{5000} \end{bmatrix} - \left(\frac{\pi^2}{2} + \frac{400617}{10000} + \frac{\sqrt{25000000\pi^4 - 4006170000\pi^2 + 3740000000\sqrt{555} + 16}}{10000}\right.\right.$$

$$\left[\begin{matrix} \frac{\pi^2}{2} - \frac{400617}{10000} - \frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000} & \frac{187}{5} \\ \sqrt{555} & -\frac{\pi^2}{2} - \frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000}}{10000} \end{matrix}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc} \frac{\pi^2}{2}-\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000} & \frac{187}{5} \\ \sqrt{555} & -\frac{\pi^2}{2}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{}}}{10000} \end{array}\right.$$

$$R_2 = R_2 - \frac{\sqrt{555}\, R_1}{\frac{\pi^2}{2}-\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}} \Longrightarrow \left[\begin{array}{c} \frac{\pi^2}{2}-\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4}}{} \\ \end{array}\right.$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc} \frac{\pi^2}{2}-\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000} & \frac{187}{5} \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} v_1 \\ v_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{374000t}{5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617}\right\}$

Hence the solution is

$$\left[\begin{array}{c} -\frac{374000t}{5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{374}{5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+}} \\ t \end{array}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} -\frac{374000t}{5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617} \\ t \end{array}\right] = t\left[\begin{array}{c} -\frac{37}{5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2}} \\ \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} -\frac{374000t}{5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{374}{5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+}} \\ 1 \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} -\frac{374000t}{5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}-400617} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{374}{5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+}} \\ 1 \end{array}\right.$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?
	algebraic m	geometric k	
$\frac{\pi^2}{2}+\frac{400617}{10000}+\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}$	1	1	No
$\frac{\pi^2}{2}+\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}$	1	1	No

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{\pi^2}{2}+\frac{400617}{10000}+\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{\pi^2}{2}+\frac{400617}{10000}+\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}\right)t} \\
&= \left[\begin{array}{c} -\frac{187}{5\left(\frac{\pi^2}{2}-\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}\right)} \\ 1 \end{array}\right] e^{\left(\frac{\pi^2}{2}+\frac{400617}{10000}+\frac{\sqrt{25000000\pi^4-4006170000\pi^2+374}}{10000}\right.}
\end{aligned}
$$

Since eigenvalue $\frac{\pi^2}{2}+\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(\frac{\pi^2}{2}+\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}\right)t} \\
&= \left[\begin{array}{c} -\frac{187}{5\left(\frac{\pi^2}{2}-\frac{400617}{10000}+\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}\right)} \\ 1 \end{array}\right] e^{\left(\frac{\pi^2}{2}+\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+374}}{10000}\right.}
\end{aligned}
$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{187\,\mathrm{e}^{\left(\frac{\pi^2}{2}+\frac{400617}{10000}+\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}\right)t}}{5\left(\frac{\pi^2}{2}-\frac{400617}{10000}-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}\right)} \\ \mathrm{e}^{\left(\frac{\pi^2}{2}+\frac{400617}{10000}+\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{10000}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{187\,\mathrm{e}^{\left(\frac{\pi^2}{2}+\frac{400617}{10000}-\sqrt{25\cdots}\right.}}{5\left(\frac{\pi^2}{2}-\frac{400617}{10000}+\sqrt{250\cdots}\right.} \\ \mathrm{e}^{\left(\frac{\pi^2}{2}+\frac{400617}{10000}-\sqrt{25000\cdots}\right.} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{555}\left(c_2\left(-\frac{400617}{5000}+\pi^2-\frac{\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{5000}\right)\mathrm{e}^{\frac{\left(5000\pi^2-\sqrt{25000000\pi^4-4006170000\pi^2+3\cdots}\right.}{100\cdots}}\right.}{\cdots} \\ c_1\mathrm{e}^{\frac{\left(5000\pi^2+\sqrt{25000000\pi^4-4006170000\pi^2+37400\cdots}\right.}{10000}} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\left(\frac{(c_1-c_2)\sqrt{25000000\pi^4-4006170000\pi^2+3740000000\sqrt{555}+160493980689}}{5000}+(c_2+c_1)\left(\pi^2-\frac{400617}{5000}\right)\right)\sqrt{555}}{1110} \\ c_2 + c_1 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 0 \\ c_2 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The following is the phase plot of the system.

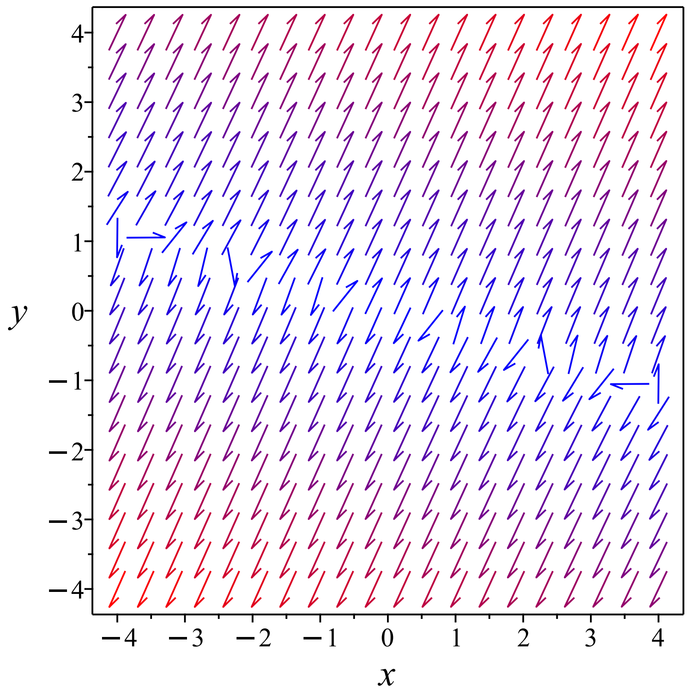
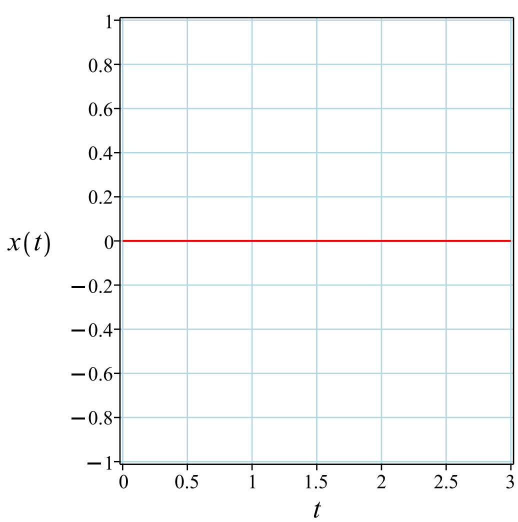
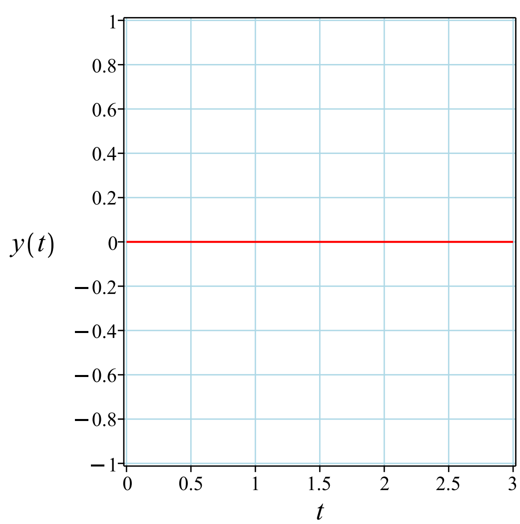


Figure 463: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(x(t),t) = Pi^2*x(t)+187/5*y(t), diff(y(t),t) = 555^(1/2)*x(t)+400617/5000*y(t),
```

$$x(t) = 0$$
$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 10

```
DSolve[{x'[t]==Pi^2*x[t]+374/10*y[t],y'[t]==Sqrt[555]*x[t]+801234/10000*y[t]},{x[0]==0,y[0]=
```

$$x(t) \to 0$$
$$y(t) \to 0$$

15.7 problem 19(i)

Internal problem ID [13149]
Internal file name [`OUTPUT/11804_Sunday_December_03_2023_07_16_47_PM_29840161/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 19(i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + y \\
y' &= -2x(t) - y
\end{aligned}
$$

15.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + \sin(t) & \sin(t) \\ -2\sin(t) & \cos(t) - \sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t)+\sin(t) & \sin(t) \\ -2\sin(t) & \cos(t)-\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(t)+\sin(t))\, c_1 + \sin(t)\, c_2 \\ -2\sin(t)\, c_1 + (\cos(t)-\sin(t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1+c_2)\sin(t) + c_1\cos(t) \\ (-2c_1-c_2)\sin(t) + c_2\cos(t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+i & 1 & 0 \\ -2 & -1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2}+\frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
i	1	1	No	$\left[\begin{array}{c} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{array}\right]$
$-i$	1	1	No	$\left[\begin{array}{c} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{it} \\ \mathrm{e}^{it} \end{array}\right] + c_2 \left[\begin{array}{c} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \left(-\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{it} + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{-it} \\ c_1 \mathrm{e}^{it} + c_2 \mathrm{e}^{-it} \end{array}\right]$$

The following is the phase plot of the system.

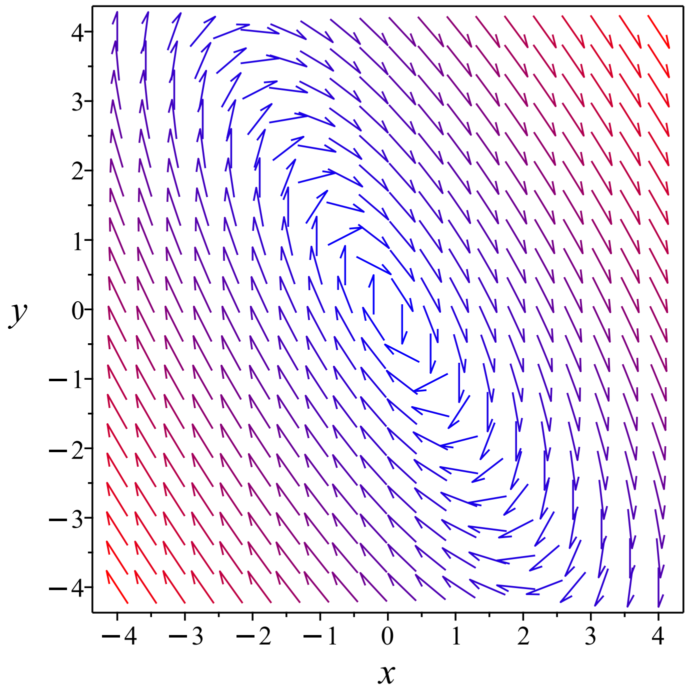


Figure 464: Phase plot

15.7.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + y, y' = -2x(t) - y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 1 & 1 \\ -2 & -1 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{x}_{1}(t) = \left[\begin{array}{c} \frac{\sin(t)}{2}-\frac{\cos(t)}{2} \\ \cos(t) \end{array}\right], \overrightarrow{x}_{2}(t) = \left[\begin{array}{c} \frac{\cos(t)}{2}+\frac{\sin(t)}{2} \\ -\sin(t) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1}\overrightarrow{x}_{1}(t) + c_{2}\overrightarrow{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} c_2\left(\frac{\cos(t)}{2}+\frac{\sin(t)}{2}\right)+c_1\left(\frac{\sin(t)}{2}-\frac{\cos(t)}{2}\right) \\ c_1\cos(t)-c_2\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(-c_1+c_2)\cos(t)}{2}+\frac{(c_1+c_2)\sin(t)}{2} \\ c_1\cos(t)-c_2\sin(t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(-c_1+c_2)\cos(t)}{2}+\frac{(c_1+c_2)\sin(t)}{2}, y = c_1\cos(t)-c_2\sin(t)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve([diff(x(t),t)=1*x(t)+1*y(t),diff(y(t),t)=-2*x(t)-y(t)],singsol=all)
```

$$x(t) = c_1\sin(t)+c_2\cos(t)$$
$$y(t) = c_1\cos(t)-c_2\sin(t)-c_1\sin(t)-c_2\cos(t)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 39

```
DSolve[{x'[t]==1*x[t]+1*y[t],y'[t]==-2*x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to c_1\cos(t)+(c_1+c_2)\sin(t)$$
$$y(t) \to c_2\cos(t)-(2c_1+c_2)\sin(t)$$

15.8 problem 19 (ii)

Internal problem ID [13150]
Internal file name [`OUTPUT/11805_Sunday_December_03_2023_07_16_47_PM_22126226/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 19 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -3x(t) + y$$
$$y' = -x(t) + y$$

15.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(3+2\sqrt{3}\right)e^{-t\left(1+\sqrt{3}\right)}}{6} + \frac{\left(3-2\sqrt{3}\right)e^{t\left(\sqrt{3}-1\right)}}{6} & -\frac{\left(-e^{t\left(\sqrt{3}-1\right)}+e^{-t\left(1+\sqrt{3}\right)}\right)\sqrt{3}}{6} \\ \frac{\left(-e^{t\left(\sqrt{3}-1\right)}+e^{-t\left(1+\sqrt{3}\right)}\right)\sqrt{3}}{6} & \frac{\left(3-2\sqrt{3}\right)e^{-t\left(1+\sqrt{3}\right)}}{6} + \frac{e^{t\left(\sqrt{3}-1\right)}\left(3+2\sqrt{3}\right)}{6} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(3+2\sqrt{3})\,\mathrm{e}^{-t(1+\sqrt{3})}}{6} + \frac{(3-2\sqrt{3})\,\mathrm{e}^{t(\sqrt{3}-1)}}{6} & -\frac{\left(-\mathrm{e}^{t(\sqrt{3}-1)}+\mathrm{e}^{-t(1+\sqrt{3})}\right)\sqrt{3}}{6} \\ \frac{\left(-\mathrm{e}^{t(\sqrt{3}-1)}+\mathrm{e}^{-t(1+\sqrt{3})}\right)\sqrt{3}}{6} & \frac{(3-2\sqrt{3})\,\mathrm{e}^{-t(1+\sqrt{3})}}{6} + \frac{\mathrm{e}^{t(\sqrt{3}-1)}\,(3+2\sqrt{3})}{6} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{(3+2\sqrt{3})\,\mathrm{e}^{-t(1+\sqrt{3})}}{6} + \frac{(3-2\sqrt{3})\,\mathrm{e}^{t(\sqrt{3}-1)}}{6}\right)c_1 - \frac{\left(-\mathrm{e}^{t(\sqrt{3}-1)}+\mathrm{e}^{-t(1+\sqrt{3})}\right)\sqrt{3}\,c_2}{6} \\ \frac{\left(-\mathrm{e}^{t(\sqrt{3}-1)}+\mathrm{e}^{-t(1+\sqrt{3})}\right)\sqrt{3}\,c_1}{6} + \left(\frac{(3-2\sqrt{3})\,\mathrm{e}^{-t(1+\sqrt{3})}}{6} + \frac{\mathrm{e}^{t(\sqrt{3}-1)}\,(3+2\sqrt{3})}{6}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left((2c_1-c_2)\sqrt{3}+3c_1\right)\mathrm{e}^{-t(1+\sqrt{3})}}{6} - \frac{\mathrm{e}^{t(\sqrt{3}-1)}\left(\left(c_1-\frac{c_2}{2}\right)\sqrt{3}-\frac{3c_1}{2}\right)}{3} \\ \frac{\left((c_1-2c_2)\sqrt{3}+3c_2\right)\mathrm{e}^{-t(1+\sqrt{3})}}{6} - \frac{\mathrm{e}^{t(\sqrt{3}-1)}\left((c_1-2c_2)\sqrt{3}-3c_2\right)}{6} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{3} - 1$$
$$\lambda_2 = -1 - \sqrt{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\sqrt{3}-1$	1	real eigenvalue
$-1-\sqrt{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - \sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix} - \left(-1-\sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \sqrt{3}-2 & 1 \\ -1 & 2+\sqrt{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \sqrt{3}-2 & 1 & 0 \\ -1 & 2+\sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\sqrt{3}-2} \Longrightarrow \left[\begin{array}{cc|c} \sqrt{3}-2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \sqrt{3}-2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{\sqrt{3}-2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{\sqrt{3}-2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{\sqrt{3}-2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{\sqrt{3}-2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{\sqrt{3}-2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{\sqrt{3}-2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}-2} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \sqrt{3} - 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix} - \left(\sqrt{3}-1\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2-\sqrt{3} & 1 & 0 \\ -1 & 2-\sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{-2-\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} -2-\sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-\sqrt{3} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2+\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2+\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2+\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2+\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2+\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2+\sqrt{3}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\sqrt{3}-1$	1	1	No	$\left[\begin{array}{c} \frac{1}{2+\sqrt{3}} \\ 1 \end{array}\right]$
$-1-\sqrt{3}$	1	1	No	$\left[\begin{array}{c} \frac{1}{2-\sqrt{3}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{3}-1$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{t\left(\sqrt{3}-1\right)} \\
&= \left[\begin{array}{c} \frac{1}{2+\sqrt{3}} \\ 1 \end{array}\right] e^{t\left(\sqrt{3}-1\right)}
\end{aligned}
$$

Since eigenvalue $-1-\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-1-\sqrt{3}\right)t} \\
&= \left[\begin{array}{c} \frac{1}{2-\sqrt{3}} \\ 1 \end{array}\right] e^{\left(-1-\sqrt{3}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \frac{\mathrm{e}^{t\left(\sqrt{3}-1\right)}}{2+\sqrt{3}} \\ \mathrm{e}^{t\left(\sqrt{3}-1\right)} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{\mathrm{e}^{\left(-1-\sqrt{3}\right)t}}{2-\sqrt{3}} \\ \mathrm{e}^{\left(-1-\sqrt{3}\right)t} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} c_2\left(2+\sqrt{3}\right)\mathrm{e}^{-t\left(1+\sqrt{3}\right)} - c_1\mathrm{e}^{t\left(\sqrt{3}-1\right)}\left(\sqrt{3}-2\right) \\ c_1\mathrm{e}^{t\left(\sqrt{3}-1\right)} + c_2\mathrm{e}^{-t\left(1+\sqrt{3}\right)} \end{array}\right]
$$

The following is the phase plot of the system.

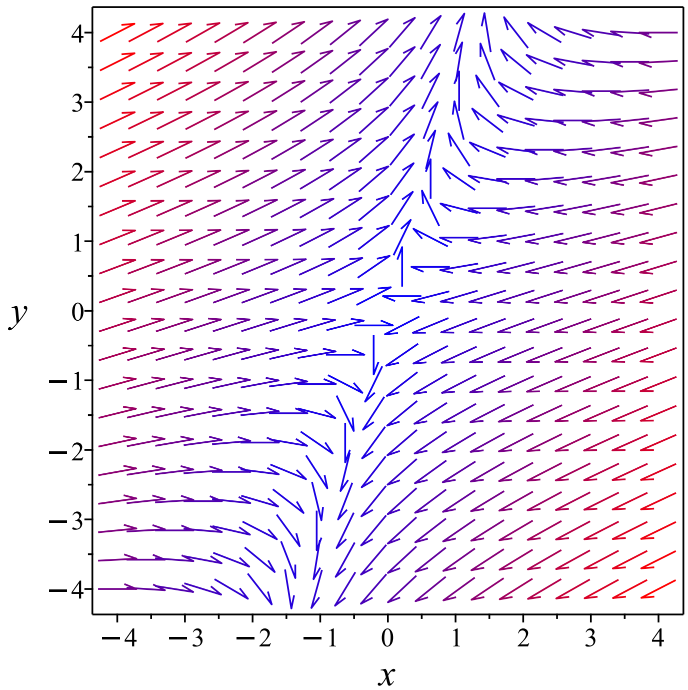


Figure 465: Phase plot

15.8.3 Maple step by step solution

Let's solve

$[x'(t) = -3x(t) + y, y' = -x(t) + y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -3 & 1 \\ -1 & 1 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}^{\prime}(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-\sqrt{3}, \left[\begin{array}{c} \frac{1}{2-\sqrt{3}} \\ 1 \end{array}\right]\right], \left[\sqrt{3}-1, \left[\begin{array}{c} \frac{1}{2+\sqrt{3}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1-\sqrt{3}, \left[\begin{array}{c} \frac{1}{2-\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{1} = \mathrm{e}^{\left(-1-\sqrt{3}\right)t} \cdot \left[\begin{array}{c} \frac{1}{2-\sqrt{3}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[\sqrt{3}-1, \left[\begin{array}{c} \frac{1}{2+\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_{2} = \mathrm{e}^{t\left(\sqrt{3}-1\right)} \cdot \left[\begin{array}{c} \frac{1}{2+\sqrt{3}} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_{1}\overrightarrow{x}_{1} + c_{2}\overrightarrow{x}_{2}$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_{1}\mathrm{e}^{\left(-1-\sqrt{3}\right)t} \cdot \left[\begin{array}{c} \frac{1}{2-\sqrt{3}} \\ 1 \end{array}\right] + c_{2}\mathrm{e}^{t\left(\sqrt{3}-1\right)} \cdot \left[\begin{array}{c} \frac{1}{2+\sqrt{3}} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1(2+\sqrt{3})\,\mathrm{e}^{-t(1+\sqrt{3})} - c_2\mathrm{e}^{t(\sqrt{3}-1)}(\sqrt{3}-2) \\ c_1\mathrm{e}^{-t(1+\sqrt{3})} + c_2\mathrm{e}^{t(\sqrt{3}-1)} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = c_1(2+\sqrt{3})\,\mathrm{e}^{-t(1+\sqrt{3})} - c_2\mathrm{e}^{t(\sqrt{3}-1)}(\sqrt{3}-2), y = c_1\mathrm{e}^{-t(1+\sqrt{3})} + c_2\mathrm{e}^{t(\sqrt{3}-1)}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 82

```
dsolve([diff(x(t),t)=-3*x(t)+1*y(t),diff(y(t),t)=-1*x(t)+1*y(t)],singsol=all)
```

$$x(t) = c_1\mathrm{e}^{(\sqrt{3}-1)t} + c_2\mathrm{e}^{-(1+\sqrt{3})t}$$
$$y(t) = c_1\mathrm{e}^{(\sqrt{3}-1)t}\sqrt{3} - c_2\mathrm{e}^{-(1+\sqrt{3})t}\sqrt{3} + 2c_1\mathrm{e}^{(\sqrt{3}-1)t} + 2c_2\mathrm{e}^{-(1+\sqrt{3})t}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 147

```
DSolve[{x'[t]==-3*x[t]+1*y[t],y'[t]==-1*x[t]+1*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{6}e^{-((1+\sqrt{3})t)}\left(c_1\left(\left(3-2\sqrt{3}\right)e^{2\sqrt{3}t} + 3 + 2\sqrt{3}\right) + \sqrt{3}c_2\left(e^{2\sqrt{3}t} - 1\right)\right)$$
$$y(t) \to \frac{1}{6}e^{-((1+\sqrt{3})t)}\left(c_2\left(\left(3+2\sqrt{3}\right)e^{2\sqrt{3}t} + 3 - 2\sqrt{3}\right) - \sqrt{3}c_1\left(e^{2\sqrt{3}t} - 1\right)\right)$$

15.9 problem 19 (iii)

Internal problem ID [13151]
Internal file name [`OUTPUT/11806_Sunday_December_03_2023_07_16_48_PM_38837053/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 19 (iii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -3x(t) + y$$
$$y' = -x(t)$$

15.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(3\sqrt{5}+5\right)e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} + \frac{\left(-3\sqrt{5}+5\right)e^{\frac{\left(\sqrt{5}-3\right)t}{2}}}{10} & -\frac{\left(-e^{\frac{\left(\sqrt{5}-3\right)t}{2}}+e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}\right)\sqrt{5}}{5} \\ \frac{\left(-e^{\frac{\left(\sqrt{5}-3\right)t}{2}}+e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}\right)\sqrt{5}}{5} & \frac{\left(-3\sqrt{5}+5\right)e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} + \frac{e^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(3\sqrt{5}+5\right)}{10} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{(3\sqrt{5}+5)\,\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{(-3\sqrt{5}+5)\,\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}}{10} & -\frac{\left(-\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}+\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}\right)\sqrt{5}}{5} \\ \frac{\left(-\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}+\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}\right)\sqrt{5}}{5} & \frac{(-3\sqrt{5}+5)\,\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}(3\sqrt{5}+5)}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{(3\sqrt{5}+5)\,\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{(-3\sqrt{5}+5)\,\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}}{10}\right)c_1 - \frac{\left(-\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}+\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}\right)\sqrt{5}\,c_2}{5} \\ \frac{\left(-\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}+\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}\right)\sqrt{5}\,c_1}{5} + \left(\frac{(-3\sqrt{5}+5)\,\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}(3\sqrt{5}+5)}{10}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((3c_1-2c_2)\sqrt{5}+5c_1\right)\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{3\,\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}\left(\left(c_1-\frac{2c_2}{3}\right)\sqrt{5}-\frac{5c_1}{3}\right)}{10} \\ \frac{\left((2c_1-3c_2)\sqrt{5}+5c_2\right)\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}\left(\left(c_1-\frac{3c_2}{2}\right)\sqrt{5}-\frac{5c_2}{2}\right)}{5} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-3 & 1\\ -1 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-3-\lambda & 1\\ -1 & -\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{\sqrt{5}}{2} - \frac{3}{2}$$
$$\lambda_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{3}{2} - \frac{\sqrt{5}}{2}$	1	real eigenvalue
$\frac{\sqrt{5}}{2} - \frac{3}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-3 & 1\\ -1 & 0\end{bmatrix} - \left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{\sqrt{5}}{2} - \frac{3}{2} & 1\\ -1 & \frac{3}{2} + \frac{\sqrt{5}}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{\sqrt{5}}{2} - \frac{3}{2} & 1 & 0\\ -1 & \frac{3}{2} + \frac{\sqrt{5}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\frac{\sqrt{5}}{2} - \frac{3}{2}} \implies \left[\begin{array}{cc|c} \frac{\sqrt{5}}{2} - \frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{\sqrt{5}}{2} - \frac{3}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{\sqrt{5}-3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{\sqrt{5}-3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{\sqrt{5}-3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}-3} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{\sqrt{5}}{2} - \frac{3}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix} - \left(\frac{\sqrt{5}}{2} - \frac{3}{2} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{2} - \frac{\sqrt{5}}{2} & 1 \\ -1 & \frac{3}{2} - \frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3}{2} - \frac{\sqrt{5}}{2} & 1 & 0 \\ -1 & \frac{3}{2} - \frac{\sqrt{5}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{-\frac{3}{2} - \frac{\sqrt{5}}{2}} \implies \left[\begin{array}{cc|c} -\frac{3}{2} - \frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3}{2} - \frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3+\sqrt{5}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$\frac{\sqrt{5}}{2}-\frac{3}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{\frac{3}{2}+\frac{\sqrt{5}}{2}} \\ 1\end{array}\right]$
$-\frac{3}{2}-\frac{\sqrt{5}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{\frac{3}{2}-\frac{\sqrt{5}}{2}} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{\sqrt{5}}{2}-\frac{3}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} \\
&= \left[\begin{array}{c}\frac{1}{\frac{3}{2}+\frac{\sqrt{5}}{2}} \\ 1\end{array}\right] e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $-\frac{3}{2}-\frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \\
&= \left[\begin{array}{c}\frac{1}{\frac{3}{2}-\frac{\sqrt{5}}{2}} \\ 1\end{array}\right] e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c}x(t) \\ y\end{array}\right] = c_1 \left[\begin{array}{c}\frac{e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t}}{\frac{3}{2}+\frac{\sqrt{5}}{2}} \\ e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t}\end{array}\right] + c_2 \left[\begin{array}{c}\frac{e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}}{\frac{3}{2}-\frac{\sqrt{5}}{2}} \\ e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}\end{array}\right]
$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_2\left(3+\sqrt{5}\right)e^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{c_1 e^{\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-3\right)}{2} \\ c_1 e^{\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 e^{-\frac{\left(3+\sqrt{5}\right)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

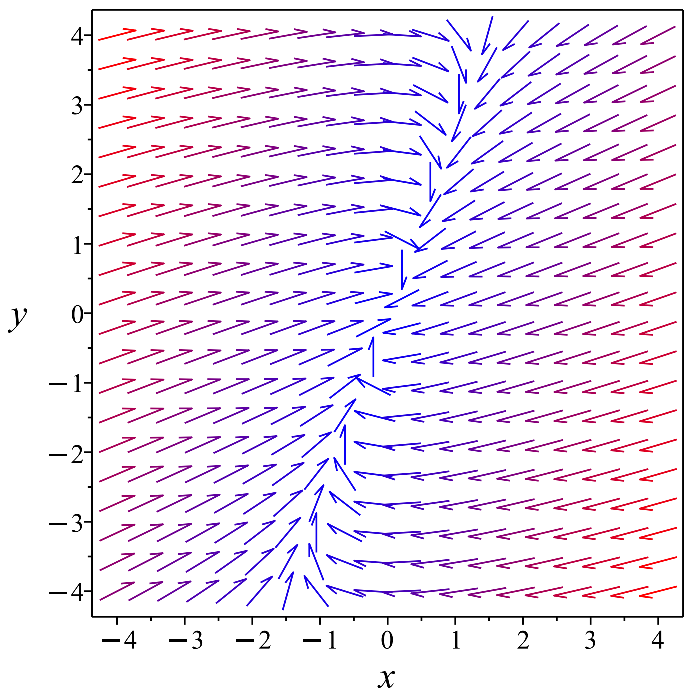


Figure 466: Phase plot

15.9.3 Maple step by step solution

Let's solve

$[x'(t) = -3x(t) + y, y' = -x(t)]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{\frac{3}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{\sqrt{5}}{2} - \frac{3}{2}, \begin{bmatrix} \frac{1}{\frac{3}{2} + \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{\frac{3}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{\left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{\frac{3}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{\sqrt{5}}{2} - \frac{3}{2}, \begin{bmatrix} \frac{1}{\frac{3}{2} + \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{\left(\frac{\sqrt{5}}{2} - \frac{3}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{\frac{3}{2} + \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\frac{3}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\frac{3}{2}+\frac{\sqrt{5}}{2}} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{c_1\left(3+\sqrt{5}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}} c_2\left(\sqrt{5}-3\right)}{2} \\ c_1 \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{c_1\left(3+\sqrt{5}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} - \frac{\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}} c_2\left(\sqrt{5}-3\right)}{2}, y = c_1 \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 68

```
dsolve([diff(x(t),t)=-3*x(t)+1*y(t),diff(y(t),t)=-1*x(t)+0*y(t)],singsol=all)
```

$$x(t) = \left(-\frac{\sqrt{5}}{2} + \frac{3}{2}\right) c_1 \mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}} + \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right) c_2 \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}$$
$$y(t) = c_1 \mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 148

```
DSolve[{x'[t]==-3*x[t]+1*y[t],y'[t]==-1*x[t]+0*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{10} e^{-\frac{1}{2}\left(3+\sqrt{5}\right)t}\left(c_1\left(\left(5-3\sqrt{5}\right)e^{\sqrt{5}t} + 5 + 3\sqrt{5}\right) + 2\sqrt{5}c_2\left(e^{\sqrt{5}t} - 1\right)\right)$$
$$y(t) \to \frac{1}{10} e^{-\frac{1}{2}\left(3+\sqrt{5}\right)t}\left(c_2\left(\left(5+3\sqrt{5}\right)e^{\sqrt{5}t} + 5 - 3\sqrt{5}\right) - 2\sqrt{5}c_1\left(e^{\sqrt{5}t} - 1\right)\right)$$

15.10 problem 19 (iv)

Internal problem ID [13152]
Internal file name [`OUTPUT/11807_Sunday_December_03_2023_07_16_48_PM_99156758/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 19 (iv).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -x(t) + y \\ y' &= -2x(t) + y \end{aligned}$$

15.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) - \sin(t) & \sin(t) \\ -2\sin(t) & \cos(t) + \sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t)-\sin(t) & \sin(t) \\ -2\sin(t) & \cos(t)+\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(t)-\sin(t))\,c_1+\sin(t)\,c_2 \\ -2\sin(t)\,c_1+(\cos(t)+\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-c_1+c_2)\sin(t)+c_1\cos(t) \\ (-2c_1+c_2)\sin(t)+c_2\cos(t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A-\lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -1-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} - (-i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+i & 1 \\ -2 & 1+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+i & 1 & 0 \\ -2 & 1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1-i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - i & 1 \\ -2 & 1 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 - i & 1 & 0 \\ -2 & 1 - i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 + i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 - i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
i	1	1	No	$\left[\begin{array}{c} \frac{1}{2}-\frac{i}{2} \\ 1 \end{array}\right]$
$-i$	1	1	No	$\left[\begin{array}{c} \frac{1}{2}+\frac{i}{2} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} \left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{it} \\ \mathrm{e}^{it} \end{array}\right] + c_2 \left[\begin{array}{c} \left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \left(\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{it} + \left(\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{-it} \\ c_1 \mathrm{e}^{it} + c_2 \mathrm{e}^{-it} \end{array}\right]$$

The following is the phase plot of the system.

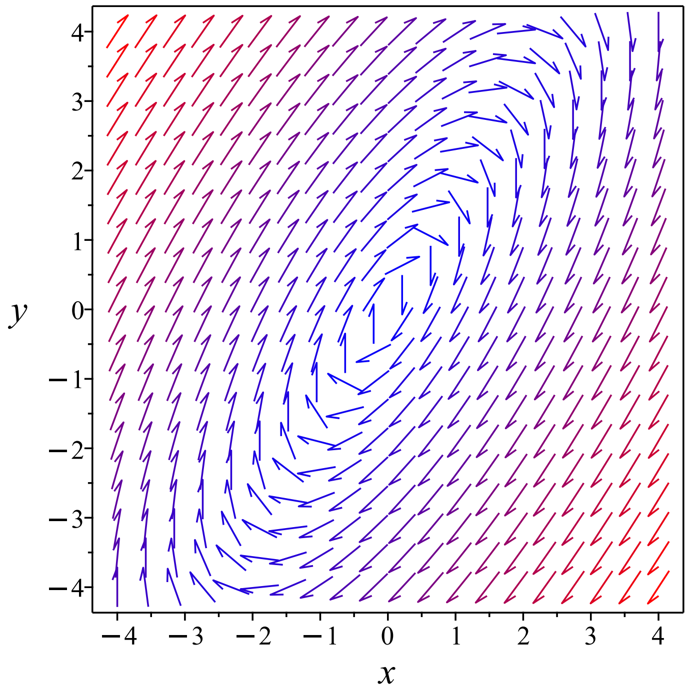


Figure 467: Phase plot

15.10.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + y, y' = -2x(t) + y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[-\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

  $$\begin{bmatrix} \left(\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_1(t) = \begin{bmatrix} \frac{\cos(t)}{2}+\frac{\sin(t)}{2} \\ \cos(t) \end{bmatrix}, \overrightarrow{x}_2(t) = \begin{bmatrix} \frac{\cos(t)}{2}-\frac{\sin(t)}{2} \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $$\overrightarrow{x} = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} c_2\left(\frac{\cos(t)}{2} - \frac{\sin(t)}{2}\right) + c_1\left(\frac{\cos(t)}{2} + \frac{\sin(t)}{2}\right) \\ c_1 \cos(t) - c_2 \sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{\cos(t)(c_1+c_2)}{2} + \frac{(c_1-c_2)\sin(t)}{2} \\ c_1 \cos(t) - c_2 \sin(t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{\cos(t)(c_1+c_2)}{2} + \frac{(c_1-c_2)\sin(t)}{2}, y = c_1 \cos(t) - c_2 \sin(t)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=-1*x(t)+1*y(t),diff(y(t),t)=-2*x(t)+1*y(t)],singsol=all)
```

$$x(t) = c_1 \sin(t) + c_2 \cos(t)$$
$$y(t) = c_1 \sin(t) - c_2 \sin(t) + c_1 \cos(t) + c_2 \cos(t)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 39

```
DSolve[{x'[t]==-1*x[t]+1*y[t],y'[t]==-2*x[t]+1*y[t]},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to c_1 \cos(t) + (c_2 - c_1)\sin(t)$$
$$y(t) \to c_2(\sin(t) + \cos(t)) - 2c_1 \sin(t)$$

15.11 problem 19 (v)

Internal problem ID [13153]
Internal file name [`OUTPUT/11808_Sunday_December_03_2023_07_16_49_PM_39563143/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 19 (v).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) \\
y' &= x(t) - y
\end{aligned}
$$

15.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{2t} & 0 \\ \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{-t}}{3} & \mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & 0 \\ \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 \\ \left(\frac{e^{2t}}{3} - \frac{e^{-t}}{3}\right)c_1 + e^{-t}c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 \\ \frac{(-c_1+3c_2)e^{-t}}{3} + \frac{e^{2t}c_1}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(2-\lambda)(-1-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -3 & 0 \end{array}\right]$$

Since the current pivot $A(1, 1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 3\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{-t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 3c_1 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-t} \end{bmatrix}
$$

The following is the phase plot of the system.

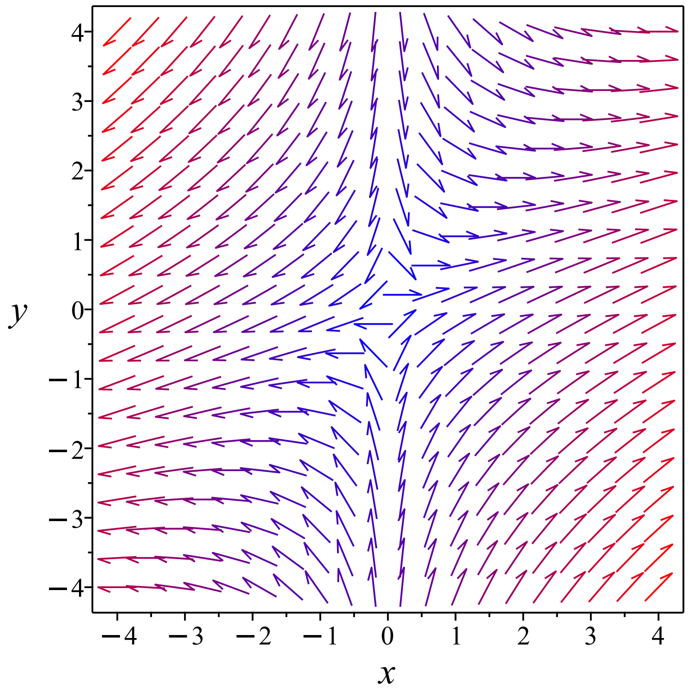


Figure 468: Phase plot

15.11.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t), y' = x(t) - y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{2t} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 3c_2 e^{2t} \\ c_1 e^{-t} + c_2 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = 3c_2 e^{2t}, y = c_1 e^{-t} + c_2 e^{2t}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve([diff(x(t),t)=2*x(t)+0*y(t),diff(y(t),t)=1*x(t)-1*y(t)],singsol=all)
```

$$x(t) = c_2 e^{2t}$$
$$y(t) = \frac{c_2 e^{2t}}{3} + e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 40

```
DSolve[{x'[t]==2*x[t]+0*y[t],y'[t]==1*x[t]-1*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to c_1 e^{2t}$$
$$y(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} - 1\right) + 3c_2\right)$$

15.12 problem 19 (vi)

Internal problem ID [13154]
Internal file name [`OUTPUT/11809_Sunday_December_03_2023_07_16_49_PM_75390535/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 19 (vi).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3x(t) + y \\
y' &= -x(t)
\end{aligned}
$$

15.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{\left(3\sqrt{5}+5\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} + \frac{\left(-3\sqrt{5}+5\right)e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}}{10} & -\frac{\left(-e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}}{5} \\ \frac{\left(-e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\right)\sqrt{5}}{5} & \frac{\left(-3\sqrt{5}+5\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{10} + \frac{e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(3\sqrt{5}+5\right)}{10} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{(3\sqrt{5}+5)\,\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{(-3\sqrt{5}+5)\,\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}}{10} & -\frac{\left(-\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}+\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}}{5} \\ \frac{\left(-\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}+\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}}{5} & \frac{(-3\sqrt{5}+5)\,\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}(3\sqrt{5}+5)}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{(3\sqrt{5}+5)\,\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{(-3\sqrt{5}+5)\,\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}}{10}\right)c_1 - \frac{\left(-\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}+\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}\,c_2}{5} \\ \frac{\left(-\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}+\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}\right)\sqrt{5}\,c_1}{5} + \left(\frac{(-3\sqrt{5}+5)\,\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}(3\sqrt{5}+5)}{10}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{((3c_1+2c_2)\sqrt{5}+5c_1)\,\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}}{10} - \frac{3\,\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}\left(\left(c_1+\frac{2c_2}{3}\right)\sqrt{5}-\frac{5c_1}{3}\right)}{10} \\ \frac{((-2c_1-3c_2)\sqrt{5}+5c_2)\,\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}}{10} + \frac{\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}\left(\left(c_1+\frac{3c_2}{2}\right)\sqrt{5}+\frac{5c_2}{2}\right)}{5} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}3 & 1\\ -1 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}3-\lambda & 1\\ -1 & -\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{3}{2} - \frac{\sqrt{5}}{2}$	1	real eigenvalue
$\frac{3}{2} + \frac{\sqrt{5}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{3}{2} - \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}3 & 1\\ -1 & 0\end{bmatrix} - \left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{3}{2} + \frac{\sqrt{5}}{2} & 1\\ -1 & \frac{\sqrt{5}}{2} - \frac{3}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{3}{2} + \frac{\sqrt{5}}{2} & 1 & 0\\ -1 & \frac{\sqrt{5}}{2} - \frac{3}{2} & 0\end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\frac{3}{2}+\frac{\sqrt{5}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{2}+\frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2}+\frac{\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3+\sqrt{5}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{3}{2} + \frac{\sqrt{5}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} - \left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2}-\frac{\sqrt{5}}{2} & 1 \\ -1 & -\frac{3}{2}-\frac{\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{3}{2}-\frac{\sqrt{5}}{2} & 1 & 0 \\ -1 & -\frac{3}{2}-\frac{\sqrt{5}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{\frac{3}{2}-\frac{\sqrt{5}}{2}} \Longrightarrow \left[\begin{array}{cc|c}\frac{3}{2}-\frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix}\frac{3}{2}-\frac{\sqrt{5}}{2} & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1 \\ v_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{\sqrt{5}-3}\right\}$

Hence the solution is

$$\begin{bmatrix}\frac{2t}{\sqrt{5}-3} \\ t\end{bmatrix} = \begin{bmatrix}\frac{2t}{\sqrt{5}-3} \\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}\frac{2t}{\sqrt{5}-3} \\ t\end{bmatrix} = t\begin{bmatrix}\frac{2}{\sqrt{5}-3} \\ 1\end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix}\frac{2t}{\sqrt{5}-3} \\ t\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{5}-3} \\ 1\end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\frac{3}{2}+\frac{\sqrt{5}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{\frac{\sqrt{5}}{2}-\frac{3}{2}}\\ 1\end{array}\right]$
$\frac{3}{2}-\frac{\sqrt{5}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{-\frac{3}{2}-\frac{\sqrt{5}}{2}}\\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{3}{2}+\frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t}\\
&= \left[\begin{array}{c}\frac{1}{\frac{\sqrt{5}}{2}-\frac{3}{2}}\\ 1\end{array}\right] e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Since eigenvalue $\frac{3}{2}-\frac{\sqrt{5}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}\\
&= \left[\begin{array}{c}\frac{1}{-\frac{3}{2}-\frac{\sqrt{5}}{2}}\\ 1\end{array}\right] e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c}x(t)\\ y\end{array}\right] = c_1 \left[\begin{array}{c}\frac{\mathrm{e}^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t}}{\frac{\sqrt{5}}{2}-\frac{3}{2}}\\ \mathrm{e}^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t}\end{array}\right] + c_2 \left[\begin{array}{c}\frac{\mathrm{e}^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{5}}{2}}\\ \mathrm{e}^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}\end{array}\right]
$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{c_1\left(3+\sqrt{5}\right)e^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} + \frac{e^{-\frac{\left(\sqrt{5}-3\right)t}{2}}c_2\left(\sqrt{5}-3\right)}{2} \\ c_1 e^{\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 e^{-\frac{\left(\sqrt{5}-3\right)t}{2}} \end{bmatrix}$$

The following is the phase plot of the system.

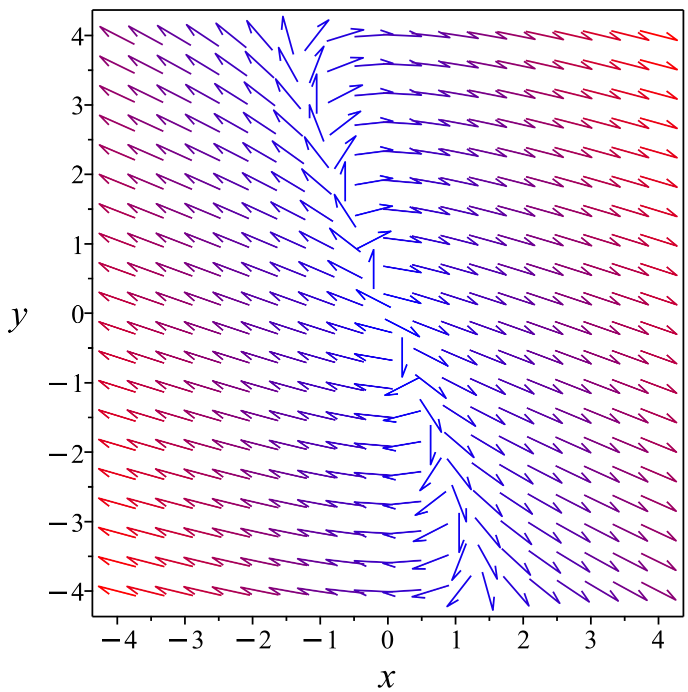


Figure 469: Phase plot

15.12.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) + y, y' = -x(t)]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{-\frac{3}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{3}{2} + \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{3}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{3}{2} - \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{-\frac{3}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{-\frac{3}{2} - \frac{\sqrt{5}}{2}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{3}{2} + \frac{\sqrt{5}}{2}, \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{3}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{\frac{\sqrt{5}}{2} - \frac{3}{2}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{-\frac{3}{2}-\frac{\sqrt{5}}{2}} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\frac{\sqrt{5}}{2}-\frac{3}{2}} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -\frac{c_2\left(3+\sqrt{5}\right)\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} + \frac{c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-3\right)}{2} \\ c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{c_2\left(3+\sqrt{5}\right)\mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}}{2} + \frac{c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}\left(\sqrt{5}-3\right)}{2}, y = c_1 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 68

```
dsolve([diff(x(t),t)=3*x(t)+1*y(t),diff(y(t),t)=-1*x(t)+0*y(t)],singsol=all)
```

$$x(t) = \left(\frac{\sqrt{5}}{2} - \frac{3}{2}\right) c_2 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}} + \left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right) c_1 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}}$$
$$y(t) = c_1 \mathrm{e}^{\frac{\left(3+\sqrt{5}\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(\sqrt{5}-3\right)t}{2}}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 148

```
DSolve[{x'[t]==3*x[t]+1*y[t],y'[t]==-1*x[t]+0*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t) \to \frac{1}{10} e^{-\frac{1}{2}\left(\sqrt{5}-3\right)t}\left(c_1\left(\left(5+3\sqrt{5}\right)e^{\sqrt{5}t} + 5 - 3\sqrt{5}\right) + 2\sqrt{5}c_2\left(e^{\sqrt{5}t} - 1\right)\right)$$
$$y(t) \to -\frac{1}{10} e^{-\frac{1}{2}\left(\sqrt{5}-3\right)t}\left(2\sqrt{5}c_1\left(e^{\sqrt{5}t} - 1\right) + c_2\left(\left(3\sqrt{5}-5\right)e^{\sqrt{5}t} - 5 - 3\sqrt{5}\right)\right)$$

15.13 problem 19 (vii)

Internal problem ID [13155]
Internal file name [`OUTPUT/11810_Sunday_December_03_2023_07_16_50_PM_48564082/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 19 (vii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= y \\
y' &= -4x(t) - 4y
\end{aligned}
$$

15.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} (2t+1)\,\mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} \\ -4t\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}(1-2t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} (2t+1)\,\mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t} \\ -4t\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (2t+1)\,\mathrm{e}^{-2t}c_1 + t\,\mathrm{e}^{-2t}c_2 \\ -4t\,\mathrm{e}^{-2t}c_1 + \mathrm{e}^{-2t}(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-2t}(2tc_1 + c_2 t + c_1) \\ (c_2(1-2t) - 4tc_1)\,\mathrm{e}^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	2	1	Yes	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

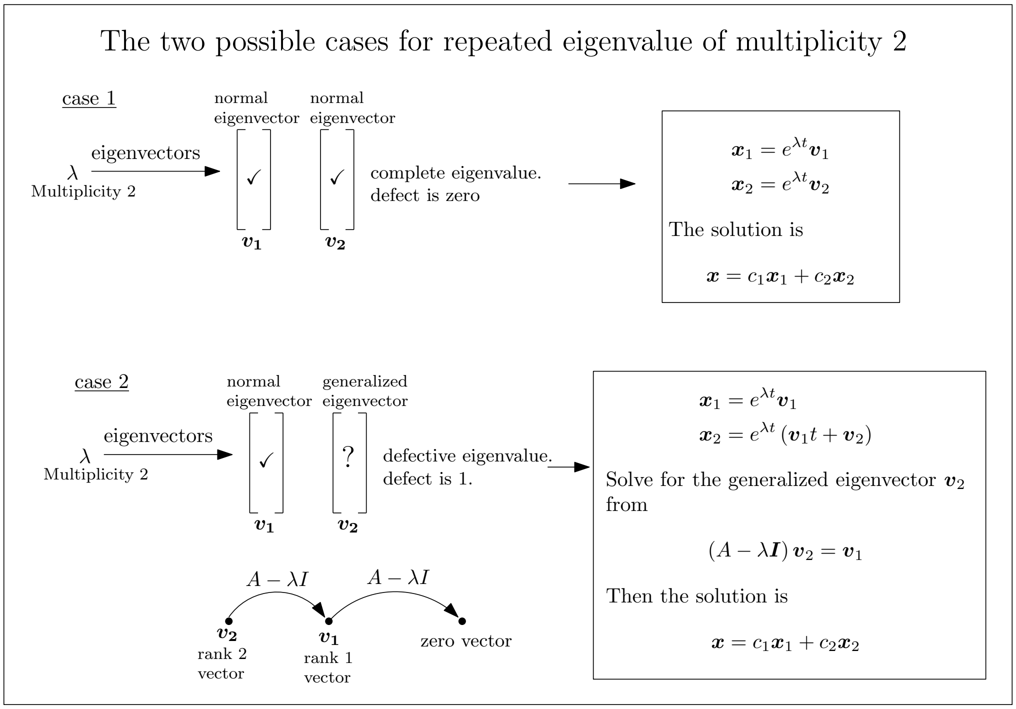


Figure 470: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{-2t}\\ &= \begin{bmatrix} -\frac{\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-2t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}\right) \mathrm{e}^{-2t}\\ &= \begin{bmatrix} -\frac{\mathrm{e}^{-2t}(-2+t)}{2} \\ \frac{\mathrm{e}^{-2t}(2t-5)}{2} \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-2t}\left(-\frac{t}{2} + 1\right) \\ \mathrm{e}^{-2t}\left(t - \frac{5}{2}\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{e}^{-2t}((-2+t)c_2 + c_1)}{2} \\ \mathrm{e}^{-2t}\left(c_1 + c_2 t - \frac{5}{2} c_2\right) \end{bmatrix}$$

The following is the phase plot of the system.

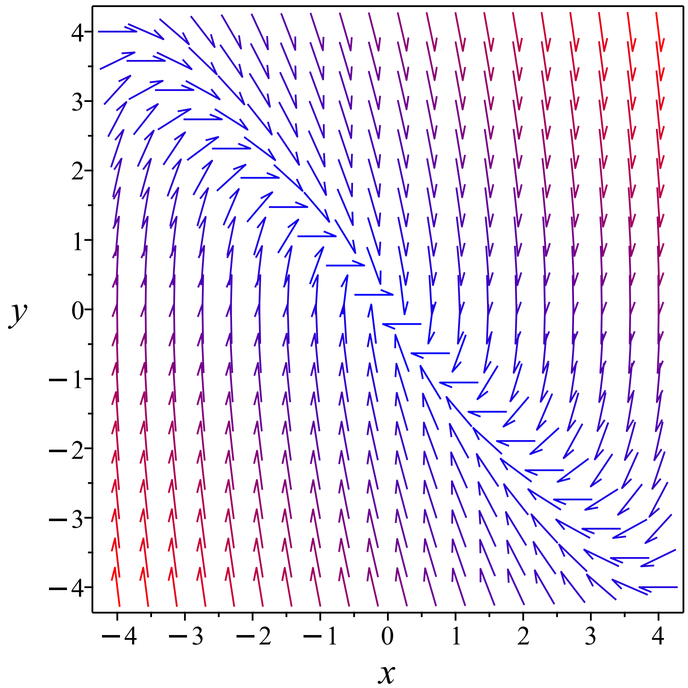


Figure 471: Phase plot

15.13.3 Maple step by step solution

Let's solve

$[x'(t) = y, y' = -4x(t) - 4y]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- System to solve

$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[-2, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $\left[-2, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$
- First solution from eigenvalue -2

  $\overrightarrow{x}_1(t) = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -2$ is the eigenvalue, an

  $\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$
- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$
- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$
- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -2

$$\left(\left[\begin{array}{cc}0 & 1 \\ -4 & -4\end{array}\right]-(-2)\cdot\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}-\frac{1}{2} \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}-\frac{1}{4} \\ 0\end{array}\right]$$

- Second solution from eigenvalue -2

$$\overrightarrow{x}_2(t)=\mathrm{e}^{-2t}\cdot\left(t\cdot\left[\begin{array}{c}-\frac{1}{2} \\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{1}{4} \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x}=c_1\overrightarrow{x}_1(t)+c_2\overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x}=c_1\mathrm{e}^{-2t}\cdot\left[\begin{array}{c}-\frac{1}{2} \\ 1\end{array}\right]+c_2\mathrm{e}^{-2t}\cdot\left(t\cdot\left[\begin{array}{c}-\frac{1}{2} \\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{1}{4} \\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right]=\left[\begin{array}{c}-\frac{\mathrm{e}^{-2t}(2c_2t+2c_1+c_2)}{4} \\ \mathrm{e}^{-2t}(c_2t+c_1)\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t)=-\frac{\mathrm{e}^{-2t}(2c_2t+2c_1+c_2)}{4}, y=\mathrm{e}^{-2t}(c_2t+c_1)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=0*x(t)+1*y(t),diff(y(t),t)=-4*x(t)-4*y(t)],singsol=all)
```

$$x(t)=(c_2t+c_1)\,\mathrm{e}^{-2t}$$
$$y(t)=-\mathrm{e}^{-2t}(2c_2t+2c_1-c_2)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 45

```
DSolve[{x'[t]==0*x[t]+1*y[t],y'[t]==-4*x[t]-4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t) \to e^{-2t}(2c_1t + c_2t + c_1)$$
$$y(t) \to e^{-2t}(c_2 - 2(2c_1 + c_2)t)$$

15.14 problem 19 (viii)

Internal problem ID [13156]
Internal file name [`OUTPUT/11811_Sunday_December_03_2023_07_16_50_PM_5340378/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 19 (viii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -3x(t) - 3y$$
$$y' = 2x(t) + y$$

15.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} \cos(\sqrt{2}\,t)\,\mathrm{e}^{-t} - \sin(\sqrt{2}\,t)\,\sqrt{2}\,\mathrm{e}^{-t} & -\frac{3\sin(\sqrt{2}\,t)\sqrt{2}\,\mathrm{e}^{-t}}{2} \\ \sin(\sqrt{2}\,t)\,\sqrt{2}\,\mathrm{e}^{-t} & \cos(\sqrt{2}\,t)\,\mathrm{e}^{-t} + \sin(\sqrt{2}\,t)\,\sqrt{2}\,\mathrm{e}^{-t} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{-t}(-\sqrt{2}\,\sin(\sqrt{2}\,t) + \cos(\sqrt{2}\,t)) & -\frac{3\sin(\sqrt{2}\,t)\sqrt{2}\,\mathrm{e}^{-t}}{2} \\ \sin(\sqrt{2}\,t)\,\sqrt{2}\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(\cos(\sqrt{2}\,t) + \sqrt{2}\,\sin(\sqrt{2}\,t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-t}\left(-\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)+\cos\left(\sqrt{2}\,t\right)\right) & -\frac{3\sin\left(\sqrt{2}\,t\right)\sqrt{2}\,\mathrm{e}^{-t}}{2} \\ \sin\left(\sqrt{2}\,t\right)\sqrt{2}\,\mathrm{e}^{-t} & \mathrm{e}^{-t}\left(\cos\left(\sqrt{2}\,t\right)+\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t}\left(-\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)+\cos\left(\sqrt{2}\,t\right)\right)c_1 - \frac{3\sin\left(\sqrt{2}\,t\right)\sqrt{2}\,\mathrm{e}^{-t}c_2}{2} \\ \sin\left(\sqrt{2}\,t\right)\sqrt{2}\,\mathrm{e}^{-t}c_1 + \mathrm{e}^{-t}\left(\cos\left(\sqrt{2}\,t\right)+\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} -\left(\left(c_1+\frac{3c_2}{2}\right)\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)-\cos\left(\sqrt{2}\,t\right)c_1\right)\mathrm{e}^{-t} \\ \left(\sqrt{2}\,(c_1+c_2)\sin\left(\sqrt{2}\,t\right)+\cos\left(\sqrt{2}\,t\right)c_2\right)\mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

15.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & -3 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i\sqrt{2} - 1$$
$$\lambda_2 = -1 - i\sqrt{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$i\sqrt{2} - 1$	1	complex eigenvalue
$-1 - i\sqrt{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix} - \left(-1 - i\sqrt{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i\sqrt{2} - 2 & -3 \\ 2 & 2 + i\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i\sqrt{2} - 2 & -3 & 0 \\ 2 & 2 + i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{i\sqrt{2} - 2} \Longrightarrow \left[\begin{array}{cc|c} i\sqrt{2} - 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i\sqrt{2} - 2 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{i\sqrt{2}-2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{i\sqrt{2}-2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{i\sqrt{2}-2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{i\sqrt{2}-2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{i\sqrt{2}-2} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i\sqrt{2} - 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix} - \left(i\sqrt{2} - 1\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 - i\sqrt{2} & -3 \\ 2 & 2 - i\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 - i\sqrt{2} & -3 & 0 \\ 2 & 2 - i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{-2 - i\sqrt{2}} \Longrightarrow \left[\begin{array}{cc|c} -2 - i\sqrt{2} & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-i\sqrt{2} & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{2+i\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{2+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2+i\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3t}{2+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{2+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{3t}{2+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{2+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{3t}{2+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{2+i\sqrt{2}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$i\sqrt{2}-1$	1	1	No	$\left[\begin{array}{c} -\frac{3}{2+i\sqrt{2}} \\ 1 \end{array}\right]$
$-1-i\sqrt{2}$	1	1	No	$\left[\begin{array}{c} -\frac{3}{2-i\sqrt{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{3\,\mathrm{e}^{\left(i\sqrt{2}-1\right)t}}{2+i\sqrt{2}} \\ \mathrm{e}^{\left(i\sqrt{2}-1\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{3\,\mathrm{e}^{\left(-1-i\sqrt{2}\right)t}}{2-i\sqrt{2}} \\ \mathrm{e}^{\left(-1-i\sqrt{2}\right)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{c_2\left(-2-i\sqrt{2}\right)\mathrm{e}^{-\left(1+i\sqrt{2}\right)t}}{2} + \frac{\mathrm{e}^{\left(i\sqrt{2}-1\right)t}c_1\left(i\sqrt{2}-2\right)}{2} \\ c_1\mathrm{e}^{\left(i\sqrt{2}-1\right)t} + c_2\mathrm{e}^{-\left(1+i\sqrt{2}\right)t} \end{array}\right]$$

The following is the phase plot of the system.

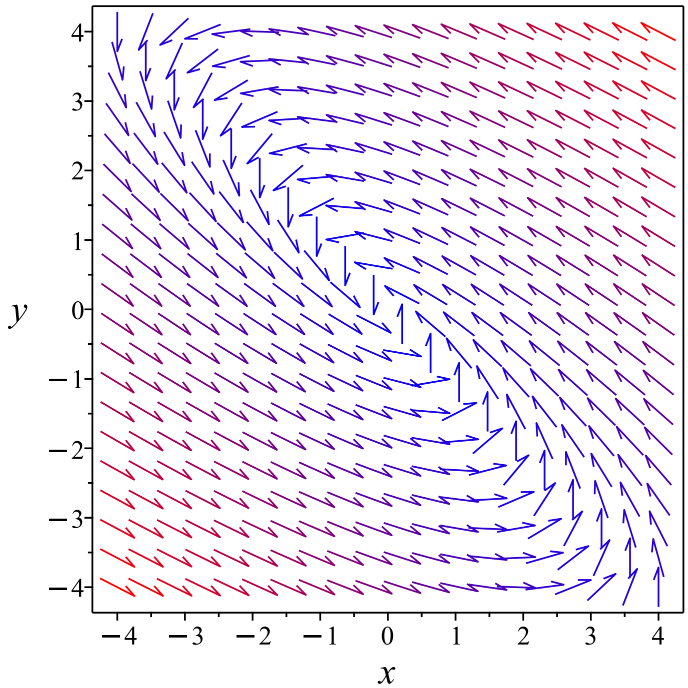


Figure 472: Phase plot

15.14.3 Maple step by step solution

Let's solve

$[x'(t) = -3x(t) - 3y, y' = 2x(t) + y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & -3 \\ 2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1 - \mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{3}{2-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{2} - 1, \begin{bmatrix} -\frac{3}{2+\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - \mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{3}{2-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-1-\mathrm{I}\sqrt{2}\right)t} \cdot \begin{bmatrix} -\frac{3}{2-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-t} \cdot \left(\cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right)\right) \cdot \begin{bmatrix} -\frac{3}{2-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{3\left(\cos\left(\sqrt{2}\,t\right)-\mathrm{I}\sin\left(\sqrt{2}\,t\right)\right)}{2-\mathrm{I}\sqrt{2}} \\ \cos\left(\sqrt{2}\,t\right) - \mathrm{I}\sin\left(\sqrt{2}\,t\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{x}_1(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\cos\left(\sqrt{2}\,t\right) - \frac{\sqrt{2}\sin\left(\sqrt{2}\,t\right)}{2} \\ \cos\left(\sqrt{2}\,t\right) \end{bmatrix}, \overrightarrow{x}_2(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{\sqrt{2}\cos\left(\sqrt{2}\,t\right)}{2} + \sin\left(\sqrt{2}\,t\right) \\ -\sin\left(\sqrt{2}\,t\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-t} \cdot \begin{bmatrix} -\cos\left(\sqrt{2}\, t\right) - \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2} \\ \cos\left(\sqrt{2}\, t\right) \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} -\frac{\sqrt{2}\, \cos\left(\sqrt{2}\, t\right)}{2} + \sin\left(\sqrt{2}\, t\right) \\ -\sin\left(\sqrt{2}\, t\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{e^{-t}\left(\left(2c_1+\sqrt{2}\, c_2\right)\cos\left(\sqrt{2}\, t\right)+\sin\left(\sqrt{2}\, t\right)\left(\sqrt{2}\, c_1-2c_2\right)\right)}{2} \\ e^{-t}\left(\cos\left(\sqrt{2}\, t\right) c_1 - c_2 \sin\left(\sqrt{2}\, t\right)\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{e^{-t}\left(\left(2c_1+\sqrt{2}\, c_2\right)\cos\left(\sqrt{2}\, t\right)+\sin\left(\sqrt{2}\, t\right)\left(\sqrt{2}\, c_1-2c_2\right)\right)}{2}, y = e^{-t}\left(\cos\left(\sqrt{2}\, t\right) c_1 - c_2 \sin\left(\sqrt{2}\, t\right)\right)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 78

```
dsolve([diff(x(t),t)=-3*x(t)-3*y(t),diff(y(t),t)=2*x(t)+1*y(t)],singsol=all)
```

$$x(t) = e^{-t}\left(c_1 \sin\left(\sqrt{2}\, t\right) + c_2 \cos\left(\sqrt{2}\, t\right)\right)$$

$$y(t) = \frac{e^{-t}\left(\sin\left(\sqrt{2}\, t\right)\sqrt{2}\, c_2 - \cos\left(\sqrt{2}\, t\right)\sqrt{2}\, c_1 - 2c_1 \sin\left(\sqrt{2}\, t\right) - 2c_2 \cos\left(\sqrt{2}\, t\right)\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 91

```
DSolve[{x'[t]==-3*x[t]-3*y[t],y'[t]==2*x[t]+1*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t) \to \frac{1}{2} e^{-t}\left(2c_1 \cos\left(\sqrt{2}t\right) - \sqrt{2}(2c_1 + 3c_2)\sin\left(\sqrt{2}t\right)\right)$$

$$y(t) \to e^{-t}\left(c_2 \cos\left(\sqrt{2}t\right) + \sqrt{2}(c_1 + c_2)\sin\left(\sqrt{2}t\right)\right)$$

15.15 problem 23

Internal problem ID [13157]
Internal file name [`OUTPUT/11812_Sunday_December_03_2023_07_16_51_PM_34869035/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 5y' + 6y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 2]$$

15.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 6 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 6y = 0$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.15.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 5\lambda\, e^{\lambda t} + 6\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{5^2 - (4)\,(1)\,(6)} \\ &= -\frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{5}{2} + \frac{1}{2}$$

$$\lambda_2 = -\frac{5}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = -2$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(-2)t} + c_2 e^{(-3)t}$$

Or

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2t} - 3c_2 \mathrm{e}^{-3t}$$

substituting $y' = 2$ and $t = 0$ in the above gives

$$2 = -2c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -2$$

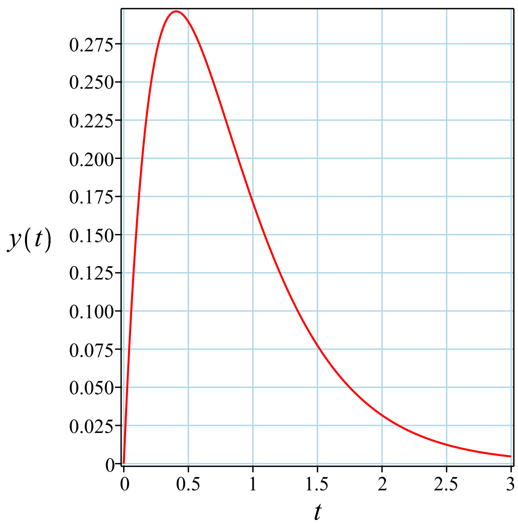
Substituting these values back in above solution results in

$$y = 2\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-3t}$$

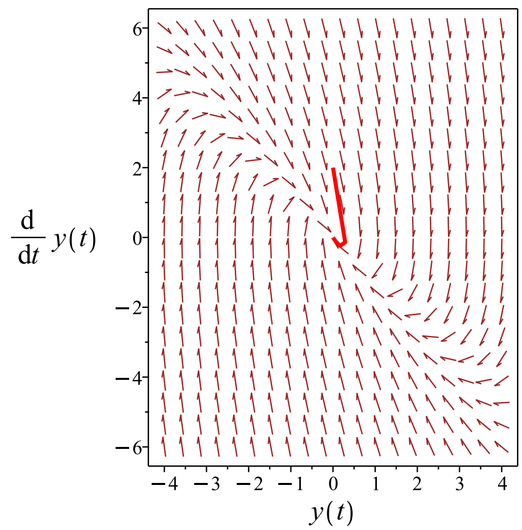
Summary

The solution(s) found are the following

$$y = 2\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-3t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-3t}$$

Verified OK.

15.15.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 403: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dt} \\ &= z_1 e^{-\frac{5t}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{5t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-5t}}{(y_1)^2}\,dt \\ &= y_1 \left(\mathrm{e}^{t}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t} \left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-2t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3\,\mathrm{e}^{-3t} c_1 - 2 c_2 \mathrm{e}^{-2t}
$$

substituting $y' = 2$ and $t = 0$ in the above gives

$$
2 = -3c_1 - 2c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

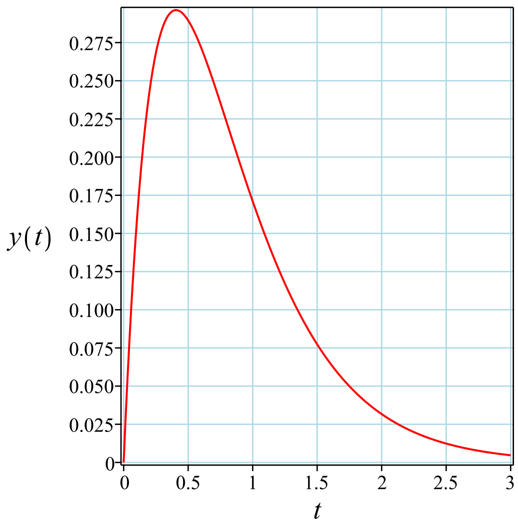
$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

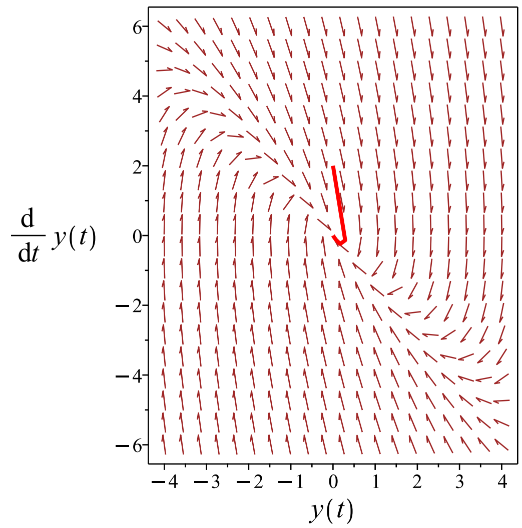
$$
y = 2\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-3t}
$$

Summary
The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-3t} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-3t}$$

Verified OK.

15.15.4 Maple step by step solution

Let's solve

$$\left[y'' + 5y' + 6y = 0, y(0) = 0, y'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -2)$

- 1st solution of the ODE

$y_1(t) = e^{-3t}$

- 2nd solution of the ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = e^{-3t} c_1 + c_2 e^{-2t}$

□ Check validity of solution $y = e^{-3t} c_1 + c_2 e^{-2t}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -3\, e^{-3t} c_1 - 2 c_2 e^{-2t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 2$

    $2 = -3c_1 - 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = -2, c_2 = 2\}$

  - Substitute constant values into general solution and simplify

    $y = 2\, e^{-2t} - 2\, e^{-3t}$

- Solution to the IVP

  $y = 2\, e^{-2t} - 2\, e^{-3t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)+5*diff(y(t),t)+6*y(t)=0,y(0) = 0, D(y)(0) = 2],y(t), singsol=all)
```

$$y(t) = -2\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 17

```
DSolve[{y''[t]+5*y'[t]+6*y[t]==0,{y[0]==0,y'[0]==2}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to 2e^{-3t}\left(e^t - 1\right)$$

15.16 problem 24

Internal problem ID [13158]
Internal file name [`OUTPUT/11813_Sunday_December_03_2023_07_16_54_PM_32728918/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 5y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = -1]$$

15.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.16.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + 5\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(5)} \\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2i \\ \lambda_2 &= -1 - 2i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -1 + 2i$$
$$\lambda_2 = -1 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $t = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\mathrm{e}^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) + \mathrm{e}^{-t}(-2c_1 \sin(2t) + 2c_2 \cos(2t))$$

substituting $y' = -1$ and $t = 0$ in the above gives

$$-1 = -c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 1$$

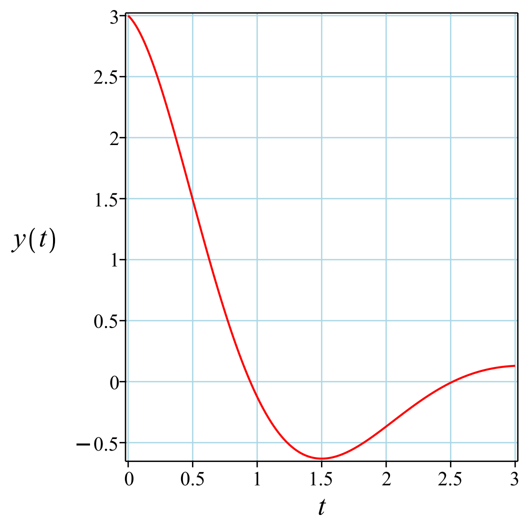
Substituting these values back in above solution results in

$$y = (3\cos(2t) + \sin(2t))\,\mathrm{e}^{-t}$$

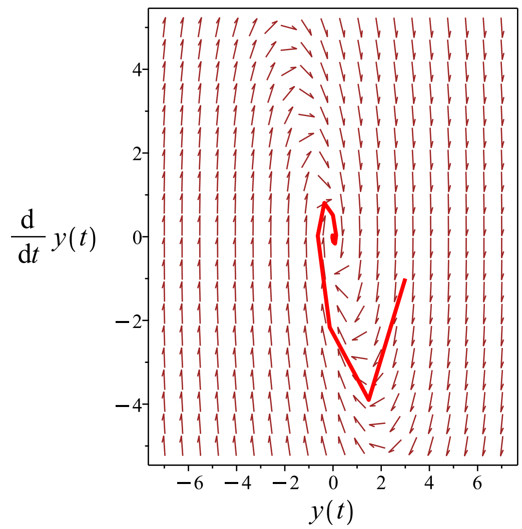
Summary

The solution(s) found are the following

$$y = (3\cos(2t) + \sin(2t))\,\mathrm{e}^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (3\cos(2t) + \sin(2t))\,\mathrm{e}^{-t}$$

Verified OK.

15.16.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 405: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dt} \\ &= z_1 e^{-t} \\ &= z_1 \left(\mathrm{e}^{-t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t} \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-2t}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t} \cos(2t)\right) + c_2 \left(\mathrm{e}^{-t} \cos(2t) \left(\frac{\tan(2t)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-t} \cos(2t) + \frac{c_2 \mathrm{e}^{-t} \sin(2t)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $t = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-t} \cos(2t) - 2c_1 \mathrm{e}^{-t} \sin(2t) - \frac{c_2 \mathrm{e}^{-t} \sin(2t)}{2} + c_2 \mathrm{e}^{-t} \cos(2t)
$$

substituting $y' = -1$ and $t = 0$ in the above gives

$$
-1 = -c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 3\,\mathrm{e}^{-t} \cos(2t) + \mathrm{e}^{-t} \sin(2t)
$$

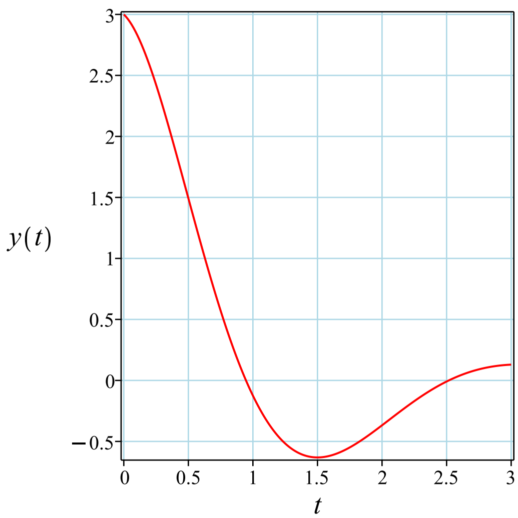
Which simplifies to

$$
y = (3\cos(2t) + \sin(2t))\,\mathrm{e}^{-t}
$$

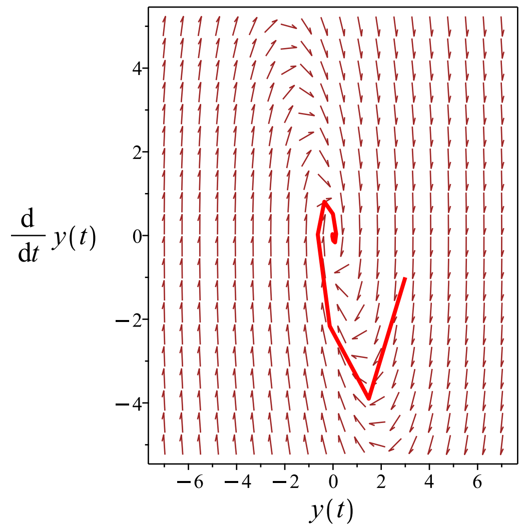
Summary

The solution(s) found are the following

$$
y = (3\cos(2t) + \sin(2t))\,\mathrm{e}^{-t} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (3\cos(2t) + \sin(2t))\,e^{-t}$$

Verified OK.

15.16.4 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 5y = 0, y(0) = 3, y'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the ODE

$y_1(t) = e^{-t} \cos(2t)$

- 2nd solution of the ODE

  $y_2(t) = e^{-t} \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t)$

☐ Check validity of solution $y = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t)$

  - Use initial condition $y(0) = 3$

    $3 = c_1$

  - Compute derivative of the solution

    $y' = -c_1 e^{-t} \cos(2t) - 2c_1 e^{-t} \sin(2t) - c_2 e^{-t} \sin(2t) + 2c_2 e^{-t} \cos(2t)$

  - Use the initial condition $y'\Big|_{\{t=0\}} = -1$

    $-1 = -c_1 + 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 1\}$

  - Substitute constant values into general solution and simplify

    $y = (3\cos(2t) + \sin(2t))\, e^{-t}$

- Solution to the IVP

  $y = (3\cos(2t) + \sin(2t))\, e^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+5*y(t)=0,y(0) = 3, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{-t}(\sin(2t) + 3\cos(2t))$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 22

```
DSolve[{y''[t]+2*y'[t]+5*y[t]==0,{y[0]==3,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to e^{-t}(\sin(2t) + 3\cos(2t))$$

15.17 problem 25

Internal problem ID [13159]
Internal file name [`OUTPUT/11814_Sunday_December_03_2023_07_16_57_PM_16437271/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

15.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.17.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $t = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t\, e^{-t}$$

substituting $y' = 1$ and $t = 0$ in the above gives

$$1 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 2$$

Substituting these values back in above solution results in

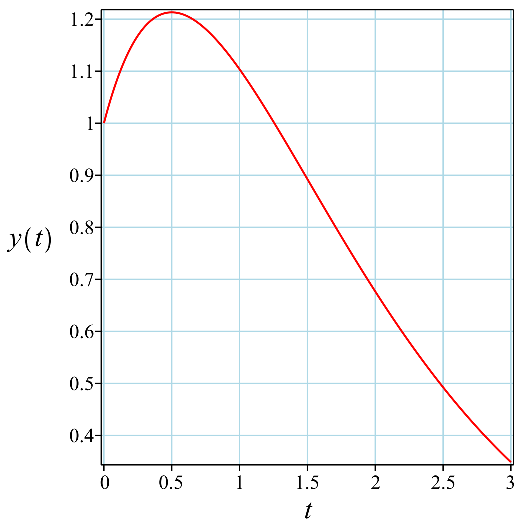
$$y = e^{-t} + 2t\, e^{-t}$$

Which simplifies to

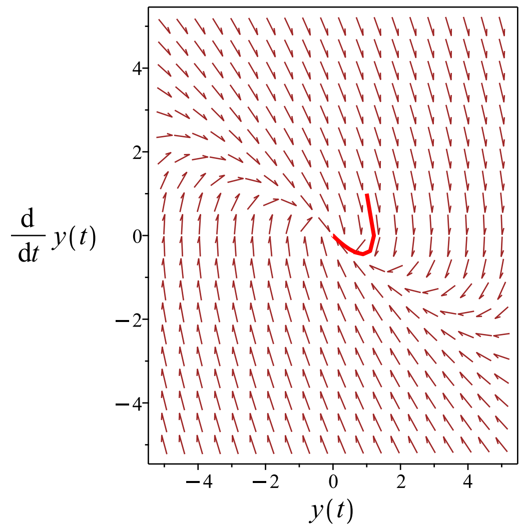
$$y = e^{-t}(2t + 1)$$

Summary
The solution(s) found are the following

$$y = e^{-t}(2t + 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-t}(2t + 1)$$

Verified OK.

15.17.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{\left(p(t)^2 + p'(t)\right) y}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int 2\,dx} \\
&= e^{t}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(e^{t} y\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$(e^t y)' = c_1$$

Integrating again gives

$$(e^t y) = c_1 t + c_2$$

Hence the solution is

$$y = \frac{c_1 t + c_2}{e^t}$$

Or

$$y = t\,e^{-t} c_1 + c_2 e^{-t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = t\,e^{-t} c_1 + c_2 e^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $t = 0$ in the above gives

$$1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{-t} - t\,e^{-t} c_1 - c_2 e^{-t}$$

substituting $y' = 1$ and $t = 0$ in the above gives

$$1 = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = e^{-t} + 2t\,e^{-t}$$

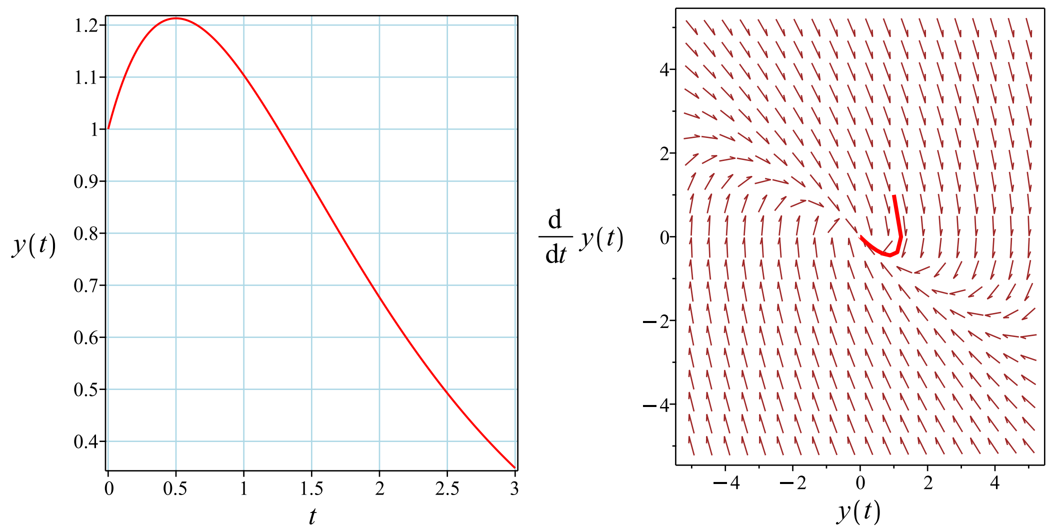
Which simplifies to

$$y = e^{-t}(2t + 1)$$

Summary

The solution(s) found are the following

$$y = e^{-t}(2t + 1) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{-t}(2t + 1)$$

Verified OK.

15.17.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 407: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt}\\ &= z_1 e^{-t}\\ &= z_1\left(\mathrm{e}^{-t}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt\\ &= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt\\ &= y_1(t)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t}(t)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-t} + c_2 t \,\mathrm{e}^{-t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $t = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} - c_2 t \,\mathrm{e}^{-t}
$$

substituting $y' = 1$ and $t = 0$ in the above gives

$$
1 = -c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \mathrm{e}^{-t} + 2t \,\mathrm{e}^{-t}
$$

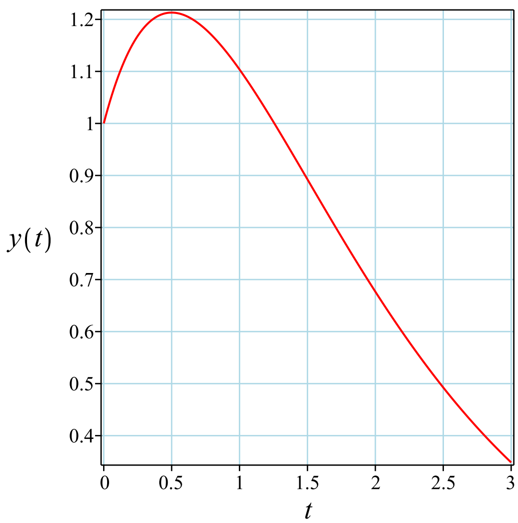
Which simplifies to

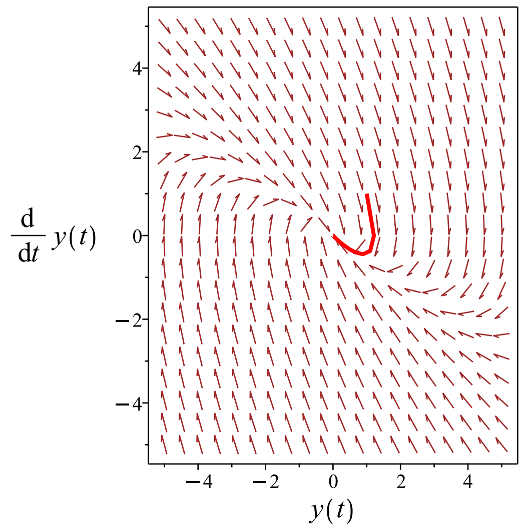
$$
y = \mathrm{e}^{-t}(2t + 1)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-t}(2t + 1) \tag{1}
$$





(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^{-t}(2t + 1)$$

Verified OK.

15.17.5 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r + 1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

$y_1(t) = \mathrm{e}^{-t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-t} + c_2 t \mathrm{e}^{-t}$

  - Use initial condition $y(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $y' = -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} - c_2 t\,\mathrm{e}^{-t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = -c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 2\}$

  - Substitute constant values into general solution and simplify

    $y = \mathrm{e}^{-t}(2t + 1)$

- Solution to the IVP

  $y = \mathrm{e}^{-t}(2t + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=0,y(0) = 1, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{-t}(2t + 1)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 16

```
DSolve[{y''[t]+2*y'[t]+y[t]==0,{y[0]==1,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow e^{-t}(2t + 1)$$

15.18 problem 26

Internal problem ID [13160]
Internal file name [`OUTPUT/11815_Sunday_December_03_2023_07_16_59_PM_10625611/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 3. Linear Systems. Review Exercises for chapter 3. page 376
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y = 0}$$

With initial conditions

$$\left[y(0) = 3, y'(0) = -\sqrt{2}\right]$$

15.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.18.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\lambda_1 = +i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$

Which simplifies to

$$\lambda_1 = i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0 \left(\cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)\right)$$

Or

$$y = \cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $t = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) c_1 + \sqrt{2}\, \cos\left(\sqrt{2}\, t\right) c_2$$

substituting $y' = -\sqrt{2}$ and $t = 0$ in the above gives

$$-\sqrt{2} = \sqrt{2}\, c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

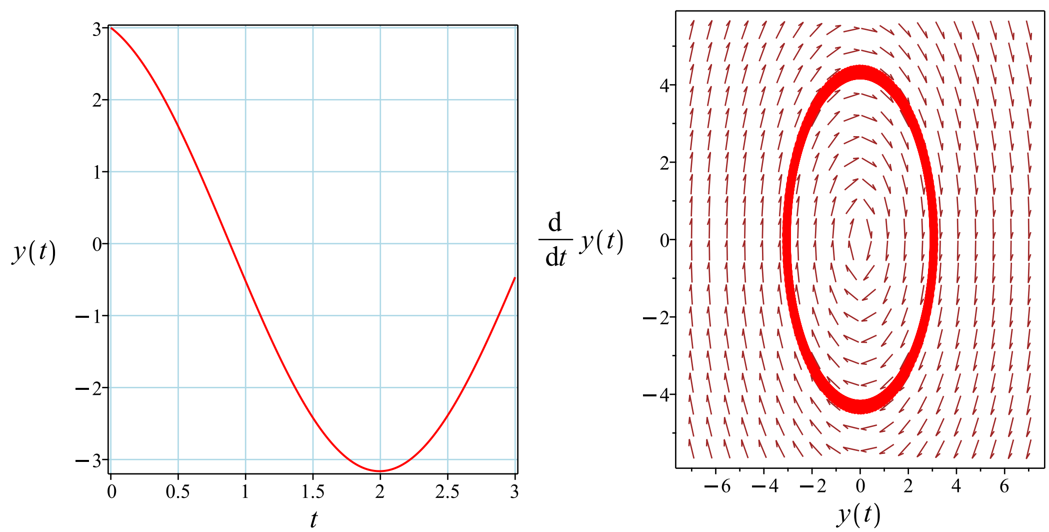
$$c_1 = 3$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = 3\cos\left(\sqrt{2}\, t\right) - \sin\left(\sqrt{2}\, t\right)$$

Summary
The solution(s) found are the following

$$y = 3\cos\left(\sqrt{2}\, t\right) - \sin\left(\sqrt{2}\, t\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3\cos\left(\sqrt{2}\, t\right) - \sin\left(\sqrt{2}\, t\right)$$

Verified OK.

15.18.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 2y'y = 0$$

Integrating the above w.r.t t gives

$$\int (y'y'' + 2y'y)\, dt = 0$$
$$\frac{y'^2}{2} + y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-2y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-2y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-2y^2 + 2c_1}} dy = \int dt$$
$$\frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\, y}{\sqrt{-2y^2+2c_1}}\right)}{2} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-2y^2 + 2c_1}} dy = \int dt$$
$$-\frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\, y}{\sqrt{-2y^2+2c_1}}\right)}{2} = t + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,y}{\sqrt{-2y^2+2c_1}}\right)}{2} = t+c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=3$ and $t=0$ in the above gives

$$\frac{\arctan\left(\frac{3}{\sqrt{-9+c_1}}\right)\sqrt{2}}{2} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \sqrt{2}\left(\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2+1\right)\sqrt{\frac{c_1}{\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2+1}} - \frac{\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2 c_1\sqrt{2}}{\sqrt{\frac{c_1}{\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2+1}}\left(\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2+1\right)}$$

substituting $y'=-\sqrt{2}$ and $t=0$ in the above gives

$$-\sqrt{2} = \frac{\cos\left(\sqrt{2}\,c_2\right)^2\sqrt{2}\,c_1}{\sqrt{\cos\left(\sqrt{2}\,c_2\right)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,y}{\sqrt{-2y^2+2c_1}}\right)}{2} = t+c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=3$ and $t=0$ in the above gives

$$-\frac{\arctan\left(\frac{3}{\sqrt{-9+c_1}}\right)\sqrt{2}}{2} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sqrt{2}\left(\tan\left(\left(t+c_3\right)\sqrt{2}\right)^2+1\right)\sqrt{\frac{c_1}{\tan\left(\left(t+c_3\right)\sqrt{2}\right)^2+1}} + \frac{\tan\left(\left(t+c_3\right)\sqrt{2}\right)^2 c_1\sqrt{2}}{\sqrt{\frac{c_1}{\tan\left(\left(t+c_3\right)\sqrt{2}\right)^2+1}}\left(\tan\left(\left(t+c_3\right)\sqrt{2}\right)^2+\right)}$$

substituting $y' = -\sqrt{2}$ and $t = 0$ in the above gives

$$-\sqrt{2} = -\frac{\cos\left(\sqrt{2}\, c_3\right)^2 \sqrt{2}\, c_1}{\sqrt{\cos\left(\sqrt{2}\, c_3\right)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 10 \\ c_3 &= -\frac{\arctan\left(3\right)\sqrt{2}}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$-\frac{\arctan\left(\frac{y}{\sqrt{-y^2+10}}\right)\sqrt{2}}{2} = t - \frac{\arctan\left(3\right)\sqrt{2}}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\arctan\left(\frac{y}{\sqrt{-y^2+10}}\right)\sqrt{2}}{2} = t - \frac{\arctan\left(3\right)\sqrt{2}}{2} \tag{1}$$

Verification of solutions

$$-\frac{\arctan\left(\frac{y}{\sqrt{-y^2+10}}\right)\sqrt{2}}{2} = t - \frac{\arctan\left(3\right)\sqrt{2}}{2}$$

Verified OK.

15.18.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -2z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 409: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\sqrt{2}\, t\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos\left(\sqrt{2}\, t\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, t\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\ &= \cos\left(\sqrt{2}\, t\right) \int \frac{1}{\cos\left(\sqrt{2}\, t\right)^2}\, dt \\ &= \cos\left(\sqrt{2}\, t\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, t\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\cos\left(\sqrt{2}\, t\right)\right) + c_2\left(\cos\left(\sqrt{2}\, t\right)\left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, t\right)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \cos\left(\sqrt{2}\, t\right) c_1 + \frac{c_2\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $t = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) c_1 + c_2 \cos\left(\sqrt{2}\, t\right)
$$

substituting $y' = -\sqrt{2}$ and $t = 0$ in the above gives

$$
-\sqrt{2} = c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= -\sqrt{2}
\end{aligned}
$$

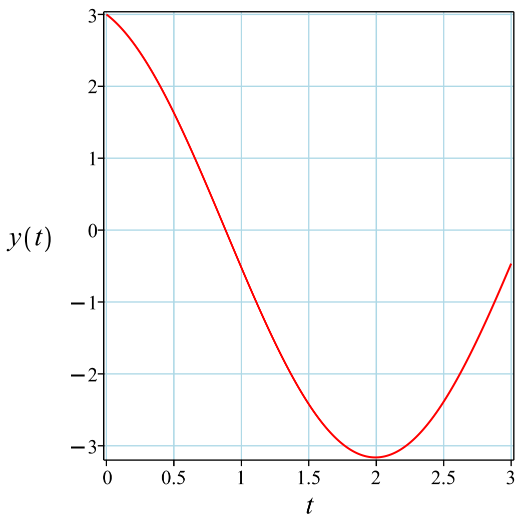
Substituting these values back in above solution results in

$$
y = 3\cos\left(\sqrt{2}\, t\right) - \sin\left(\sqrt{2}\, t\right)
$$

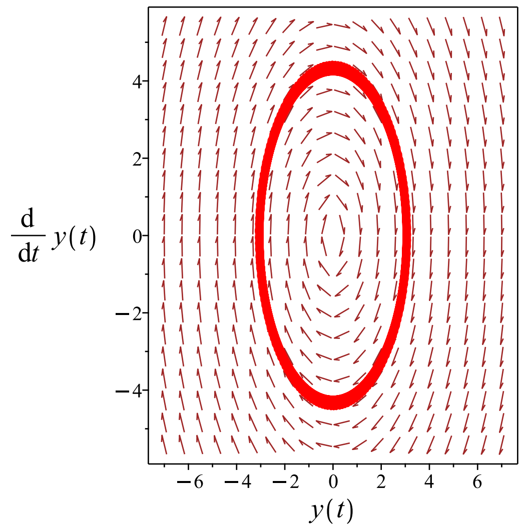
Summary

The solution(s) found are the following

$$
y = 3\cos\left(\sqrt{2}\, t\right) - \sin\left(\sqrt{2}\, t\right) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\cos\left(\sqrt{2}\,t\right) - \sin\left(\sqrt{2}\,t\right)$$

Verified OK.

15.18.5 Maple step by step solution

Let's solve

$\left[y'' + 2y = 0, y(0) = 3, y'\Big|_{\{t=0\}} = -\sqrt{2}\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}\right)$

- 1st solution of the ODE

$y_1(t) = \cos\left(\sqrt{2}\, t\right)$

- 2nd solution of the ODE

  $y_2(t) = \sin\left(\sqrt{2}\, t\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = \cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)$

□ Check validity of solution $y = \cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)$

  - Use initial condition $y(0) = 3$

    $3 = c_1$

  - Compute derivative of the solution

    $y' = -\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) c_1 + \sqrt{2}\, \cos\left(\sqrt{2}\, t\right) c_2$

  - Use the initial condition $y'\Big|_{\{t=0\}} = -\sqrt{2}$

    $-\sqrt{2} = \sqrt{2}\, c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = -1\}$

  - Substitute constant values into general solution and simplify

    $y = 3\cos\left(\sqrt{2}\, t\right) - \sin\left(\sqrt{2}\, t\right)$

- Solution to the IVP

  $y = 3\cos\left(\sqrt{2}\, t\right) - \sin\left(\sqrt{2}\, t\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)+2*y(t)=0,y(0) = 3, D(y)(0) = -sqrt(2)],y(t), singsol=all)
```

$$y(t) = -\sin\left(\sqrt{2}\, t\right) + 3\cos\left(\sqrt{2}\, t\right)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 26

```
DSolve[{y''[t]+2*y[t]==0,{y[0]==3,y'[0]==-Sqrt[2]}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 3\cos\left(\sqrt{2}t\right) - \sin\left(\sqrt{2}t\right)$$

16 Chapter 4. Forcing and Resonance. Section 4.1 page 399

16.1 problem 1

Internal problem ID [13161]
Internal file name [`OUTPUT/11816_Sunday_December_03_2023_07_17_02_PM_73348288/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - y' - 6y = e^{4t}}$$

16.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -1, C = -6, f(t) = e^{4t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -1, C = -6$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - \lambda\,\mathrm{e}^{\lambda t} - 6\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-6)}\\ &= \frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{5}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(3)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{4t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{3t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{4t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{4t} = e^{4t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{4t}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3t} + c_2 e^{-2t}\right) + \left(\frac{e^{4t}}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3t} + c_2 e^{-2t} + \frac{e^{4t}}{6} \tag{1}$$

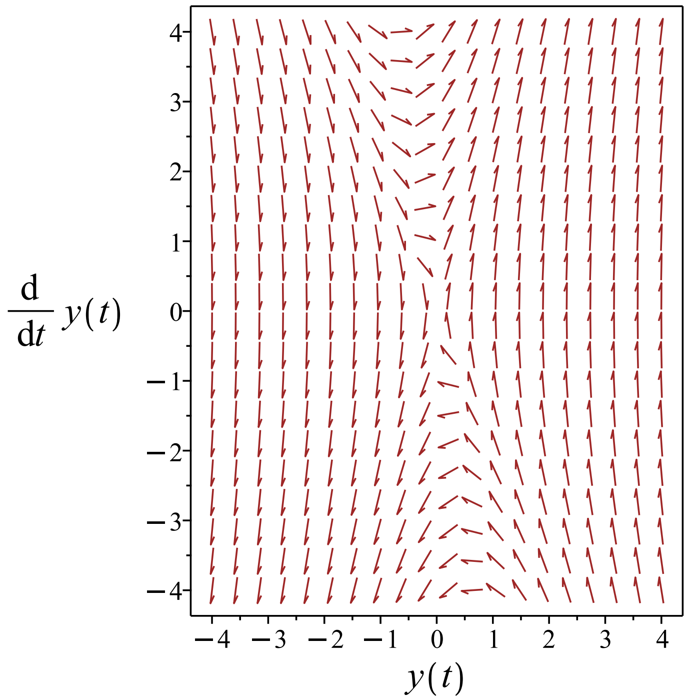


Figure 482: Slope field plot

Verification of solutions

$$y = c_1 e^{3t} + c_2 e^{-2t} + \frac{e^{4t}}{6}$$

Verified OK.

16.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{25z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 411: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = e^{-\frac{5t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\, dt} \\ &= z_1 e^{\frac{t}{2}} \\ &= z_1 \left(e^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^t}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{e^{5t}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t} \left(\frac{\mathrm{e}^{5t}}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' - y' - 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2t} + \frac{c_2 \mathrm{e}^{3t}}{5}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{4t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{4t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{3t}}{5}, \mathrm{e}^{-2t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{4t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{4t} = e^{4t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{4t}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + \frac{c_2 e^{3t}}{5}\right) + \left(\frac{e^{4t}}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2t} + \frac{c_2 e^{3t}}{5} + \frac{e^{4t}}{6} \tag{1}$$

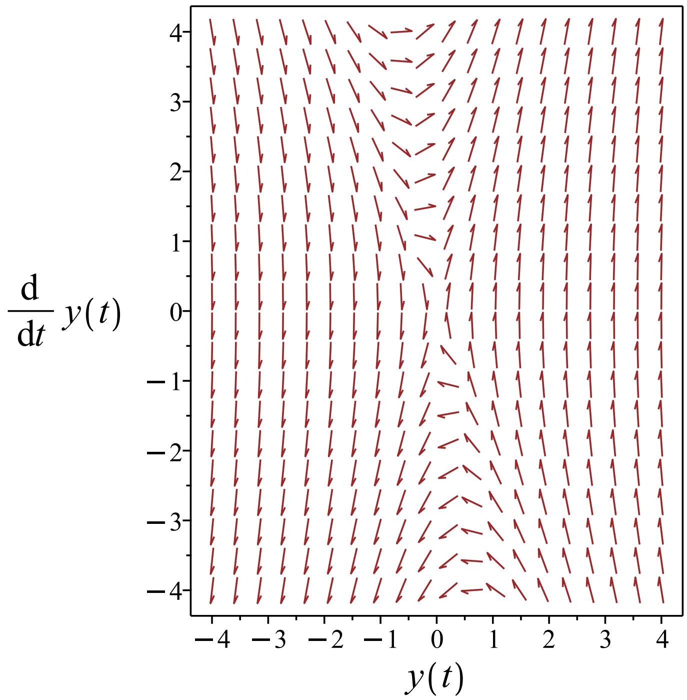


Figure 483: Slope field plot

Verification of solutions

$$y = c_1 e^{-2t} + \frac{c_2 e^{3t}}{5} + \frac{e^{4t}}{6}$$

Verified OK.

16.1.3 Maple step by step solution

Let's solve

$y'' - y' - 6y = e^{4t}$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 - r - 6 = 0$

- Factor the characteristic polynomial

$(r + 2)(r - 3) = 0$

- Roots of the characteristic polynomial

$r = (-2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{3t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathrm{e}^{4t}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{3t} \\ -2\,\mathrm{e}^{-2t} & 3\,\mathrm{e}^{3t} \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 5\,\mathrm{e}^{t}$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{(\mathrm{e}^{5t}(\int \mathrm{e}^{t}dt) - (\int \mathrm{e}^{6t}dt))\mathrm{e}^{-2t}}{5}$

  ○ Compute integrals

  $y_p(t) = \frac{\mathrm{e}^{4t}}{6}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{3t} + \frac{\mathrm{e}^{4t}}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(t),t$2)-diff(y(t),t)-6*y(t)=exp(4*t),y(t), singsol=all)
```

$$y(t) = \frac{(e^{6t} + 6c_2e^{5t} + 6c_1)\,e^{-2t}}{6}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 31

```
DSolve[y''[t]-y'[t]-6*y[t]==Exp[4*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{4t}}{6} + c_1e^{-2t} + c_2e^{3t}$$

16.2 problem 2

Internal problem ID [13162]
Internal file name [`OUTPUT/11817_Sunday_December_03_2023_07_17_04_PM_49833966/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 6y' + 8y = 2\,\mathrm{e}^{-3t}}$$

16.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 6, C = 8, f(t) = 2\,\mathrm{e}^{-3t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 6, C = 8$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda\, e^{\lambda t} + 8\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{6^2 - (4)\,(1)\,(8)}\\ &= -3 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 1\\ \lambda_2 &= -3 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-2)t} + c_2 e^{(-4)t}\end{aligned}$$

Or

$$y = c_1 e^{-2t} + c_2 e^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2t} + c_2 e^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-4t}, \mathrm{e}^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-3t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \mathrm{e}^{-3t} = 2\,\mathrm{e}^{-3t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\,\mathrm{e}^{-3t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}\right) + \left(-2\,\mathrm{e}^{-3t}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t} - 2\,\mathrm{e}^{-3t} \tag{1}$$

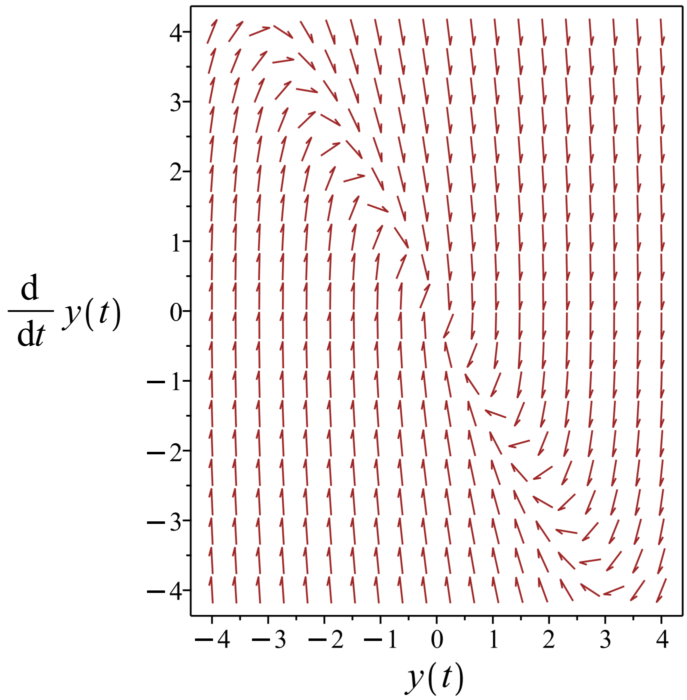


Figure 484: Slope field plot

Verification of solutions

$$y = c_1 e^{-2t} + c_2 e^{-4t} - 2\,e^{-3t}$$

Verified OK.

16.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 413: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dt}\\ &= z_1 e^{-3t}\\ &= z_1\left(\mathrm{e}^{-3t}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dt}}{(y_1)^2}\,dt\\ &= y_1 \int \frac{e^{-6t}}{(y_1)^2}\,dt\\ &= y_1\left(\frac{\mathrm{e}^{2t}}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 6y' + 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2\,\mathrm{e}^{-3t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-3t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{-2t}}{2}, \mathrm{e}^{-4t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-3t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \mathrm{e}^{-3t} = 2\,\mathrm{e}^{-3t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\,\mathrm{e}^{-3t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2}\right) + \left(-2\,\mathrm{e}^{-3t}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2} - 2\,\mathrm{e}^{-3t} \tag{1}$$

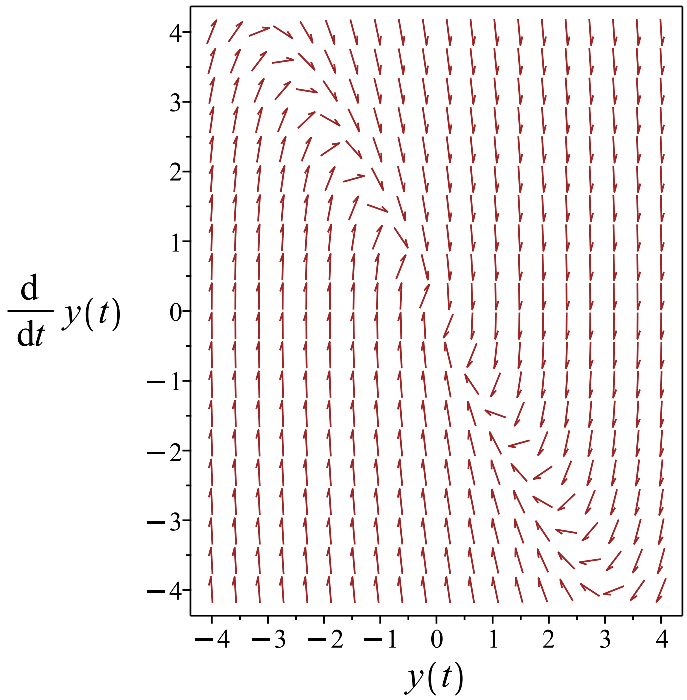


Figure 485: Slope field plot

Verification of solutions

$$y = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2} - 2\,e^{-3t}$$

Verified OK.

16.2.3 Maple step by step solution

Let's solve

$y'' + 6y' + 8y = 2\,e^{-3t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r + 4)(r + 2) = 0$

- Roots of the characteristic polynomial

$r = (-4, -2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 2\,\mathrm{e}^{-3t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-4t} & \mathrm{e}^{-2t} \\ -4\,\mathrm{e}^{-4t} & -2\,\mathrm{e}^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{-6t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\mathrm{e}^{-4t}\left(\int \mathrm{e}^{t} dt\right) + \mathrm{e}^{-2t}\left(\int \mathrm{e}^{-t} dt\right)$

  - Compute integrals

    $y_p(t) = -2\,\mathrm{e}^{-3t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-2t} - 2\,\mathrm{e}^{-3t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(t),t$2)+6*diff(y(t),t)+8*y(t)=2*exp(-3*t),y(t), singsol=all)
```

$$y(t) = -\frac{\left(\mathrm{e}^{-2t}c_1 + 4\,\mathrm{e}^{-t} - 2c_2\right)\mathrm{e}^{-2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 27

```
DSolve[y''[t]+6*y'[t]+8*y[t]==2*Exp[-3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-4t}\left(-2e^t + c_2e^{2t} + c_1\right)$$

16.3 problem 3

Internal problem ID [13163]
Internal file name [`OUTPUT/11818_Sunday_December_03_2023_07_17_06_PM_16485240/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - y' - 2y = 5\,e^{3t}}$$

16.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -1, C = -2, f(t) = 5\,e^{3t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - \lambda\, \mathrm{e}^{\lambda t} - 2\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-2)} \\
&= \frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
y &= c_1 e^{(2)t} + c_2 e^{(-1)t}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\,\mathrm{e}^{3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-t}, \mathrm{e}^{2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{3t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \mathrm{e}^{3t} = 5\,\mathrm{e}^{3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5\,\mathrm{e}^{3t}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-t}\right) + \left(\frac{5\,\mathrm{e}^{3t}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{-t} + \frac{5\,\mathrm{e}^{3t}}{4} \tag{1}$$

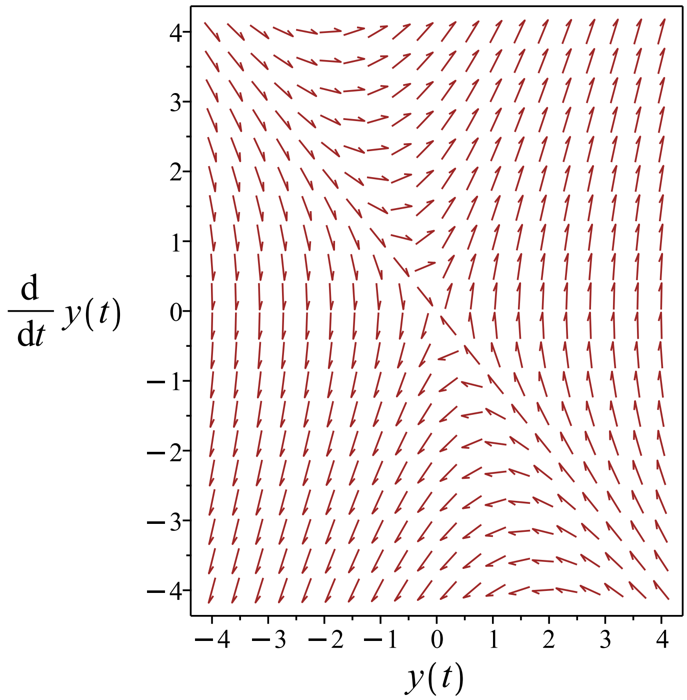


Figure 486: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{2t} + c_2 e^{-t} + \frac{5\,e^{3t}}{4}$$

Verified OK.

16.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 415: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{3t}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dt} \\
&= z_1 e^{\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{\mathrm{e}^{3t}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t} \left(\frac{\mathrm{e}^{3t}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' - y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{2t}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
5\,\mathrm{e}^{3t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{3t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2t}}{3}, \mathrm{e}^{-t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{3t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{3t} = 5\,e^{3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5\,e^{3t}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + \frac{c_2 e^{2t}}{3}\right) + \left(\frac{5\,e^{3t}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-t} + \frac{c_2 e^{2t}}{3} + \frac{5\,e^{3t}}{4} \tag{1}$$

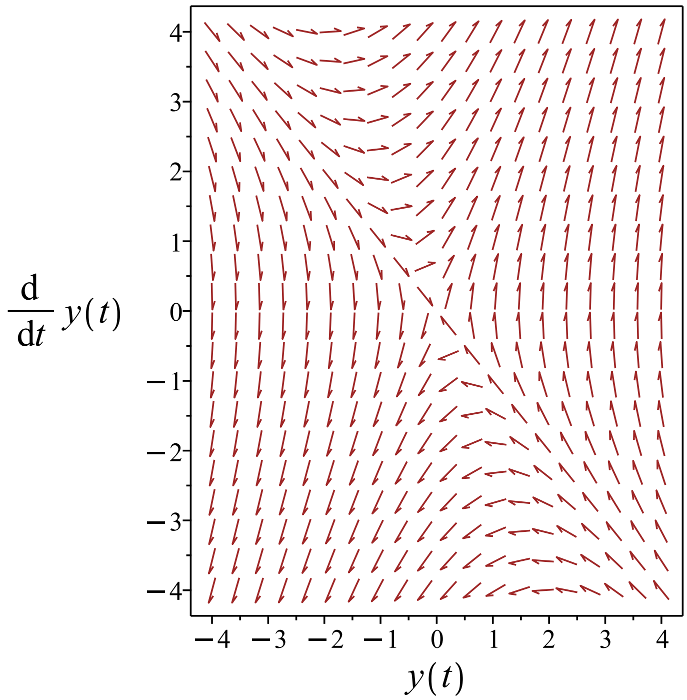


Figure 487: Slope field plot

Verification of solutions

$$y = c_1 e^{-t} + \frac{c_2 e^{2t}}{3} + \frac{5\,e^{3t}}{4}$$

Verified OK.

16.3.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = 5\,e^{3t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

$r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-t} + c_2 e^{2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 5\,e^{3t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2\,e^{2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 3\,e^{t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{5\,e^{-t}\left(\int e^{4t}dt\right)}{3} + \frac{5\,e^{2t}\left(\int e^{t}dt\right)}{3}$

  - Compute integrals

    $y_p(t) = \frac{5\,e^{3t}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t} + c_2 e^{2t} + \frac{5\,e^{3t}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(t),t$2)-diff(y(t),t)-2*y(t)=5*exp(3*t),y(t), singsol=all)
```

$$y(t) = c_2 e^{-t} + c_1 e^{2t} + \frac{5\, e^{3t}}{4}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 31

```
DSolve[y''[t]-y'[t]-2*y[t]==5*Exp[3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{5e^{3t}}{4} + c_1 e^{-t} + c_2 e^{2t}$$

16.4 problem 4

Internal problem ID [13164]
Internal file name [`OUTPUT/11819_Sunday_December_03_2023_07_17_08_PM_79892913/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 13y = e^{-t}}$$

16.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 13, f(t) = e^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 13$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} + 13\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(13)}\\ &= -2 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 3i\\ \lambda_2 &= -2 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 3i\\ \lambda_2 &= -2 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t} \cos(3t), e^{-2t} \sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 e^{-t} = e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-t}}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))\right) + \left(\frac{e^{-t}}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t)) + \frac{e^{-t}}{10} \tag{1}$$

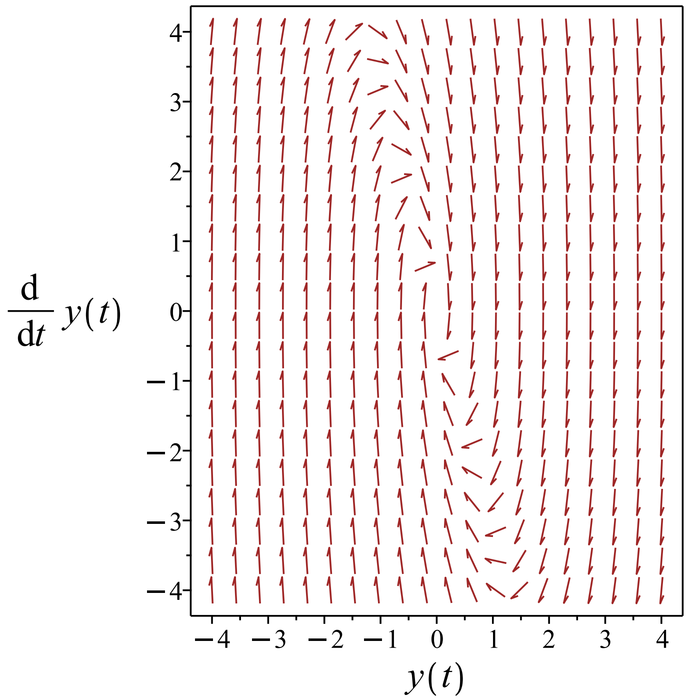


Figure 488: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t)) + \frac{e^{-t}}{10}$$

Verified OK.

16.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 417: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1 \left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t} \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{\tan(3t)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-2t} \cos(3t)\right) + c_2 \left(e^{-2t} \cos(3t) \left(\frac{\tan(3t)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' + 13y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = e^{-2t} \cos(3t) c_1 + \frac{e^{-2t} \sin(3t) c_2}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^{-t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{-t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{e^{-2t} \cos(3t), \frac{e^{-2t} \sin(3t)}{3}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^{-t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1\mathrm{e}^{-t} = \mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\mathrm{e}^{-t}}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2t} \cos(3t)\, c_1 + \frac{\mathrm{e}^{-2t} \sin(3t)\, c_2}{3}\right) + \left(\frac{\mathrm{e}^{-t}}{10}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{-2t} \cos(3t)\, c_1 + \frac{\mathrm{e}^{-2t} \sin(3t)\, c_2}{3} + \frac{\mathrm{e}^{-t}}{10} \tag{1}$$

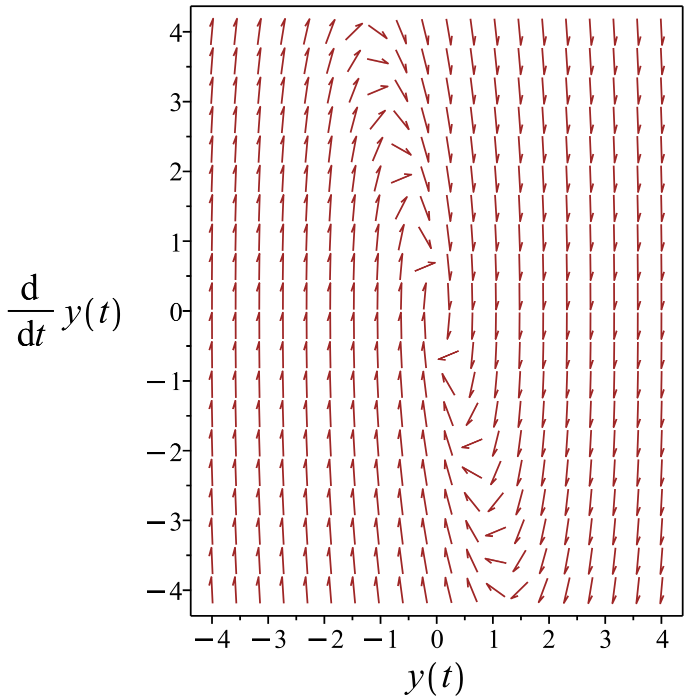


Figure 489: Slope field plot

Verification of solutions

$$y = e^{-2t} \cos(3t) c_1 + \frac{e^{-2t} \sin(3t) c_2}{3} + \frac{e^{-t}}{10}$$

Verified OK.

16.4.3 Maple step by step solution

Let's solve

$y'' + 4y' + 13y = e^{-t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-2 - 3\,\mathrm{I}, -2 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t} \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t} \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-2t} \cos(3t)\, c_1 + \mathrm{e}^{-2t} \sin(3t)\, c_2 + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathrm{e}^{-t}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} \cos(3t) & \mathrm{e}^{-2t} \sin(3t) \\ -2\,\mathrm{e}^{-2t} \cos(3t) - 3\,\mathrm{e}^{-2t} \sin(3t) & -2\,\mathrm{e}^{-2t} \sin(3t) + 3\,\mathrm{e}^{-2t} \cos(3t) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,\mathrm{e}^{-4t}$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\mathrm{e}^{-2t}\left(\cos(3t)\left(\int \mathrm{e}^{t} \sin(3t)dt\right) - \sin(3t)\left(\int \mathrm{e}^{t} \cos(3t)dt\right)\right)}{3}$

  ○ Compute integrals

  $y_p(t) = \frac{\mathrm{e}^{-t}}{10}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-2t} \cos(3t)\, c_1 + \mathrm{e}^{-2t} \sin(3t)\, c_2 + \frac{\mathrm{e}^{-t}}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(t),t$2)+4*diff(y(t),t)+13*y(t)=exp(-t),y(t), singsol=all)
```

$$y(t) = c_2 e^{-2t} \sin(3t) + c_1 e^{-2t} \cos(3t) + \frac{e^{-t}}{10}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 34

```
DSolve[y''[t]+4*y'[t]+13*y[t]==Exp[-t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{10} e^{-2t} \left(e^t + 10c_2 \cos(3t) + 10c_1 \sin(3t)\right)$$

16.5 problem 5

Internal problem ID [13165]
Internal file name [`OUTPUT/11820_Sunday_December_03_2023_07_17_13_PM_4275737/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 4y' + 13y = -3\,\mathrm{e}^{-2t}}$$

16.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 13, f(t) = -3\,\mathrm{e}^{-2t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 13$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 4\lambda \,\mathrm{e}^{\lambda t} + 13 \,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 13$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(13)} \\
&= -2 \pm 3i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -2 + 3i \\
\lambda_2 &= -2 - 3i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -2 + 3i \\
\lambda_2 &= -2 - 3i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-3\,\mathrm{e}^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2t}\cos(3t),\mathrm{e}^{-2t}\sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1\mathrm{e}^{-2t} = -3\,\mathrm{e}^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\mathrm{e}^{-2t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2t}(c_1\cos(3t) + c_2\sin(3t))\right) + \left(-\frac{\mathrm{e}^{-2t}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2t}(c_1\cos(3t) + c_2\sin(3t)) - \frac{\mathrm{e}^{-2t}}{3} \tag{1}$$

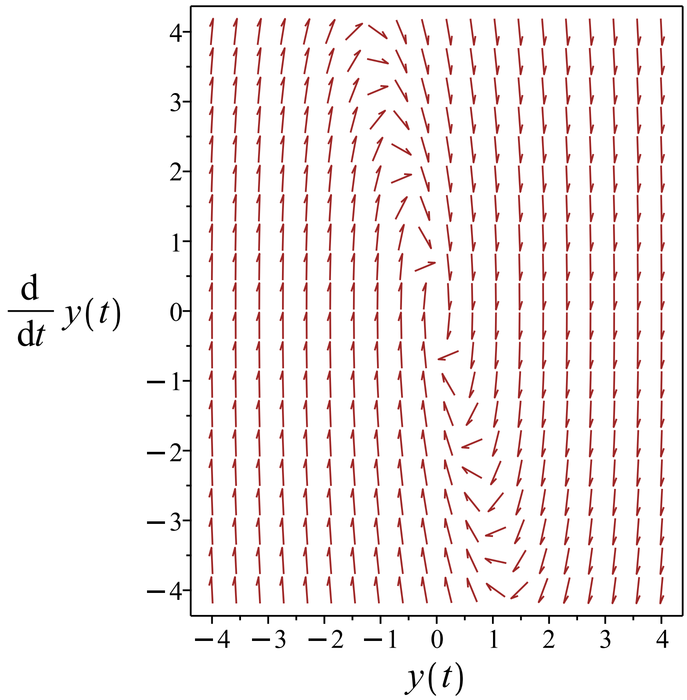


Figure 490: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t)) - \frac{e^{-2t}}{3}$$

Verified OK.

16.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 419: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\ &= z_1 e^{-2t} \\ &= z_1 \left(\mathrm{e}^{-2t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t} \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\ &= y_1 \left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2t} \cos(3t)\right) + c_2 \left(\mathrm{e}^{-2t} \cos(3t) \left(\frac{\tan(3t)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' + 13y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-2t} \cos(3t)\, c_1 + \frac{\mathrm{e}^{-2t} \sin(3t)\, c_2}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
-3\,\mathrm{e}^{-2t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-2t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-2t} \cos(3t), \frac{\mathrm{e}^{-2t} \sin(3t)}{3}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-2t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^{-2t} = -3\,e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{-2t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}\cos(3t)\,c_1 + \frac{e^{-2t}\sin(3t)\,c_2}{3}\right) + \left(-\frac{e^{-2t}}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2t}\cos(3t)\,c_1 + \frac{e^{-2t}\sin(3t)\,c_2}{3} - \frac{e^{-2t}}{3} \tag{1}$$

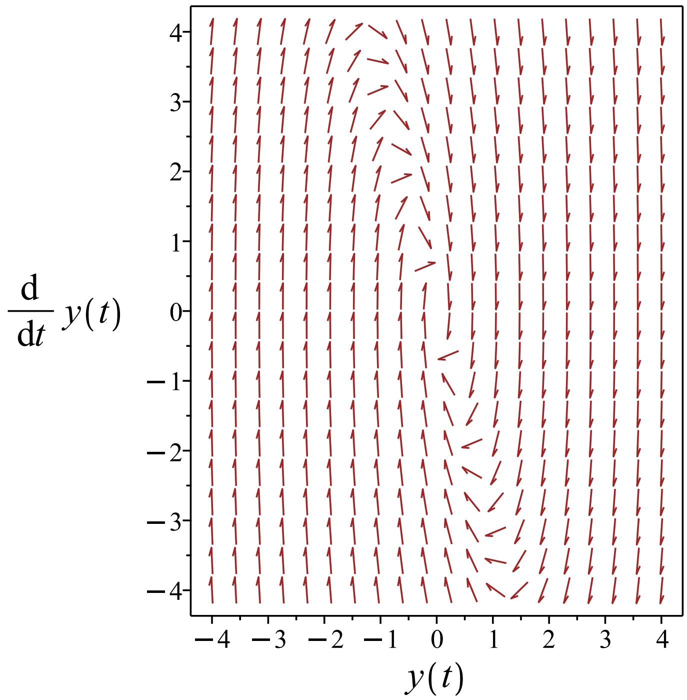


Figure 491: Slope field plot

Verification of solutions

$$y = e^{-2t}\cos(3t) c_1 + \frac{e^{-2t}\sin(3t) c_2}{3} - \frac{e^{-2t}}{3}$$

Verified OK.

16.5.3 Maple step by step solution

Let's solve

$y'' + 4y' + 13y = -3\,e^{-2t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-2 - 3\,\mathrm{I}, -2 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t} \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t} \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-2t} \cos(3t)\, c_1 + \mathrm{e}^{-2t} \sin(3t)\, c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = -3\,\mathrm{e}^{-2t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} \cos(3t) & \mathrm{e}^{-2t} \sin(3t) \\ -2\,\mathrm{e}^{-2t} \cos(3t) - 3\,\mathrm{e}^{-2t} \sin(3t) & -2\,\mathrm{e}^{-2t} \sin(3t) + 3\,\mathrm{e}^{-2t} \cos(3t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,\mathrm{e}^{-4t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \mathrm{e}^{-2t}\left(\cos(3t)\left(\int \sin(3t)\, dt\right) - \sin(3t)\left(\int \cos(3t)\, dt\right)\right)$

- ○ Compute integrals

  $y_p(t) = -\frac{\mathrm{e}^{-2t}}{3}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-2t} \cos(3t)\, c_1 + \mathrm{e}^{-2t} \sin(3t)\, c_2 - \frac{\mathrm{e}^{-2t}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(t),t$2)+4*diff(y(t),t)+13*y(t)=-3*exp(-2*t),y(t), singsol=all)
```

$$y(t) = \frac{e^{-2t}(3c_1 \cos(3t) + 3c_2 \sin(3t) - 1)}{3}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 32

```
DSolve[y''[t]+4*y'[t]+13*y[t]==-3*Exp[-2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{3}e^{-2t}(3c_2 \cos(3t) + 3c_1 \sin(3t) - 1)$$

16.6 problem 6

Internal problem ID [13166]
Internal file name [`OUTPUT/11821_Sunday_December_03_2023_07_17_16_PM_98285266/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 7y' + 10y = e^{-2t}}$$

16.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 7, C = 10, f(t) = e^{-2t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 7y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 7, C = 10$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 7\lambda\, e^{\lambda t} + 10\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 7\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 7, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-7}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{7^2 - (4)\,(1)\,(10)}\\ &= -\frac{7}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{7}{2} + \frac{3}{2}\\ \lambda_2 &= -\frac{7}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -5\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-2)t} + c_2 e^{(-5)t}\end{aligned}$$

Or

$$y = c_1 e^{-2t} + c_2 e^{-5t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2t} + c_2 e^{-5t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-5t}, e^{-2t}\}$$

Since e^{-2t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,e^{-2t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t\,e^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{-2t} = e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{t\,e^{-2t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-5t}\right) + \left(\frac{t\,e^{-2t}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2t} + c_2 e^{-5t} + \frac{t\,e^{-2t}}{3} \tag{1}$$

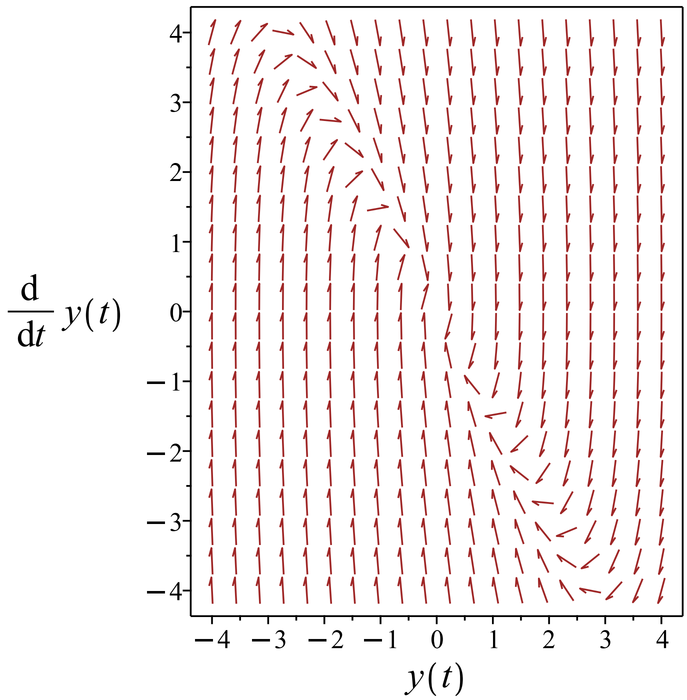


Figure 492: Slope field plot

Verification of solutions

$$y = c_1 e^{-2t} + c_2 e^{-5t} + \frac{t\,e^{-2t}}{3}$$

Verified OK.

16.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 7y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 7 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 421: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{3t}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7}{1}\,dt} \\
&= z_1 e^{-\frac{7t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{7t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-7t}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{\mathrm{e}^{3t}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-5t}\right) + c_2 \left(\mathrm{e}^{-5t} \left(\frac{\mathrm{e}^{3t}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 7y' + 10y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-5t} + \frac{c_2 \mathrm{e}^{-2t}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-5t}$$
$$y_2 = \frac{\mathrm{e}^{-2t}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-5t} & \frac{\mathrm{e}^{-2t}}{3} \\ \frac{d}{dt}\left(\mathrm{e}^{-5t}\right) & \frac{d}{dt}\left(\frac{\mathrm{e}^{-2t}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-5t} & \frac{\mathrm{e}^{-2t}}{3} \\ -5\,\mathrm{e}^{-5t} & -\frac{2\,\mathrm{e}^{-2t}}{3} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-5t}\right)\left(-\frac{2\,\mathrm{e}^{-2t}}{3}\right) - \left(\frac{\mathrm{e}^{-2t}}{3}\right)\left(-5\,\mathrm{e}^{-5t}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-5t}\mathrm{e}^{-2t}$$

Which simplifies to

$$W = e^{-7t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-4t}}{3}}{e^{-7t}} dt$$

Which simplifies to

$$u_1 = -\int \frac{e^{3t}}{3} dt$$

Hence

$$u_1 = -\frac{e^{3t}}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-5t} e^{-2t}}{e^{-7t}} dt$$

Which simplifies to

$$u_2 = \int 1 dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{e^{-5t} e^{3t}}{9} + \frac{t\, e^{-2t}}{3}$$

Which simplifies to

$$y_p(t) = \frac{e^{-2t}(-1 + 3t)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-5t} + \frac{c_2 e^{-2t}}{3}\right) + \left(\frac{e^{-2t}(-1 + 3t)}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-5t} + \frac{c_2 e^{-2t}}{3} + \frac{e^{-2t}(-1 + 3t)}{9} \tag{1}$$

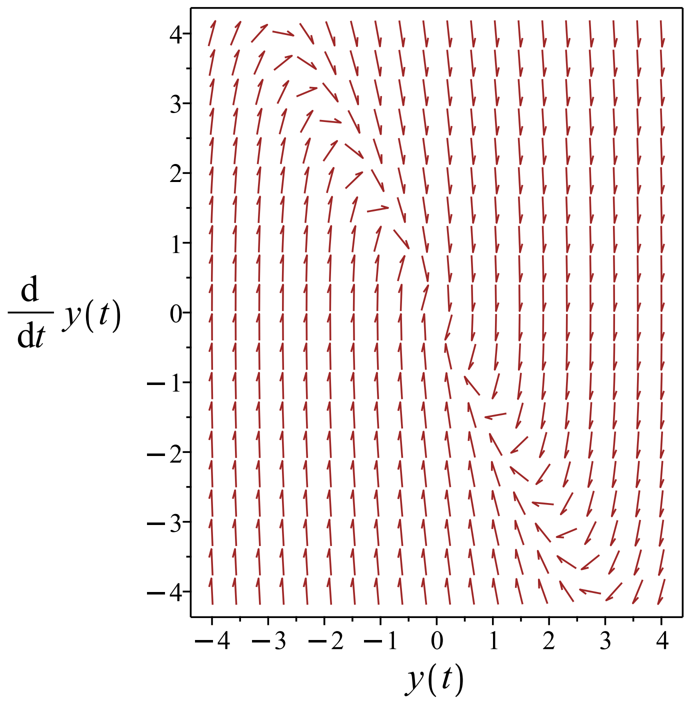


Figure 493: Slope field plot

Verification of solutions

$$y = c_1 e^{-5t} + \frac{c_2 e^{-2t}}{3} + \frac{e^{-2t}(-1 + 3t)}{9}$$

Verified OK.

16.6.3 Maple step by step solution

Let's solve

$y'' + 7y' + 10y = e^{-2t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 7r + 10 = 0$

- Factor the characteristic polynomial

  $(r+5)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-5, -2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-5t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-5t} + c_2 e^{-2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = e^{-2t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-5t} & e^{-2t} \\ -5\,e^{-5t} & -2\,e^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 3\,e^{-7t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{e^{-5t}\left(\int e^{3t} dt\right)}{3} + \frac{e^{-2t}\left(\int 1 dt\right)}{3}$

  - Compute integrals

    $y_p(t) = \frac{e^{-2t}(-1+3t)}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-5t} + c_2 e^{-2t} + \frac{e^{-2t}(-1+3t)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(t),t$2)+7*diff(y(t),t)+10*y(t)=exp(-2*t),y(t), singsol=all)
```

$$y(t) = \frac{(t + 3c_1)\,\mathrm{e}^{-2t}}{3} + c_2\mathrm{e}^{-5t}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 31

```
DSolve[y''[t]+7*y'[t]+10*y[t]==Exp[-2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-5t}\left(e^{3t}\left(\frac{t}{3} - \frac{1}{9} + c_2\right) + c_1\right)$$

16.7 problem 7

Internal problem ID [13167]
Internal file name [`OUTPUT/11822_Sunday_December_03_2023_07_17_18_PM_89583894/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 5y' + 4y = e^{4t}}$$

16.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -5, C = 4, f(t) = e^{4t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - 5y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -5, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 5\lambda\,\mathrm{e}^{\lambda t} + 4\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 5\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-5^2 - (4)\,(1)\,(4)}\\ &= \frac{5}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{5}{2} + \frac{3}{2}$$
$$\lambda_2 = \frac{5}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 4$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(4)t} + c_2 e^{(1)t}$$

Or

$$y = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{4t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{t}, e^{4t}\}$$

Since e^{4t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{e^{4t}t\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{4t} t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{4t} = e^{4t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{4t}t}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{4t} + c_2 e^{t}\right) + \left(\frac{e^{4t}t}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{4t} + c_2 e^{t} + \frac{e^{4t} t}{3} \tag{1}$$

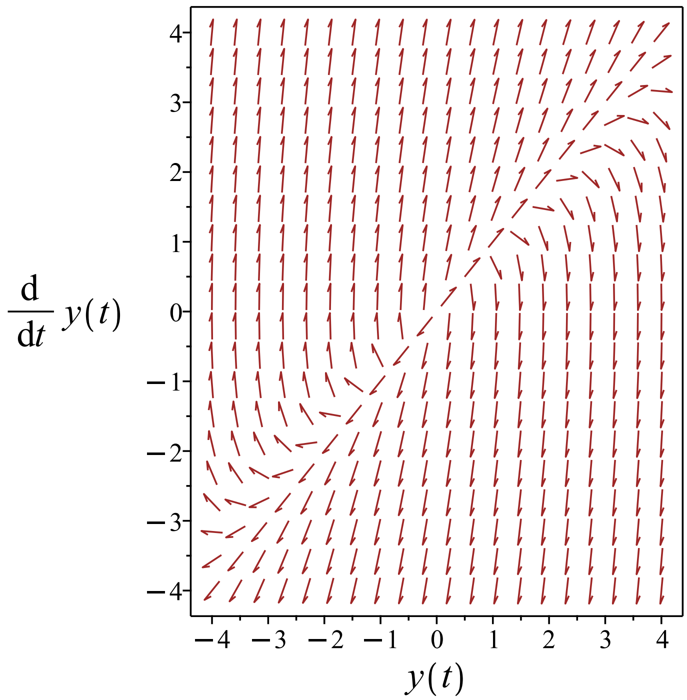


Figure 494: Slope field plot

Verification of solutions

$$y = c_1 e^{4t} + c_2 e^{t} + \frac{e^{4t} t}{3}$$

Verified OK.

16.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 423: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{3t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dt} \\
&= z_1 e^{\frac{5t}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{5t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{5t}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{\mathrm{e}^{3t}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (\mathrm{e}^t) + c_2 \left(\mathrm{e}^t \left(\frac{\mathrm{e}^{3t}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - 5y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^t + \frac{c_2 \mathrm{e}^{4t}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^t$$

$$y_2 = \frac{\mathrm{e}^{4t}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^t & \frac{\mathrm{e}^{4t}}{3} \\ \frac{d}{dt}(\mathrm{e}^t) & \frac{d}{dt}\left(\frac{\mathrm{e}^{4t}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^t & \frac{\mathrm{e}^{4t}}{3} \\ \mathrm{e}^t & \frac{4\,\mathrm{e}^{4t}}{3} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^t)\left(\frac{4\,\mathrm{e}^{4t}}{3}\right) - \left(\frac{\mathrm{e}^{4t}}{3}\right)(\mathrm{e}^t)$$

Which simplifies to

$$W = \mathrm{e}^t \mathrm{e}^{4t}$$

Which simplifies to

$$W = e^{5t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{8t}}{3}}{e^{5t}} dt$$

Which simplifies to

$$u_1 = -\int \frac{e^{3t}}{3} dt$$

Hence

$$u_1 = -\frac{e^{3t}}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^t e^{4t}}{e^{5t}} dt$$

Which simplifies to

$$u_2 = \int 1 dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{e^t e^{3t}}{9} + \frac{e^{4t} t}{3}$$

Which simplifies to

$$y_p(t) = \frac{e^{4t}(-1 + 3t)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^t + \frac{c_2 e^{4t}}{3}\right) + \left(\frac{e^{4t}(-1 + 3t)}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^t + \frac{c_2 e^{4t}}{3} + \frac{e^{4t}(-1 + 3t)}{9} \tag{1}$$

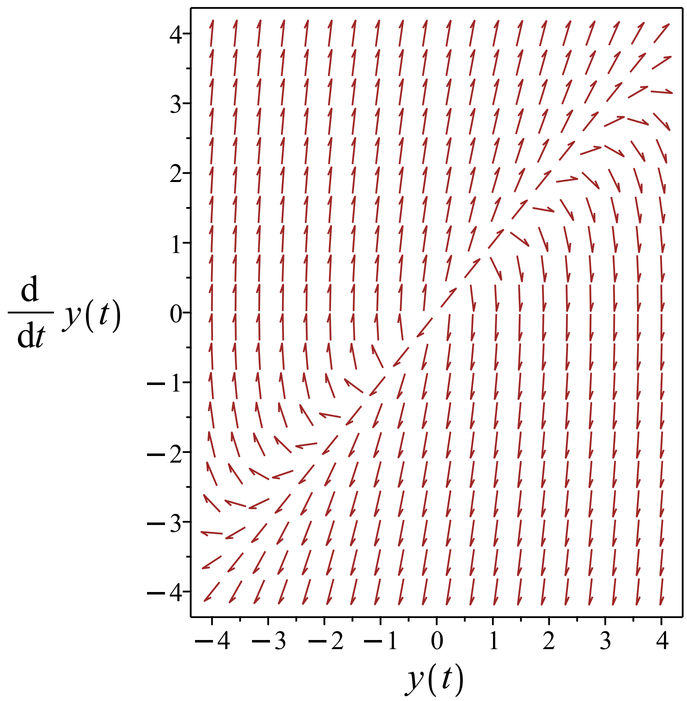


Figure 495: Slope field plot

Verification of solutions

$$y = c_1 e^t + \frac{c_2 e^{4t}}{3} + \frac{e^{4t}(-1 + 3t)}{9}$$

Verified OK.

16.7.3 Maple step by step solution

Let's solve

$y'' - 5y' + 4y = e^{4t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 5r + 4 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (1, 4)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^t$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{4t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{4t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathrm{e}^{4t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^t & \mathrm{e}^{4t} \\ \mathrm{e}^t & 4\,\mathrm{e}^{4t} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,\mathrm{e}^{5t}$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\mathrm{e}^t\left(\int \mathrm{e}^{3t}dt\right)}{3} + \frac{\mathrm{e}^{4t}\left(\int 1 dt\right)}{3}$

  ◦ Compute integrals

  $y_p(t) = \frac{\mathrm{e}^{4t}(-1+3t)}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{4t} + \frac{\mathrm{e}^{4t}(-1+3t)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(t),t$2)-5*diff(y(t),t)+4*y(t)=exp(4*t),y(t), singsol=all)
```

$$y(t) = \frac{(t + 3c_2)\,\mathrm{e}^{4t}}{3} + c_1\mathrm{e}^{t}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 29

```
DSolve[y''[t]-5*y'[t]+4*y[t]==Exp[4*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1e^t + e^{4t}\left(\frac{t}{3} - \frac{1}{9} + c_2\right)$$

16.8 problem 8

Internal problem ID [13168]
Internal file name [`OUTPUT/11823_Sunday_December_03_2023_07_17_20_PM_58380412/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' - 6y = 4\,\mathrm{e}^{-3t}}$$

16.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 1, C = -6, f(t) = 4\,\mathrm{e}^{-3t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 6 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-6)}\\ &= -\frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{5}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(2)t} + c_2 e^{(-3)t}\end{aligned}$$

Or

$$y = c_1 e^{2t} + c_2 e^{-3t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2t} + c_2 e^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{-3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3t}, \mathrm{e}^{2t}\}$$

Since e^{-3t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,\mathrm{e}^{-3t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t\,\mathrm{e}^{-3t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1\mathrm{e}^{-3t} = 4\,\mathrm{e}^{-3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4t\,\mathrm{e}^{-3t}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{-3t}\right) + \left(-\frac{4t\,\mathrm{e}^{-3t}}{5}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2t} + c_2 e^{-3t} - \frac{4t\,e^{-3t}}{5} \tag{1}$$

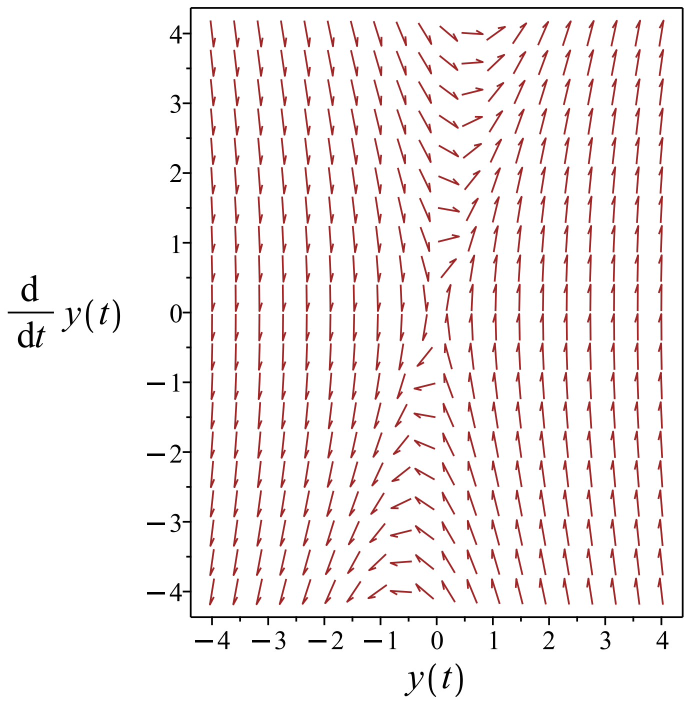


Figure 496: Slope field plot

Verification of solutions

$$y = c_1 e^{2t} + c_2 e^{-3t} - \frac{4t\,e^{-3t}}{5}$$

Verified OK.

16.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{25 z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 425: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{5t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{1}{1} \, dt} \\ &= z_1 e^{-\frac{t}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-t}}{(y_1)^2}\,dt \\ &= y_1 \left(\frac{\mathrm{e}^{5t}}{5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t} \left(\frac{\mathrm{e}^{5t}}{5}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + y' - 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-3t} c_1 + \frac{c_2 \mathrm{e}^{2t}}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{-3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{2t}}{5}, \mathrm{e}^{-3t}\right\}$$

Since e^{-3t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,\mathrm{e}^{-3t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t\,\mathrm{e}^{-3t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1\mathrm{e}^{-3t} = 4\,\mathrm{e}^{-3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4t\,\mathrm{e}^{-3t}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-3t}c_1 + \frac{c_2\mathrm{e}^{2t}}{5}\right) + \left(-\frac{4t\,\mathrm{e}^{-3t}}{5}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-3t}c_1 + \frac{c_2 e^{2t}}{5} - \frac{4t\,e^{-3t}}{5} \tag{1}$$

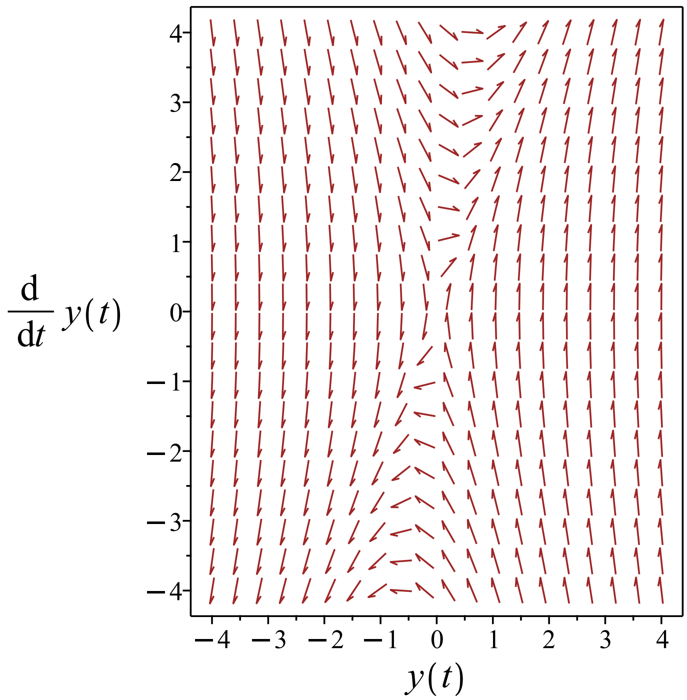


Figure 497: Slope field plot

Verification of solutions

$$y = e^{-3t}c_1 + \frac{c_2 e^{2t}}{5} - \frac{4t\,e^{-3t}}{5}$$

Verified OK.

16.8.3 Maple step by step solution

Let's solve

$y'' + y' - 6y = 4\,e^{-3t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r - 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-3t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{-3t} c_1 + c_2 e^{2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 4\,e^{-3t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-3t} & e^{2t} \\ -3\,e^{-3t} & 2\,e^{2t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 5\,e^{-t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{4\left(-e^{5t}\left(\int e^{-5t} dt\right) + \int 1 dt\right) e^{-3t}}{5}$

- ○ Compute integrals

  $y_p(t) = -\frac{4(5t+1) e^{-3t}}{25}$

- Substitute particular solution into general solution to ODE

  $y = e^{-3t} c_1 + c_2 e^{2t} - \frac{4(5t+1) e^{-3t}}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(t),t$2)+diff(y(t),t)-6*y(t)=4*exp(-3*t),y(t), singsol=all)
```

$$y(t) = \frac{(5c_1 e^{5t} + 5c_2 - 4t)\, e^{-3t}}{5}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 32

```
DSolve[y''[t]+y'[t]-6*y[t]==4*Exp[-3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{25} e^{-3t} \left(-20t + 25c_2 e^{5t} - 4 + 25c_1\right)$$

16.9 problem 9

Internal problem ID [13169]
Internal file name [`OUTPUT/11824_Sunday_December_03_2023_07_17_22_PM_69823683/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 6y' + 8y = e^{-t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 8 \\ F &= e^{-t} \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 8y = e^{-t}$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.9.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 6, C = 8, f(t) = e^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 6, C = 8$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda e^{\lambda t} + 8 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(8)}\\ &= -3 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = -3 + 1$$
$$\lambda_2 = -3 - 1$$

Which simplifies to

$$\lambda_1 = -2$$
$$\lambda_2 = -4$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(-2)t} + c_2 e^{(-4)t}$$

Or

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{-t} = e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-4t}\right) + \left(\frac{e^{-t}}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + c_2 e^{-4t} + \frac{e^{-t}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2t} - 4c_2 e^{-4t} - \frac{e^{-t}}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 4c_2 - \frac{1}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{2}$$
$$c_2 = \frac{1}{6}$$

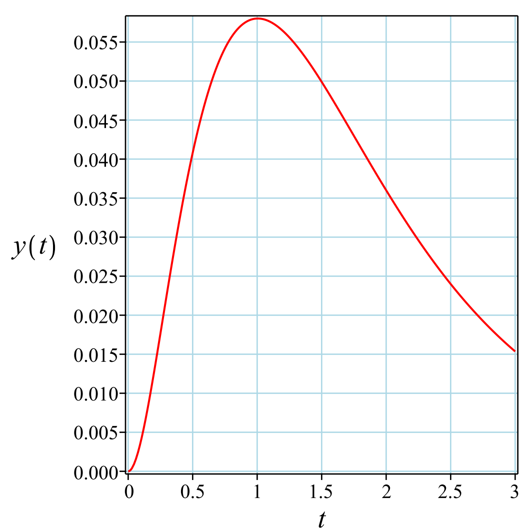
Substituting these values back in above solution results in

$$y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$$

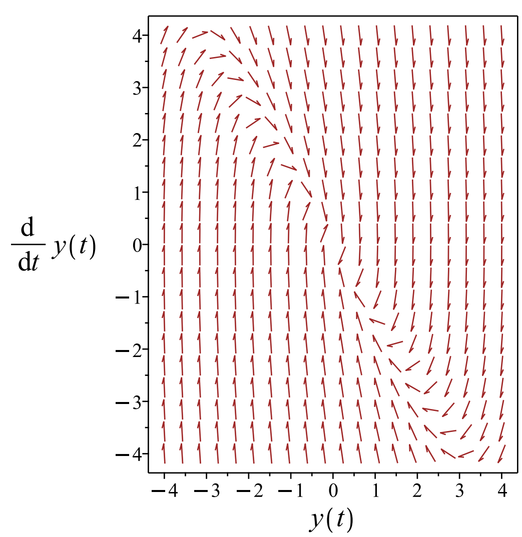
Summary

The solution(s) found are the following

$$y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$$

Verified OK.

16.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 427: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-3t} \\
&= z_1 \left(\mathrm{e}^{-3t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-6t}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 6y' + 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-2t}}{2}, e^{-4t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{-t} = e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2}\right) + \left(\frac{e^{-t}}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2} + \frac{e^{-t}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} + \frac{1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 e^{-4t} - c_2 e^{-2t} - \frac{e^{-t}}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_1 - c_2 - \frac{1}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{6}$$
$$c_2 = -1$$

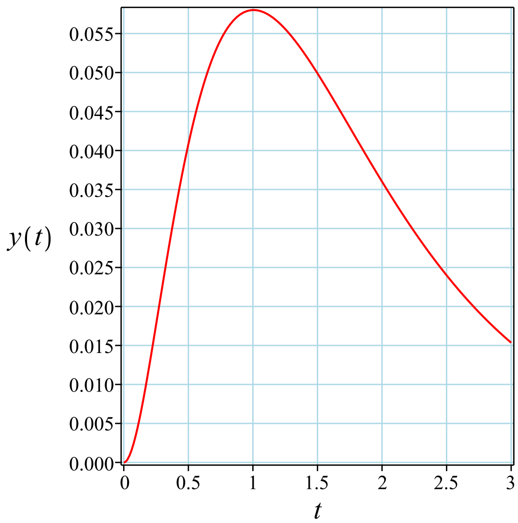
Substituting these values back in above solution results in

$$y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$$

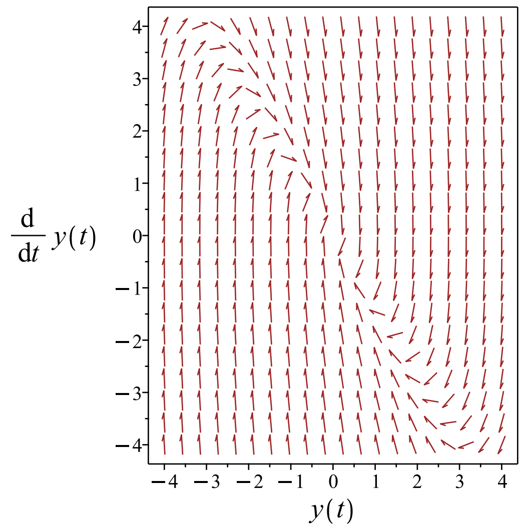
Summary

The solution(s) found are the following

$$y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$$

Verified OK.

16.9.4 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 8y = e^{-t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r + 4)(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -2)$

- 1st solution of the homogeneous ODE

$y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = e^{-t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -4\,e^{-4t} & -2\,e^{-2t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-6t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^{-4t}\left(\int e^{3t}dt\right)}{2} + \frac{e^{-2t}\left(\int e^{t}dt\right)}{2}$

- ○ Compute integrals

  $y_p(t) = \frac{e^{-t}}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + \frac{e^{-t}}{3}$

☐ Check validity of solution $y = c_1 e^{-4t} + c_2 e^{-2t} + \frac{e^{-t}}{3}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 + \frac{1}{3}$

- ○ Compute derivative of the solution

  $y' = -4c_1 e^{-4t} - 2c_2 e^{-2t} - \frac{e^{-t}}{3}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -4c_1 - 2c_2 - \frac{1}{3}$

- Solve for c_1 and c_2

  $\{c_1 = \frac{1}{6}, c_2 = -\frac{1}{2}\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$

- Solution to the IVP

  $y = -\frac{e^{-2t}}{2} + \frac{e^{-t}}{3} + \frac{e^{-4t}}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+8*y(t)=exp(-t),y(0) = 0, D(y)(0) = 0],y(t), singsol=al
```

$$y(t) = \frac{(2\,e^{3t} - 3\,e^{2t} + 1)\,e^{-4t}}{6}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 28

```
DSolve[{y''[t]+6*y'[t]+8*y[t]==Exp[-t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -
```

$$y(t) \to \frac{1}{6}e^{-4t}\left(e^t - 1\right)^2\left(2e^t + 1\right)$$

16.10 problem 10

Internal problem ID [13170]
Internal file name [`OUTPUT/11825_Sunday_December_03_2023_07_17_25_PM_8196642/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 7y' + 12y = 3\,\mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

16.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 7 \\ q(t) &= 12 \\ F &= 3\,\mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$y'' + 7y' + 12y = 3\,\mathrm{e}^{-t}$$

The domain of $p(t) = 7$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 12$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 3\,\mathrm{e}^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.10.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 7, C = 12, f(t) = 3\,\mathrm{e}^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 7y' + 12y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 7, C = 12$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 7\lambda\,\mathrm{e}^{\lambda t} + 12\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 7\lambda + 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 7, C = 12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-7}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{7^2 - (4)(1)(12)} \\ &= -\frac{7}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{7}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{7}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 \\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ y &= c_1 e^{(-3)t} + c_2 e^{(-4)t}\end{aligned}$$

Or

$$y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}, e^{-3t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{-t} = 3\,e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-3t} c_1 + c_2 e^{-4t}\right) + \left(\frac{e^{-t}}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-3t} c_1 + c_2 e^{-4t} + \frac{e^{-t}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$2 = c_1 + c_2 + \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,\mathrm{e}^{-3t} c_1 - 4c_2 \mathrm{e}^{-4t} - \frac{\mathrm{e}^{-t}}{2}$$

substituting $y' = 1$ and $t = 0$ in the above gives

$$1 = -3c_1 - 4c_2 - \frac{1}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{15}{2}$$
$$c_2 = -6$$

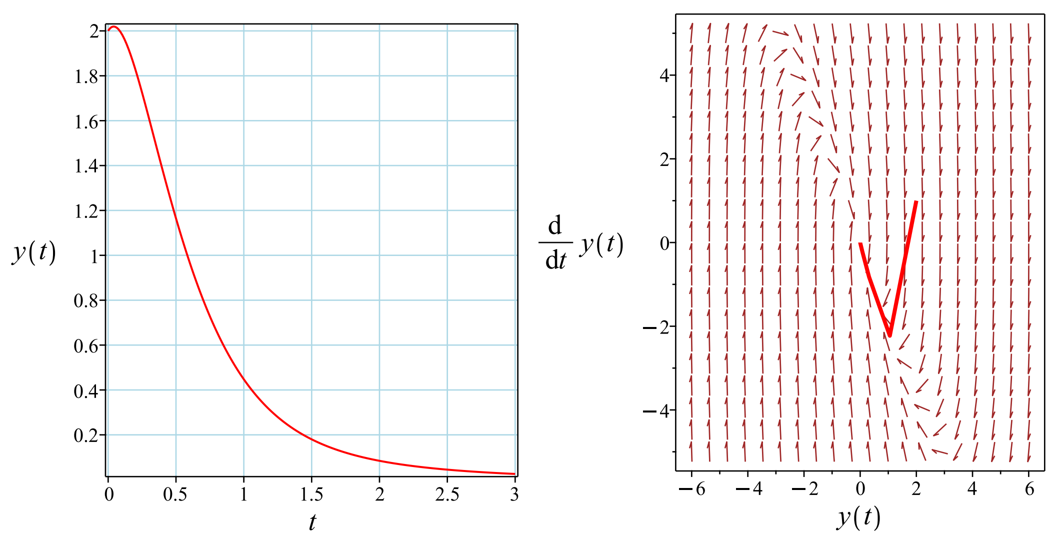
Substituting these values back in above solution results in

$$y = \frac{15\,\mathrm{e}^{-3t}}{2} - 6\,\mathrm{e}^{-4t} + \frac{\mathrm{e}^{-t}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{15\,\mathrm{e}^{-3t}}{2} - 6\,\mathrm{e}^{-4t} + \frac{\mathrm{e}^{-t}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{15\,\mathrm{e}^{-3t}}{2} - 6\,\mathrm{e}^{-4t} + \frac{\mathrm{e}^{-t}}{2}$$

Verified OK.

16.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 7y' + 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 7 \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 429: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7}{1}\,dt} \\
&= z_1 e^{-\frac{7t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{7t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-7t}}{(y_1)^2}\,dt \\
&= y_1 \left(\mathrm{e}^{t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t}\left(\mathrm{e}^{t}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+7y'+12y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-4t} + c_2 e^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\, e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}, e^{-3t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{-t} = 3\, e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4t} + c_2 e^{-3t}\right) + \left(\frac{e^{-t}}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-4t} + c_2 e^{-3t} + \frac{e^{-t}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$2 = c_1 + c_2 + \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 e^{-4t} - 3c_2 e^{-3t} - \frac{e^{-t}}{2}$$

substituting $y' = 1$ and $t = 0$ in the above gives

$$1 = -4c_1 - 3c_2 - \frac{1}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -6 \\ c_2 &= \frac{15}{2} \end{aligned}$$

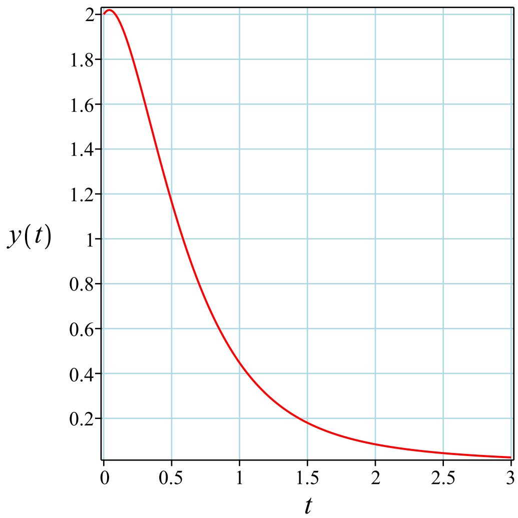
Substituting these values back in above solution results in

$$y = \frac{15\,e^{-3t}}{2} - 6\,e^{-4t} + \frac{e^{-t}}{2}$$

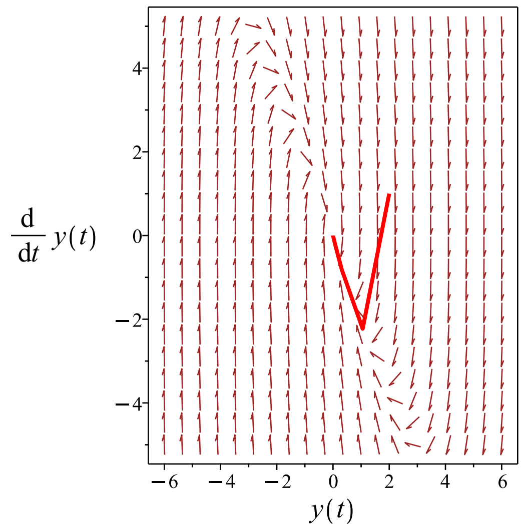
Summary

The solution(s) found are the following

$$y = \frac{15\,e^{-3t}}{2} - 6\,e^{-4t} + \frac{e^{-t}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{15\,e^{-3t}}{2} - 6\,e^{-4t} + \frac{e^{-t}}{2}$$

Verified OK.

16.10.4 Maple step by step solution

Let's solve

$$\left[y'' + 7y' + 12y = 3\,e^{-t}, y(0) = 2, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 7r + 12 = 0$

- Factor the characteristic polynomial

  $(r+4)\,(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -3)$

- 1st solution of the homogeneous ODE

$y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{-3t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 3\,e^{-t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-3t} \\ -4\,e^{-4t} & -3\,e^{-3t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = e^{-7t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -3\,e^{-4t}\left(\int e^{3t}dt\right) + 3\,e^{-3t}\left(\int e^{2t}dt\right)$

  - Compute integrals

    $y_p(t) = \frac{e^{-t}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-3t} + \frac{e^{-t}}{2}$

☐ Check validity of solution $y = c_1 e^{-4t} + c_2 e^{-3t} + \frac{e^{-t}}{2}$

  - Use initial condition $y(0) = 2$

    $2 = c_1 + c_2 + \frac{1}{2}$

  - Compute derivative of the solution

    $y' = -4c_1 e^{-4t} - 3c_2 e^{-3t} - \frac{e^{-t}}{2}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = -4c_1 - 3c_2 - \frac{1}{2}$

- Solve for c_1 and c_2

  $\{c_1 = -6, c_2 = \frac{15}{2}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{15\,e^{-3t}}{2} - 6\,e^{-4t} + \frac{e^{-t}}{2}$

- Solution to the IVP

  $y = \frac{15\,e^{-3t}}{2} - 6\,e^{-4t} + \frac{e^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+7*diff(y(t),t)+12*y(t)=3*exp(-t),y(0) = 2, D(y)(0) = 1],y(t), singsol
```

$$y(t) = \frac{15\,e^{-3t}}{2} - 6\,e^{-4t} + \frac{e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 26

```
DSolve[{y''[t]+7*y'[t]+12*y[t]==3*Exp[-t],{y[0]==2,y'[0]==1}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{1}{2}e^{-4t}\left(15e^t + e^{3t} - 12\right)$$

16.11 problem 11

Internal problem ID [13171]
Internal file name [`OUTPUT/11826_Sunday_December_03_2023_07_17_49_PM_2894948/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 4y' + 13y = -3\,\mathrm{e}^{-2t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}
p(t) &= 4 \\
q(t) &= 13 \\
F &= -3\,\mathrm{e}^{-2t}
\end{aligned}$$

Hence the ode is

$$y'' + 4y' + 13y = -3\,\mathrm{e}^{-2t}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -3\,\mathrm{e}^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.11.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 13, f(t) = -3\,\mathrm{e}^{-2t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 13$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 4\lambda\,\mathrm{e}^{\lambda t} + 13\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(13)} \\ &= -2 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 3i \\ \lambda_2 &= -2 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 3i \\ \lambda_2 &= -2 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-3\,\mathrm{e}^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}\cos(3t), e^{-2t}\sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^{-2t} = -3\, e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{-2t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}(c_1\cos(3t) + c_2\sin(3t))\right) + \left(-\frac{e^{-2t}}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-2t}(c_1\cos(3t) + c_2\sin(3t)) - \frac{e^{-2t}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{3} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\,\mathrm{e}^{-2t}(c_1\cos(3t) + c_2\sin(3t)) + \mathrm{e}^{-2t}(-3c_1\sin(3t) + 3c_2\cos(3t)) + \frac{2\,\mathrm{e}^{-2t}}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 + \frac{2}{3} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{3}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{\mathrm{e}^{-2t}\cos(3t)}{3} - \frac{\mathrm{e}^{-2t}}{3}$$

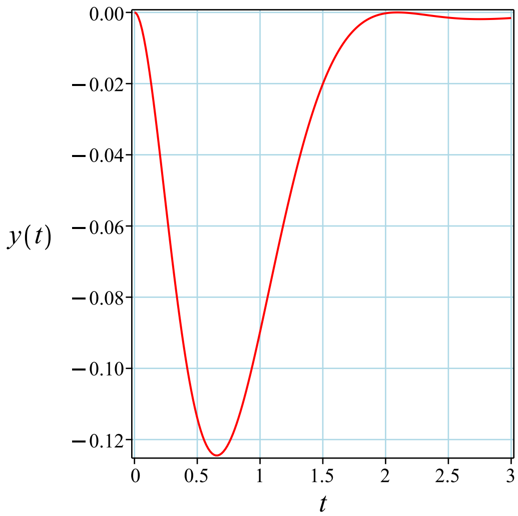
Which simplifies to

$$y = \frac{\mathrm{e}^{-2t}(-1 + \cos(3t))}{3}$$

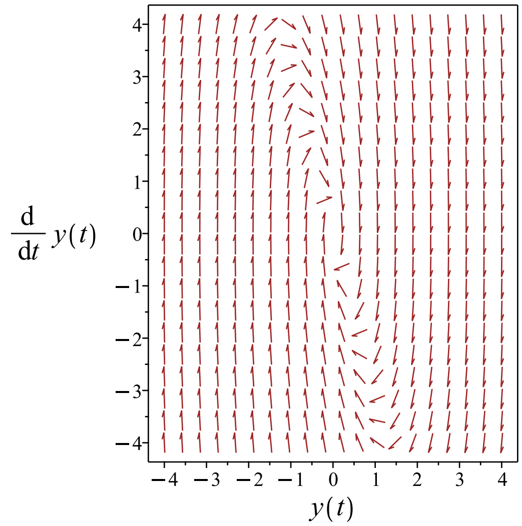
Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-2t}(-1 + \cos(3t))}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-2t}(-1+\cos(3t))}{3}$$

Verified OK.

16.11.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -9$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 431: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\&= 0 - 0\\&= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt}\\&= z_1 e^{-2t}\\&= z_1\left(\mathrm{e}^{-2t}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t}\cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt\\&= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt\\&= y_1\left(\frac{\tan(3t)}{3}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2t} \cos(3t)\right) + c_2 \left(\mathrm{e}^{-2t} \cos(3t) \left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 13y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-2t} \cos(3t)\, c_1 + \frac{\mathrm{e}^{-2t} \sin(3t)\, c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-3\,\mathrm{e}^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-2t} \cos(3t), \frac{\mathrm{e}^{-2t} \sin(3t)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1\mathrm{e}^{-2t} = -3\,\mathrm{e}^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\mathrm{e}^{-2t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2t}\cos(3t)\,c_1 + \frac{\mathrm{e}^{-2t}\sin(3t)\,c_2}{3}\right) + \left(-\frac{\mathrm{e}^{-2t}}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-2t}\cos(3t)\,c_1 + \frac{\mathrm{e}^{-2t}\sin(3t)\,c_2}{3} - \frac{\mathrm{e}^{-2t}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{3} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\,\mathrm{e}^{-2t}\cos(3t)\,c_1 - 3\,\mathrm{e}^{-2t}\sin(3t)\,c_1 - \frac{2\,\mathrm{e}^{-2t}\sin(3t)\,c_2}{3} + \mathrm{e}^{-2t}\cos(3t)\,c_2 + \frac{2\,\mathrm{e}^{-2t}}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 + \frac{2}{3} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{3}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{e^{-2t}\cos(3t)}{3} - \frac{e^{-2t}}{3}$$

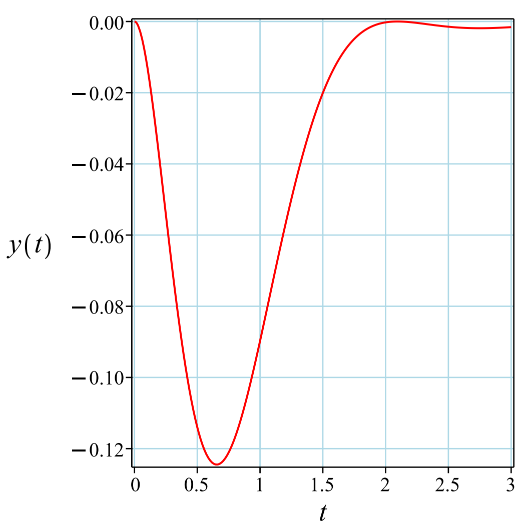
Which simplifies to

$$y = \frac{e^{-2t}(-1+\cos(3t))}{3}$$

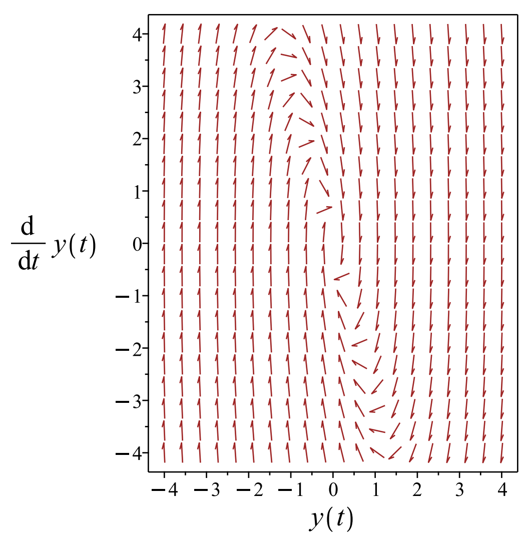
Summary

The solution(s) found are the following

$$y = \frac{e^{-2t}(-1+\cos(3t))}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-2t}(-1+\cos(3t))}{3}$$

Verified OK.

16.11.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 13y = -3\,\mathrm{e}^{-2t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - 3\,\mathrm{I}, -2 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}\cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t}\sin(3t)$

- General solution of the ODE

  $y = c_1y_1(t) + c_2y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-2t}\cos(3t)\,c_1 + \mathrm{e}^{-2t}\sin(3t)\,c_2 + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = -3\,\mathrm{e}^{-2t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t}\cos(3t) & \mathrm{e}^{-2t}\sin(3t) \\ -2\,\mathrm{e}^{-2t}\cos(3t) - 3\,\mathrm{e}^{-2t}\sin(3t) & -2\,\mathrm{e}^{-2t}\sin(3t) + 3\,\mathrm{e}^{-2t}\cos(3t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,\mathrm{e}^{-4t}$

- ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = e^{-2t}\left(\cos(3t)\left(\int \sin(3t)\,dt\right) - \sin(3t)\left(\int \cos(3t)\,dt\right)\right)$

- Compute integrals

  $y_p(t) = -\frac{e^{-2t}}{3}$

- Substitute particular solution into general solution to ODE

  $y = e^{-2t}\cos(3t)\,c_1 + e^{-2t}\sin(3t)\,c_2 - \frac{e^{-2t}}{3}$

□ Check validity of solution $y = e^{-2t}\cos(3t)\,c_1 + e^{-2t}\sin(3t)\,c_2 - \frac{e^{-2t}}{3}$

- Use initial condition $y(0) = 0$

  $0 = -\frac{1}{3} + c_1$

- Compute derivative of the solution

  $y' = -2\,e^{-2t}\cos(3t)\,c_1 - 3\,e^{-2t}\sin(3t)\,c_1 - 2\,e^{-2t}\sin(3t)\,c_2 + 3\,e^{-2t}\cos(3t)\,c_2 + \frac{2\,e^{-2t}}{3}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -2c_1 + \frac{2}{3} + 3c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{3}, c_2 = 0\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^{-2t}(-1+\cos(3t))}{3}$

- Solution to the IVP

  $y = \frac{e^{-2t}(-1+\cos(3t))}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+13*y(t)=-3*exp(-2*t),y(0) = 0, D(y)(0) = 0],y(t), sing
```

$$y(t) = \frac{e^{-2t}(\cos(3t) - 1)}{3}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 20

```
DSolve[{y''[t]+4*y'[t]+13*y[t]==-3*Exp[-2*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolut
```

$$y(t) \to \frac{1}{3}e^{-2t}(\cos(3t) - 1)$$

16.12 problem 12

Internal problem ID [13172]
Internal file name [`OUTPUT/11827_Sunday_December_03_2023_07_17_53_PM_92935846/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''+7y'+10y = \mathrm{e}^{-2t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(t)y'+q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 7 \\ q(t) &= 10 \\ F &= \mathrm{e}^{-2t} \end{aligned}$$

Hence the ode is

$$y'' + 7y' + 10y = e^{-2t}$$

The domain of $p(t) = 7$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.12.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 7, C = 10, f(t) = e^{-2t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 7y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 7, C = 10$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 7\lambda e^{\lambda t} + 10 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 7\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 7, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-7}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{7^2 - (4)(1)(10)} \\ &= -\frac{7}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{7}{2} + \frac{3}{2} \\ \lambda_2 &= -\frac{7}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 \\ \lambda_2 &= -5\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ y &= c_1 e^{(-2)t} + c_2 e^{(-5)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-5t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-5t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-5t}, e^{-2t}\}$$

Since e^{-2t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,e^{-2t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t\,e^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{-2t} = e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{t\,e^{-2t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-5t}\right) + \left(\frac{t\,e^{-2t}}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + c_2 e^{-5t} + \frac{t\, e^{-2t}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2t} - 5c_2 e^{-5t} + \frac{e^{-2t}}{3} - \frac{2t\, e^{-2t}}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 5c_2 + \frac{1}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{9}$$
$$c_2 = \frac{1}{9}$$

Substituting these values back in above solution results in

$$y = -\frac{e^{-2t}}{9} + \frac{e^{-5t}}{9} + \frac{t\, e^{-2t}}{3}$$

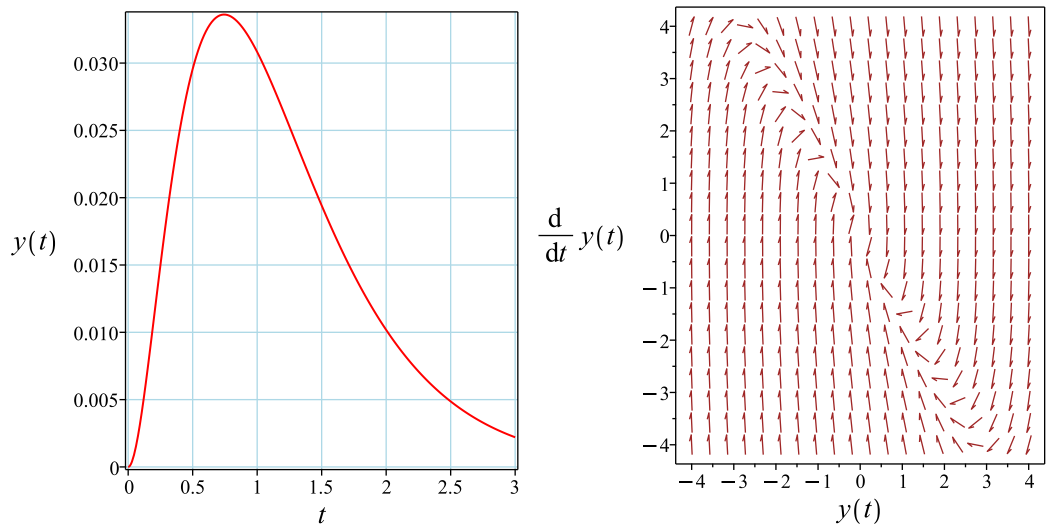
Which simplifies to

$$y = \frac{e^{-2t}(-1+3t)}{9} + \frac{e^{-5t}}{9}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2t}(-1+3t)}{9} + \frac{e^{-5t}}{9} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{-2t}(-1+3t)}{9} + \frac{e^{-5t}}{9}$$

Verified OK.

16.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 7y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 7 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{3t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7}{1}\,dt} \\ &= z_1 e^{-\frac{7t}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{7t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{7}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-7t}}{(y_1)^2}\,dt \\ &= y_1 \left(\frac{\mathrm{e}^{3t}}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-5t}\right) + c_2\left(\mathrm{e}^{-5t}\left(\frac{\mathrm{e}^{3t}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 7y' + 10y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-5t} + \frac{c_2 \mathrm{e}^{-2t}}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-5t} \\
y_2 &= \frac{\mathrm{e}^{-2t}}{3}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}
$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-5t} & \frac{e^{-2t}}{3} \\ \frac{d}{dt}\left(e^{-5t}\right) & \frac{d}{dt}\left(\frac{e^{-2t}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-5t} & \frac{e^{-2t}}{3} \\ -5\,e^{-5t} & -\frac{2\,e^{-2t}}{3} \end{vmatrix}$$

Therefore

$$W = \left(e^{-5t}\right)\left(-\frac{2\,e^{-2t}}{3}\right) - \left(\frac{e^{-2t}}{3}\right)\left(-5\,e^{-5t}\right)$$

Which simplifies to

$$W = e^{-5t}e^{-2t}$$

Which simplifies to

$$W = e^{-7t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-4t}}{3}}{e^{-7t}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{e^{3t}}{3}dt$$

Hence

$$u_1 = -\frac{e^{3t}}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-5t}e^{-2t}}{e^{-7t}}\,dt$$

Which simplifies to

$$u_2 = \int 1 dt$$

Hence

$$u_2 = t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{e^{-5t} e^{3t}}{9} + \frac{t\, e^{-2t}}{3}$$

Which simplifies to

$$y_p(t) = \frac{e^{-2t}(-1 + 3t)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-5t} + \frac{c_2 e^{-2t}}{3}\right) + \left(\frac{e^{-2t}(-1 + 3t)}{9}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-5t} + \frac{c_2 e^{-2t}}{3} + \frac{e^{-2t}(-1 + 3t)}{9} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{3} - \frac{1}{9} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -5c_1 e^{-5t} - \frac{2c_2 e^{-2t}}{3} - \frac{2\, e^{-2t}(-1 + 3t)}{9} + \frac{e^{-2t}}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -5c_1 - \frac{2c_2}{3} + \frac{5}{9} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{9}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

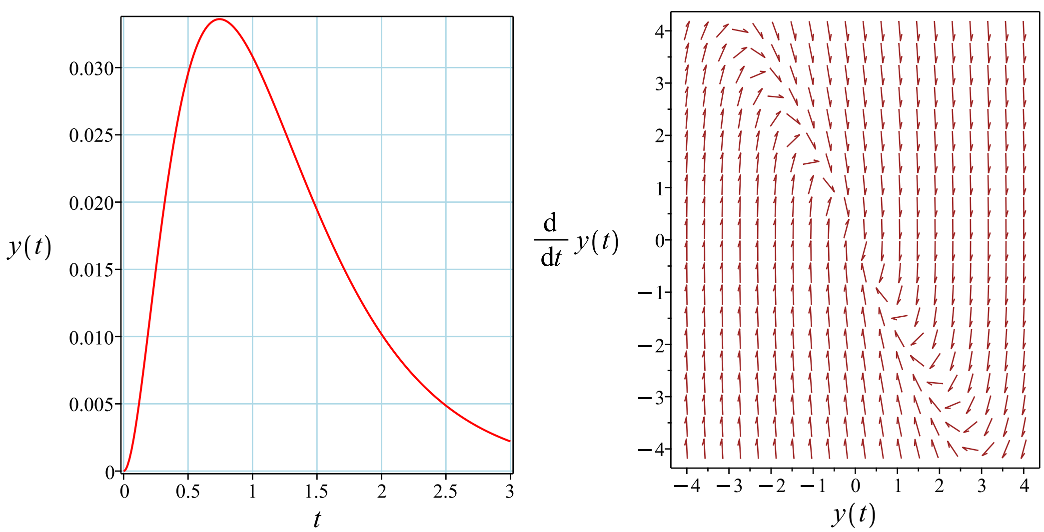
$$y = -\frac{e^{-2t}}{9} + \frac{e^{-5t}}{9} + \frac{t\,e^{-2t}}{3}$$

Which simplifies to

$$y = \frac{e^{-2t}(-1+3t)}{9} + \frac{e^{-5t}}{9}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-2t}(-1+3t)}{9} + \frac{e^{-5t}}{9} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{-2t}(-1+3t)}{9} + \frac{e^{-5t}}{9}$$

Verified OK.

16.12.4 Maple step by step solution

Let's solve

$$\left[y'' + 7y' + 10y = e^{-2t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 7r + 10 = 0$

- Factor the characteristic polynomial

  $(r + 5)(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-5, -2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-5t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-5t} + c_2 e^{-2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = e^{-2t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-5t} & e^{-2t} \\ -5\,e^{-5t} & -2\,e^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 3\,e^{-7t}$

  - Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{\mathrm{e}^{-5t}\left(\int \mathrm{e}^{3t}dt\right)}{3} + \frac{\mathrm{e}^{-2t}\left(\int 1dt\right)}{3}$$

- Compute integrals

$$y_p(t) = \frac{\mathrm{e}^{-2t}(-1+3t)}{9}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-5t} + c_2\mathrm{e}^{-2t} + \frac{\mathrm{e}^{-2t}(-1+3t)}{9}$$

□ Check validity of solution $y = c_1\mathrm{e}^{-5t} + c_2\mathrm{e}^{-2t} + \frac{\mathrm{e}^{-2t}(-1+3t)}{9}$

- Use initial condition $y(0) = 0$

$$0 = c_1 + c_2 - \frac{1}{9}$$

- Compute derivative of the solution

$$y' = -5c_1\mathrm{e}^{-5t} - 2c_2\mathrm{e}^{-2t} - \frac{2\,\mathrm{e}^{-2t}(-1+3t)}{9} + \frac{\mathrm{e}^{-2t}}{3}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -5c_1 - 2c_2 + \frac{5}{9}$$

- Solve for c_1 and c_2

$$\left\{c_1 = \frac{1}{9}, c_2 = 0\right\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{\mathrm{e}^{-2t}(-1+3t)}{9} + \frac{\mathrm{e}^{-5t}}{9}$$

- Solution to the IVP

$$y = \frac{\mathrm{e}^{-2t}(-1+3t)}{9} + \frac{\mathrm{e}^{-5t}}{9}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(y(t),t$2)+7*diff(y(t),t)+10*y(t)=exp(-2*t),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \frac{(3t-1)\,\mathrm{e}^{-2t}}{9} + \frac{\mathrm{e}^{-5t}}{9}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 27

```
DSolve[{y''[t]+7*y'[t]+10*y[t]==Exp[-2*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{1}{9}e^{-5t}\left(e^{3t}(3t-1)+1\right)$$

16.13 problem 13

Internal problem ID [13173]
Internal file name [OUTPUT/11828_Sunday_December_03_2023_07_18_11_PM_73207322/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 3y = \mathrm{e}^{-\frac{t}{2}}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3 \\ F &= \mathrm{e}^{-\frac{t}{2}} \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 3y = e^{-\frac{t}{2}}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-\frac{t}{2}}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.13.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 3, f(t) = e^{-\frac{t}{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 3 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(3)}\\ &= -2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 1\\ \lambda_2 &= -2 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-1)t} + c_2 e^{(-3)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-3t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-\frac{t}{2}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\mathrm{e}^{-\frac{t}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-\frac{t}{2}}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\frac{5A_1 e^{-\frac{t}{2}}}{4} = e^{-\frac{t}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\, e^{-\frac{t}{2}}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 e^{-3t}\right) + \left(\frac{4\, e^{-\frac{t}{2}}}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-t} + c_2 e^{-3t} + \frac{4\, e^{-\frac{t}{2}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{4}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-t} - 3c_2 e^{-3t} - \frac{2\,e^{-\frac{t}{2}}}{5}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 - 3c_2 - \frac{2}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = \frac{1}{5}$$

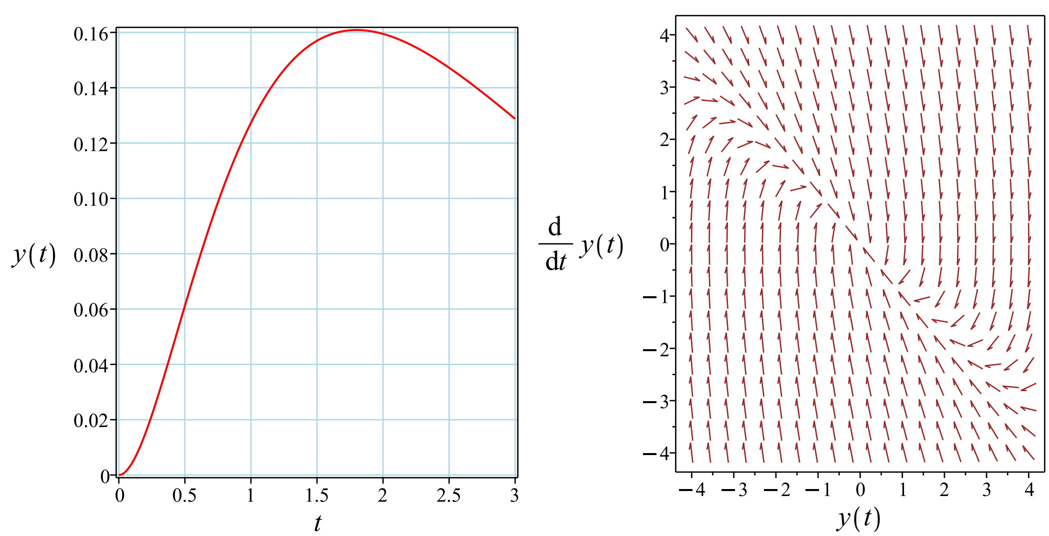
Substituting these values back in above solution results in

$$y = -e^{-t} + \frac{e^{-3t}}{5} + \frac{4\,e^{-\frac{t}{2}}}{5}$$

Summary

The solution(s) found are the following

$$y = -e^{-t} + \frac{e^{-3t}}{5} + \frac{4\,e^{-\frac{t}{2}}}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -e^{-t} + \frac{e^{-3t}}{5} + \frac{4\,e^{-\frac{t}{2}}}{5}$$

Verified OK.

16.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4}{1} \, dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-2t} \\
&= z_1\left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dt}}{\left(y_1\right)^2}\, dt \\
&= y_1 \int \frac{e^{-4t}}{\left(y_1\right)^2}\, dt \\
&= y_1\left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3t}\right) + c_2\left(\mathrm{e}^{-3t}\left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-3t} c_1 + \frac{c_2 \mathrm{e}^{-t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-\frac{t}{2}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{e^{-\frac{t}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-t}}{2}, e^{-3t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-\frac{t}{2}}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\frac{5A_1 e^{-\frac{t}{2}}}{4} = e^{-\frac{t}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\,e^{-\frac{t}{2}}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-3t} c_1 + \frac{c_2 e^{-t}}{2}\right) + \left(\frac{4\,e^{-\frac{t}{2}}}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-3t} c_1 + \frac{c_2 e^{-t}}{2} + \frac{4 e^{-\frac{t}{2}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} + \frac{4}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3 e^{-3t} c_1 - \frac{c_2 e^{-t}}{2} - \frac{2 e^{-\frac{t}{2}}}{5}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 - \frac{c_2}{2} - \frac{2}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{5}$$
$$c_2 = -2$$

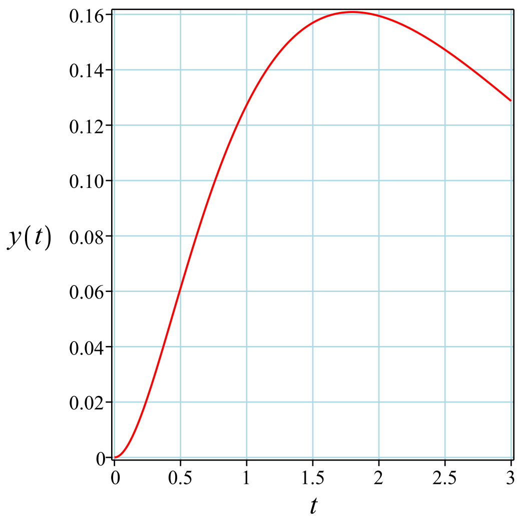
Substituting these values back in above solution results in

$$y = -e^{-t} + \frac{e^{-3t}}{5} + \frac{4 e^{-\frac{t}{2}}}{5}$$

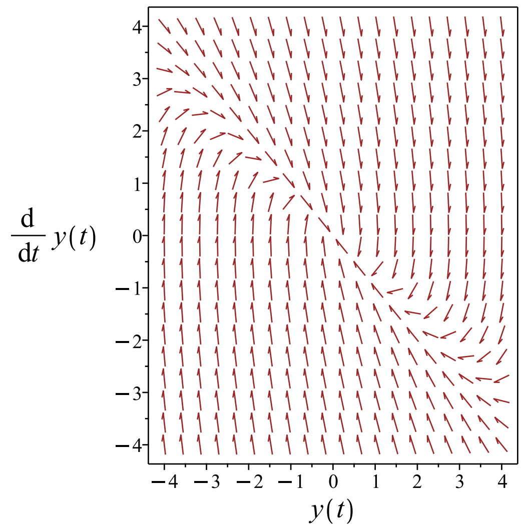
Summary

The solution(s) found are the following

$$y = -e^{-t} + \frac{e^{-3t}}{5} + \frac{4 e^{-\frac{t}{2}}}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -e^{-t} + \frac{e^{-3t}}{5} + \frac{4\,e^{-\frac{t}{2}}}{5}$$

Verified OK.

16.13.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 3y = e^{-\frac{t}{2}}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r+3)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -1)$

- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-3t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathrm{e}^{-\frac{t}{2}}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t} & \mathrm{e}^{-t} \\ -3\,\mathrm{e}^{-3t} & -\mathrm{e}^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{-4t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\mathrm{e}^{-3t}\left(\int \mathrm{e}^{\frac{5t}{2}} dt\right)}{2} + \frac{\mathrm{e}^{-t}\left(\int \mathrm{e}^{\frac{t}{2}} dt\right)}{2}$

  - Compute integrals

    $y_p(t) = \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{5}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-t} + \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{5}$

☐ Check validity of solution $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-t} + \frac{4\mathrm{e}^{-\frac{t}{2}}}{5}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2 + \frac{4}{5}$

  - Compute derivative of the solution

    $y' = -3\,\mathrm{e}^{-3t} c_1 - c_2 \mathrm{e}^{-t} - \frac{2\,\mathrm{e}^{-\frac{t}{2}}}{5}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$0=-3c_1-c_2-\frac{2}{5}$

- Solve for c_1 and c_2

  $\left\{c_1=\frac{1}{5}, c_2=-1\right\}$

- Substitute constant values into general solution and simplify

  $y=-\mathrm{e}^{-t}+\frac{\mathrm{e}^{-3t}}{5}+\frac{4\,\mathrm{e}^{-\frac{t}{2}}}{5}$

- Solution to the IVP

  $y=-\mathrm{e}^{-t}+\frac{\mathrm{e}^{-3t}}{5}+\frac{4\,\mathrm{e}^{-\frac{t}{2}}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+3*y(t)=exp(-t/2),y(0) = 0, D(y)(0) = 0],y(t), singsol=
```

$$y(t)=\frac{\mathrm{e}^{-3t}}{5}-\mathrm{e}^{-t}+\frac{4\,\mathrm{e}^{-\frac{t}{2}}}{5}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 32

```
DSolve[{y''[t]+4*y'[t]+3*y[t]==Exp[-t/2],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions
```

$$y(t)\to \frac{1}{5}e^{-3t}\left(-5e^{2t}+4e^{5t/2}+1\right)$$

16.14 problem 14

Internal problem ID [13174]
Internal file name [`OUTPUT/11829_Sunday_December_03_2023_07_18_14_PM_54963002/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 3y = e^{-2t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3 \\ F &= e^{-2t} \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 3y = e^{-2t}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.14.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 3, f(t) = e^{-2t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 3 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(3)} \\ &= -2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 1 \\ \lambda_2 &= -2 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ y &= c_1 e^{(-1)t} + c_2 e^{(-3)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-3t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-2t} = e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -e^{-2t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-t} + c_2 e^{-3t}) + (-e^{-2t}) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-t} + c_2 e^{-3t} - e^{-2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-t} - 3c_2 e^{-3t} + 2 e^{-2t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 - 3c_2 + 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = \frac{1}{2}$$

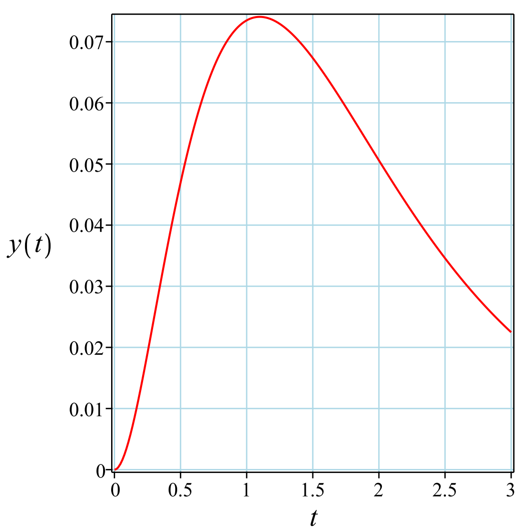
Substituting these values back in above solution results in

$$y = \frac{e^{-t}}{2} + \frac{e^{-3t}}{2} - e^{-2t}$$

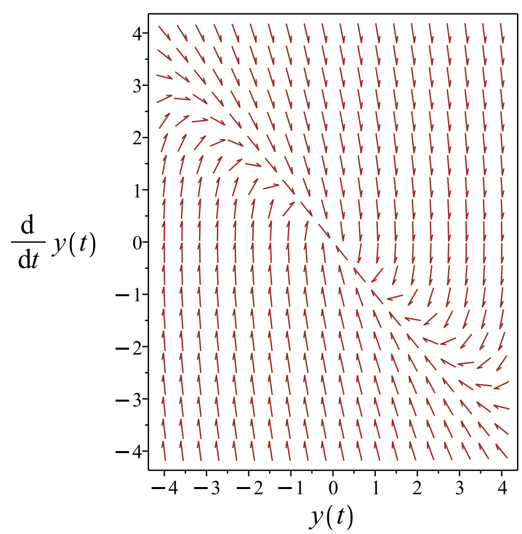
Summary

The solution(s) found are the following

$$y = \frac{e^{-t}}{2} + \frac{e^{-3t}}{2} - e^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-t}}{2} + \frac{e^{-3t}}{2} - e^{-2t}$$

Verified OK.

16.14.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4}{1} \, dt}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-2t} \\
&= z_1 \left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dt}}{\left(y_1\right)^2}\, dt \\
&= y_1 \int \frac{e^{-4t}}{\left(y_1\right)^2}\, dt \\
&= y_1 \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-3t} c_1 + \frac{c_2 \mathrm{e}^{-t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-t}}{2}, e^{-3t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-2t} = e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -e^{-2t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-3t} c_1 + \frac{c_2 e^{-t}}{2}\right) + \left(-e^{-2t}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-3t}c_1 + \frac{c_2 e^{-t}}{2} - e^{-2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,e^{-3t}c_1 - \frac{c_2 e^{-t}}{2} + 2\,e^{-2t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 - \frac{c_2}{2} + 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = 1$$

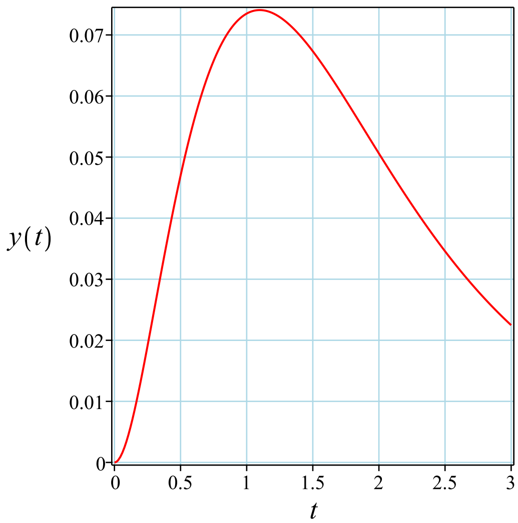
Substituting these values back in above solution results in

$$y = \frac{e^{-t}}{2} + \frac{e^{-3t}}{2} - e^{-2t}$$

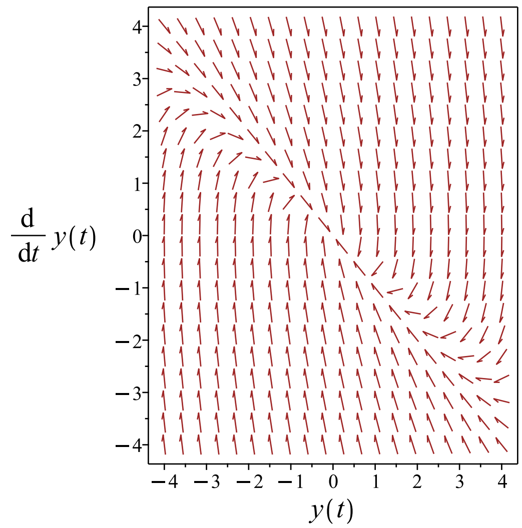
Summary

The solution(s) found are the following

$$y = \frac{e^{-t}}{2} + \frac{e^{-3t}}{2} - e^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-t}}{2} + \frac{e^{-3t}}{2} - e^{-2t}$$

Verified OK.

16.14.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 3y = e^{-2t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r + 3)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -1)$

- 1st solution of the homogeneous ODE

$y_1(t) = e^{-3t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{-3t} c_1 + c_2 e^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = e^{-2t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -3\,e^{-3t} & -e^{-t} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-4t}$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^{-3t}\left(\int e^{t} dt\right)}{2} + \frac{e^{-t}\left(\int e^{-t} dt\right)}{2}$

  ◦ Compute integrals

  $y_p(t) = -e^{-2t}$

- Substitute particular solution into general solution to ODE

  $y = e^{-3t} c_1 + c_2 e^{-t} - e^{-2t}$

☐ Check validity of solution $y = e^{-3t} c_1 + c_2 e^{-t} - e^{-2t}$

  ◦ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - 1$

  ◦ Compute derivative of the solution

  $y' = -3\,e^{-3t} c_1 - c_2 e^{-t} + 2\,e^{-2t}$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -3c_1 - c_2 + 2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{2}, c_2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^{-t}}{2} + \frac{e^{-3t}}{2} - e^{-2t}$

- Solution to the IVP

  $y = \frac{e^{-t}}{2} + \frac{e^{-3t}}{2} - e^{-2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+3*y(t)=exp(-2*t),y(0) = 0, D(y)(0) = 0],y(t), singsol=
```

$$y(t) = \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} - e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 21

```
DSolve[{y''[t]+4*y'[t]+3*y[t]==Exp[-2*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to \frac{1}{2}e^{-3t}\left(e^t - 1\right)^2$$

16.15 problem 15

Internal problem ID [13175]
Internal file name [OUTPUT/11830_Sunday_December_03_2023_07_18_17_PM_24795958/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 3y = e^{-4t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3 \\ F &= e^{-4t} \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 3y = e^{-4t}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-4t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.15.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 3, f(t) = e^{-4t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 3 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(3)}\\ &= -2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 1\\ \lambda_2 &= -2 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-1)t} + c_2 e^{(-3)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-3t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-4t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-4t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{-4t} = e^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-4t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 e^{-3t}\right) + \left(\frac{e^{-4t}}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-t} + c_2 e^{-3t} + \frac{e^{-4t}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-t} - 3c_2 e^{-3t} - \frac{4\,e^{-4t}}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 - 3c_2 - \frac{4}{3} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{6}$$
$$c_2 = -\frac{1}{2}$$

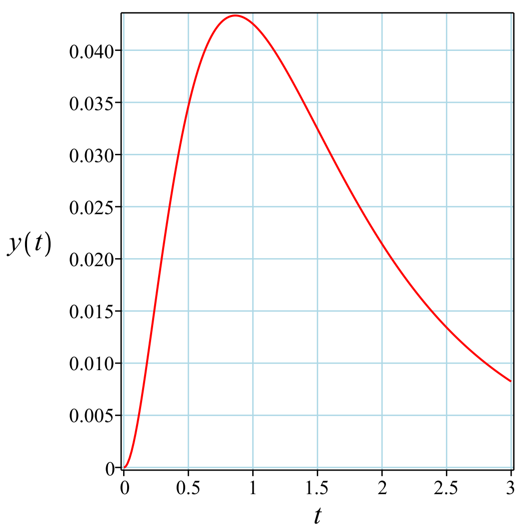
Substituting these values back in above solution results in

$$y = \frac{e^{-t}}{6} - \frac{e^{-3t}}{2} + \frac{e^{-4t}}{3}$$

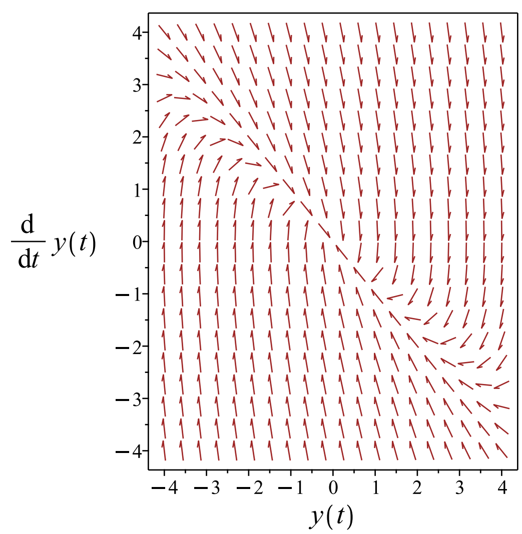
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^{-t}}{6} - \frac{e^{-3t}}{2} + \frac{e^{-4t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{-t}}{6} - \frac{e^{-3t}}{2} + \frac{e^{-4t}}{3}$$

Verified OK.

16.15.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt}
\end{aligned}
$$

$$= z_1 e^{-2t}$$
$$= z_1 \left(\mathrm{e}^{-2t}\right)$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt$$
$$= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt$$
$$= y_1 \left(\frac{\mathrm{e}^{2t}}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-3t} c_1 + \frac{c_2 \mathrm{e}^{-t}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-4t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-t}}{2}, e^{-3t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-4t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{-4t} = e^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-4t}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-3t}c_1 + \frac{c_2 e^{-t}}{2}\right) + \left(\frac{e^{-4t}}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-3t} c_1 + \frac{c_2 e^{-t}}{2} + \frac{e^{-4t}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} + \frac{1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\, e^{-3t} c_1 - \frac{c_2 e^{-t}}{2} - \frac{4\, e^{-4t}}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 - \frac{c_2}{2} - \frac{4}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{1}{2} \\ c_2 &= \frac{1}{3} \end{aligned}$$

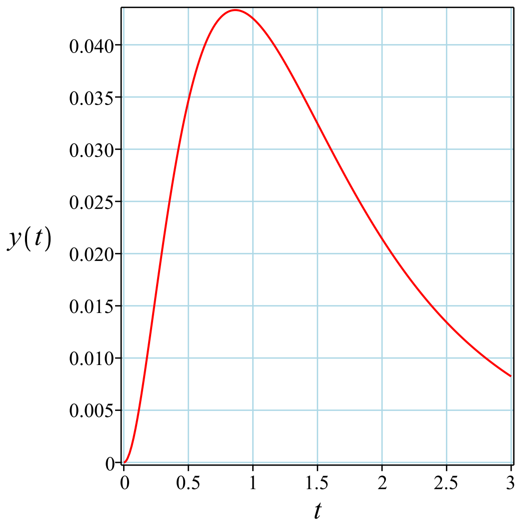
Substituting these values back in above solution results in

$$y = \frac{e^{-t}}{6} - \frac{e^{-3t}}{2} + \frac{e^{-4t}}{3}$$

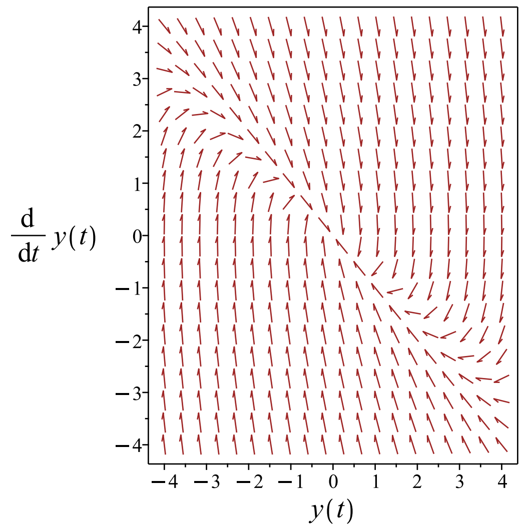
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^{-t}}{6} - \frac{e^{-3t}}{2} + \frac{e^{-4t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{-t}}{6} - \frac{e^{-3t}}{2} + \frac{e^{-4t}}{3}$$

Verified OK.

16.15.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 3y = e^{-4t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r + 3)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -1)$

- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-3t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathrm{e}^{-4t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t} & \mathrm{e}^{-t} \\ -3\,\mathrm{e}^{-3t} & -\mathrm{e}^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{-4t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\mathrm{e}^{-3t}\left(\int \mathrm{e}^{-t}dt\right)}{2} + \frac{\mathrm{e}^{-t}\left(\int \mathrm{e}^{-3t}dt\right)}{2}$

  - Compute integrals

    $y_p(t) = \frac{\mathrm{e}^{-4t}}{3}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-t} + \frac{\mathrm{e}^{-4t}}{3}$

☐ Check validity of solution $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-t} + \frac{\mathrm{e}^{-4t}}{3}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2 + \frac{1}{3}$

  - Compute derivative of the solution

    $y' = -3\,\mathrm{e}^{-3t} c_1 - c_2 \mathrm{e}^{-t} - \frac{4\,\mathrm{e}^{-4t}}{3}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -3c_1 - c_2 - \frac{4}{3}$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{2}, c_2 = \frac{1}{6}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^{-t}}{6} - \frac{e^{-3t}}{2} + \frac{e^{-4t}}{3}$

- Solution to the IVP

  $y = \frac{e^{-t}}{6} - \frac{e^{-3t}}{2} + \frac{e^{-4t}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+3*y(t)=exp(-4*t),y(0) = 0, D(y)(0) = 0],y(t), singsol=
```

$$y(t) = -\frac{e^{-3t}}{2} + \frac{e^{-t}}{6} + \frac{e^{-4t}}{3}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 26

```
DSolve[{y''[t]+4*y'[t]+3*y[t]==Exp[-4*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to \frac{1}{6}e^{-4t}(e^t - 1)^2 (e^t + 2)$$

16.16 problem 16

Internal problem ID [13176]
Internal file name [`OUTPUT/11831_Sunday_December_03_2023_07_18_20_PM_5833124/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 20y = e^{-\frac{t}{2}}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 4$$
$$q(t) = 20$$
$$F = e^{-\frac{t}{2}}$$

Hence the ode is

$$y'' + 4y' + 20y = e^{-\frac{t}{2}}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 20$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-\frac{t}{2}}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.16.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 20, f(t) = e^{-\frac{t}{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 20$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 20 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(20)} \\ &= -2 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 4i \\ \lambda_2 &= -2 - 4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 4i \\ \lambda_2 &= -2 - 4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-\frac{t}{2}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{e^{-\frac{t}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4t)\,e^{-2t}, \sin(4t)\,e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-\frac{t}{2}}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\frac{73A_1 e^{-\frac{t}{2}}}{4} = e^{-\frac{t}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{73}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\,e^{-\frac{t}{2}}}{73}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}(c_1\cos(4t) + c_2\sin(4t))\right) + \left(\frac{4\,e^{-\frac{t}{2}}}{73}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-2t}(c_1\cos(4t) + c_2\sin(4t)) + \frac{4\,e^{-\frac{t}{2}}}{73} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{4}{73} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\,\mathrm{e}^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \mathrm{e}^{-2t}(-4c_1 \sin(4t) + 4c_2 \cos(4t)) - \frac{2\,\mathrm{e}^{-\frac{t}{2}}}{73}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - \frac{2}{73} + 4c_2 \tag{2A}$$

Equations $\{1\mathrm{A},2\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{4}{73} \\ c_2 &= -\frac{3}{146} \end{aligned}$$

Substituting these values back in above solution results in

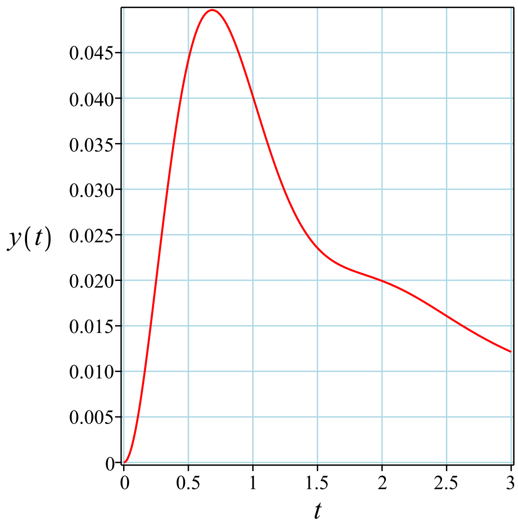
$$y = -\frac{4\cos(4t)\,\mathrm{e}^{-2t}}{73} - \frac{3\sin(4t)\,\mathrm{e}^{-2t}}{146} + \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{73}$$

Which simplifies to

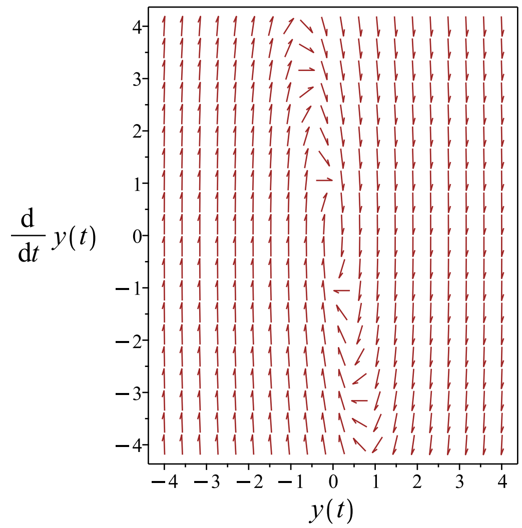
$$y = \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{73} + \frac{(-8\cos(4t) - 3\sin(4t))\,\mathrm{e}^{-2t}}{146}$$

Summary
The solution(s) found are the following

$$y = \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{73} + \frac{(-8\cos(4t) - 3\sin(4t))\,\mathrm{e}^{-2t}}{146} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{73} + \frac{(-8\cos(4t) - 3\sin(4t))\,\mathrm{e}^{-2t}}{146}$$

Verified OK.

16.16.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$
$$B = 4 \tag{3}$$
$$C = 20$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -16 \\ t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -16z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 441: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(4t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt}\\ &= z_1 e^{-2t}\\ &= z_1\left(\mathrm{e}^{-2t}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(4t)\,\mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt\\ &= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt\\ &= y_1\left(\frac{\tan(4t)}{4}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(4t)\, \mathrm{e}^{-2t}\right) + c_2 \left(\cos(4t)\, \mathrm{e}^{-2t} \left(\frac{\tan(4t)}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' + 20y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(4t)\, \mathrm{e}^{-2t} + \frac{\mathrm{e}^{-2t} c_2 \sin(4t)}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{-\frac{t}{2}}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
\left[\left\{\mathrm{e}^{-\frac{t}{2}}\right\}\right]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\cos(4t)\, \mathrm{e}^{-2t}, \frac{\sin(4t)\, \mathrm{e}^{-2t}}{4}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-\frac{t}{2}}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\frac{73A_1\mathrm{e}^{-\frac{t}{2}}}{4} = \mathrm{e}^{-\frac{t}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{73}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{73}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(4t)\,\mathrm{e}^{-2t} + \frac{\mathrm{e}^{-2t} c_2 \sin(4t)}{4}\right) + \left(\frac{4\,\mathrm{e}^{-\frac{t}{2}}}{73}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(4t)\,\mathrm{e}^{-2t} + \frac{\mathrm{e}^{-2t} c_2 \sin(4t)}{4} + \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{73} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{4}{73} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 \sin(4t)\,\mathrm{e}^{-2t} - 2c_1 \cos(4t)\,\mathrm{e}^{-2t} - \frac{\mathrm{e}^{-2t} c_2 \sin(4t)}{2} + \mathrm{e}^{-2t} c_2 \cos(4t) - \frac{2\,\mathrm{e}^{-\frac{t}{2}}}{73}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{2}{73} - 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{4}{73}$$
$$c_2 = -\frac{6}{73}$$

Substituting these values back in above solution results in

$$y = -\frac{4\cos(4t)\,e^{-2t}}{73} - \frac{3\sin(4t)\,e^{-2t}}{146} + \frac{4\,e^{-\frac{t}{2}}}{73}$$

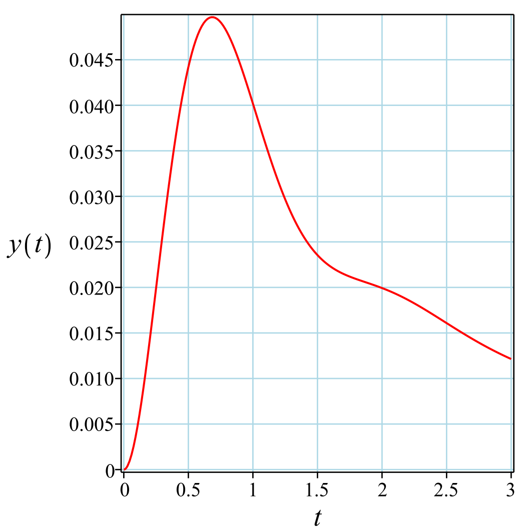
Which simplifies to

$$y = \frac{4\,e^{-\frac{t}{2}}}{73} + \frac{(-8\cos(4t) - 3\sin(4t))\,e^{-2t}}{146}$$

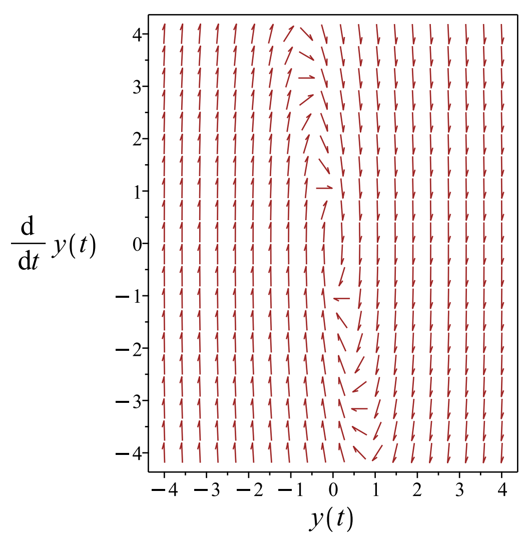
Summary

The solution(s) found are the following

$$y = \frac{4\,e^{-\frac{t}{2}}}{73} + \frac{(-8\cos(4t) - 3\sin(4t))\,e^{-2t}}{146} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4\,e^{-\frac{t}{2}}}{73} + \frac{(-8\cos(4t) - 3\sin(4t))\,e^{-2t}}{146}$$

Verified OK.

16.16.4 Maple step by step solution

Let's solve

$$\left[y''+4y'+20y=\mathrm{e}^{-\frac{t}{2}}, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4r+20=0$

- Use quadratic formula to solve for r

  $r=\frac{(-4)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r=(-2-4\,\mathrm{I}, -2+4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(4t)\,\mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(4t)\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(4t)\,\mathrm{e}^{-2t}+\mathrm{e}^{-2t}c_2\sin(4t)+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\mathrm{e}^{-\frac{t}{2}}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t))=\begin{bmatrix}\cos(4t)\,\mathrm{e}^{-2t} & \sin(4t)\,\mathrm{e}^{-2t}\\ -4\sin(4t)\,\mathrm{e}^{-2t}-2\cos(4t)\,\mathrm{e}^{-2t} & 4\cos(4t)\,\mathrm{e}^{-2t}-2\sin(4t)\,\mathrm{e}^{-2t}\end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t))=4\,\mathrm{e}^{-4t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{e^{-2t}\left(\cos(4t)\left(\int \sin(4t)e^{\frac{3t}{2}}dt\right)-\sin(4t)\left(\int \cos(4t)e^{\frac{3t}{2}}dt\right)\right)}{4}$$

- Compute integrals

$$y_p(t) = \frac{4\,e^{-\frac{t}{2}}}{73}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(4t)\,e^{-2t} + e^{-2t}c_2\sin(4t) + \frac{4\,e^{-\frac{t}{2}}}{73}$$

☐ Check validity of solution $y = c_1\cos(4t)\,e^{-2t} + e^{-2t}c_2\sin(4t) + \frac{4e^{-\frac{t}{2}}}{73}$

- Use initial condition $y(0) = 0$

$$0 = c_1 + \frac{4}{73}$$

- Compute derivative of the solution

$$y' = -4c_1\sin(4t)\,e^{-2t} - 2c_1\cos(4t)\,e^{-2t} - 2\,e^{-2t}c_2\sin(4t) + 4\,e^{-2t}c_2\cos(4t) - \frac{2\,e^{-\frac{t}{2}}}{73}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -2c_1 - \frac{2}{73} + 4c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{4}{73}, c_2 = -\frac{3}{146}\right\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{4\,e^{-\frac{t}{2}}}{73} + \frac{(-8\cos(4t)-3\sin(4t))e^{-2t}}{146}$$

- Solution to the IVP

$$y = \frac{4\,e^{-\frac{t}{2}}}{73} + \frac{(-8\cos(4t)-3\sin(4t))e^{-2t}}{146}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+20*y(t)=exp(-t/2),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \frac{4\,\mathrm{e}^{-\frac{t}{2}}}{73} + \frac{(-3\sin(4t) - 8\cos(4t))\,\mathrm{e}^{-2t}}{146}$$

✓ Solution by Mathematica

Time used: 0.259 (sec). Leaf size: 36

```
DSolve[{y''[t]+4*y'[t]+20*y[t]==Exp[-t/2],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{1}{146} e^{-2t} \left(8e^{3t/2} - 3\sin(4t) - 8\cos(4t)\right)$$

16.17 problem 17

Internal problem ID [13177]
Internal file name [`OUTPUT/11832_Sunday_December_03_2023_07_18_27_PM_39793253/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 20y = e^{-2t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 20 \\ F &= e^{-2t} \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 20y = e^{-2t}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 20$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.17.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 20, f(t) = e^{-2t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 20$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 20 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(20)}\\ &= -2 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 4i\\ \lambda_2 &= -2 - 4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 4i\\ \lambda_2 &= -2 - 4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4t)\,e^{-2t}, \sin(4t)\,e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1 e^{-2t} = e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{16}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-2t}}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))\right) + \left(\frac{e^{-2t}}{16}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{e^{-2t}}{16} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{16} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\,\mathrm{e}^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \mathrm{e}^{-2t}(-4c_1 \sin(4t) + 4c_2 \cos(4t)) - \frac{\mathrm{e}^{-2t}}{8}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - \frac{1}{8} + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{16}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -\frac{\cos(4t)\,\mathrm{e}^{-2t}}{16} + \frac{\mathrm{e}^{-2t}}{16}$$

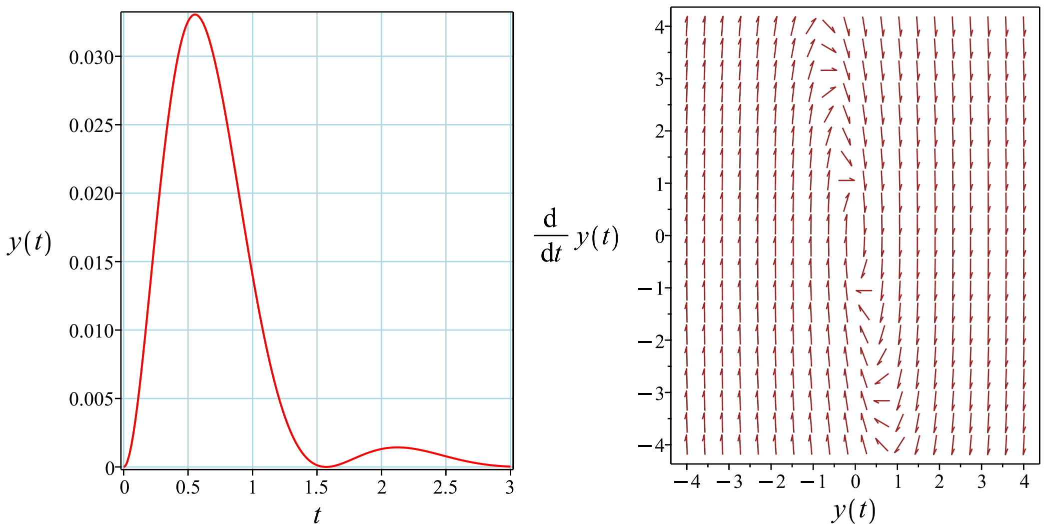
Which simplifies to

$$y = -\frac{\mathrm{e}^{-2t}(-1 + \cos(4t))}{16}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-2t}(-1 + \cos(4t))}{16} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{e^{-2t}(-1+\cos(4t))}{16}$$

Verified OK.

16.17.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 20y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -16$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -16z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 443: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(4t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\ &= z_1 e^{-2t} \\ &= z_1 \left(\mathrm{e}^{-2t}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(4t)\,\mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\ &= y_1 \left(\frac{\tan(4t)}{4}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(4t)\, \mathrm{e}^{-2t}\right) + c_2 \left(\cos(4t)\, \mathrm{e}^{-2t} \left(\frac{\tan(4t)}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' + 20y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(4t)\, \mathrm{e}^{-2t} + \frac{\mathrm{e}^{-2t} c_2 \sin(4t)}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{-2t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-2t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\cos(4t)\, \mathrm{e}^{-2t}, \frac{\sin(4t)\, \mathrm{e}^{-2t}}{4}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-2t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1 e^{-2t} = e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{16}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-2t}}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(4t) e^{-2t} + \frac{e^{-2t} c_2 \sin(4t)}{4}\right) + \left(\frac{e^{-2t}}{16}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(4t) e^{-2t} + \frac{e^{-2t} c_2 \sin(4t)}{4} + \frac{e^{-2t}}{16} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{16} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 \sin(4t) e^{-2t} - 2c_1 \cos(4t) e^{-2t} - \frac{e^{-2t} c_2 \sin(4t)}{2} + e^{-2t} c_2 \cos(4t) - \frac{e^{-2t}}{8}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{8} - 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{16}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -\frac{\cos(4t)\,e^{-2t}}{16} + \frac{e^{-2t}}{16}$$

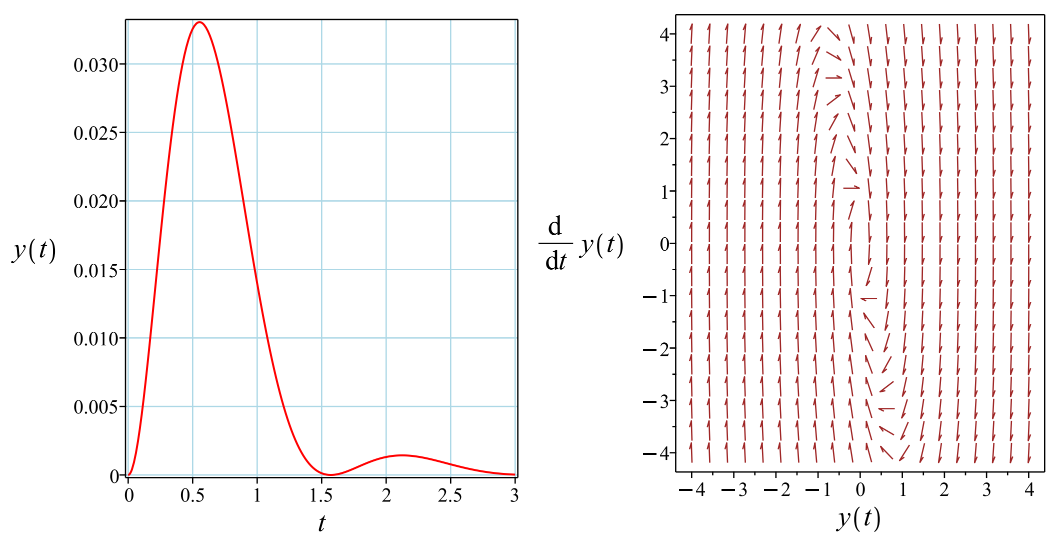
Which simplifies to

$$y = -\frac{e^{-2t}(-1 + \cos(4t))}{16}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-2t}(-1 + \cos(4t))}{16} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-2t}(-1 + \cos(4t))}{16}$$

Verified OK.

16.17.4 Maple step by step solution

Let's solve

$$\left[y''+4y'+20y=\mathrm{e}^{-2t}, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4r+20=0$

- Use quadratic formula to solve for r

  $r=\frac{(-4)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r=(-2-4\,\mathrm{I},-2+4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(4t)\,\mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(4t)\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(4t)\,\mathrm{e}^{-2t}+\mathrm{e}^{-2t}c_2\sin(4t)+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\mathrm{e}^{-2t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t),y_2(t))=\begin{bmatrix}\cos(4t)\,\mathrm{e}^{-2t} & \sin(4t)\,\mathrm{e}^{-2t}\\ -4\sin(4t)\,\mathrm{e}^{-2t}-2\cos(4t)\,\mathrm{e}^{-2t} & 4\cos(4t)\,\mathrm{e}^{-2t}-2\sin(4t)\,\mathrm{e}^{-2t}\end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t),y_2(t))=4\,\mathrm{e}^{-4t}$

  ◦ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{e^{-2t}\left(\cos(4t)\left(\int \sin(4t)dt\right)-\sin(4t)\left(\int \cos(4t)dt\right)\right)}{4}$$

- Compute integrals

  $$y_p(t) = \frac{e^{-2t}}{16}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 \cos(4t)\, e^{-2t} + e^{-2t} c_2 \sin(4t) + \frac{e^{-2t}}{16}$$

- □ Check validity of solution $y = c_1 \cos(4t)\, e^{-2t} + e^{-2t} c_2 \sin(4t) + \frac{e^{-2t}}{16}$
  - Use initial condition $y(0) = 0$

    $$0 = c_1 + \frac{1}{16}$$

  - Compute derivative of the solution

    $$y' = -4c_1 \sin(4t)\, e^{-2t} - 2c_1 \cos(4t)\, e^{-2t} - 2\, e^{-2t} c_2 \sin(4t) + 4\, e^{-2t} c_2 \cos(4t) - \frac{e^{-2t}}{8}$$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $$0 = -2c_1 - \frac{1}{8} + 4c_2$$

  - Solve for c_1 and c_2

    $$\left\{c_1 = -\frac{1}{16}, c_2 = 0\right\}$$

  - Substitute constant values into general solution and simplify

    $$y = -\frac{e^{-2t}(-1+\cos(4t))}{16}$$

- Solution to the IVP

  $$y = -\frac{e^{-2t}(-1+\cos(4t))}{16}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+20*y(t)=exp(-2*t),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = -\frac{e^{-2t}(-1 + \cos(4t))}{16}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 20

```
DSolve[{y''[t]+4*y'[t]+20*y[t]==Exp[-2*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{1}{8}e^{-2t}\sin^2(2t)$$

16.18 problem 18

Internal problem ID [13178]
Internal file name [`OUTPUT/11833_Sunday_December_03_2023_07_18_35_PM_22075003/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 20y = e^{-4t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 20 \\ F &= e^{-4t} \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 20y = e^{-4t}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 20$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-4t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.18.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 20, f(t) = e^{-4t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 20$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 20 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(20)} \\ &= -2 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 4i \\ \lambda_2 &= -2 - 4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 4i \\ \lambda_2 &= -2 - 4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-4t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4t)\,e^{-2t}, \sin(4t)\,e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-4t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$20A_1 e^{-4t} = e^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{20}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-4t}}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))\right) + \left(\frac{e^{-4t}}{20}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{e^{-4t}}{20} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{20} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\,\mathrm{e}^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \mathrm{e}^{-2t}(-4c_1 \sin(4t) + 4c_2 \cos(4t)) - \frac{\mathrm{e}^{-4t}}{5}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - \frac{1}{5} + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{20}$$
$$c_2 = \frac{1}{40}$$

Substituting these values back in above solution results in

$$y = -\frac{\cos(4t)\,\mathrm{e}^{-2t}}{20} + \frac{\sin(4t)\,\mathrm{e}^{-2t}}{40} + \frac{\mathrm{e}^{-4t}}{20}$$

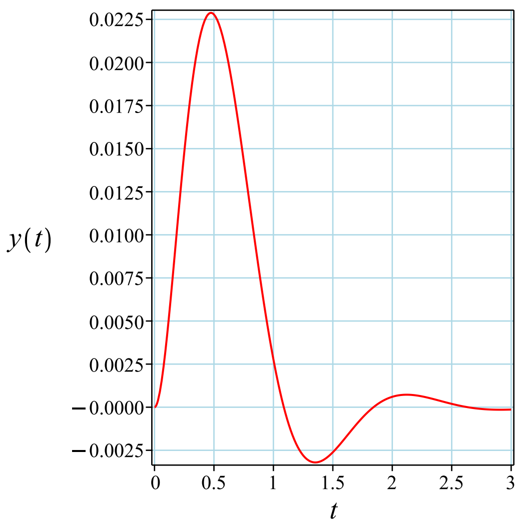
Which simplifies to

$$y = \frac{(-2\cos(4t) + \sin(4t))\,\mathrm{e}^{-2t}}{40} + \frac{\mathrm{e}^{-4t}}{20}$$

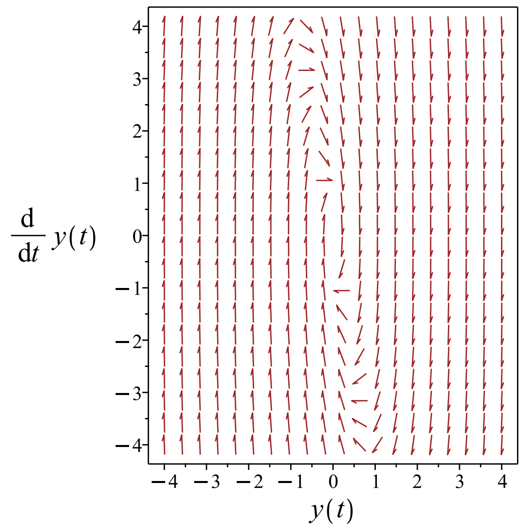
Summary

The solution(s) found are the following

$$y = \frac{(-2\cos(4t) + \sin(4t))\,\mathrm{e}^{-2t}}{40} + \frac{\mathrm{e}^{-4t}}{20} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(-2\cos(4t) + \sin(4t))\, e^{-2t}}{40} + \frac{e^{-4t}}{20}$$

Verified OK.

16.18.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -16$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -16z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 445: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -16$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(4t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1 \left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(4t)\,\mathrm{e}^{-2t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{\tan(4t)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(4t)\, \mathrm{e}^{-2t}\right) + c_2 \left(\cos(4t)\, \mathrm{e}^{-2t} \left(\frac{\tan(4t)}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' + 20y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(4t)\, \mathrm{e}^{-2t} + \frac{\mathrm{e}^{-2t} c_2 \sin(4t)}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{-4t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-4t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\cos(4t)\, \mathrm{e}^{-2t}, \frac{\sin(4t)\, \mathrm{e}^{-2t}}{4}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-4t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$20A_1 e^{-4t} = e^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{20}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-4t}}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(4t) e^{-2t} + \frac{e^{-2t} c_2 \sin(4t)}{4}\right) + \left(\frac{e^{-4t}}{20}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(4t) e^{-2t} + \frac{e^{-2t} c_2 \sin(4t)}{4} + \frac{e^{-4t}}{20} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{20} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 \sin(4t) e^{-2t} - 2c_1 \cos(4t) e^{-2t} - \frac{e^{-2t} c_2 \sin(4t)}{2} + e^{-2t} c_2 \cos(4t) - \frac{e^{-4t}}{5}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{5} - 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{20} \\
c_2 &= \frac{1}{10}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -\frac{\cos(4t)\,\mathrm{e}^{-2t}}{20} + \frac{\sin(4t)\,\mathrm{e}^{-2t}}{40} + \frac{\mathrm{e}^{-4t}}{20}
$$

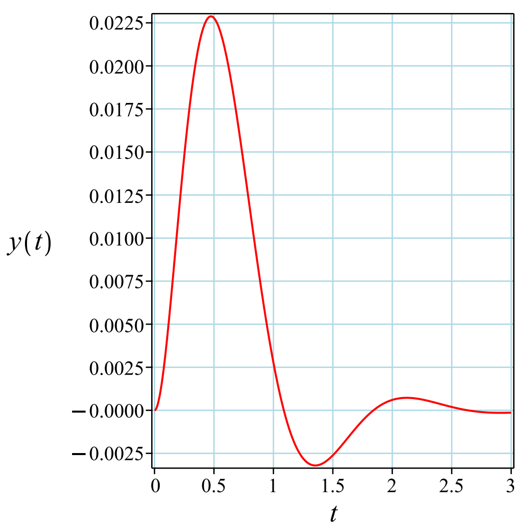
Which simplifies to

$$
y = \frac{(-2\cos(4t) + \sin(4t))\,\mathrm{e}^{-2t}}{40} + \frac{\mathrm{e}^{-4t}}{20}
$$

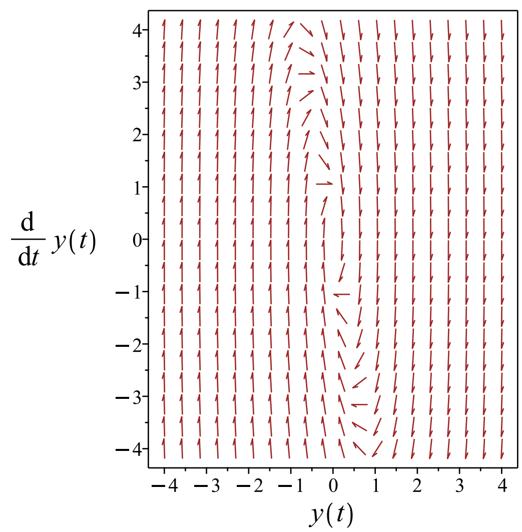
Summary

The solution(s) found are the following

$$
y = \frac{(-2\cos(4t) + \sin(4t))\,\mathrm{e}^{-2t}}{40} + \frac{\mathrm{e}^{-4t}}{20} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \frac{(-2\cos(4t) + \sin(4t))\,\mathrm{e}^{-2t}}{40} + \frac{\mathrm{e}^{-4t}}{20}
$$

Verified OK.

16.18.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 20y = e^{-4t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 20 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - 4\,\mathrm{I}, -2 + 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(4t)\, e^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(4t)\, e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4t)\, e^{-2t} + e^{-2t} c_2 \sin(4t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = e^{-4t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(4t)\, e^{-2t} & \sin(4t)\, e^{-2t} \\ -4\sin(4t)\, e^{-2t} - 2\cos(4t)\, e^{-2t} & 4\cos(4t)\, e^{-2t} - 2\sin(4t)\, e^{-2t} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 4\, e^{-4t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{e^{-2t} \left(\cos(4t) \left(\int \sin(4t) e^{-2t} dt\right) - \sin(4t) \left(\int \cos(4t) e^{-2t} dt\right)\right)}{4}$$

- Compute integrals

$$y_p(t) = \frac{e^{-4t}}{20}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(4t) e^{-2t} + e^{-2t} c_2 \sin(4t) + \frac{e^{-4t}}{20}$$

□ Check validity of solution $y = c_1 \cos(4t) e^{-2t} + e^{-2t} c_2 \sin(4t) + \frac{e^{-4t}}{20}$

- Use initial condition $y(0) = 0$

$$0 = c_1 + \frac{1}{20}$$

- Compute derivative of the solution

$$y' = -4c_1 \sin(4t) e^{-2t} - 2c_1 \cos(4t) e^{-2t} - 2 e^{-2t} c_2 \sin(4t) + 4 e^{-2t} c_2 \cos(4t) - \frac{e^{-4t}}{5}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -2c_1 - \frac{1}{5} + 4c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{1}{20}, c_2 = \frac{1}{40}\right\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{(-2\cos(4t) + \sin(4t)) e^{-2t}}{40} + \frac{e^{-4t}}{20}$$

- Solution to the IVP

$$y = \frac{(-2\cos(4t) + \sin(4t)) e^{-2t}}{40} + \frac{e^{-4t}}{20}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+20*y(t)=exp(-4*t),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \frac{(\sin(4t) - 2\cos(4t))\,e^{-2t}}{40} + \frac{e^{-4t}}{20}$$

✓ Solution by Mathematica

Time used: 0.18 (sec). Leaf size: 37

```
DSolve[{y''[t]+4*y'[t]+20*y[t]==Exp[-4*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{1}{40}e^{-4t}\left(e^{2t}\sin(4t) - 2e^{2t}\cos(4t) + 2\right)$$

16.19 problem 19

Internal problem ID [13179]
Internal file name [`OUTPUT/11834_Sunday_December_03_2023_07_18_42_PM_42690196/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + y = \mathrm{e}^{-t}}$$

16.19.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 2, C = 1, f(t) = \mathrm{e}^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-t} + c_2 t\, e^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\, e^{-t}, e^{-t}\}$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,\mathrm{e}^{-t}\}]$$

Since $t\,\mathrm{e}^{-t}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t^2\mathrm{e}^{-t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t^2 \mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{-t} = \mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{t^2\mathrm{e}^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t}\right) + \left(\frac{t^2\mathrm{e}^{-t}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-t}(c_2 t + c_1) + \frac{t^2\mathrm{e}^{-t}}{2}$$

Summary
The solution(s) found are the following

$$y = e^{-t}(c_2 t + c_1) + \frac{t^2 e^{-t}}{2} \tag{1}$$

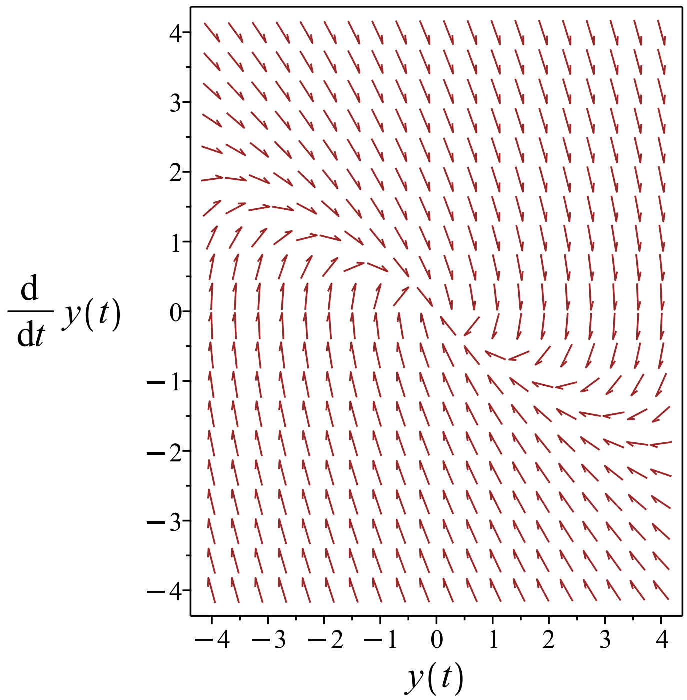


Figure 518: Slope field plot

Verification of solutions

$$y = e^{-t}(c_2 t + c_1) + \frac{t^2 e^{-t}}{2}$$

Verified OK.

16.19.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\,y' + \frac{\left(p(t)^2 + p'(t)\right) y}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int 2\,dx} \\
&= \mathrm{e}^{t}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \mathrm{e}^{t}\mathrm{e}^{-t} \\
\left(\mathrm{e}^{t}y\right)'' &= \mathrm{e}^{t}\mathrm{e}^{-t}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{t}y\right)' = t + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{t}y\right) = \frac{t(t+2c_1)}{2} + c_2
$$

Hence the solution is

$$
y = \frac{\frac{t(t+2c_1)}{2} + c_2}{\mathrm{e}^{t}}
$$

Or

$$
y = t\,\mathrm{e}^{-t}c_1 + \frac{t^2\mathrm{e}^{-t}}{2} + c_2\mathrm{e}^{-t}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = t\,\mathrm{e}^{-t}c_1 + \frac{t^2\mathrm{e}^{-t}}{2} + c_2\mathrm{e}^{-t} \tag{1}
$$

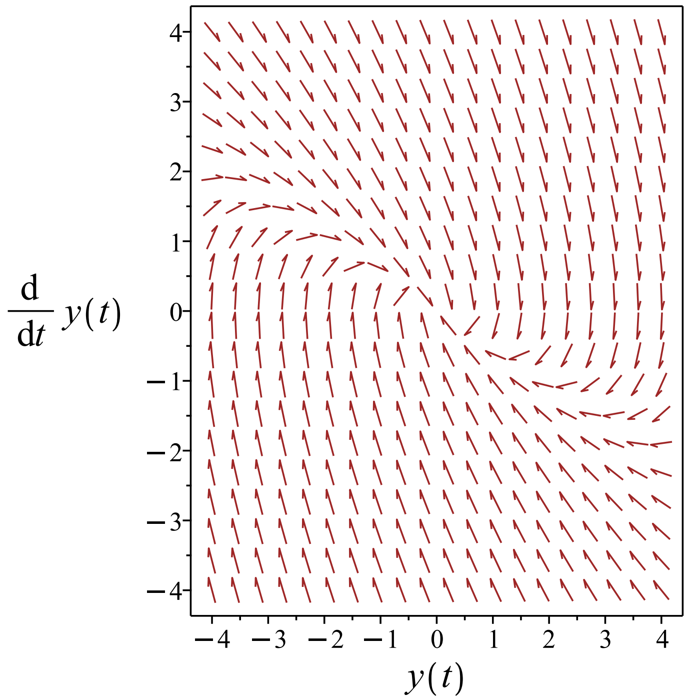


Figure 519: Slope field plot

Verification of solutions

$$y = t\,\mathrm{e}^{-t}c_1 + \frac{t^2\mathrm{e}^{-t}}{2} + c_2\mathrm{e}^{-t}$$

Verified OK.

16.19.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 447: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\ &= z_1 e^{-t} \\ &= z_1 \left(\mathrm{e}^{-t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt \\ &= y_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{-t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{t\,\mathrm{e}^{-t}, \mathrm{e}^{-t}\}
$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{t\,\mathrm{e}^{-t}\}]
$$

Since $t\,\mathrm{e}^{-t}$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{t^2 \mathrm{e}^{-t}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t^2 \mathrm{e}^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{-t} = \mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{t^2 \mathrm{e}^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t}\right) + \left(\frac{t^2 \mathrm{e}^{-t}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-t}(c_2 t + c_1) + \frac{t^2 \mathrm{e}^{-t}}{2}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-t}(c_2 t + c_1) + \frac{t^2 \mathrm{e}^{-t}}{2} \tag{1}$$

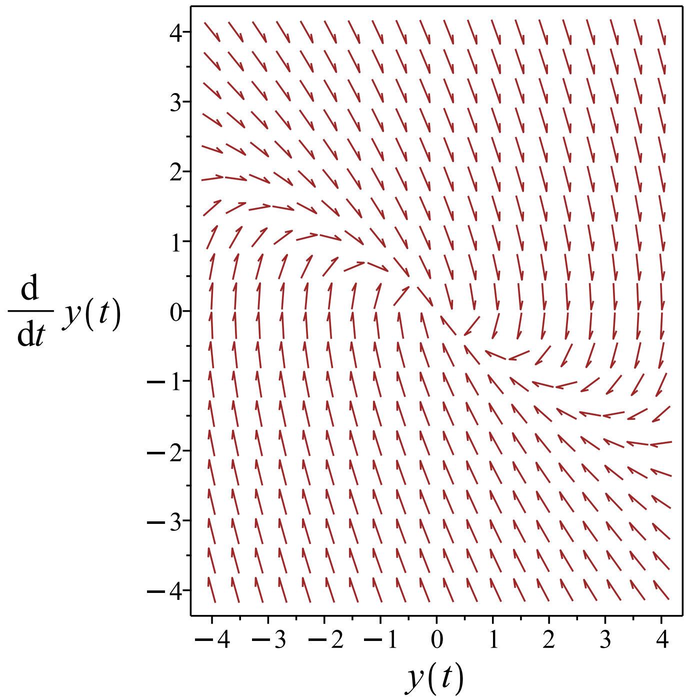


Figure 520: Slope field plot

Verification of solutions

$$y = e^{-t}(c_2 t + c_1) + \frac{t^2 e^{-t}}{2}$$

Verified OK.

16.19.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = e^{-t}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r + 1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathrm{e}^{-t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \mathrm{e}^{-t}\left(-\left(\int t dt\right) + \left(\int 1 dt\right) t\right)$

  - Compute integrals

    $y_p(t) = \frac{t^2 \mathrm{e}^{-t}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_2 t\,\mathrm{e}^{-t} + c_1 \mathrm{e}^{-t} + \frac{t^2 \mathrm{e}^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(t),t$2)+2*diff(y(t),t)+y(t)=exp(-t),y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{-t}\left(\frac{1}{2}t^2 + c_1 t + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 27

```
DSolve[y''[t]+2*y'[t]+y[t]==Exp[-t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{-t}(t^2 + 2c_2 t + 2c_1)$$

16.20 problem 21

Internal problem ID [13180]
Internal file name [`OUTPUT/11835_Sunday_December_03_2023_07_18_45_PM_22357976/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 5y' + 4y = 5}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -5 \\ q(t) &= 4 \\ F &= 5 \end{aligned}$$

Hence the ode is

$$y'' - 5y' + 4y = 5$$

The domain of $p(t) = -5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.20.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -5, C = 4, f(t) = 5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - 5y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -5, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 5\lambda\, e^{\lambda t} + 4\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 5\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(4)}\\ &= \frac{5}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{5}{2} + \frac{3}{2}\\ \lambda_2 &= \frac{5}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(4)t} + c_2 e^{(1)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^t, e^{4t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 = 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{4t} + c_2 e^t\right) + \left(\frac{5}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{4t} + c_2 e^t + \frac{5}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{5}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 4c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = 4c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{5}{12}$$
$$c_2 = -\frac{5}{3}$$

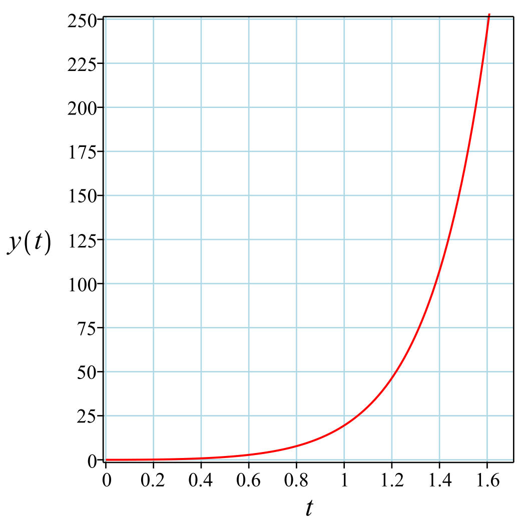
Substituting these values back in above solution results in

$$y = \frac{5}{4} + \frac{5\,\mathrm{e}^{4t}}{12} - \frac{5\,\mathrm{e}^{t}}{3}$$

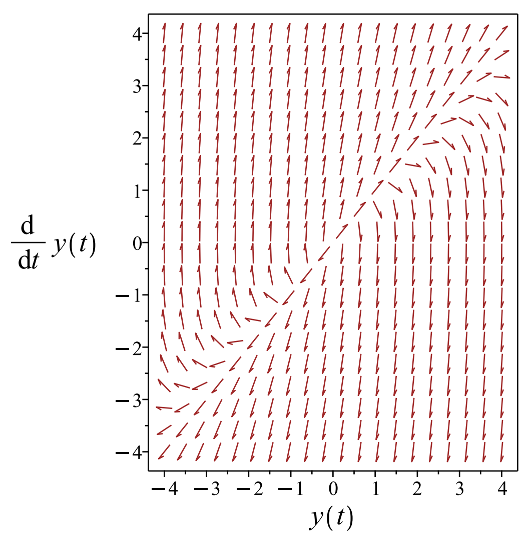
Summary

The solution(s) found are the following

$$y = \frac{5}{4} + \frac{5\,\mathrm{e}^{4t}}{12} - \frac{5\,\mathrm{e}^{t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5}{4} + \frac{5\,e^{4t}}{12} - \frac{5\,e^{t}}{3}$$

Verified OK.

16.20.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{9z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 449: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{3t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dt} \\ &= z_1 e^{\frac{5t}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{5t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{5t}}{(y_1)^2}\,dt \\ &= y_1 \left(\frac{\mathrm{e}^{3t}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{t}\right) + c_2 \left(\mathrm{e}^{t} \left(\frac{\mathrm{e}^{3t}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' - 5y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^t + \frac{c_2 e^{4t}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{4t}}{3}, e^t\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 = 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^t + \frac{c_2 \mathrm{e}^{4t}}{3}\right) + \left(\frac{5}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^t + \frac{c_2 \mathrm{e}^{4t}}{3} + \frac{5}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{3} + \frac{5}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^t + \frac{4c_2 \mathrm{e}^{4t}}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{4c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

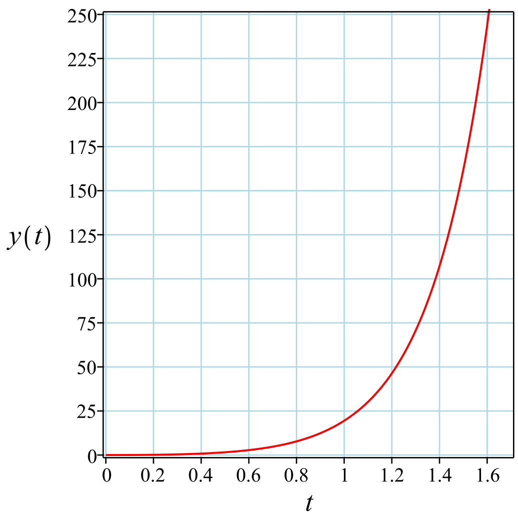
$$\begin{aligned} c_1 &= -\frac{5}{3} \\ c_2 &= \frac{5}{4} \end{aligned}$$

Substituting these values back in above solution results in

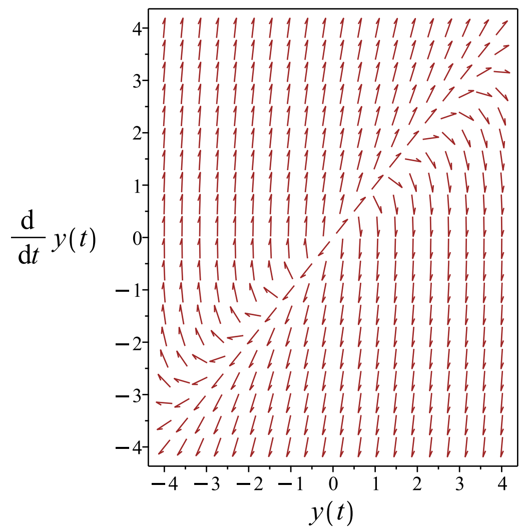
$$y = \frac{5}{4} + \frac{5\,\mathrm{e}^{4t}}{12} - \frac{5\,\mathrm{e}^t}{3}$$

Summary
The solution(s) found are the following

$$y = \frac{5}{4} + \frac{5\,e^{4t}}{12} - \frac{5\,e^{t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5}{4} + \frac{5\,e^{4t}}{12} - \frac{5\,e^{t}}{3}$$

Verified OK.

16.20.4 Maple step by step solution

Let's solve

$$\left[y'' - 5y' + 4y = 5, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 4 = 0$

- Factor the characteristic polynomial

$(r-1)(r-4)=0$

- Roots of the characteristic polynomial

  $r=(1,4)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^t$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{4t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^t+c_2\mathrm{e}^{4t}+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=5\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t),y_2(t))=\begin{bmatrix}\mathrm{e}^t & \mathrm{e}^{4t}\\ \mathrm{e}^t & 4\,\mathrm{e}^{4t}\end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t),y_2(t))=3\,\mathrm{e}^{5t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t)=-\frac{5\,\mathrm{e}^t\left(\int \mathrm{e}^{-t}dt\right)}{3}+\frac{5\,\mathrm{e}^{4t}\left(\int \mathrm{e}^{-4t}dt\right)}{3}$

- ○ Compute integrals

  $y_p(t)=\frac{5}{4}$

- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^t+c_2\mathrm{e}^{4t}+\frac{5}{4}$

☐ Check validity of solution $y=c_1\mathrm{e}^t+c_2\mathrm{e}^{4t}+\frac{5}{4}$

- ○ Use initial condition $y(0)=0$

  $0=c_1+c_2+\frac{5}{4}$

- ○ Compute derivative of the solution

$$y' = c_1 e^t + 4c_2 e^{4t}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = c_1 + 4c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{5}{3}, c_2 = \frac{5}{12}\right\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{5}{4} + \frac{5\,e^{4t}}{12} - \frac{5\,e^t}{3}$$

- Solution to the IVP

$$y = \frac{5}{4} + \frac{5\,e^{4t}}{12} - \frac{5\,e^t}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$2)-5*diff(y(t),t)+4*y(t)=5,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{5\,e^{4t}}{12} - \frac{5\,e^t}{3} + \frac{5}{4}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 21

```
DSolve[{y''[t]-5*y'[t]+4*y[t]==5,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to \frac{5}{12}\left(-4e^t + e^{4t} + 3\right)$$

16.21 problem 22

Internal problem ID [13181]
Internal file name [`OUTPUT/11836_Sunday_December_03_2023_07_18_47_PM_63944662/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 5y' + 6y = 2}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 6 \\ F &= 2 \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 6y = 2$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.21.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 5, C = 6, f(t) = 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{5^2 - (4)(1)(6)}\\ &= -\frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{5}{2} + \frac{1}{2}$$
$$\lambda_2 = -\frac{5}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = -2$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(-2)t} + c_2 e^{(-3)t}$$

Or

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 = 2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-3t}\right) + \left(\frac{1}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2t} - 3c_2 \mathrm{e}^{-3t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = \frac{2}{3}$$

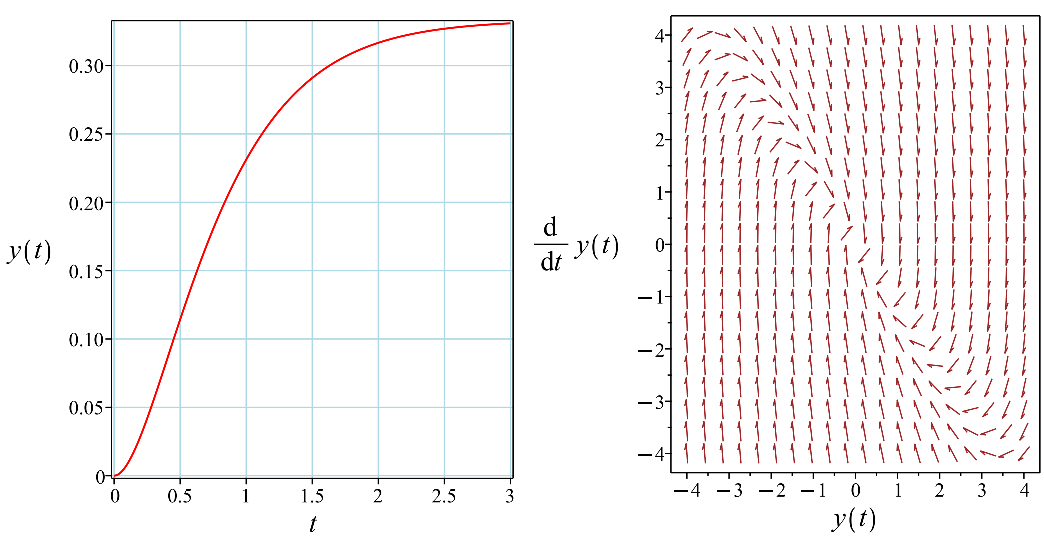
Substituting these values back in above solution results in

$$y = \frac{1}{3} - \mathrm{e}^{-2t} + \frac{2\,\mathrm{e}^{-3t}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3} - \mathrm{e}^{-2t} + \frac{2\,\mathrm{e}^{-3t}}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{3} - e^{-2t} + \frac{2\,e^{-3t}}{3}$$

Verified OK.

16.21.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 451: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dt} \\
&= z_1 e^{-\frac{5t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-5t}}{(y_1)^2}\,dt \\
&= y_1 \left(\mathrm{e}^{t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t}\left(\mathrm{e}^{t}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+5y'+6y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-3t}c_1 + c_2\mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3t}, \mathrm{e}^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 = 2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{-3t}c_1 + c_2 e^{-2t}\right) + \left(\frac{1}{3}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = e^{-3t}c_1 + c_2 e^{-2t} + \frac{1}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 + c_2 + \frac{1}{3} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3\,e^{-3t}c_1 - 2c_2 e^{-2t}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = -3c_1 - 2c_2 \tag{2A}
$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{2}{3} \\
c_2 &= -1
\end{aligned}
$$

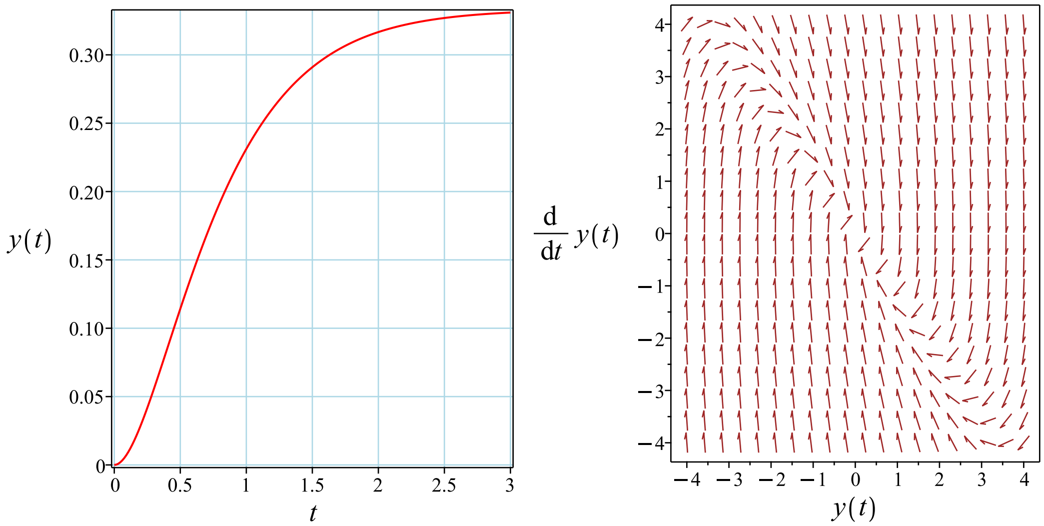
Substituting these values back in above solution results in

$$
y = \frac{1}{3} - e^{-2t} + \frac{2\,e^{-3t}}{3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{3} - e^{-2t} + \frac{2\,e^{-3t}}{3} \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{3} - e^{-2t} + \frac{2\,e^{-3t}}{3}$$

Verified OK.

16.21.4 Maple step by step solution

Let's solve

$$\left[y'' + 5y' + 6y = 2, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 6 = 0$
- Factor the characteristic polynomial

  $(r + 3)(r + 2) = 0$
- Roots of the characteristic polynomial

  $r = (-3, -2)$
- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-3t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))} dt\right), f(t) = 2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t} & \mathrm{e}^{-2t} \\ -3\,\mathrm{e}^{-3t} & -2\,\mathrm{e}^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-5t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -2\,\mathrm{e}^{-3t}\left(\int \mathrm{e}^{3t} dt\right) + 2\,\mathrm{e}^{-2t}\left(\int \mathrm{e}^{2t} dt\right)$

  - Compute integrals

    $y_p(t) = \frac{1}{3}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-2t} + \frac{1}{3}$

☐ Check validity of solution $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-2t} + \frac{1}{3}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2 + \frac{1}{3}$

  - Compute derivative of the solution

    $y' = -3\,\mathrm{e}^{-3t} c_1 - 2 c_2 \mathrm{e}^{-2t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -3 c_1 - 2 c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{2}{3}, c_2 = -1\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{1}{3} - e^{-2t} + \frac{2\,e^{-3t}}{3}$

- Solution to the IVP

  $y = \frac{1}{3} - e^{-2t} + \frac{2\,e^{-3t}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(y(t),t$2)+5*diff(y(t),t)+6*y(t)=2,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{2\,e^{-3t}}{3} - e^{-2t} + \frac{1}{3}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 26

```
DSolve[{y''[t]+5*y'[t]+6*y[t]==2,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to \frac{1}{3}e^{-3t}(e^t - 1)^2 (e^t + 2)$$

16.22 problem 23

Internal problem ID [13182]
Internal file name [`OUTPUT/11837_Sunday_December_03_2023_07_18_51_PM_10590537/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 10y = 10}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 10 \\ F &= 10 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 10y = 10$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.22.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 2, C = 10, f(t) = 10$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 10$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 2\lambda\,\mathrm{e}^{\lambda t} + 10\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(10)} \\ &= -1 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 3i \\ \lambda_2 &= -1 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 3i \\ \lambda_2 &= -1 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

1

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-t} \cos(3t), e^{-t} \sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 = 10$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-t}(c_1 \cos(3t) + c_2 \sin(3t))\right) + (1) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t)) + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = 1 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -e^{-t}(c_1 \cos(3t) + c_2 \sin(3t)) + e^{-t}(-3c_1 \sin(3t) + 3c_2 \cos(3t))$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = -\frac{1}{3}$$

Substituting these values back in above solution results in

$$y = 1 - \frac{e^{-t} \sin(3t)}{3} - e^{-t} \cos(3t)$$

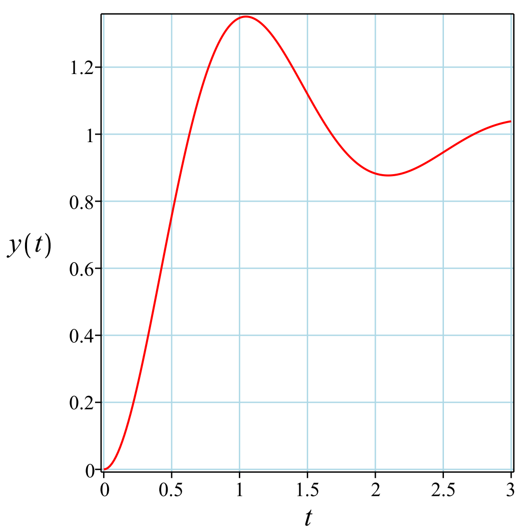
Which simplifies to

$$y = 1 + \frac{(-3\cos(3t) - \sin(3t))\, e^{-t}}{3}$$

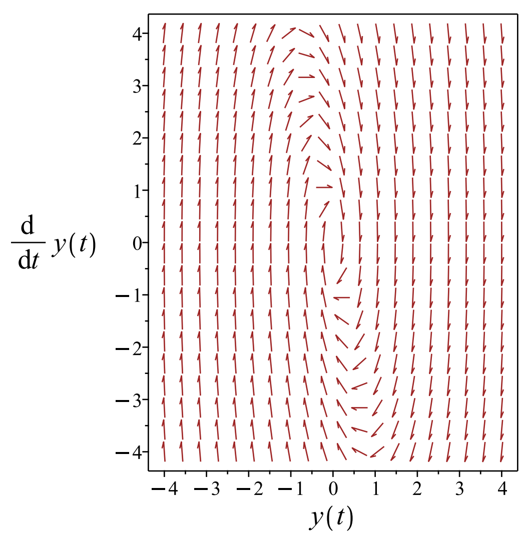
Summary

The solution(s) found are the following

$$y = 1 + \frac{(-3\cos(3t) - \sin(3t))\, e^{-t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + \frac{(-3\cos(3t) - \sin(3t))\,\mathrm{e}^{-t}}{3}$$

Verified OK.

16.22.3 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} y'' + 2y' + 10y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1 \\ B &= 2 \tag{3} \\ C &= 10 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*} s &= -9 \\ t &= 1 \end{align*}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 453: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\
&= z_1 e^{-t} \\
&= z_1 \left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-t} \cos(3t)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{\tan(3t)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t} \cos(3t)\right) + c_2 \left(\mathrm{e}^{-t} \cos(3t) \left(\frac{\tan(3t)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+2y'+10y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-t}\cos(3t)+\frac{\mathrm{e}^{-t}\sin(3t)\,c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-t}\cos(3t),\frac{\mathrm{e}^{-t}\sin(3t)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 = 10$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-t} \cos(3t) + \frac{e^{-t} \sin(3t) c_2}{3}\right) + (1)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{-t} \cos(3t) + \frac{e^{-t} \sin(3t) c_2}{3} + 1 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = 1 + c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 e^{-t} \cos(3t) - 3c_1 e^{-t} \sin(3t) - \frac{e^{-t} \sin(3t) c_2}{3} + e^{-t} \cos(3t) c_2
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = -c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -1 \\
c_2 &= -1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 1 - \frac{e^{-t} \sin(3t)}{3} - e^{-t} \cos(3t)
$$

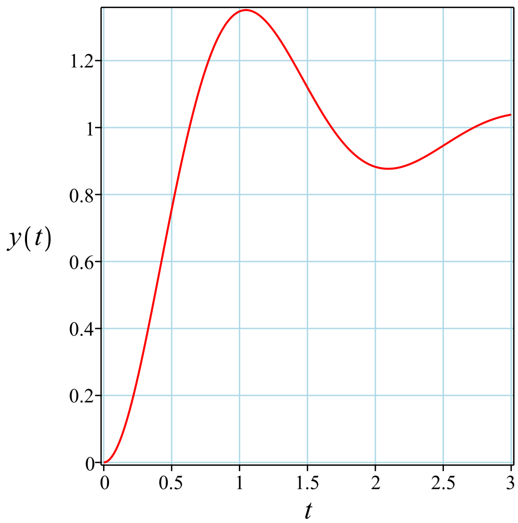
Which simplifies to

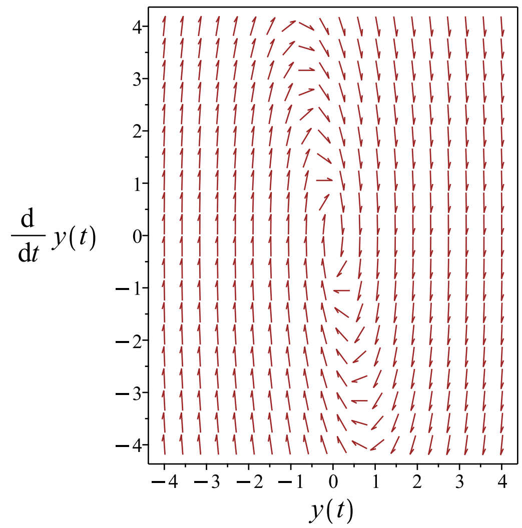
$$
y = 1 + \frac{(-3\cos(3t) - \sin(3t)) e^{-t}}{3}
$$

Summary

The solution(s) found are the following

$$
y = 1 + \frac{(-3\cos(3t) - \sin(3t)) e^{-t}}{3} \tag{1}
$$





(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 1 + \frac{(-3\cos(3t) - \sin(3t))\,e^{-t}}{3}$$

Verified OK.

16.22.4 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 10y = 10, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 3\,\mathrm{I}, -1 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = e^{-t} \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t} \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-t} \cos(3t) + e^{-t} \sin(3t) c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 10\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t}\cos(3t) & e^{-t}\sin(3t) \\ -e^{-t}\cos(3t) - 3e^{-t}\sin(3t) & -e^{-t}\sin(3t) + 3e^{-t}\cos(3t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 3e^{-2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{10e^{-t}\left(\cos(3t)\left(\int e^t \sin(3t)dt\right) - \sin(3t)\left(\int e^t \cos(3t)dt\right)\right)}{3}$

  - Compute integrals

    $y_p(t) = 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t} \cos(3t) + e^{-t} \sin(3t) c_2 + 1$

☐ Check validity of solution $y = c_1 e^{-t} \cos(3t) + e^{-t} \sin(3t) c_2 + 1$

  - Use initial condition $y(0) = 0$

    $0 = 1 + c_1$

  - Compute derivative of the solution

    $y' = -c_1 e^{-t} \cos(3t) - 3c_1 e^{-t} \sin(3t) - e^{-t} \sin(3t) c_2 + 3e^{-t} \cos(3t) c_2$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -c_1 + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = -\frac{1}{3}\}$

- Substitute constant values into general solution and simplify

  $y = 1 + \frac{(-3\cos(3t) - \sin(3t))e^{-t}}{3}$

- Solution to the IVP

  $y = 1 + \frac{(-3\cos(3t) - \sin(3t))e^{-t}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+10*y(t)=10,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = 1 + \frac{(-3\cos(3t) - \sin(3t))\,e^{-t}}{3}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 32

```
DSolve[{y''[t]+2*y'[t]+10*y[t]==10,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to \frac{1}{3}e^{-t}(3e^t - \sin(3t) - 3\cos(3t))$$

16.23 problem 24

Internal problem ID [13183]
Internal file name [`OUTPUT/11838_Sunday_December_03_2023_07_18_58_PM_41684427/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + 6y = -8}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 6 \\ F &= -8 \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 6y = -8$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.23.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 6, f(t) = -8$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 6$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} + 6\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(6)}\\ &= -2 \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + i\sqrt{2}\\ \lambda_2 &= -2 - i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} - 2\\ \lambda_2 &= -2 - i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}\left(\cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2t}\left(\cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-2t} \cos\left(\sqrt{2}\, t\right), e^{-2t} \sin\left(\sqrt{2}\, t\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 = -8$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}\left(\cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)\right)\right) + \left(-\frac{4}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-2t}\left(\cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)\right) - \frac{4}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 - \frac{4}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\,\mathrm{e}^{-2t}\left(\cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right)\right) + \mathrm{e}^{-2t}\left(-\sqrt{2}\,\sin\left(\sqrt{2}\,t\right) c_1 + \sqrt{2}\,\cos\left(\sqrt{2}\,t\right) c_2\right)$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 + \sqrt{2}\,c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{4}{3}$$
$$c_2 = \frac{4\sqrt{2}}{3}$$

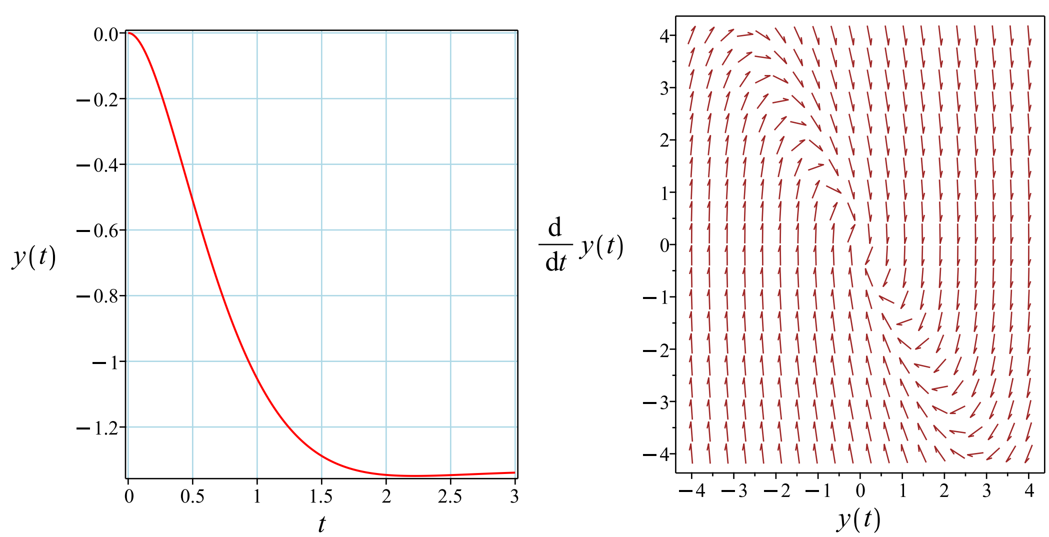
Substituting these values back in above solution results in

$$y = \frac{4\,\mathrm{e}^{-2t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{4\,\mathrm{e}^{-2t}\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{3} - \frac{4}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{4\,\mathrm{e}^{-2t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{4\,\mathrm{e}^{-2t}\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{3} - \frac{4}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{4\,\mathrm{e}^{-2t} \cos\left(\sqrt{2}\,t\right)}{3} + \frac{4\,\mathrm{e}^{-2t} \sin\left(\sqrt{2}\,t\right) \sqrt{2}}{3} - \frac{4}{3}$$

Verified OK.

16.23.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -2z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 455: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\sqrt{2}\, t\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1 \left(\mathrm{e}^{-2t}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{\sqrt{2}\,\tan\left(\sqrt{2}\, t\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right)\right) + c_2 \left(\mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right) \left(\frac{\sqrt{2}\,\tan\left(\sqrt{2}\, t\right)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+4y'+6y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right)+\frac{c_2 \mathrm{e}^{-2t} \sin\left(\sqrt{2}\, t\right) \sqrt{2}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right), \frac{\mathrm{e}^{-2t} \sin\left(\sqrt{2}\, t\right) \sqrt{2}}{2}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 = -8$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-2t} \cos\left(\sqrt{2}\, t\right) + \frac{c_2 e^{-2t} \sin\left(\sqrt{2}\, t\right) \sqrt{2}}{2}\right) + \left(-\frac{4}{3}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{-2t} \cos\left(\sqrt{2}\, t\right) + \frac{c_2 e^{-2t} \sin\left(\sqrt{2}\, t\right) \sqrt{2}}{2} - \frac{4}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 - \frac{4}{3} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 e^{-2t} \cos\left(\sqrt{2}\, t\right) - c_1 e^{-2t} \sqrt{2} \sin\left(\sqrt{2}\, t\right) - c_2 e^{-2t} \sin\left(\sqrt{2}\, t\right) \sqrt{2} + c_2 e^{-2t} \cos\left(\sqrt{2}\, t\right)
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = -2c_1 + c_2 \tag{2A}
$$

Equations $\{1\text{A}, 2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{4}{3} \\
c_2 &= \frac{8}{3}
\end{aligned}
$$

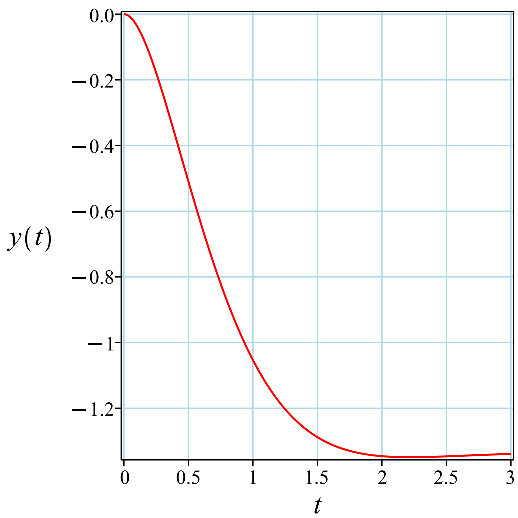
Substituting these values back in above solution results in

$$
y = \frac{4\, e^{-2t} \cos\left(\sqrt{2}\, t\right)}{3} + \frac{4\, e^{-2t} \sin\left(\sqrt{2}\, t\right) \sqrt{2}}{3} - \frac{4}{3}
$$

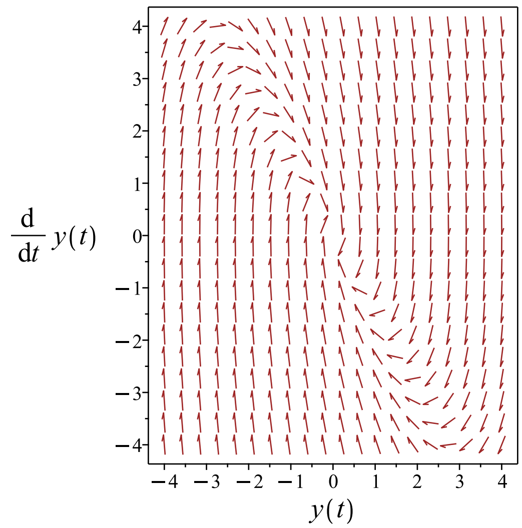
Summary

The solution(s) found are the following

$$
y = \frac{4\, e^{-2t} \cos\left(\sqrt{2}\, t\right)}{3} + \frac{4\, e^{-2t} \sin\left(\sqrt{2}\, t\right) \sqrt{2}}{3} - \frac{4}{3} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{4\,\mathrm{e}^{-2t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{4\,\mathrm{e}^{-2t}\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{3} - \frac{4}{3}$$

Verified OK.

16.23.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 6y = -8, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 6 = 0$
- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm\left(\sqrt{-8}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(-2 - \mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2} - 2\right)$
- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t} \sin\left(\sqrt{2}\, t\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right) + \mathrm{e}^{-2t} c_2 \sin\left(\sqrt{2}\, t\right) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = -8\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right) & \mathrm{e}^{-2t} \sin\left(\sqrt{2}\, t\right) \\ -2\,\mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right) - \mathrm{e}^{-2t} \sin\left(\sqrt{2}\, t\right)\sqrt{2} & -2\,\mathrm{e}^{-2t} \sin\left(\sqrt{2}\, t\right) + \mathrm{e}^{-2t}\sqrt{2} \cos\left(\sqrt{\right. \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = \sqrt{2}\,\mathrm{e}^{-4t}$

- ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = 4\sqrt{2}\,\mathrm{e}^{-2t}\left(\cos\left(\sqrt{2}\, t\right)\left(\int \mathrm{e}^{2t} \sin\left(\sqrt{2}\, t\right) dt\right) - \sin\left(\sqrt{2}\, t\right)\left(\int \mathrm{e}^{2t} \cos\left(\sqrt{2}\, t\right) dt\right)\right)$

- ◦ Compute integrals

  $y_p(t) = -\frac{4}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right) + \mathrm{e}^{-2t} c_2 \sin\left(\sqrt{2}\, t\right) - \frac{4}{3}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right) + \mathrm{e}^{-2t} c_2 \sin\left(\sqrt{2}\, t\right) - \frac{4}{3}$

- ◦ Use initial condition $y(0) = 0$

  $0 = c_1 - \frac{4}{3}$

- ◦ Compute derivative of the solution

  $y' = -2c_1 \mathrm{e}^{-2t} \cos\left(\sqrt{2}\, t\right) - c_1 \mathrm{e}^{-2t}\sqrt{2} \sin\left(\sqrt{2}\, t\right) - 2\,\mathrm{e}^{-2t} c_2 \sin\left(\sqrt{2}\, t\right) + \mathrm{e}^{-2t} c_2 \sqrt{2} \cos\left(\sqrt{2}\, t\right)$

- ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -2c_1 + \sqrt{2}\, c_2$

- Solve for c_1 and c_2

  $$\left\{c_1 = \frac{4}{3}, c_2 = \frac{4\sqrt{2}}{3}\right\}$$

- Substitute constant values into general solution and simplify

  $$y = \frac{4\,e^{-2t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{4\,e^{-2t}\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{3} - \frac{4}{3}$$

- Solution to the IVP

  $$y = \frac{4\,e^{-2t}\cos\left(\sqrt{2}\,t\right)}{3} + \frac{4\,e^{-2t}\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{3} - \frac{4}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+6*y(t)=-8,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{4\,e^{-2t}\sin\left(\sqrt{2}\,t\right)\sqrt{2}}{3} + \frac{4\,e^{-2t}\cos\left(\sqrt{2}\,t\right)}{3} - \frac{4}{3}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 44

```
DSolve[{y''[t]+4*y'[t]+6*y[t]==-8,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to \frac{4}{3}e^{-2t}\left(-e^{2t} + \sqrt{2}\sin\left(\sqrt{2}t\right) + \cos\left(\sqrt{2}t\right)\right)$$

16.24 problem 25

Internal problem ID [13184]
Internal file name [`OUTPUT/11839_Sunday_December_03_2023_07_19_05_PM_62662093/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 9y = \mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= \mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$y'' + 9y = e^{-t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.24.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 9, f(t) = e^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 9 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{0}(c_1 \cos(3t) + c_2 \sin(3t))$$

Or

$$y = c_1 \cos(3t) + c_2 \sin(3t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3t) + c_2 \sin(3t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3t), \sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 e^{-t} = e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-t}}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3t) + c_2 \sin(3t)) + \left(\frac{e^{-t}}{10}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{e^{-t}}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{10} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3t) + 3c_2 \cos(3t) - \frac{e^{-t}}{10}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{10} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{10}$$
$$c_2 = \frac{1}{30}$$

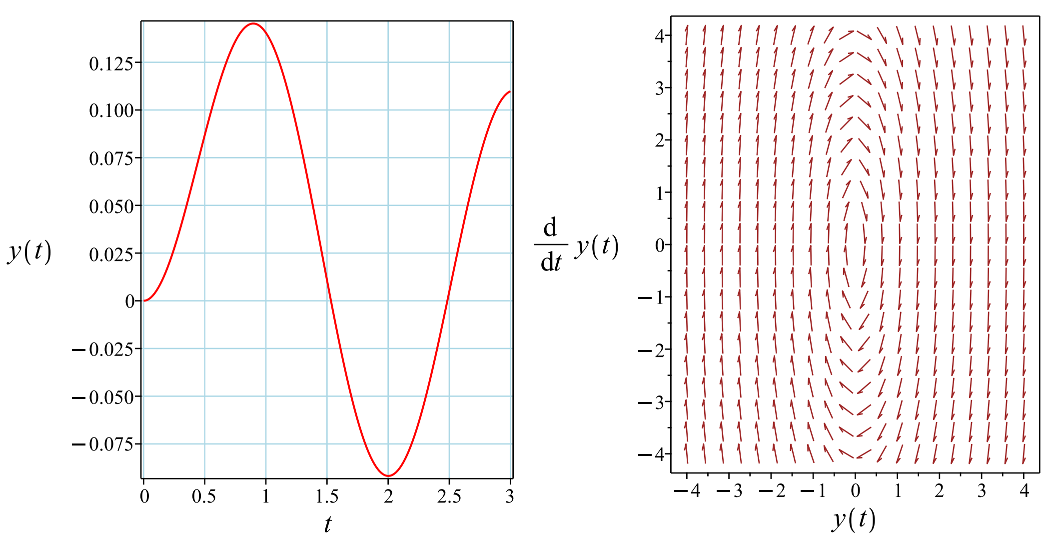
Substituting these values back in above solution results in

$$y = -\frac{\cos(3t)}{10} + \frac{\sin(3t)}{30} + \frac{e^{-t}}{10}$$

Summary

The solution(s) found are the following

$$y = -\frac{\cos(3t)}{10} + \frac{\sin(3t)}{30} + \frac{e^{-t}}{10} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\cos(3t)}{10} + \frac{\sin(3t)}{30} + \frac{e^{-t}}{10}$$

Verified OK.

16.24.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 457: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= \cos(3t) \int \frac{1}{\cos(3t)^2}\,dt \\ &= \cos(3t)\left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3t)) + c_2\left(\cos(3t)\left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+9y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\cos(3t)+\frac{c_2\sin(3t)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3t)}{3},\cos(3t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1e^{-t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1e^{-t}=e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=\frac{e^{-t}}{10}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}\right) + \left(\frac{e^{-t}}{10}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} + \frac{e^{-t}}{10} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 + \frac{1}{10} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3c_1 \sin(3t) + c_2 \cos(3t) - \frac{e^{-t}}{10}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = -\frac{1}{10} + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{10} \\
c_2 &= \frac{1}{10}
\end{aligned}
$$

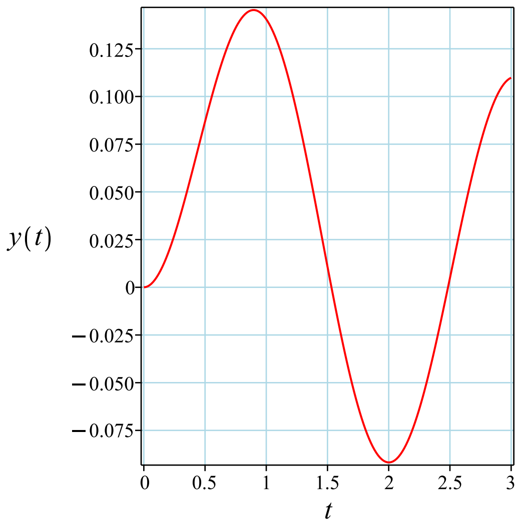
Substituting these values back in above solution results in

$$
y = -\frac{\cos(3t)}{10} + \frac{\sin(3t)}{30} + \frac{e^{-t}}{10}
$$

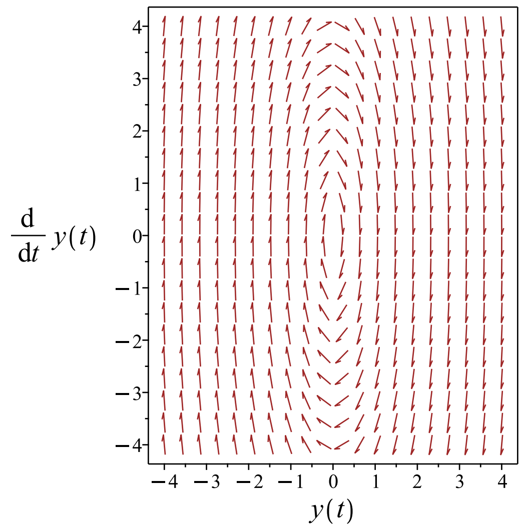
Summary

The solution(s) found are the following

$$
y = -\frac{\cos(3t)}{10} + \frac{\sin(3t)}{30} + \frac{e^{-t}}{10} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\cos(3t)}{10} + \frac{\sin(3t)}{30} + \frac{e^{-t}}{10}$$

Verified OK.

16.24.4 Maple step by step solution

Let's solve

$\left[y'' + 9y = e^{-t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathrm{e}^{-t}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$

- ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(3t)\left(\int \mathrm{e}^{-t} \sin(3t) dt\right)}{3} + \frac{\sin(3t)\left(\int \mathrm{e}^{-t} \cos(3t) dt\right)}{3}$

- ◦ Compute integrals

  $y_p(t) = \frac{\mathrm{e}^{-t}}{10}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\mathrm{e}^{-t}}{10}$

☐ Check validity of solution $y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\mathrm{e}^{-t}}{10}$

- ◦ Use initial condition $y(0) = 0$

  $0 = c_1 + \frac{1}{10}$

- ◦ Compute derivative of the solution

  $y' = -3c_1 \sin(3t) + 3c_2 \cos(3t) - \frac{\mathrm{e}^{-t}}{10}$

- ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -\frac{1}{10} + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = -\frac{1}{10}, c_2 = \frac{1}{30}\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\cos(3t)}{10} + \frac{\sin(3t)}{30} + \frac{e^{-t}}{10}$

- Solution to the IVP

  $y = -\frac{\cos(3t)}{10} + \frac{\sin(3t)}{30} + \frac{e^{-t}}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+9*y(t)=exp(-t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{\sin(3t)}{30} - \frac{\cos(3t)}{10} + \frac{e^{-t}}{10}$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 33

```
DSolve[{y''[t]+9*y[t]==Exp[-t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{30}e^{-t}\left(e^t \sin(3t) - 3e^t \cos(3t) + 3\right)$$

16.25 problem 26

Internal problem ID [13185]
Internal file name [OUTPUT/11840_Sunday_December_03_2023_07_19_09_PM_11004804/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y = 2\,\mathrm{e}^{-2t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= 2\,\mathrm{e}^{-2t} \end{aligned}$$

Hence the ode is

$$y'' + 4y = 2\,\mathrm{e}^{-2t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2\,\mathrm{e}^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.25.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = 2\,\mathrm{e}^{-2t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 4\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 e^{-2t} = 2\,e^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-2t}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2t) + c_2 \sin(2t)) + \left(\frac{e^{-2t}}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{e^{-2t}}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{4} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{e^{-2t}}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{2} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{4}$$
$$c_2 = \frac{1}{4}$$

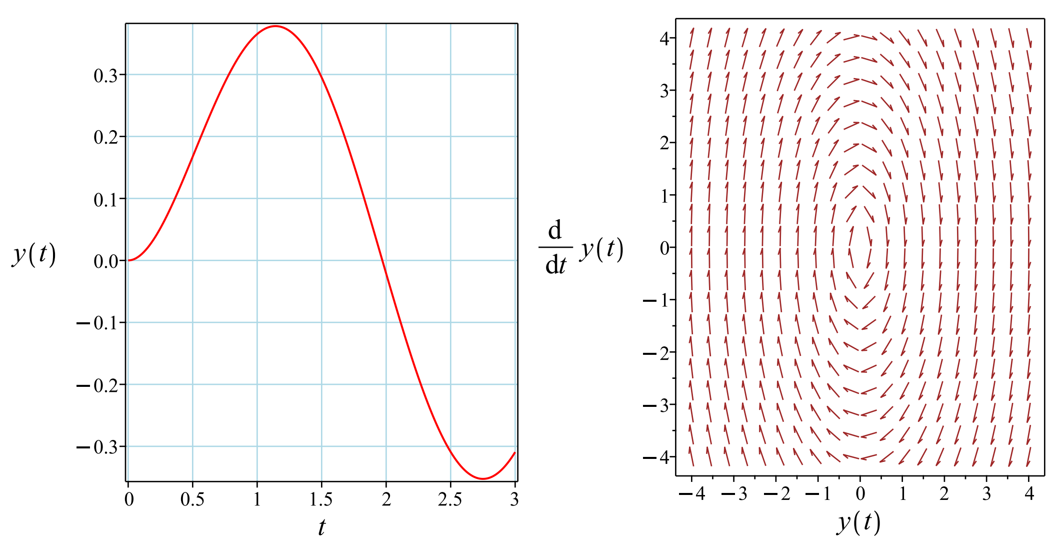
Substituting these values back in above solution results in

$$y = -\frac{\cos(2t)}{4} + \frac{\sin(2t)}{4} + \frac{e^{-2t}}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{\cos(2t)}{4} + \frac{\sin(2t)}{4} + \frac{e^{-2t}}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\cos(2t)}{4} + \frac{\sin(2t)}{4} + \frac{e^{-2t}}{4}$$

Verified OK.

16.25.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 459: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= \cos(2t) \int \frac{1}{\cos(2t)^2}\,dt \\ &= \cos(2t) \left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\cos(2t)+\frac{c_2\sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-2t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2},\cos(2t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1\mathrm{e}^{-2t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1\mathrm{e}^{-2t}=2\,\mathrm{e}^{-2t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=\frac{\mathrm{e}^{-2t}}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + \left(\frac{e^{-2t}}{4}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} + \frac{e^{-2t}}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{4} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2t) + c_2 \cos(2t) - \frac{e^{-2t}}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{4} \\
c_2 &= \frac{1}{2}
\end{aligned}
$$

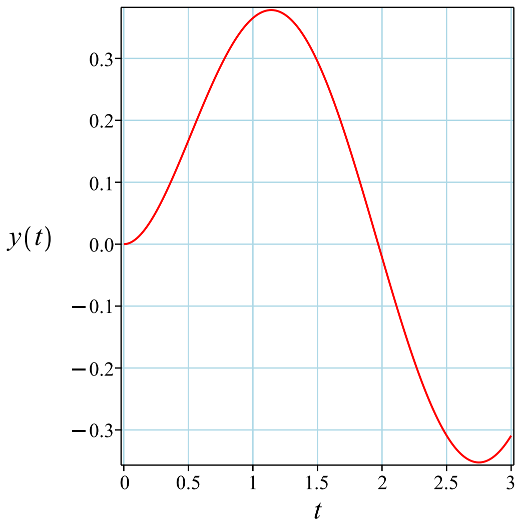
Substituting these values back in above solution results in

$$y = -\frac{\cos(2t)}{4} + \frac{\sin(2t)}{4} + \frac{e^{-2t}}{4}$$

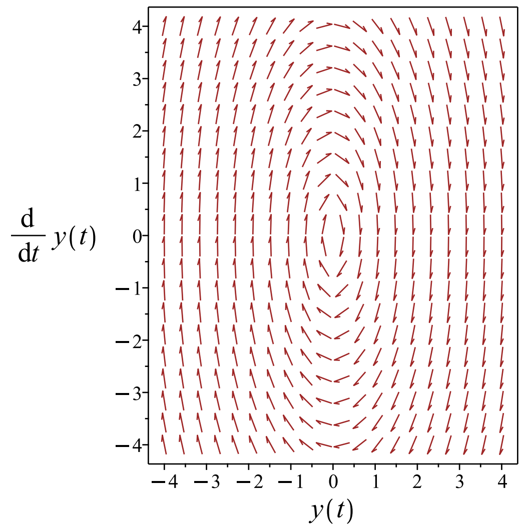
Summary

The solution(s) found are the following

$$y = -\frac{\cos(2t)}{4} + \frac{\sin(2t)}{4} + \frac{e^{-2t}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\cos(2t)}{4} + \frac{\sin(2t)}{4} + \frac{e^{-2t}}{4}$$

Verified OK.

16.25.4 Maple step by step solution

Let's solve

$\left[y'' + 4y = 2\,e^{-2t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 2\,\mathrm{e}^{-2t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\cos(2t)\left(\int \mathrm{e}^{-2t} \sin(2t)\, dt\right) + \sin(2t)\left(\int \mathrm{e}^{-2t} \cos(2t)\, dt\right)$

- ○ Compute integrals

  $y_p(t) = \frac{\mathrm{e}^{-2t}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\mathrm{e}^{-2t}}{4}$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\mathrm{e}^{-2t}}{4}$

- ○ Use initial condition $y(0) = 0$

  $0 = \frac{1}{4} + c_1$

- ○ Compute derivative of the solution

  $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{\mathrm{e}^{-2t}}{2}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -\frac{1}{2} + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = -\frac{1}{4}, c_2 = \frac{1}{4}\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\cos(2t)}{4} + \frac{\sin(2t)}{4} + \frac{e^{-2t}}{4}$

- Solution to the IVP

  $y = -\frac{\cos(2t)}{4} + \frac{\sin(2t)}{4} + \frac{e^{-2t}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+4*y(t)=2*exp(-2*t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{\sin(2t)}{4} - \frac{\cos(2t)}{4} + \frac{e^{-2t}}{4}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 25

```
DSolve[{y''[t]+4*y[t]==2*Exp[-2*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to \frac{1}{4}(e^{-2t} + \sin(2t) - \cos(2t))$$

16.26 problem 27

Internal problem ID [13186]
Internal file name [`OUTPUT/11841_Sunday_December_03_2023_07_19_13_PM_43696469/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + 2y = -3}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 2 \\ F &= -3 \end{aligned}$$

Hence the ode is

$$y'' + 2y = -3$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.26.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 2, f(t) = -3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(2)}\\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0\left(\cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right)\right)$$

Or

$$y = \cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{2}\, t\right), \sin\left(\sqrt{2}\, t\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = -3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)\right) + \left(-\frac{3}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right) - \frac{3}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{3}{2} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) c_1 + \sqrt{2}\, \cos\left(\sqrt{2}\, t\right) c_2$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \sqrt{2}\, c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{3}{2} \\ c_2 &= 0 \end{aligned}$$

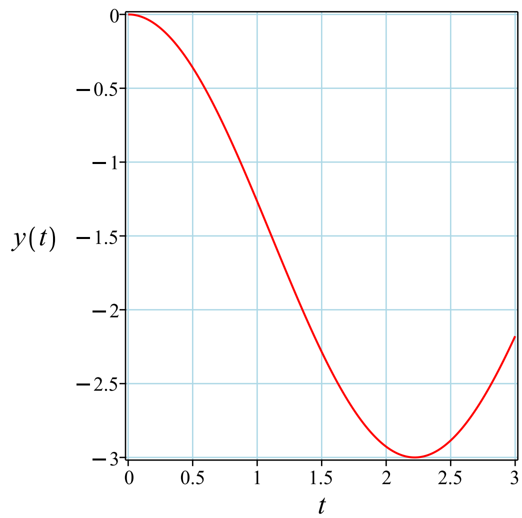
Substituting these values back in above solution results in

$$y = \frac{3 \cos\left(\sqrt{2}\, t\right)}{2} - \frac{3}{2}$$

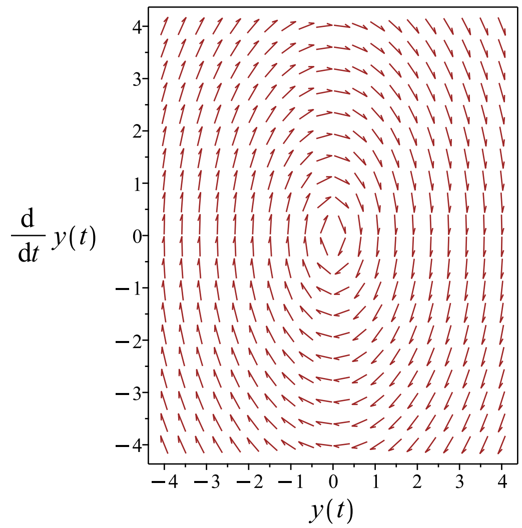
Summary

The solution(s) found are the following

$$y = \frac{3 \cos\left(\sqrt{2}\, t\right)}{2} - \frac{3}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{3\cos\left(\sqrt{2}\, t\right)}{2} - \frac{3}{2}$$

Verified OK.

16.26.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 2y'y + 3y' = 0$$

Integrating the above w.r.t t gives

$$\int (y'y'' + 2y'y + 3y')\, dt = 0$$
$$\frac{y'^2}{2} + y^2 + 3y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-2y^2 - 6y + 2c_1} \tag{1}$$
$$y' = -\sqrt{-2y^2 - 6y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-2y^2+2c_1-6y}}dy = \int dt$$

$$\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\left(y+\frac{3}{2}\right)}{\sqrt{-2y^2-6y+2c_1}}\right)}{2} = t+c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-2y^2+2c_1-6y}}dy = \int dt$$

$$-\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\left(y+\frac{3}{2}\right)}{\sqrt{-2y^2-6y+2c_1}}\right)}{2} = t+c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\left(y+\frac{3}{2}\right)}{\sqrt{-2y^2-6y+2c_1}}\right)}{2} = t+c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=0$ and $t=0$ in the above gives

$$\frac{\arctan\left(\frac{3}{2\sqrt{c_1}}\right)\sqrt{2}}{2} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2\left(9+4c_1\right)\sqrt{2}}{2\sqrt{\frac{9+4c_1}{\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2+1}}\left(\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2+1\right)} + \frac{\sqrt{\frac{9+4c_1}{\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2+1}}\,\sqrt{2}\left(\tan\left(\left(t+c_2\right)\sqrt{2}\right)^2+1\right)}{2}$$

substituting $y'=0$ and $t=0$ in the above gives

$$0 = \frac{\cos\left(\sqrt{2}\,c_2\right)^2\left(9+4c_1\right)\sqrt{2}}{2\sqrt{\cos\left(\sqrt{2}\,c_2\right)^2\left(9+4c_1\right)}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,(y+\frac{3}{2})}{\sqrt{-2y^2-6y+2c_1}}\right)}{2} = t + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$-\frac{\arctan\left(\frac{3}{2\sqrt{c_1}}\right)\sqrt{2}}{2} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\tan\left((t+c_3)\sqrt{2}\right)^2 (9+4c_1)\sqrt{2}}{2\sqrt{\frac{9+4c_1}{\tan\left((t+c_3)\sqrt{2}\right)^2+1}}\left(\tan\left((t+c_3)\sqrt{2}\right)^2+1\right)} - \frac{\sqrt{\frac{9+4c_1}{\tan\left((t+c_3)\sqrt{2}\right)^2+1}}\,\sqrt{2}\left(\tan\left((t+c_3)\sqrt{2}\right)^2+1\right)}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{\cos\left(\sqrt{2}\,c_3\right)^2 (9+4c_1)\sqrt{2}}{2\sqrt{\cos\left(\sqrt{2}\,c_3\right)^2 (9+4c_1)}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

16.26.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -2z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 461: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\sqrt{2}\, t\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos\left(\sqrt{2}\, t\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, t\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\
&= \cos\left(\sqrt{2}\, t\right) \int \frac{1}{\cos\left(\sqrt{2}\, t\right)^2}\, dt \\
&= \cos\left(\sqrt{2}\, t\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, t\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\sqrt{2}\, t\right)\right) + c_2 \left(\cos\left(\sqrt{2}\, t\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, t\right)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos\left(\sqrt{2}\, t\right) c_1 + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{2}, \cos\left(\sqrt{2}\,t\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = -3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\sqrt{2}\,t\right) c_1 + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{2}\right) + \left(-\frac{3}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \cos\left(\sqrt{2}\, t\right) c_1 + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{2} - \frac{3}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{3}{2} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) c_1 + c_2 \cos\left(\sqrt{2}\, t\right)$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3}{2}$$
$$c_2 = 0$$

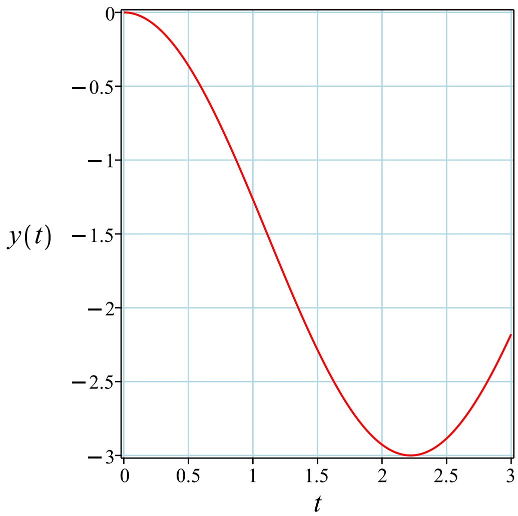
Substituting these values back in above solution results in

$$y = \frac{3\cos\left(\sqrt{2}\, t\right)}{2} - \frac{3}{2}$$

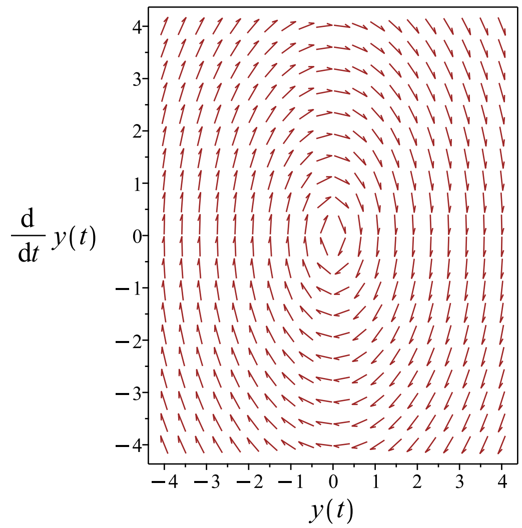
Summary

The solution(s) found are the following

$$y = \frac{3\cos\left(\sqrt{2}\, t\right)}{2} - \frac{3}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3\cos\left(\sqrt{2}\,t\right)}{2} - \frac{3}{2}$$

Verified OK.

16.26.5 Maple step by step solution

Let's solve

$$\left[y'' + 2y = -3, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos\left(\sqrt{2}\,t\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin\left(\sqrt{2}\,t\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))} dt\right), f(t) = -3\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t)\,, y_2(t)) = \left[\begin{array}{cc} \cos\left(\sqrt{2}\,t\right) & \sin\left(\sqrt{2}\,t\right) \\ -\sqrt{2}\,\sin\left(\sqrt{2}\,t\right) & \sqrt{2}\,\cos\left(\sqrt{2}\,t\right) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(t)\,, y_2(t)) = \sqrt{2}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{3\sqrt{2}\left(\cos\left(\sqrt{2}\,t\right)\left(\int \sin\left(\sqrt{2}\,t\right)dt\right) - \sin\left(\sqrt{2}\,t\right)\left(\int \cos\left(\sqrt{2}\,t\right)dt\right)\right)}{2}$

  - Compute integrals

    $y_p(t) = -\frac{3}{2}$

- Substitute particular solution into general solution to ODE

  $y = \cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right) - \frac{3}{2}$

□ Check validity of solution $y = \cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right) - \frac{3}{2}$

  - Use initial condition $y(0) = 0$

    $0 = -\frac{3}{2} + c_1$

  - Compute derivative of the solution

    $y' = -\sqrt{2}\,\sin\left(\sqrt{2}\,t\right) c_1 + \sqrt{2}\,\cos\left(\sqrt{2}\,t\right) c_2$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = \sqrt{2}\,c_2$

- Solve for c_1 and c_2

  $\{c_1 = \frac{3}{2}, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $y = \frac{3\cos(\sqrt{2}\,t)}{2} - \frac{3}{2}$

- Solution to the IVP

  $y = \frac{3\cos(\sqrt{2}\,t)}{2} - \frac{3}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(t),t$2)+2*y(t)=-3,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{3}{2} + \frac{3\cos(\sqrt{2}\,t)}{2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 17

```
DSolve[{y''[t]+2*y[t]==-3,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -3\sin^2\left(\frac{t}{\sqrt{2}}\right)$$

16.27 problem 28

Internal problem ID [13187]
Internal file name [`OUTPUT/11842_Sunday_December_03_2023_07_19_17_PM_68362535/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y = e^t}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= e^t \end{aligned}$$

Hence the ode is

$$y'' + 4y = e^t$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.27.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = e^t$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^t = e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^t}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2t) + c_2 \sin(2t)) + \left(\frac{e^t}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{e^t}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{5} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{e^t}{5}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{5} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{5}$$
$$c_2 = -\frac{1}{10}$$

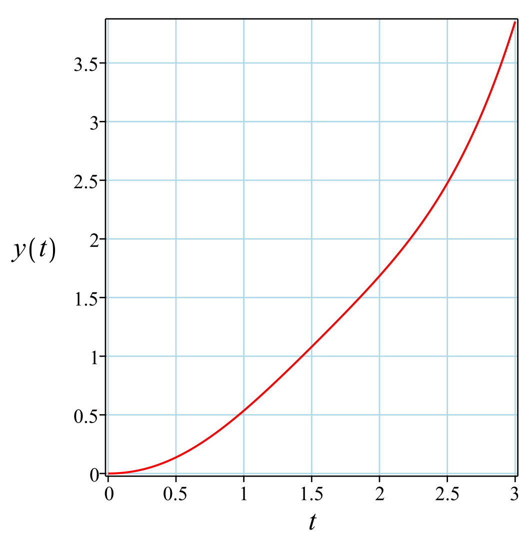
Substituting these values back in above solution results in

$$y = \frac{e^t}{5} - \frac{\cos(2t)}{5} - \frac{\sin(2t)}{10}$$

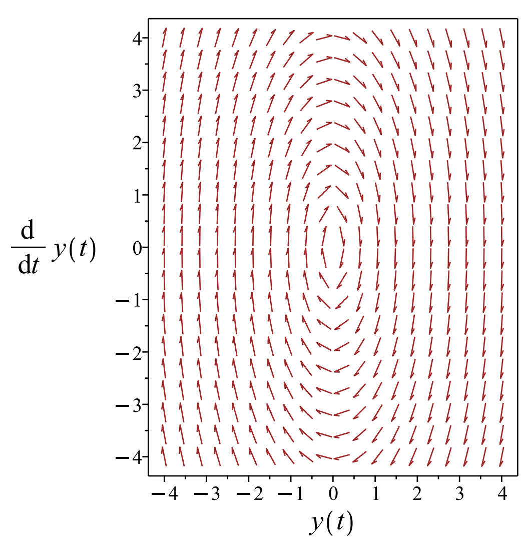
Summary

The solution(s) found are the following

$$y = \frac{e^t}{5} - \frac{\cos(2t)}{5} - \frac{\sin(2t)}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^t}{5} - \frac{\cos(2t)}{5} - \frac{\sin(2t)}{10}$$

Verified OK.

16.27.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 463: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(2t)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dt\\ &= \cos(2t) \int \frac{1}{\cos(2t)^2}\,dt\\ &= \cos(2t)\left(\frac{\tan(2t)}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2}, \cos(2t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^t = e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^t}{5}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + \left(\frac{e^t}{5}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} + \frac{e^t}{5} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = \frac{1}{5} + c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \sin(2t) + c_2 \cos(2t) + \frac{e^t}{5}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = \frac{1}{5} + c_2 \tag{2A}
$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{5} \\
c_2 &= -\frac{1}{5}
\end{aligned}
$$

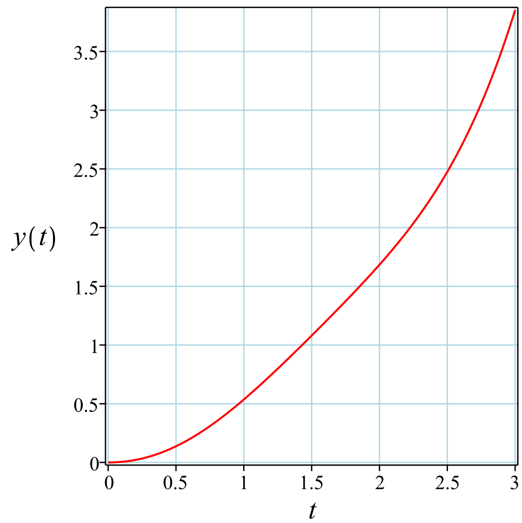
Substituting these values back in above solution results in

$$
y = \frac{e^t}{5} - \frac{\cos(2t)}{5} - \frac{\sin(2t)}{10}
$$

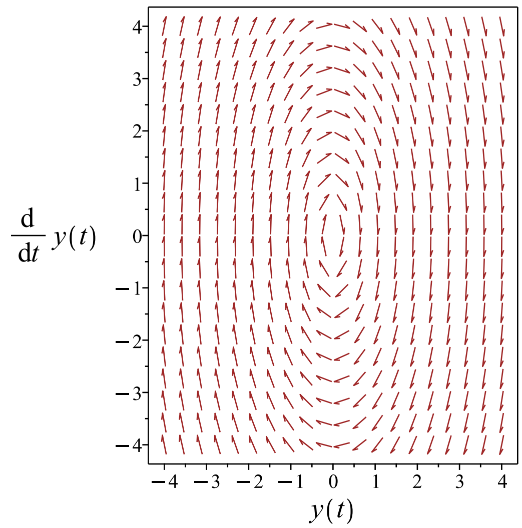
Summary

The solution(s) found are the following

$$
y = \frac{e^t}{5} - \frac{\cos(2t)}{5} - \frac{\sin(2t)}{10} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^t}{5} - \frac{\cos(2t)}{5} - \frac{\sin(2t)}{10}$$

Verified OK.

16.27.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y = e^t, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathrm{e}^t\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{array}\right]$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(2t)\left(\int \mathrm{e}^t \sin(2t) dt\right)}{2} + \frac{\sin(2t)\left(\int \mathrm{e}^t \cos(2t) dt\right)}{2}$

- Compute integrals

  $y_p(t) = \frac{\mathrm{e}^t}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\mathrm{e}^t}{5}$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\mathrm{e}^t}{5}$

- Use initial condition $y(0) = 0$

  $0 = \frac{1}{5} + c_1$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{\mathrm{e}^t}{5}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \frac{1}{5} + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{5}, c_2 = -\frac{1}{10}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^t}{5} - \frac{\cos(2t)}{5} - \frac{\sin(2t)}{10}$

- Solution to the IVP

  $y = \frac{e^t}{5} - \frac{\cos(2t)}{5} - \frac{\sin(2t)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)+4*y(t)=exp(t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{\sin(2t)}{10} - \frac{\cos(2t)}{5} + \frac{e^t}{5}$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 27

```
DSolve[{y''[t]+4*y[t]==Exp[t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{10}\left(2e^t - \sin(2t) - 2\cos(2t)\right)$$

16.28 problem 29

Internal problem ID [13188]
Internal file name [`OUTPUT/11843_Sunday_December_03_2023_07_19_21_PM_11694891/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + 9y = 6}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= 6 \end{aligned}$$

Hence the ode is

$$y'' + 9y = 6$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.28.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 9, f(t) = 6$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 9\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(3t) + c_2 \sin(3t))$$

Or

$$y = c_1 \cos(3t) + c_2 \sin(3t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3t) + c_2 \sin(3t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

1

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3t), \sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 = 6$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3t) + c_2 \sin(3t)) + \left(\frac{2}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{2}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{2}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3t) + 3c_2 \cos(3t)$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{2}{3}$$
$$c_2 = 0$$

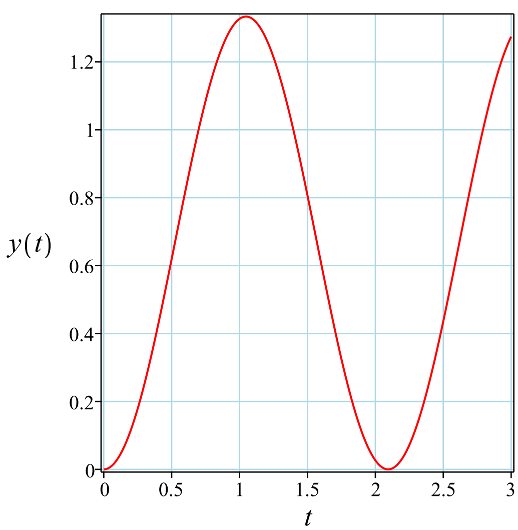
Substituting these values back in above solution results in

$$y = \frac{2}{3} - \frac{2\cos(3t)}{3}$$

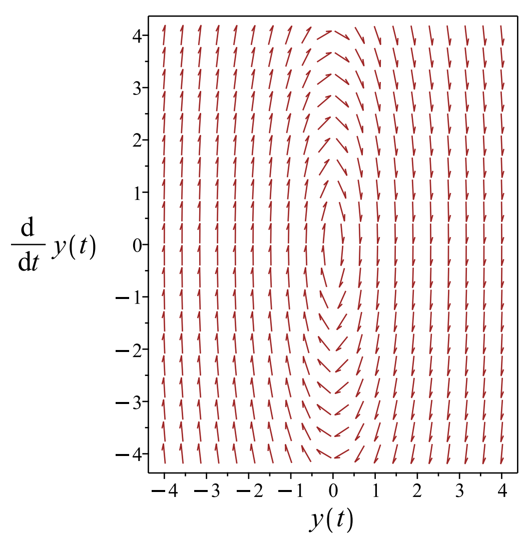
Summary

The solution(s) found are the following

$$y = \frac{2}{3} - \frac{2\cos(3t)}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{3} - \frac{2\cos(3t)}{3}$$

Verified OK.

16.28.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 9y'y - 6y' = 0$$

Integrating the above w.r.t t gives

$$\int (y'y'' + 9y'y - 6y')\,dt = 0$$
$$\frac{y'^2}{2} + \frac{9y^2}{2} - 6y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-9y^2 + 12y + 2c_1} \tag{1}$$
$$y' = -\sqrt{-9y^2 + 12y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-9y^2 + 2c_1 + 12y}}dy = \int dt$$
$$\frac{\arctan\left(\frac{3y-2}{\sqrt{-9y^2+12y+2c_1}}\right)}{3} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-9y^2 + 2c_1 + 12y}}dy = \int dt$$
$$-\frac{\arctan\left(\frac{3y-2}{\sqrt{-9y^2+12y+2c_1}}\right)}{3} = t + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{3y-2}{\sqrt{-9y^2+12y+2c_1}}\right)}{3} = t + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$-\frac{\arctan\left(\frac{\sqrt{2}}{\sqrt{c_1}}\right)}{3} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\left(3\tan\left(3t+3c_2\right)^2+3\right)\sqrt{2}\,\sqrt{\frac{c_1+2}{\tan(3t+3c_2)^2+1}}}{3} - \frac{\tan\left(3t+3c_2\right)^2\sqrt{2}\left(c_1+2\right)\left(3\tan\left(3t+3c_2\right)^2+3\right)}{3\sqrt{\frac{c_1+2}{\tan(3t+3c_2)^2+1}}\,\left(\tan\left(3t+3c_2\right)^2+1\right)^2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{\cos\left(3c_2\right)^2\left(c_1+2\right)\sqrt{2}}{\sqrt{\cos\left(3c_2\right)^2\left(c_1+2\right)}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\arctan\left(\frac{3y-2}{\sqrt{-9y^2+12y+2c_1}}\right)}{3} = t + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$\frac{\arctan\left(\frac{\sqrt{2}}{\sqrt{c_1}}\right)}{3} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(3\tan\left(3t+3c_3\right)^2+3\right)\sqrt{2}\,\sqrt{\frac{c_1+2}{\tan(3t+3c_3)^2+1}}}{3} + \frac{\tan\left(3t+3c_3\right)^2\sqrt{2}\left(c_1+2\right)\left(3\tan\left(3t+3c_3\right)^2+3\right)}{3\sqrt{\frac{c_1+2}{\tan(3t+3c_3)^2+1}}\,\left(\tan\left(3t+3c_3\right)^2+1\right)^2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{\cos(3c_3)^2 \sqrt{2}\,(-2 - c_1)}{\sqrt{\cos(3c_3)^2 (c_1 + 2)}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

16.28.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 465: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= \cos(3t) \int \frac{1}{\cos(3t)^2}\,dt \\ &= \cos(3t)\left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3t)) + c_2\left(\cos(3t)\left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+9y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3t)}{3}, \cos(3t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 = 6$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}\right) + \left(\frac{2}{3}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} + \frac{2}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 + \frac{2}{3} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3c_1 \sin(3t) + c_2 \cos(3t)
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

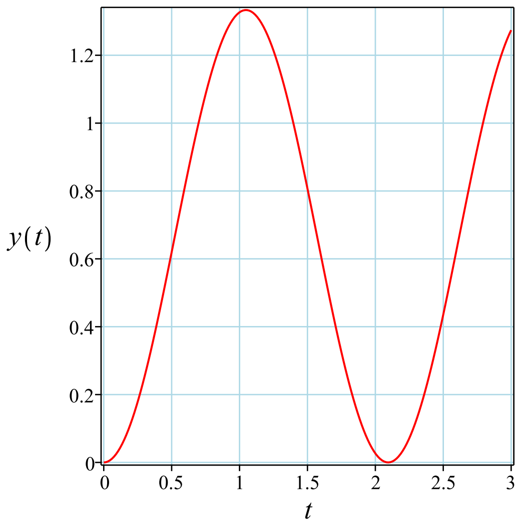
$$
\begin{aligned}
c_1 &= -\frac{2}{3} \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

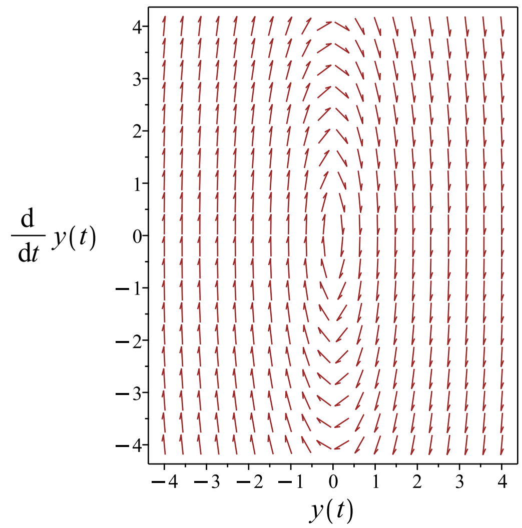
$$
y = \frac{2}{3} - \frac{2\cos(3t)}{3}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{2}{3} - \frac{2\cos(3t)}{3} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{3} - \frac{2\cos(3t)}{3}$$

Verified OK.

16.28.5 Maple step by step solution

Let's solve

$$\left[y'' + 9y = 6, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 6\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -2\cos(3t) \left(\int \sin(3t)\, dt\right) + 2\sin(3t) \left(\int \cos(3t)\, dt\right)$

- Compute integrals

  $y_p(t) = \frac{2}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{2}{3}$

☐ Check validity of solution $y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{2}{3}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + \frac{2}{3}$

- Compute derivative of the solution

  $y' = -3c_1 \sin(3t) + 3c_2 \cos(3t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = -\frac{2}{3}, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = \frac{2}{3} - \frac{2\cos(3t)}{3}$

- Solution to the IVP

  $y = \frac{2}{3} - \frac{2\cos(3t)}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([diff(y(t),t$2)+9*y(t)=6,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{2}{3} - \frac{2\cos(3t)}{3}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 17

```
DSolve[{y''[t]+9*y[t]==6,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{4}{3}\sin^2\left(\frac{3t}{2}\right)$$

16.29 problem 30

Internal problem ID [13189]
Internal file name [`OUTPUT/11844_Sunday_December_03_2023_07_19_25_PM_12119771/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y = -\mathrm{e}^{t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 2 \\ F &= -\mathrm{e}^{t} \end{aligned}$$

Hence the ode is

$$y'' + 2y = -e^t$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -e^t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.29.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 2, f(t) = -e^t$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0\left(\cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)\right)$$

Or

$$y = \cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{2}\,t\right), \sin\left(\sqrt{2}\,t\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^t = -e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^t}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right)\right) + \left(-\frac{e^t}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right) - \frac{e^t}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{3} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sqrt{2}\, \sin\left(\sqrt{2}\, t\right) c_1 + \sqrt{2}\, \cos\left(\sqrt{2}\, t\right) c_2 - \frac{e^t}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \sqrt{2}\, c_2 - \frac{1}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{3}$$
$$c_2 = \frac{\sqrt{2}}{6}$$

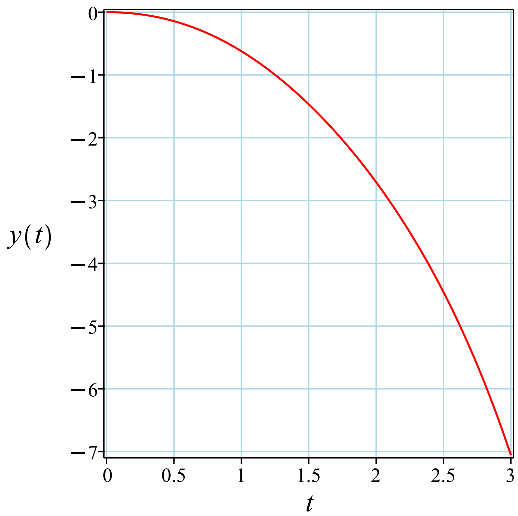
Substituting these values back in above solution results in

$$y = \frac{\cos\left(\sqrt{2}\, t\right)}{3} + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{6} - \frac{e^t}{3}$$

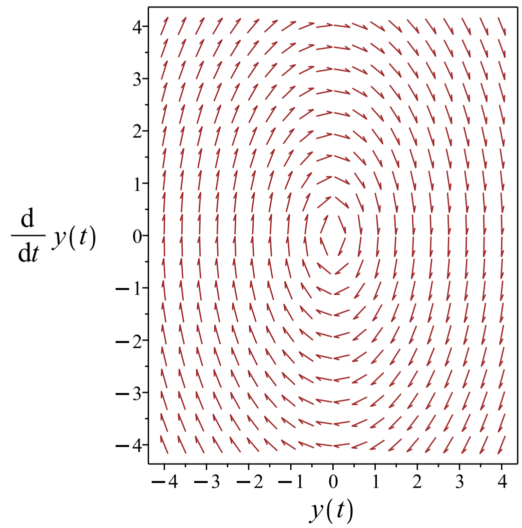
Summary

The solution(s) found are the following

$$y = \frac{\cos\left(\sqrt{2}\, t\right)}{3} + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{6} - \frac{e^t}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\cos\left(\sqrt{2}\, t\right)}{3} + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{6} - \frac{e^{t}}{3}$$

Verified OK.

16.29.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -2z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 467: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\sqrt{2}\, t\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos\left(\sqrt{2}\, t\right)\end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, t\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\ &= \cos\left(\sqrt{2}\, t\right) \int \frac{1}{\cos\left(\sqrt{2}\, t\right)^2}\, dt \\ &= \cos\left(\sqrt{2}\, t\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, t\right)}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2\\ &= c_1\left(\cos\left(\sqrt{2}\,t\right)\right) + c_2\left(\cos\left(\sqrt{2}\,t\right)\left(\frac{\sqrt{2}\,\tan\left(\sqrt{2}\,t\right)}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos\left(\sqrt{2}\,t\right)c_1 + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\mathrm{e}^t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{2}, \cos\left(\sqrt{2}\,t\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^t = -e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^t}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\sqrt{2}\,t\right) c_1 + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{2}\right) + \left(-\frac{e^t}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \cos\left(\sqrt{2}\,t\right) c_1 + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{2} - \frac{e^t}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{3} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sqrt{2}\,\sin\left(\sqrt{2}\,t\right) c_1 + c_2 \cos\left(\sqrt{2}\,t\right) - \frac{e^t}{3}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = c_2 - \frac{1}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{3}$$
$$c_2 = \frac{1}{3}$$

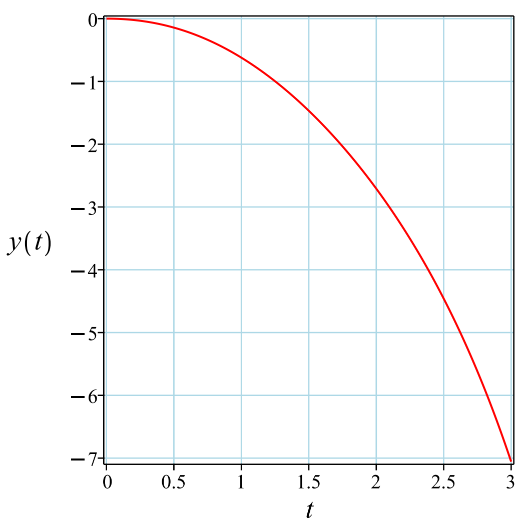
Substituting these values back in above solution results in

$$y = \frac{\cos\left(\sqrt{2}\, t\right)}{3} + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{6} - \frac{e^{t}}{3}$$

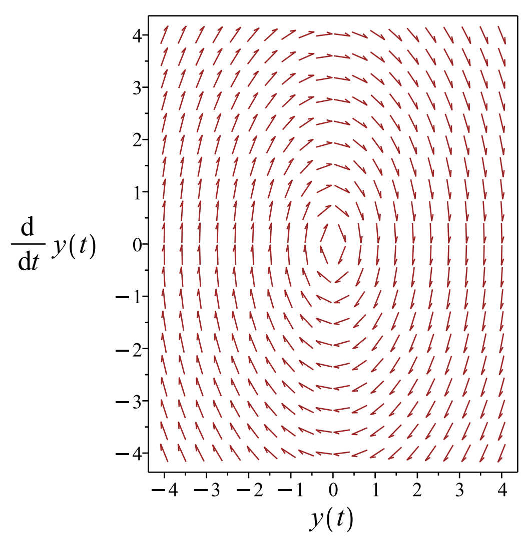
Summary

The solution(s) found are the following

$$y = \frac{\cos\left(\sqrt{2}\, t\right)}{3} + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{6} - \frac{e^{t}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\cos\left(\sqrt{2}\, t\right)}{3} + \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, t\right)}{6} - \frac{e^{t}}{3}$$

Verified OK.

16.29.4 Maple step by step solution

Let's solve

$$\left[y'' + 2y = -\mathrm{e}^{t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos\left(\sqrt{2}\, t\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin\left(\sqrt{2}\, t\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\sqrt{2}\, t\right) c_1 + c_2 \sin\left(\sqrt{2}\, t\right) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = -\mathrm{e}^{t}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos\left(\sqrt{2}\, t\right) & \sin\left(\sqrt{2}\, t\right) \\ -\sqrt{2}\sin\left(\sqrt{2}\, t\right) & \sqrt{2}\cos\left(\sqrt{2}\, t\right) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = \sqrt{2}$

- ◦ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{\sqrt{2}\left(\sin\left(\sqrt{2}\,t\right)\left(\int \mathrm{e}^t \cos\left(\sqrt{2}\,t\right)dt\right) - \cos\left(\sqrt{2}\,t\right)\left(\int \mathrm{e}^t \sin\left(\sqrt{2}\,t\right)dt\right)\right)}{2}$$

- ○ Compute integrals

  $y_p(t) = -\frac{\mathrm{e}^t}{3}$

- ● Substitute particular solution into general solution to ODE

  $y = \cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right) - \frac{\mathrm{e}^t}{3}$

- ☐ Check validity of solution $y = \cos\left(\sqrt{2}\,t\right) c_1 + c_2 \sin\left(\sqrt{2}\,t\right) - \frac{\mathrm{e}^t}{3}$

- ○ Use initial condition $y(0) = 0$

  $0 = -\frac{1}{3} + c_1$

- ○ Compute derivative of the solution

  $y' = -\sqrt{2}\,\sin\left(\sqrt{2}\,t\right) c_1 + \sqrt{2}\,\cos\left(\sqrt{2}\,t\right) c_2 - \frac{\mathrm{e}^t}{3}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \sqrt{2}\,c_2 - \frac{1}{3}$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{3}, c_2 = \frac{\sqrt{2}}{6}\right\}$

- ○ Substitute constant values into general solution and simplify

  $y = \frac{\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{6} - \frac{\mathrm{e}^t}{3}$

- ● Solution to the IVP

  $y = \frac{\cos\left(\sqrt{2}\,t\right)}{3} + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{6} - \frac{\mathrm{e}^t}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)+2*y(t)=-exp(t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,t\right)}{6} + \frac{\cos\left(\sqrt{2}\,t\right)}{3} - \frac{e^{t}}{3}$$

✓ Solution by Mathematica
Time used: 0.028 (sec). Leaf size: 39

```
DSolve[{y''[t]+2*y[t]==-Exp[t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{6}\left(-2e^{t} + \sqrt{2}\sin\left(\sqrt{2}t\right) + 2\cos\left(\sqrt{2}t\right)\right)$$

16.30 problem 31

Internal problem ID [13190]
Internal file name [OUTPUT/11845_Sunday_December_03_2023_07_19_29_PM_37466865/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y = -3t^2 + 2t + 3}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

16.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= -3t^2 + 2t + 3 \end{aligned}$$

Hence the ode is

$$y'' + 4y = -3t^2 + 2t + 3$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -3t^2 + 2t + 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.30.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = -3t^2 + 2t + 3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\,e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2 + t + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3 t^2 + 4A_2 t + 4A_1 + 2A_3 = -3t^2 + 2t + 3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{9}{8}, A_2 = \frac{1}{2}, A_3 = -\frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3}{4}t^2 + \frac{1}{2}t + \frac{9}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2t) + c_2 \sin(2t)) + \left(-\frac{3}{4}t^2 + \frac{1}{2}t + \frac{9}{8}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{3t^2}{4} + \frac{t}{2} + \frac{9}{8} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$2 = c_1 + \frac{9}{8} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{3t}{2} + \frac{1}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{2} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{7}{8}$$
$$c_2 = -\frac{1}{4}$$

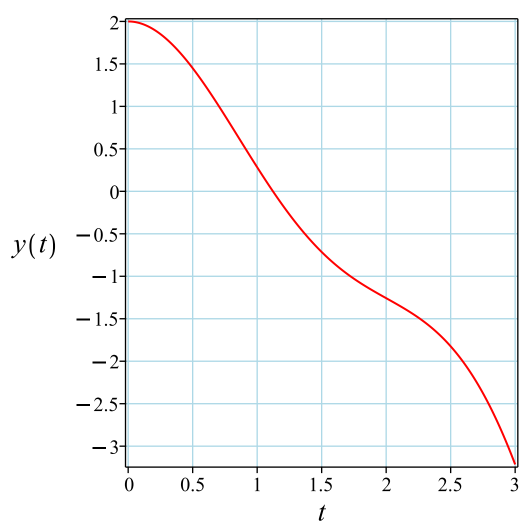
Substituting these values back in above solution results in

$$y = \frac{9}{8} + \frac{7\cos(2t)}{8} - \frac{\sin(2t)}{4} - \frac{3t^2}{4} + \frac{t}{2}$$

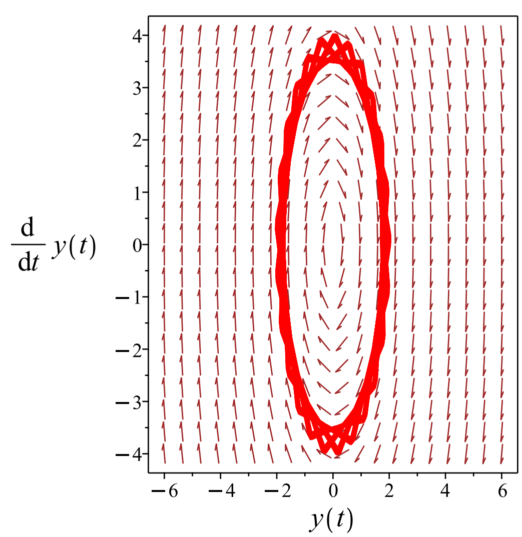
Summary

The solution(s) found are the following

$$y = \frac{9}{8} + \frac{7\cos(2t)}{8} - \frac{\sin(2t)}{4} - \frac{3t^2}{4} + \frac{t}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{9}{8} + \frac{7\cos(2t)}{8} - \frac{\sin(2t)}{4} - \frac{3t^2}{4} + \frac{t}{2}$$

Verified OK.

16.30.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 469: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= \cos(2t) \int \frac{1}{\cos(2t)^2}\,dt \\ &= \cos(2t)\left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2+t+1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2}, \cos(2t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3t^2 + 4A_2t + 4A_1 + 2A_3 = -3t^2 + 2t + 3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{9}{8}, A_2 = \frac{1}{2}, A_3 = -\frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3}{4}t^2 + \frac{1}{2}t + \frac{9}{8}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + \left(-\frac{3}{4}t^2 + \frac{1}{2}t + \frac{9}{8}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} - \frac{3t^2}{4} + \frac{t}{2} + \frac{9}{8} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $t = 0$ in the above gives

$$
2 = c_1 + \frac{9}{8} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \sin(2t) + c_2 \cos(2t) - \frac{3t}{2} + \frac{1}{2}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = \frac{1}{2} + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{7}{8} \\
c_2 &= -\frac{1}{2}
\end{aligned}
$$

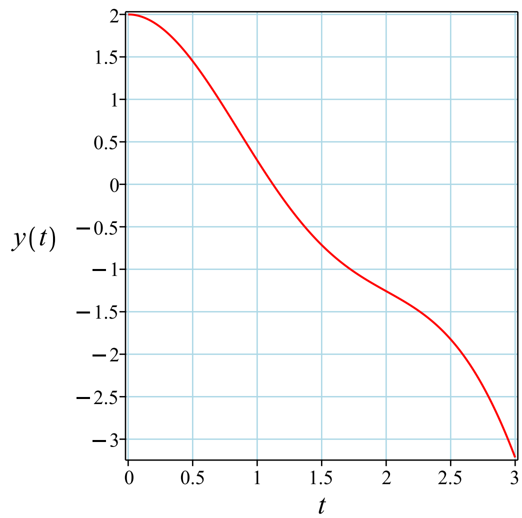
Substituting these values back in above solution results in

$$
y = \frac{9}{8} + \frac{7\cos(2t)}{8} - \frac{\sin(2t)}{4} - \frac{3t^2}{4} + \frac{t}{2}
$$

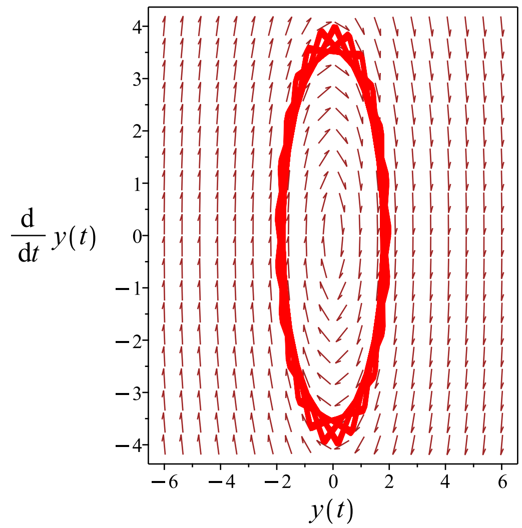
Summary

The solution(s) found are the following

$$
y = \frac{9}{8} + \frac{7\cos(2t)}{8} - \frac{\sin(2t)}{4} - \frac{3t^2}{4} + \frac{t}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{9}{8} + \frac{7\cos(2t)}{8} - \frac{\sin(2t)}{4} - \frac{3t^2}{4} + \frac{t}{2}$$

Verified OK.

16.30.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y = -3t^2 + 2t + 3, y(0) = 2, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = -3t^2 + 2t + 3\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{\cos(2t)\left(\int \sin(2t)(3t^2-2t-3)dt\right)}{2} - \frac{\sin(2t)\left(\int \cos(2t)(3t^2-2t-3)dt\right)}{2}$

- Compute integrals

  $y_p(t) = -\frac{3}{4}t^2 + \frac{1}{2}t + \frac{9}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{3t^2}{4} + \frac{t}{2} + \frac{9}{8}$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{3t^2}{4} + \frac{t}{2} + \frac{9}{8}$

- Use initial condition $y(0) = 2$

  $2 = c_1 + \frac{9}{8}$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{3t}{2} + \frac{1}{2}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \frac{1}{2} + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = \frac{7}{8}, c_2 = -\frac{1}{4}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{9}{8} + \frac{7\cos(2t)}{8} - \frac{\sin(2t)}{4} - \frac{3t^2}{4} + \frac{t}{2}$

- Solution to the IVP

  $y = \frac{9}{8} + \frac{7\cos(2t)}{8} - \frac{\sin(2t)}{4} - \frac{3t^2}{4} + \frac{t}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve([diff(y(t),t$2)+4*y(t)=-3*t^2+2*t+3,y(0) = 2, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{\sin(2t)}{4} + \frac{7\cos(2t)}{8} - \frac{3t^2}{4} + \frac{t}{2} + \frac{9}{8}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 31

```
DSolve[{y''[t]+4*y[t]==-3*t^2+2*t+3,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> T
```

$$y(t) \to \frac{1}{8}\left(-6t^2 + 4t - 2\sin(2t) - 9\cos(2t) + 9\right)$$

16.31 problem 32

Internal problem ID [13191]
Internal file name [`OUTPUT/11846_Sunday_December_03_2023_07_19_35_PM_55571381/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 2y' = 3t + 2}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 0 \\ F &= 3t + 2 \end{aligned}$$

Hence the ode is

$$y'' + 2y' = 3t + 2$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = 3t + 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.31.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 2, C = 0, f(t) = 3t + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 0$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 2\lambda\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(0)}\\ &= -1 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 1\\ \lambda_2 &= -1 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(0)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 \mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2t}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t, t^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 t^2 + A_1 t$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4tA_2 + 2A_1 + 2A_2 = 3t + 2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = \frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3}{4}t^2 + \frac{1}{4}t$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{-2t}\right) + \left(\frac{3}{4}t^2 + \frac{1}{4}t\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2 e^{-2t} + \frac{3t^2}{4} + \frac{t}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_2 e^{-2t} + \frac{3t}{2} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_2 + \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{8}$$
$$c_2 = \frac{1}{8}$$

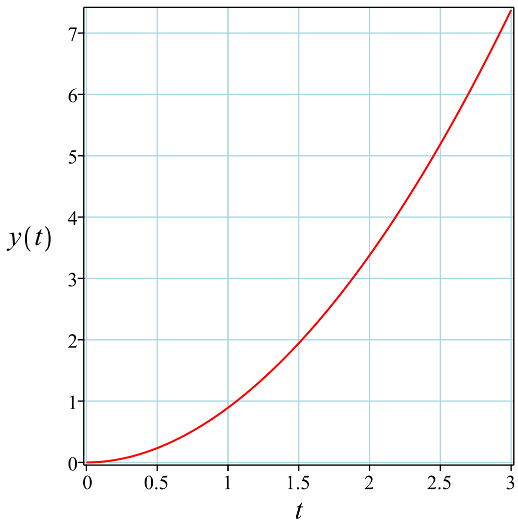
Substituting these values back in above solution results in

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

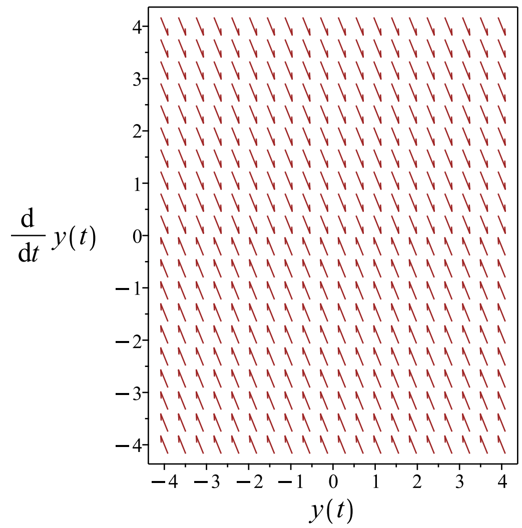
Summary

The solution(s) found are the following

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Verified OK.

16.31.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (y'' + 2y')\, dt = \int (3t + 2)\, dt$$
$$y' + 2y = \frac{3}{2}t^2 + 2t + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 2$$
$$q(t) = \frac{3}{2}t^2 + 2t + c_1$$

Hence the ode is

$$y' + 2y = \frac{3}{2}t^2 + 2t + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dt} \\ &= e^{2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\ \frac{d}{dt}\left(e^{2t}y\right) &= \left(e^{2t}\right)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\ d\left(e^{2t}y\right) &= \left(\frac{(3t^2 + 2c_1 + 4t)\,e^{2t}}{2}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2t}y &= \int \frac{(3t^2 + 2c_1 + 4t)\,e^{2t}}{2}\,dt \\ e^{2t}y &= \frac{(6t^2 + 4c_1 + 2t - 1)\,e^{2t}}{8} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$y = \frac{e^{-2t}(6t^2 + 4c_1 + 2t - 1)\,e^{2t}}{8} + c_2 e^{-2t}$$

which simplifies to

$$y = \frac{3t^2}{4} + \frac{c_1}{2} + \frac{t}{4} - \frac{1}{8} + c_2 e^{-2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{3t^2}{4} + \frac{c_1}{2} + \frac{t}{4} - \frac{1}{8} + c_2 e^{-2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{8} + \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_2 e^{-2t} + \frac{3t}{2} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_2 + \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{1}{8}$$

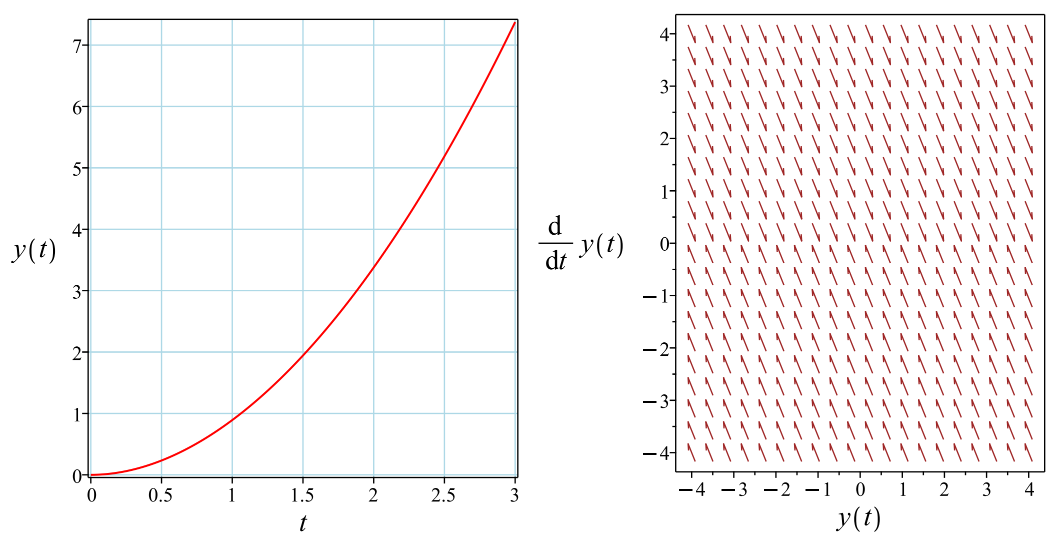
Substituting these values back in above solution results in

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Verified OK.

16.31.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$p'(t) + 2p(t) - 3t - 2 = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dt}\\ &= e^{2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu p) &= (\mu)(3t+2)\\ \frac{d}{dt}\left(e^{2t}p\right) &= \left(e^{2t}\right)(3t+2)\\ d\left(e^{2t}p\right) &= \left((3t+2)\,e^{2t}\right)\,dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2t}p &= \int (3t+2)\,e^{2t}\,dt\\ e^{2t}p &= \frac{(1+6t)\,e^{2t}}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$p(t) = \frac{e^{-2t}(1+6t)\,e^{2t}}{4} + c_1 e^{-2t}$$

which simplifies to

$$p(t) = \frac{3t}{2} + \frac{1}{4} + c_1 e^{-2t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{4} + c_1$$

$$c_1 = -\frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$p(t) = -\frac{e^{-2t}}{4} + \frac{3t}{2} + \frac{1}{4}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{e^{-2t}}{4} + \frac{3t}{2} + \frac{1}{4}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{e^{-2t}}{4} + \frac{3t}{2} + \frac{1}{4}\, dt \\ &= \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4} + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{8} + c_2$$

$$c_2 = -\frac{1}{8}$$

Substituting c_2 found above in the general solution gives

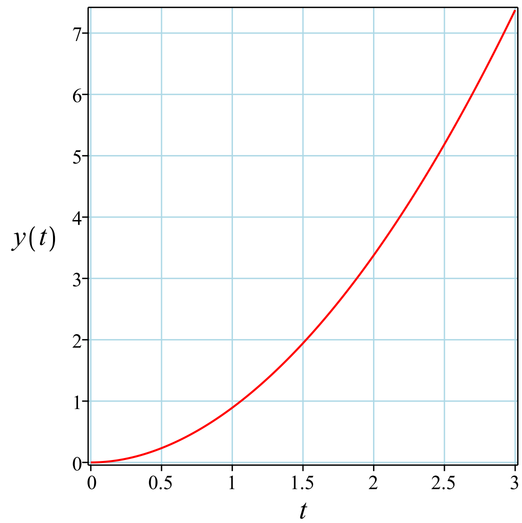
$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Initial conditions are used to solve for the constants of integration.

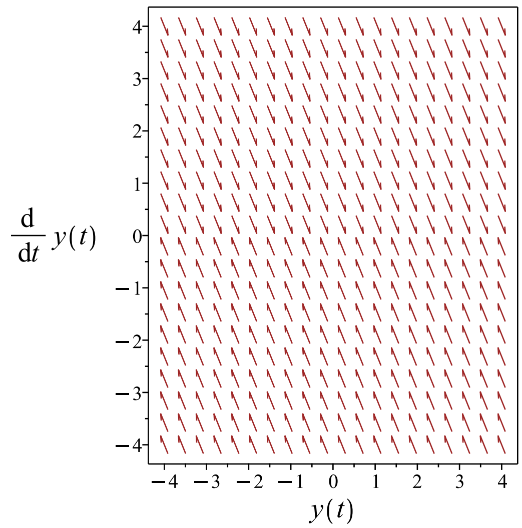
Summary

The solution(s) found are the following

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Verified OK.

16.31.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2y' = 3t + 2$$

Integrating both sides of the ODE w.r.t t gives

$$\int (y'' + 2y')\, dt = \int (3t + 2)\, dt$$

$$y' + 2y = \frac{3}{2}t^2 + 2t + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= 2 \\
q(t) &= \frac{3}{2}t^2 + 2t + c_1
\end{aligned}
$$

Hence the ode is

$$
y' + 2y = \frac{3}{2}t^2 + 2t + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2 dt} \\
&= e^{2t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\
\frac{d}{dt}\left(e^{2t} y\right) &= \left(e^{2t}\right)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\
d\left(e^{2t} y\right) &= \left(\frac{(3t^2 + 2c_1 + 4t)\, e^{2t}}{2}\right) dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{2t} y &= \int \frac{(3t^2 + 2c_1 + 4t)\, e^{2t}}{2}\, dt \\
e^{2t} y &= \frac{(6t^2 + 4c_1 + 2t - 1)\, e^{2t}}{8} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$
y = \frac{e^{-2t}(6t^2 + 4c_1 + 2t - 1)\, e^{2t}}{8} + c_2 e^{-2t}
$$

which simplifies to

$$
y = \frac{3t^2}{4} + \frac{c_1}{2} + \frac{t}{4} - \frac{1}{8} + c_2 e^{-2t}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{3t^2}{4} + \frac{c_1}{2} + \frac{t}{4} - \frac{1}{8} + c_2 e^{-2t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{8} + \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_2 e^{-2t} + \frac{3t}{2} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_2 + \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{1}{8}$$

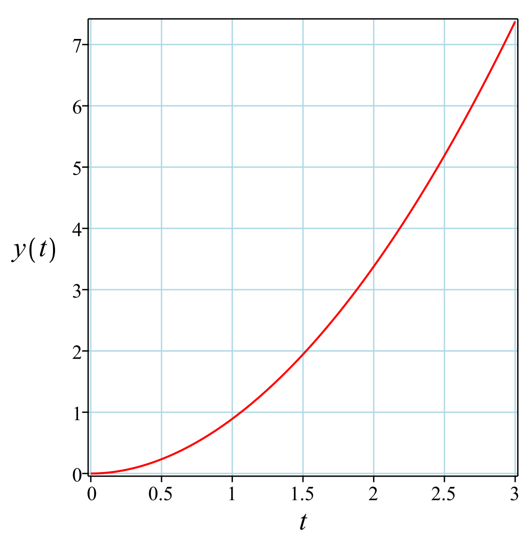
Substituting these values back in above solution results in

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

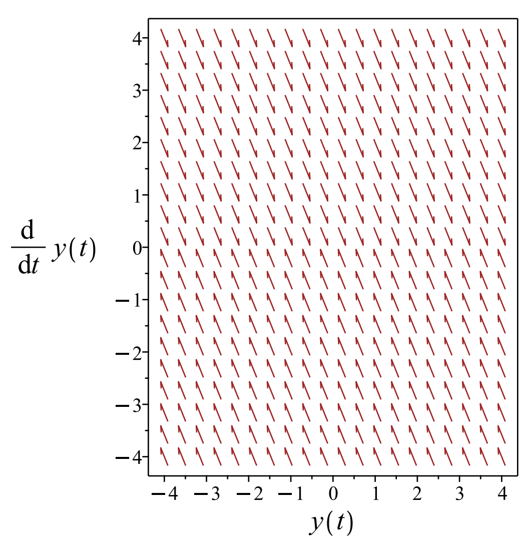
Summary

The solution(s) found are the following

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Verified OK.

16.31.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 471: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\
&= z_1 e^{-t} \\
&= z_1\left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt \\
&= y_1\left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2t}\right) + c_2\left(\mathrm{e}^{-2t}\left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+2y'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\mathrm{e}^{-2t}+\frac{c_2}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\mathrm{e}^{-2t}$$
$$y_2=\frac{1}{2}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(t)}{aW(t)} \tag{2}$$
$$u_2=\int\frac{y_1f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} \mathrm{e}^{-2t} & \frac{1}{2} \\ \frac{d}{dt}\left(\mathrm{e}^{-2t}\right) & \frac{d}{dt}\left(\frac{1}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2t} & \frac{1}{2} \\ -2\,e^{-2t} & 0 \end{vmatrix}$$

Therefore

$$W = \left(e^{-2t}\right)(0) - \left(\frac{1}{2}\right)\left(-2\,e^{-2t}\right)$$

Which simplifies to

$$W = e^{-2t}$$

Which simplifies to

$$W = e^{-2t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{3t}{2}+1}{e^{-2t}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{(3t+2)\,e^{2t}}{2}dt$$

Hence

$$u_1 = -\frac{(1+6t)\,e^{2t}}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2t}(3t+2)}{e^{-2t}}\,dt$$

Which simplifies to

$$u_2 = \int (3t+2)\,dt$$

Hence

$$u_2 = \frac{3}{2}t^2 + 2t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{e^{-2t}(1+6t)\,e^{2t}}{8} + \frac{3t^2}{4} + t$$

Which simplifies to

$$y_p(t) = -\frac{1}{8} + \frac{1}{4}t + \frac{3}{4}t^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + \frac{c_2}{2}\right) + \left(-\frac{1}{8} + \frac{1}{4}t + \frac{3}{4}t^2\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + \frac{c_2}{2} - \frac{1}{8} + \frac{t}{4} + \frac{3t^2}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} - \frac{1}{8} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2t} + \frac{1}{4} + \frac{3t}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 + \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{1}{8} \\ c_2 &= 0 \end{aligned}$$

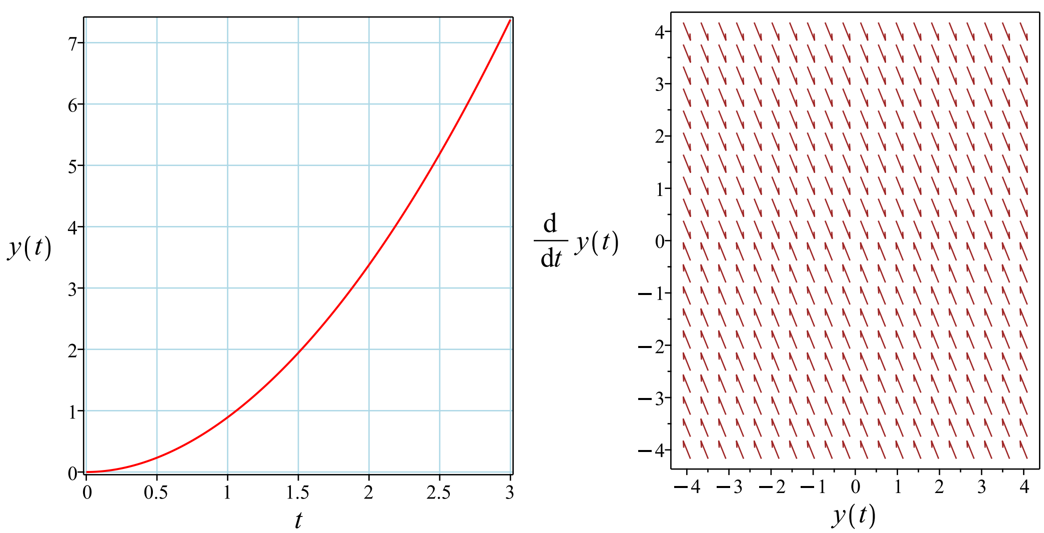
Substituting these values back in above solution results in

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Verified OK.

16.31.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 2 \\
r(x) &= 0 \\
s(x) &= 3t + 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(t)\, y' + (q(t) - p'(t))\, y)' = s(x)
$$

Integrating gives

$$
p(t)\, y' + (q(t) - p'(t))\, y = \int s(t)\, dt
$$

Substituting the above values for p, q, r, s gives

$$
y' + 2y = \int 3t + 2\, dt
$$

We now have a first order ode to solve which is

$$
y' + 2y = \frac{3}{2}t^2 + 2t + c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(t)y = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= 2 \\
q(t) &= \frac{3}{2}t^2 + 2t + c_1
\end{aligned}
$$

Hence the ode is

$$y' + 2y = \frac{3}{2}t^2 + 2t + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dt} \\ &= e^{2t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\ \frac{d}{dt}\left(e^{2t}y\right) &= \left(e^{2t}\right)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\ d\left(e^{2t}y\right) &= \left(\frac{(3t^2 + 2c_1 + 4t)\,e^{2t}}{2}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2t}y &= \int \frac{(3t^2 + 2c_1 + 4t)\,e^{2t}}{2}\, dt \\ e^{2t}y &= \frac{(6t^2 + 4c_1 + 2t - 1)\,e^{2t}}{8} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2t}$ results in

$$y = \frac{e^{-2t}(6t^2 + 4c_1 + 2t - 1)\,e^{2t}}{8} + c_2 e^{-2t}$$

which simplifies to

$$y = \frac{3t^2}{4} + \frac{c_1}{2} + \frac{t}{4} - \frac{1}{8} + c_2 e^{-2t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{3t^2}{4} + \frac{c_1}{2} + \frac{t}{4} - \frac{1}{8} + c_2 e^{-2t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{1}{8} + \frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_2 e^{-2t} + \frac{3t}{2} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_2 + \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{1}{8}$$

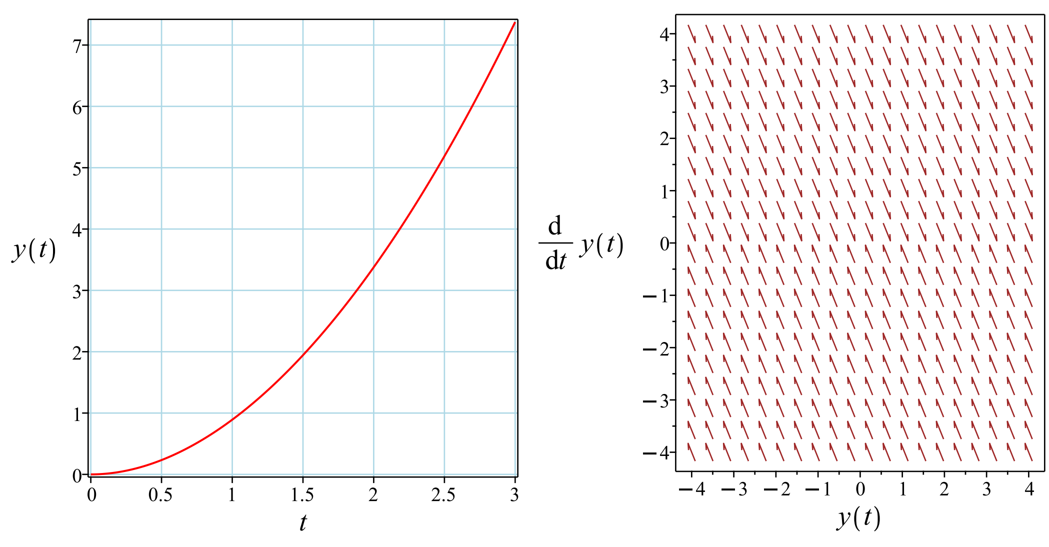
Substituting these values back in above solution results in

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Verified OK.

16.31.8 Maple step by step solution

Let's solve

$$\left[y''+2y'=3t+2, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+2r=0$

- Factor the characteristic polynomial

  $r(r+2)=0$

- Roots of the characteristic polynomial

  $r=(-2,0)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=1$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-2t}+c_2+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=3t+2\right]$$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t),y_2(t))=\begin{bmatrix}\mathrm{e}^{-2t} & 1\\ -2\,\mathrm{e}^{-2t} & 0\end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t),y_2(t))=2\,\mathrm{e}^{-2t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{e^{-2t}\left(\int (3t+2)e^{2t}dt\right)}{2} + \frac{\left(\int (3t+2)dt\right)}{2}$$

- Compute integrals

$$y_p(t) = -\frac{1}{8} + \frac{1}{4}t + \frac{3}{4}t^2$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-2t} + c_2 - \frac{1}{8} + \frac{t}{4} + \frac{3t^2}{4}$$

☐ Check validity of solution $y = c_1 e^{-2t} + c_2 - \frac{1}{8} + \frac{t}{4} + \frac{3t^2}{4}$

- Use initial condition $y(0) = 0$

$$0 = c_1 + c_2 - \frac{1}{8}$$

- Compute derivative of the solution

$$y' = -2c_1 e^{-2t} + \frac{1}{4} + \frac{3t}{2}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -2c_1 + \frac{1}{4}$$

- Solve for c_1 and c_2

$$\left\{c_1 = \frac{1}{8}, c_2 = 0\right\}$$

- Substitute constant values into general solution and simplify

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

- Solution to the IVP

$$y = -\frac{1}{8} + \frac{e^{-2t}}{8} + \frac{3t^2}{4} + \frac{t}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -2*_b(_a)+3*_a+2, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)=3*t+2,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{3t^2}{4} + \frac{e^{-2t}}{8} + \frac{t}{4} - \frac{1}{8}$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 24

```
DSolve[{y''[t]+2*y'[t]==3*t+2,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{8}\left(6t^2 + 2t + e^{-2t} - 1\right)$$

16.32 problem 33

Internal problem ID [13192]
Internal file name [`OUTPUT/11847_Sunday_December_03_2023_07_19_37_PM_90849126/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 4y' = 3t + 2}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 0 \\ F &= 3t + 2 \end{aligned}$$

Hence the ode is

$$y'' + 4y' = 3t + 2$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = 3t + 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.32.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 0, f(t) = 3t + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 0$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(0)}\\ &= -2 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 2\\ \lambda_2 &= -2 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(0)t} + c_2 e^{(-4)t}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-4t}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t, t^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 t^2 + A_1 t$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8tA_2 + 4A_1 + 2A_2 = 3t + 2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{16}, A_2 = \frac{3}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3}{8}t^2 + \frac{5}{16}t$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{-4t}\right) + \left(\frac{3}{8}t^2 + \frac{5}{16}t\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2 e^{-4t} + \frac{3t^2}{8} + \frac{5t}{16} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_2 e^{-4t} + \frac{3t}{4} + \frac{5}{16}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_2 + \frac{5}{16} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{5}{64}$$
$$c_2 = \frac{5}{64}$$

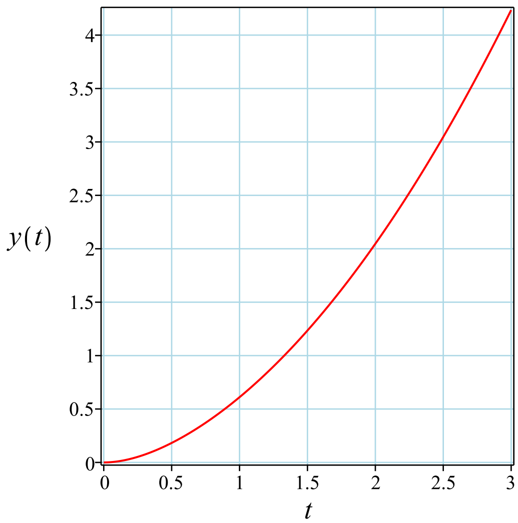
Substituting these values back in above solution results in

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

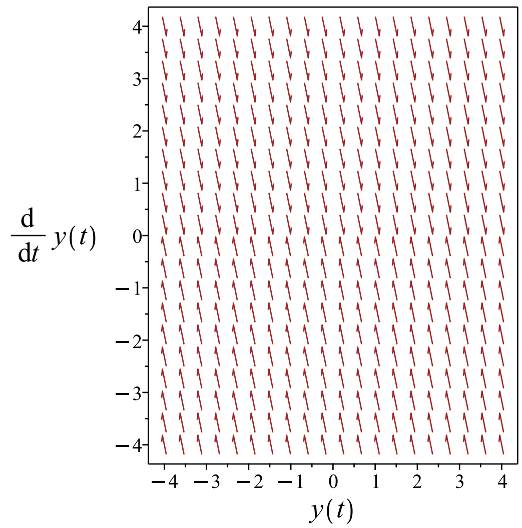
Summary

The solution(s) found are the following

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

Verified OK.

16.32.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (y'' + 4y')\,dt = \int (3t + 2)\,dt$$

$$4y + y' = \frac{3}{2}t^2 + 2t + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = 4$$
$$q(t) = \frac{3}{2}t^2 + 2t + c_1$$

Hence the ode is

$$4y + y' = \frac{3}{2}t^2 + 2t + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 4dt}\\ &= e^{4t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{3}{2}t^2 + 2t + c_1\right)\\ \frac{d}{dt}\left(e^{4t}y\right) &= \left(e^{4t}\right)\left(\frac{3}{2}t^2 + 2t + c_1\right)\\ d\left(e^{4t}y\right) &= \left(\frac{(3t^2 + 2c_1 + 4t)\,e^{4t}}{2}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{4t}y &= \int \frac{(3t^2 + 2c_1 + 4t)\,e^{4t}}{2}\, dt\\ e^{4t}y &= \frac{(24t^2 + 16c_1 + 20t - 5)\,e^{4t}}{64} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{4t}$ results in

$$y = \frac{e^{-4t}(24t^2 + 16c_1 + 20t - 5)\,e^{4t}}{64} + c_2 e^{-4t}$$

which simplifies to

$$y = \frac{3t^2}{8} + \frac{c_1}{4} + \frac{5t}{16} - \frac{5}{64} + c_2 e^{-4t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{3t^2}{8} + \frac{c_1}{4} + \frac{5t}{16} - \frac{5}{64} + c_2 e^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{5}{64} + \frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_2 e^{-4t} + \frac{3t}{4} + \frac{5}{16}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_2 + \frac{5}{16} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{5}{64}$$

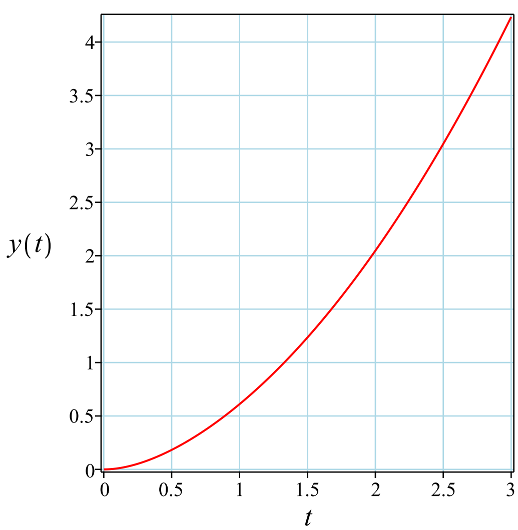
Substituting these values back in above solution results in

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

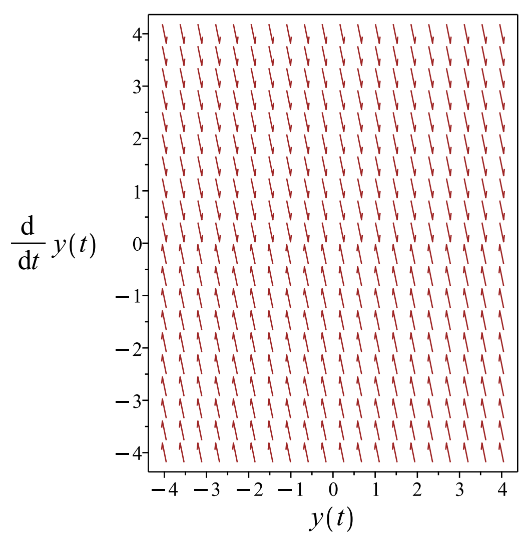
Summary

The solution(s) found are the following

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

Verified OK.

16.32.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$p'(t) + 4p(t) - 3t - 2 = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 4dt} \\ &= e^{4t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu p) &= (\mu)(3t + 2) \\ \frac{d}{dt}\left(e^{4t}p\right) &= \left(e^{4t}\right)(3t + 2) \\ d\left(e^{4t}p\right) &= \left(e^{4t}(3t + 2)\right)\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{4t}p &= \int e^{4t}(3t + 2)\, dt \\ e^{4t}p &= \frac{(12t + 5)\, e^{4t}}{16} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{4t}$ results in

$$p(t) = \frac{e^{-4t}(12t + 5)\, e^{4t}}{16} + c_1 e^{-4t}$$

which simplifies to

$$p(t) = \frac{3t}{4} + \frac{5}{16} + c_1 e^{-4t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{5}{16} + c_1$$

$$c_1 = -\frac{5}{16}$$

Substituting c_1 found above in the general solution gives

$$p(t) = -\frac{5\,\mathrm{e}^{-4t}}{16} + \frac{3t}{4} + \frac{5}{16}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{5\,\mathrm{e}^{-4t}}{16} + \frac{3t}{4} + \frac{5}{16}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{5\,\mathrm{e}^{-4t}}{16} + \frac{3t}{4} + \frac{5}{16}\,\mathrm{d}t \\ &= \frac{5\,\mathrm{e}^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16} + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $t = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{5}{64} + c_2$$

$$c_2 = -\frac{5}{64}$$

Substituting c_2 found above in the general solution gives

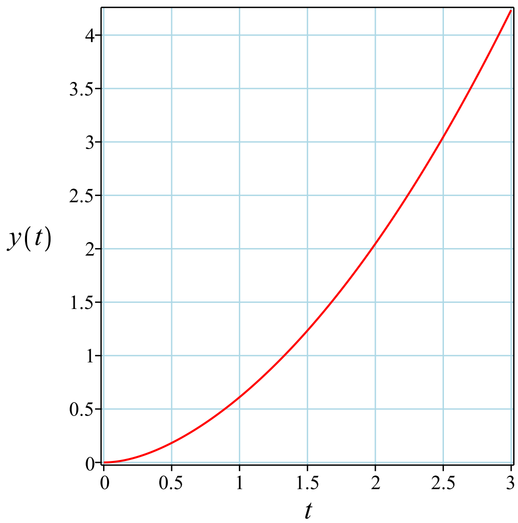
$$y = -\frac{5}{64} + \frac{5\,\mathrm{e}^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

Initial conditions are used to solve for the constants of integration.

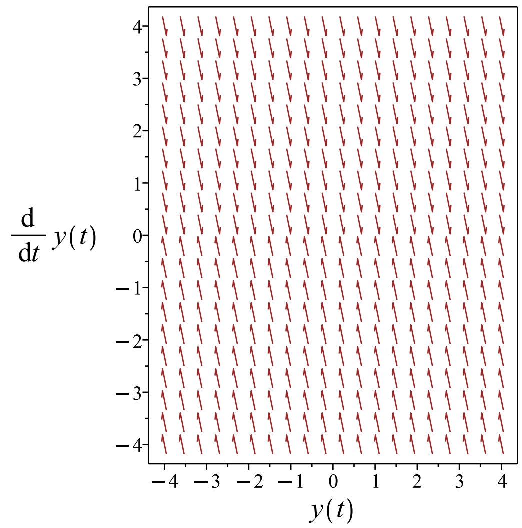
Summary

The solution(s) found are the following

$$y = -\frac{5}{64} + \frac{5\,\mathrm{e}^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

Verified OK.

16.32.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 4y' = 3t + 2$$

Integrating both sides of the ODE w.r.t t gives

$$\int (y'' + 4y')\, dt = \int (3t + 2)\, dt$$

$$4y + y' = \frac{3}{2}t^2 + 2t + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= 4 \\
q(t) &= \frac{3}{2}t^2 + 2t + c_1
\end{aligned}
$$

Hence the ode is

$$
4y + y' = \frac{3}{2}t^2 + 2t + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 4dt} \\
&= e^{4t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\
\frac{d}{dt}\left(e^{4t}y\right) &= \left(e^{4t}\right)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\
d\left(e^{4t}y\right) &= \left(\frac{(3t^2 + 2c_1 + 4t)\, e^{4t}}{2}\right) dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{4t}y &= \int \frac{(3t^2 + 2c_1 + 4t)\, e^{4t}}{2}\, dt \\
e^{4t}y &= \frac{(24t^2 + 16c_1 + 20t - 5)\, e^{4t}}{64} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{4t}$ results in

$$
y = \frac{e^{-4t}(24t^2 + 16c_1 + 20t - 5)\, e^{4t}}{64} + c_2 e^{-4t}
$$

which simplifies to

$$
y = \frac{3t^2}{8} + \frac{c_1}{4} + \frac{5t}{16} - \frac{5}{64} + c_2 e^{-4t}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{3t^2}{8} + \frac{c_1}{4} + \frac{5t}{16} - \frac{5}{64} + c_2 e^{-4t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{5}{64} + \frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_2 \mathrm{e}^{-4t} + \frac{3t}{4} + \frac{5}{16}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_2 + \frac{5}{16} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{5}{64}$$

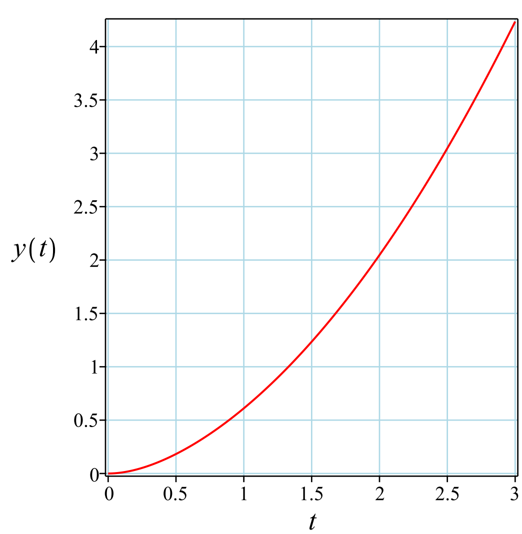
Substituting these values back in above solution results in

$$y = -\frac{5}{64} + \frac{5\,\mathrm{e}^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

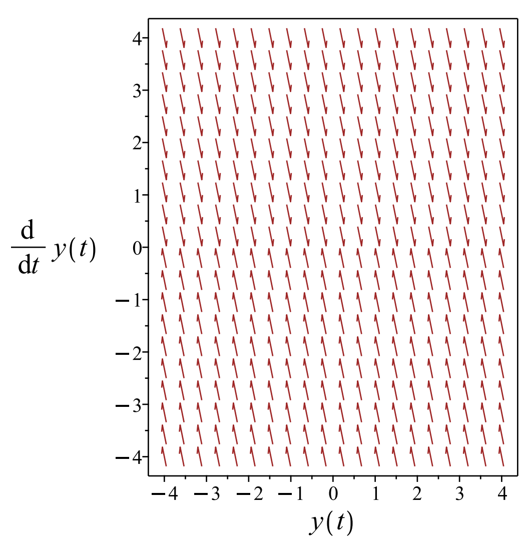
Summary

The solution(s) found are the following

$$y = -\frac{5}{64} + \frac{5\,\mathrm{e}^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{5}{64} + \frac{5\,\mathrm{e}^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

Verified OK.

16.32.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 473: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-2t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1\left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\
&= y_1\left(\frac{\mathrm{e}^{4t}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-4t}\right) + c_2\left(\mathrm{e}^{-4t}\left(\frac{\mathrm{e}^{4t}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+4y'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-4t}+\frac{c_2}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-4t} \\ y_2 &= \frac{1}{4} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-4t} & \frac{1}{4} \\ \frac{d}{dt}\left(\mathrm{e}^{-4t}\right) & \frac{d}{dt}\left(\frac{1}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-4t} & \frac{1}{4} \\ -4\,\mathrm{e}^{-4t} & 0 \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-4t}\right)(0) - \left(\frac{1}{4}\right)\left(-4\,\mathrm{e}^{-4t}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-4t}$$

Which simplifies to

$$W = \mathrm{e}^{-4t}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{3t}{4} + \frac{1}{2}}{\mathrm{e}^{-4t}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{4t}(3t+2)}{4}dt$$

Hence

$$u_1 = -\frac{(12t+5)\,\mathrm{e}^{4t}}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-4t}(3t+2)}{\mathrm{e}^{-4t}}\,dt$$

Which simplifies to

$$u_2 = \int (3t+2)\,dt$$

Hence

$$u_2 = \frac{3}{2}t^2 + 2t$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{e^{-4t}(12t+5)\,e^{4t}}{64} + \frac{3t^2}{8} + \frac{t}{2}$$

Which simplifies to

$$y_p(t) = \frac{5}{16}t - \frac{5}{64} + \frac{3}{8}t^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4t} + \frac{c_2}{4}\right) + \left(\frac{5}{16}t - \frac{5}{64} + \frac{3}{8}t^2\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-4t} + \frac{c_2}{4} + \frac{5t}{16} - \frac{5}{64} + \frac{3t^2}{8} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{4} - \frac{5}{64} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 e^{-4t} + \frac{5}{16} + \frac{3t}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_1 + \frac{5}{16} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

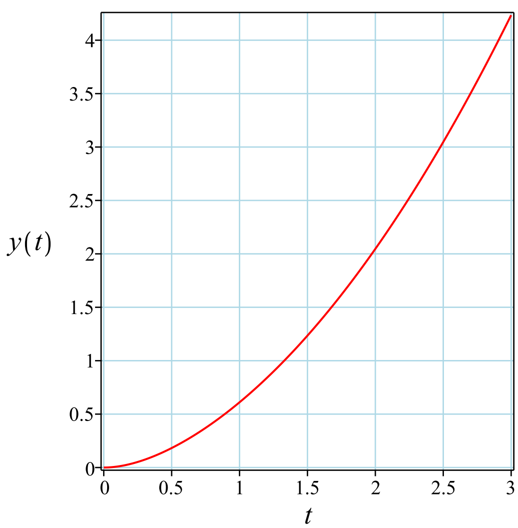
$$\begin{aligned} c_1 &= \frac{5}{64} \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

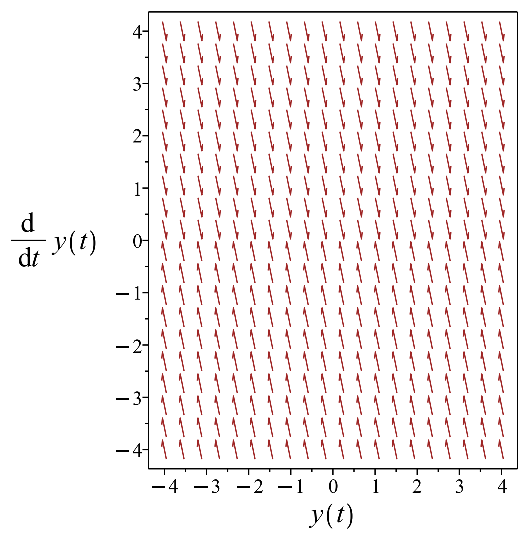
$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

Summary
The solution(s) found are the following

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

Verified OK.

16.32.7 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 4 \\
r(x) &= 0 \\
s(x) &= 3t + 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(t)\, y' + (q(t) - p'(t))\, y)' = s(x)
$$

Integrating gives

$$
p(t)\, y' + (q(t) - p'(t))\, y = \int s(t)\; dt
$$

Substituting the above values for p, q, r, s gives

$$
4y + y' = \int 3t + 2\, dt
$$

We now have a first order ode to solve which is

$$
4y + y' = \frac{3}{2}t^2 + 2t + c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(t)y = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= 4 \\
q(t) &= \frac{3}{2}t^2 + 2t + c_1
\end{aligned}
$$

Hence the ode is

$$4y + y' = \frac{3}{2}t^2 + 2t + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 4dt} \\ &= \mathrm{e}^{4t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{4t}y\right) &= \left(\mathrm{e}^{4t}\right)\left(\frac{3}{2}t^2 + 2t + c_1\right) \\ \mathrm{d}\left(\mathrm{e}^{4t}y\right) &= \left(\frac{(3t^2 + 2c_1 + 4t)\,\mathrm{e}^{4t}}{2}\right)\,\mathrm{d}t\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{4t}y &= \int \frac{(3t^2 + 2c_1 + 4t)\,\mathrm{e}^{4t}}{2}\,\mathrm{d}t \\ \mathrm{e}^{4t}y &= \frac{(24t^2 + 16c_1 + 20t - 5)\,\mathrm{e}^{4t}}{64} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{4t}$ results in

$$y = \frac{\mathrm{e}^{-4t}(24t^2 + 16c_1 + 20t - 5)\,\mathrm{e}^{4t}}{64} + c_2\mathrm{e}^{-4t}$$

which simplifies to

$$y = \frac{3t^2}{8} + \frac{c_1}{4} + \frac{5t}{16} - \frac{5}{64} + c_2\mathrm{e}^{-4t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{3t^2}{8} + \frac{c_1}{4} + \frac{5t}{16} - \frac{5}{64} + c_2\mathrm{e}^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{5}{64} + \frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_2 e^{-4t} + \frac{3t}{4} + \frac{5}{16}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_2 + \frac{5}{16} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{5}{64}$$

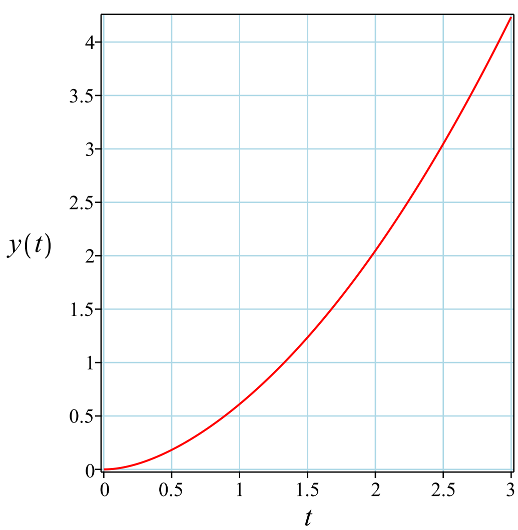
Substituting these values back in above solution results in

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

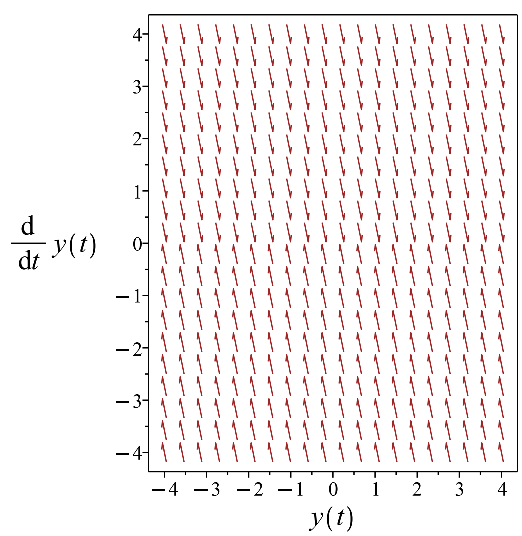
<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$$

Verified OK.

16.32.8 Maple step by step solution

Let's solve

$\left[y'' + 4y' = 3t + 2, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r = 0$

- Factor the characteristic polynomial

  $r(r + 4) = 0$

- Roots of the characteristic polynomial

  $r = (-4, 0)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = 1$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 3t + 2\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & 1 \\ -4e^{-4t} & 0 \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 4e^{-4t}$

- ○ Substitute functions into equation for $y_p(t)$

$y_p(t) = -\frac{e^{-4t}\left(\int e^{4t}(3t+2)dt\right)}{4} + \frac{\left(\int (3t+2)dt\right)}{4}$

- Compute integrals

  $y_p(t) = \frac{5}{16}t - \frac{5}{64} + \frac{3}{8}t^2$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 + \frac{5t}{16} - \frac{5}{64} + \frac{3t^2}{8}$

☐ Check validity of solution $y = c_1 e^{-4t} + c_2 + \frac{5t}{16} - \frac{5}{64} + \frac{3t^2}{8}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - \frac{5}{64}$

- Compute derivative of the solution

  $y' = -4c_1 e^{-4t} + \frac{5}{16} + \frac{3t}{4}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -4c_1 + \frac{5}{16}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{5}{64}, c_2 = 0\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$

- Solution to the IVP

  $y = -\frac{5}{64} + \frac{5\,e^{-4t}}{64} + \frac{3t^2}{8} + \frac{5t}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -4*_b(_a)+3*_a+2, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)=3*t+2,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{3t^2}{8} + \frac{5\,e^{-4t}}{64} + \frac{5t}{16} - \frac{5}{64}$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 26

```
DSolve[{y''[t]+4*y'[t]==3*t+2,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{64}\left(24t^2 + 20t + 5e^{-4t} - 5\right)$$

16.33 problem 34

Internal problem ID [13193]
Internal file name [OUTPUT/11848_Sunday_December_03_2023_07_19_39_PM_4184284/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' + 2y = t^2}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= 2 \\ F &= t^2 \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = t^2$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.33.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 3, C = 2, f(t) = t^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3^2 - (4)(1)(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-1)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 t^2 + 2A_2 t + 6tA_3 + 2A_1 + 3A_2 + 2A_3 = t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{7}{4}, A_2 = -\frac{3}{2}, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 e^{-2t}\right) + \left(\frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-t} + c_2 e^{-2t} + \frac{t^2}{2} - \frac{3t}{2} + \frac{7}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{7}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \mathrm{e}^{-t} - 2c_2 \mathrm{e}^{-2t} + t - \frac{3}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 - 2c_2 - \frac{3}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = \frac{1}{4}$$

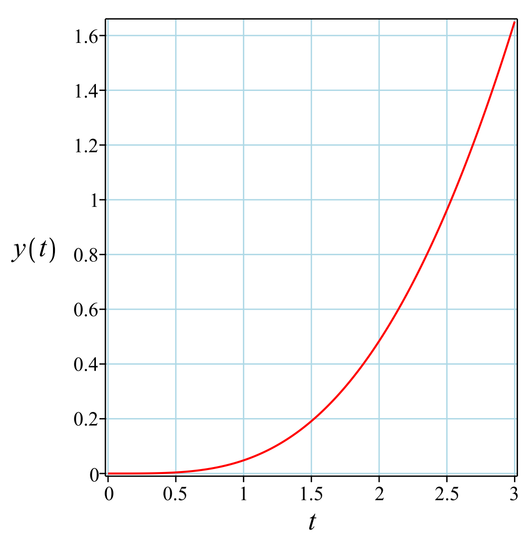
Substituting these values back in above solution results in

$$y = \frac{7}{4} - 2\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{4} + \frac{t^2}{2} - \frac{3t}{2}$$

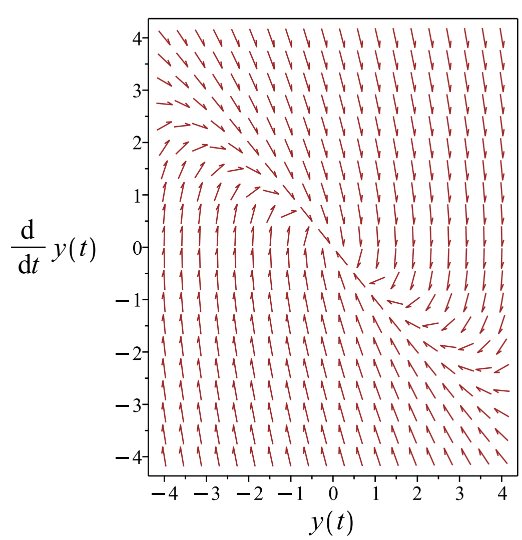
Summary

The solution(s) found are the following

$$y = \frac{7}{4} - 2\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{4} + \frac{t^2}{2} - \frac{3t}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{7}{4} - 2\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{4} + \frac{t^2}{2} - \frac{3t}{2}$$

Verified OK.

16.33.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 475: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt} \\ &= z_1 e^{-\frac{3t}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-3t}}{(y_1)^2}\,dt \\ &= y_1 \left(\mathrm{e}^{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2t}\right) + c_2\left(\mathrm{e}^{-2t}\left(\mathrm{e}^{t}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+3y'+2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2t} + c_2 e^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 t^2 + 2A_2 t + 6tA_3 + 2A_1 + 3A_2 + 2A_3 = t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{7}{4}, A_2 = -\frac{3}{2}, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t}\right) + \left(\frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + \frac{t^2}{2} - \frac{3t}{2} + \frac{7}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 + c_2 + \frac{7}{4} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2t} - c_2 \mathrm{e}^{-t} + t - \frac{3}{2}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = -2c_1 - c_2 - \frac{3}{2} \tag{2A}
$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{1}{4} \\
c_2 &= -2
\end{aligned}
$$

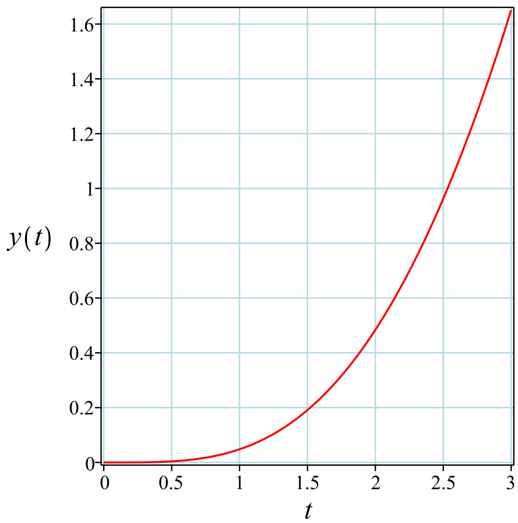
Substituting these values back in above solution results in

$$
y = \frac{7}{4} - 2\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{4} + \frac{t^2}{2} - \frac{3t}{2}
$$

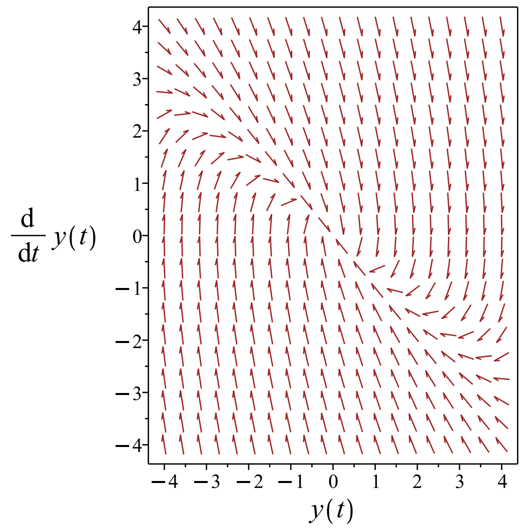
Summary

The solution(s) found are the following

$$
y = \frac{7}{4} - 2\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{4} + \frac{t^2}{2} - \frac{3t}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{7}{4} - 2\,e^{-t} + \frac{e^{-2t}}{4} + \frac{t^2}{2} - \frac{3t}{2}$$

Verified OK.

16.33.4 Maple step by step solution

Let's solve

$\left[y'' + 3y' + 2y = t^2, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = t^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-3t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\mathrm{e}^{-2t} \left(\int \mathrm{e}^{2t} t^2 dt\right) + \mathrm{e}^{-t} \left(\int t^2 \mathrm{e}^{t} dt\right)$

  - Compute integrals

    $y_p(t) = \frac{1}{2} t^2 - \frac{3}{2} t + \frac{7}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + \frac{t^2}{2} - \frac{3t}{2} + \frac{7}{4}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + \frac{t^2}{2} - \frac{3t}{2} + \frac{7}{4}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2 + \frac{7}{4}$

  - Compute derivative of the solution

    $y' = -2c_1 \mathrm{e}^{-2t} - c_2 \mathrm{e}^{-t} + t - \frac{3}{2}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -2c_1 - c_2 - \frac{3}{2}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{4}, c_2 = -2\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{7}{4} - 2\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{4} + \frac{t^2}{2} - \frac{3t}{2}$

- Solution to the IVP

  $y = \frac{7}{4} - 2\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{4} + \frac{t^2}{2} - \frac{3t}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=t^2,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{7}{4} - \frac{3t}{2} + \frac{t^2}{2} + \frac{\mathrm{e}^{-2t}}{4} - 2\,\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 37

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==t^2,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to \frac{1}{4}e^{-2t}\left(e^{2t}\left(2t^2 - 6t + 7\right) - 8e^t + 1\right)$$

16.34 problem 35

Internal problem ID [13194]
Internal file name [`OUTPUT/11849_Sunday_December_03_2023_07_19_43_PM_85429344/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y = t - \frac{1}{20}t^2}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= t - \frac{1}{20}t^2 \end{aligned}$$

Hence the ode is

$$y'' + 4y = t - \frac{1}{20}t^2$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t - \frac{1}{20}t^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.34.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = t - \frac{1}{20}t^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2 + t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3 t^2 + 4A_2 t + 4A_1 + 2A_3 = t - \frac{1}{20} t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{160}, A_2 = \frac{1}{4}, A_3 = -\frac{1}{80}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{80} t^2 + \frac{1}{4} t + \frac{1}{160}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2t) + c_2 \sin(2t)) + \left(-\frac{1}{80} t^2 + \frac{1}{4} t + \frac{1}{160}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{t^2}{80} + \frac{t}{4} + \frac{1}{160} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{1}{160} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{t}{40} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{4} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{160}$$
$$c_2 = -\frac{1}{8}$$

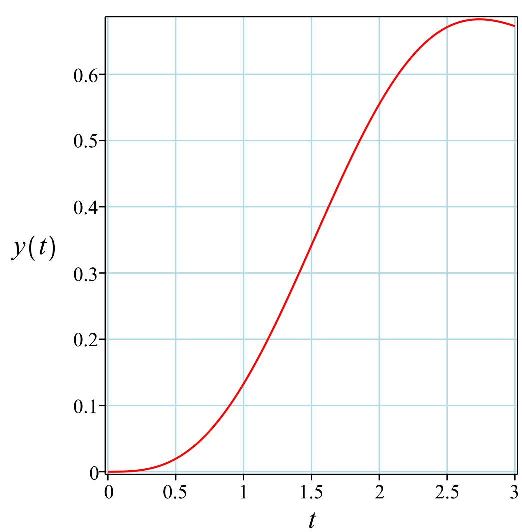
Substituting these values back in above solution results in

$$y = \frac{1}{160} - \frac{\cos(2t)}{160} - \frac{\sin(2t)}{8} - \frac{t^2}{80} + \frac{t}{4}$$

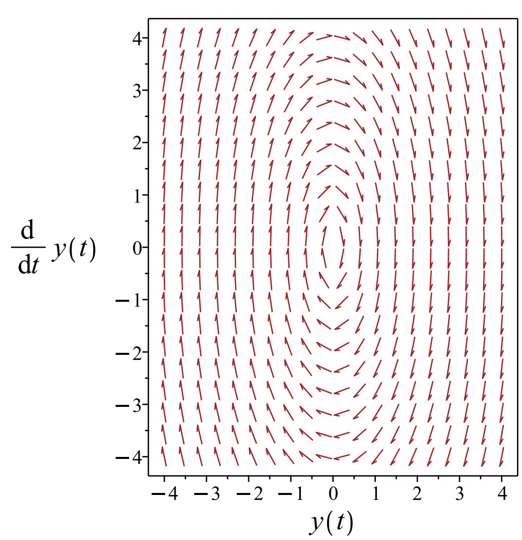
Summary

The solution(s) found are the following

$$y = \frac{1}{160} - \frac{\cos(2t)}{160} - \frac{\sin(2t)}{8} - \frac{t^2}{80} + \frac{t}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{160} - \frac{\cos(2t)}{160} - \frac{\sin(2t)}{8} - \frac{t^2}{80} + \frac{t}{4}$$

Verified OK.

16.34.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 477: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= \cos(2t) \int \frac{1}{\cos(2t)^2}\,dt \\ &= \cos(2t)\left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2+t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2}, \cos(2t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3t^2 + 4A_2t + 4A_1 + 2A_3 = t - \frac{1}{20}t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{160}, A_2 = \frac{1}{4}, A_3 = -\frac{1}{80}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{80}t^2 + \frac{1}{4}t + \frac{1}{160}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + \left(-\frac{1}{80}t^2 + \frac{1}{4}t + \frac{1}{160}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} - \frac{t^2}{80} + \frac{t}{4} + \frac{1}{160} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 + \frac{1}{160} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \sin(2t) + c_2 \cos(2t) - \frac{t}{40} + \frac{1}{4}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = \frac{1}{4} + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{160} \\
c_2 &= -\frac{1}{4}
\end{aligned}
$$

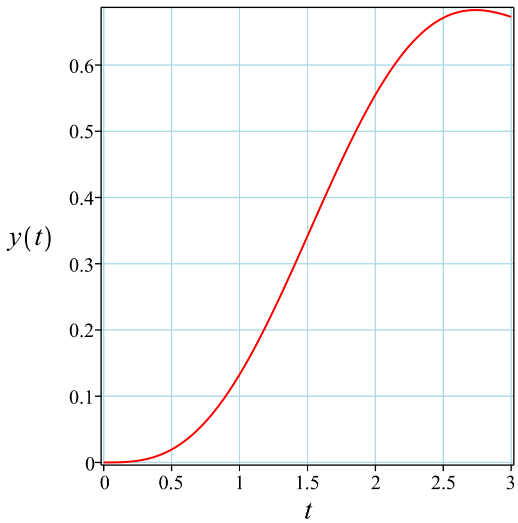
Substituting these values back in above solution results in

$$
y = \frac{1}{160} - \frac{\cos(2t)}{160} - \frac{\sin(2t)}{8} - \frac{t^2}{80} + \frac{t}{4}
$$

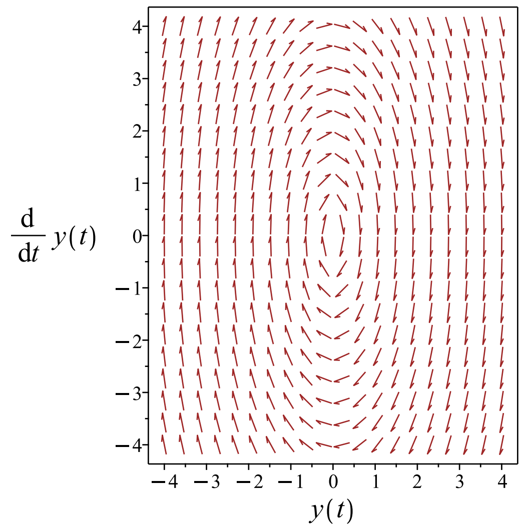
Summary

The solution(s) found are the following

$$
y = \frac{1}{160} - \frac{\cos(2t)}{160} - \frac{\sin(2t)}{8} - \frac{t^2}{80} + \frac{t}{4} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{160} - \frac{\cos(2t)}{160} - \frac{\sin(2t)}{8} - \frac{t^2}{80} + \frac{t}{4}$$

Verified OK.

16.34.4 Maple step by step solution

Let's solve

$\left[y'' + 4y = t - \frac{1}{20}t^2, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))} dt\right), f(t) = t - \frac{1}{20}t^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{\cos(2t)\left(\int \sin(2t) t(-20+t) dt\right)}{40} - \frac{\sin(2t)\left(\int \cos(2t) t(-20+t) dt\right)}{40}$

  - Compute integrals

    $y_p(t) = -\frac{1}{80}t^2 + \frac{1}{4}t + \frac{1}{160}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{t^2}{80} + \frac{t}{4} + \frac{1}{160}$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{t^2}{80} + \frac{t}{4} + \frac{1}{160}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + \frac{1}{160}$

  - Compute derivative of the solution

    $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{t}{40} + \frac{1}{4}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = \frac{1}{4} + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{160}, c_2 = -\frac{1}{8}\right\}$
- Substitute constant values into general solution and simplify

  $y = \frac{1}{160} - \frac{\cos(2t)}{160} - \frac{\sin(2t)}{8} - \frac{t^2}{80} + \frac{t}{4}$

- Solution to the IVP

  $y = \frac{1}{160} - \frac{\cos(2t)}{160} - \frac{\sin(2t)}{8} - \frac{t^2}{80} + \frac{t}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve([diff(y(t),t$2)+4*y(t)=t-t^2/20,y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{\sin(2t)}{8} - \frac{\cos(2t)}{160} - \frac{t^2}{80} + \frac{t}{4} + \frac{1}{160}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 31

```
DSolve[{y''[t]+4*y[t]==t-t^2/20,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{160}\left(-2t^2 + 40t - 20\sin(2t) - \cos(2t) + 1\right)$$

16.35 problem 37

Internal problem ID [13195]
Internal file name [`OUTPUT/11850_Sunday_December_03_2023_07_19_47_PM_98969422/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 5y' + 6y = 4 + \mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.35.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 6 \\ F &= 4 + \mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 6y = 4 + e^{-t}$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 4 + e^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.35.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 5, C = 6, f(t) = 4 + e^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{5^2 - (4)(1)(6)}\\ &= -\frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-2)t} + c_2 e^{(-3)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 + \mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^{-t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 e^{-t} + 6A_1 = 4 + e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{3}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2}{3} + \frac{e^{-t}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-3t}\right) + \left(\frac{2}{3} + \frac{e^{-t}}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + c_2 e^{-3t} + \frac{2}{3} + \frac{e^{-t}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{7}{6} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} - \frac{e^{-t}}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 3c_2 - \frac{1}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3$$
$$c_2 = \frac{11}{6}$$

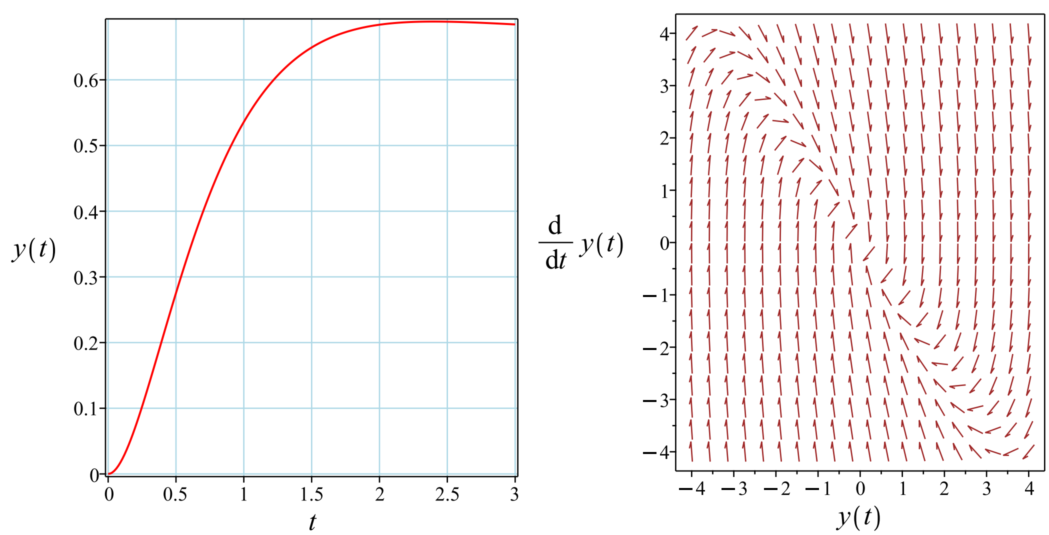
Substituting these values back in above solution results in

$$y = \frac{2}{3} - 3\,e^{-2t} + \frac{11\,e^{-3t}}{6} + \frac{e^{-t}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{3} - 3\,e^{-2t} + \frac{11\,e^{-3t}}{6} + \frac{e^{-t}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y=\frac{2}{3}-3\,\mathrm{e}^{-2t}+\frac{11\,\mathrm{e}^{-3t}}{6}+\frac{\mathrm{e}^{-t}}{2}$$

Verified OK.

16.35.3 Solving using Kovacic algorithm

Writing the ode as

$$y''+5y'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1\\B&=5\\C&=6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=1\\t&=4\end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 479: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dt} \\ &= z_1 e^{-\frac{5t}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{5t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-5t}}{(y_1)^2}\,dt \\ &= y_1 \left(\mathrm{e}^{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t}\left(\mathrm{e}^{t}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+5y'+6y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-3t}c_1 + c_2\mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4+\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\mathrm{e}^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3t}, \mathrm{e}^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2\mathrm{e}^{-t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2\mathrm{e}^{-t} + 6A_1 = 4 + \mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{3}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2}{3} + \frac{\mathrm{e}^{-t}}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{-3t}c_1 + c_2e^{-2t}\right) + \left(\frac{2}{3} + \frac{e^{-t}}{2}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = e^{-3t}c_1 + c_2e^{-2t} + \frac{2}{3} + \frac{e^{-t}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 + c_2 + \frac{7}{6} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3\,e^{-3t}c_1 - 2c_2e^{-2t} - \frac{e^{-t}}{2}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = -3c_1 - 2c_2 - \frac{1}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{11}{6} \\
c_2 &= -3
\end{aligned}
$$

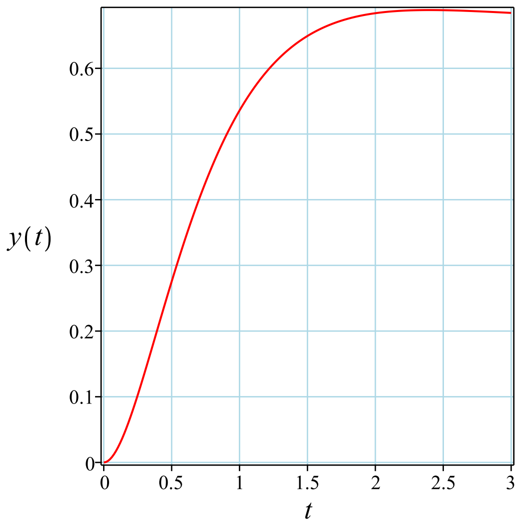
Substituting these values back in above solution results in

$$
y = \frac{2}{3} - 3\,e^{-2t} + \frac{11\,e^{-3t}}{6} + \frac{e^{-t}}{2}
$$

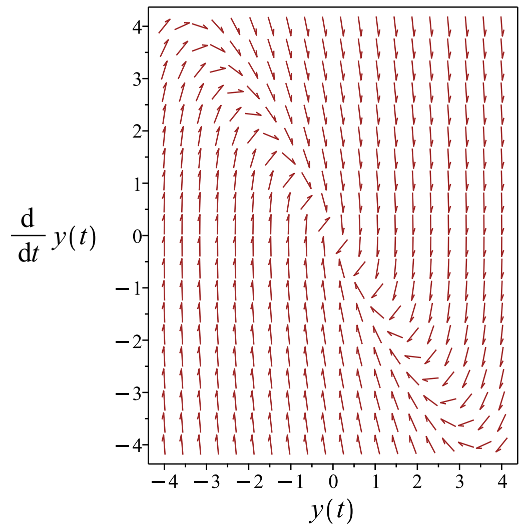
Summary

The solution(s) found are the following

$$
y = \frac{2}{3} - 3\,e^{-2t} + \frac{11\,e^{-3t}}{6} + \frac{e^{-t}}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{3} - 3\,\mathrm{e}^{-2t} + \frac{11\,\mathrm{e}^{-3t}}{6} + \frac{\mathrm{e}^{-t}}{2}$$

Verified OK.

16.35.4 Maple step by step solution

Let's solve

$$\left[y'' + 5y' + 6y = 4 + \mathrm{e}^{-t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 6 = 0$
- Factor the characteristic polynomial

  $(r + 3)(r + 2) = 0$
- Roots of the characteristic polynomial

  $r = (-3, -2)$
- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-3t}$

- 2nd solution of the homogeneous ODE

$y_2(t) = \mathrm{e}^{-2t}$

- General solution of the ODE

$y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

$y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 4 + \mathrm{e}^{-t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t} & \mathrm{e}^{-2t} \\ -3\,\mathrm{e}^{-3t} & -2\,\mathrm{e}^{-2t} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-5t}$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\mathrm{e}^{-3t}\left(\int \left(4\,\mathrm{e}^{3t} + \mathrm{e}^{2t}\right) dt\right) + \mathrm{e}^{-2t}\left(\int \left(4\,\mathrm{e}^{2t} + \mathrm{e}^{t}\right) dt\right)$

  ◦ Compute integrals

  $y_p(t) = \frac{2}{3} + \frac{\mathrm{e}^{-t}}{2}$

- Substitute particular solution into general solution to ODE

$y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-2t} + \frac{2}{3} + \frac{\mathrm{e}^{-t}}{2}$

☐ Check validity of solution $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-2t} + \frac{2}{3} + \frac{\mathrm{e}^{-t}}{2}$

  ◦ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 + \frac{7}{6}$

  ◦ Compute derivative of the solution

  $y' = -3\,\mathrm{e}^{-3t} c_1 - 2 c_2 \mathrm{e}^{-2t} - \frac{\mathrm{e}^{-t}}{2}$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -3 c_1 - 2 c_2 - \frac{1}{2}$

- Solve for c_1 and c_2

  $\{c_1 = \frac{11}{6}, c_2 = -3\}$

- Substitute constant values into general solution and simplify

  $y = \frac{2}{3} - 3\,\mathrm{e}^{-2t} + \frac{11\,\mathrm{e}^{-3t}}{6} + \frac{\mathrm{e}^{-t}}{2}$

- Solution to the IVP

  $y = \frac{2}{3} - 3\,\mathrm{e}^{-2t} + \frac{11\,\mathrm{e}^{-3t}}{6} + \frac{\mathrm{e}^{-t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve([diff(y(t),t$2)+5*diff(y(t),t)+6*y(t)=4+exp(-t),y(0) = 0, D(y)(0) = 0],y(t), singsol=
```

$$y(t) = \frac{11\,\mathrm{e}^{-3t}}{6} - 3\,\mathrm{e}^{-2t} + \frac{\mathrm{e}^{-t}}{2} + \frac{2}{3}$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 28

```
DSolve[{y''[t]+5*y'[t]+6*y[t]==4+Exp[-t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to \frac{1}{6}e^{-3t}\left(e^t - 1\right)^2\left(4e^t + 11\right)$$

16.36 problem 38

Internal problem ID [13196]
Internal file name [`OUTPUT/11851_Sunday_December_03_2023_07_19_50_PM_86148490/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' + 2y = e^{-t} - 4}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.36.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= 2 \\ F &= e^{-t} - 4 \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = e^{-t} - 4$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-t} - 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.36.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 3, C = 2, f(t) = e^{-t} - 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3^2 - (4)(1)(2)} \\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ y &= c_1 e^{(-1)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-t} - 4$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{-t}\}$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{1\}, \{t\,e^{-t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 t\,e^{-t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 e^{-t} + 2A_1 = e^{-t} - 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2 + t\,e^{-t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 e^{-2t}\right) + \left(-2 + t\,e^{-t}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-t} + c_2 e^{-2t} - 2 + t\,e^{-t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 - 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-t} - 2c_2 e^{-2t} + e^{-t} - t\, e^{-t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 - 2c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = t\, e^{-t} - 2 + 3\, e^{-t} - e^{-2t}$$

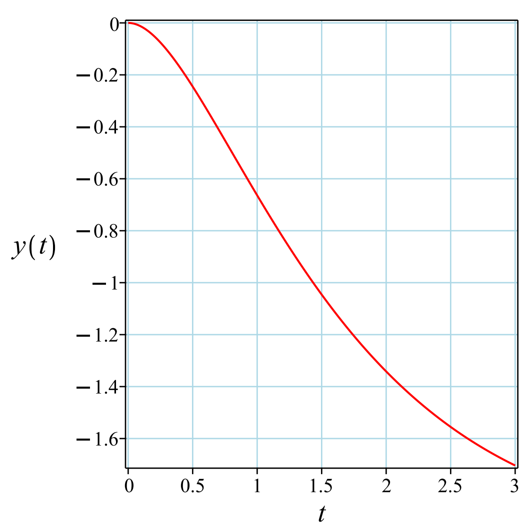
Which simplifies to

$$y = (3 + t)\, e^{-t} - e^{-2t} - 2$$

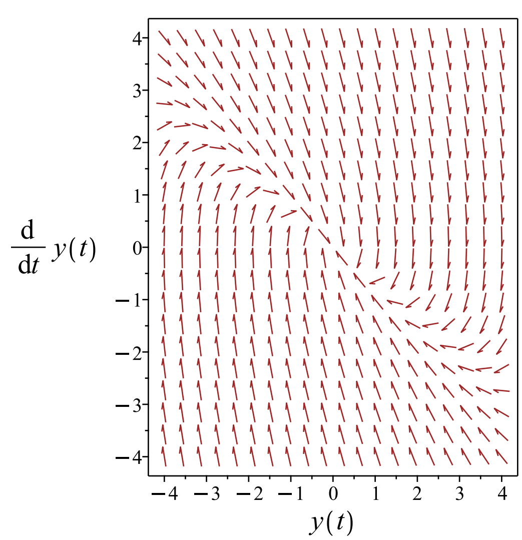
Summary

The solution(s) found are the following

$$y = (3 + t)\, e^{-t} - e^{-2t} - 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (3+t)\,\mathrm{e}^{-t} - \mathrm{e}^{-2t} - 2$$

Verified OK.

16.36.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 481: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt} \\
&= z_1 e^{-\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-3t}}{(y_1)^2}\,dt \\
&= y_1 \left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t} \left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-t} - 4$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\mathrm{e}^{-t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2t}, \mathrm{e}^{-t}\}$$

Since e^{-t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{1\}, \{t\,\mathrm{e}^{-t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 t\,\mathrm{e}^{-t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 \mathrm{e}^{-t} + 2A_1 = \mathrm{e}^{-t} - 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2 + t\,\mathrm{e}^{-t}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t}\right) + \left(-2 + t\,\mathrm{e}^{-t}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} - 2 + t\,\mathrm{e}^{-t} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = c_1 + c_2 - 2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2t} - c_2 \mathrm{e}^{-t} + \mathrm{e}^{-t} - t\,\mathrm{e}^{-t}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = -2c_1 - c_2 + 1 \tag{2A}
$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -1 \\
c_2 &= 3
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = t\,\mathrm{e}^{-t} - 2 + 3\,\mathrm{e}^{-t} - \mathrm{e}^{-2t}
$$

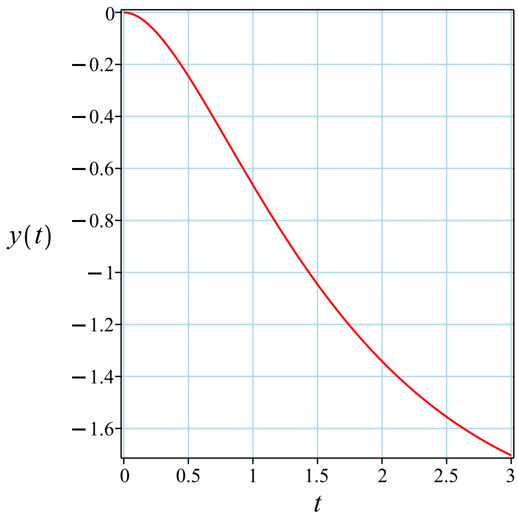
Which simplifies to

$$
y = (3 + t)\,\mathrm{e}^{-t} - \mathrm{e}^{-2t} - 2
$$

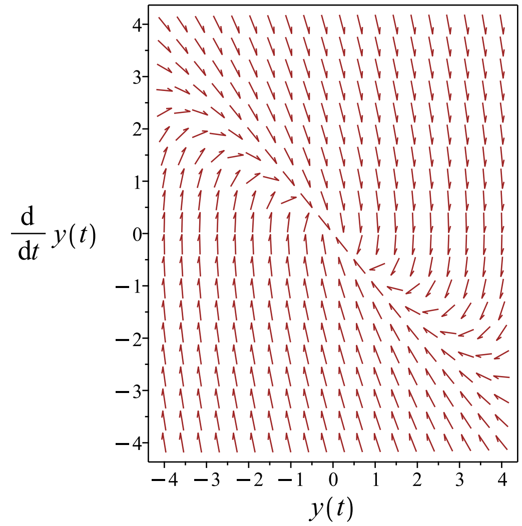
Summary

The solution(s) found are the following

$$
y = (3 + t)\,\mathrm{e}^{-t} - \mathrm{e}^{-2t} - 2 \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (3+t)\,\mathrm{e}^{-t} - \mathrm{e}^{-2t} - 2$$

Verified OK.

16.36.4 Maple step by step solution

Let's solve

$\left[y'' + 3y' + 2y = \mathrm{e}^{-t} - 4, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

$y_1(t) = e^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = e^{-t} - 4\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{-t} \\ -2\,e^{-2t} & -e^{-t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-3t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^{-2t}\left(\int \left(-4\,e^{2t} + e^{t}\right) dt\right) + e^{-t}\left(\int \left(1 - 4\,e^{t}\right) dt\right)$

- ○ Compute integrals

  $y_p(t) = -2 + e^{-t}(t - 1)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} - 2 + e^{-t}(t - 1)$

☐ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{-t} - 2 + e^{-t}(t - 1)$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - 3$

- ○ Compute derivative of the solution

  $y' = -2c_1 e^{-2t} - c_2 e^{-t} - e^{-t}(t - 1) + e^{-t}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2 - 2c_1 - c_2$

- Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = 4\}$

- Substitute constant values into general solution and simplify

  $y = (3 + t)\, e^{-t} - e^{-2t} - 2$

- Solution to the IVP

  $y = (3 + t)\, e^{-t} - e^{-2t} - 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 30

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=exp(-t)-4,y(0) = 0, D(y)(0) = 0],y(t), singsol=
```

$$y(t) = -\left(2\, e^{2t} + \ln\left(e^{-t}\right) e^{t} - 3\, e^{t} + 1\right) e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 23

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==Exp[-t]-4,{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to e^{-t}(t + 3) - e^{-2t} - 2$$

16.37 problem 39

Internal problem ID [13197]
Internal file name [`OUTPUT/11852_Sunday_December_03_2023_07_19_53_PM_65703945/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 6y' + 8y = 2t + e^{-t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.37.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 8 \\ F &= 2t + e^{-t} \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 8y = 2t + e^{-t}$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2t + e^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.37.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 6, C = 8, f(t) = 2t + e^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 6, C = 8$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda e^{\lambda t} + 8 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 8$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(8)} \\
&= -3 \pm 1
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -3 + 1 \\
\lambda_2 &= -3 - 1
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -2 \\
\lambda_2 &= -4
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
y &= c_1 e^{(-2)t} + c_2 e^{(-4)t}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2t + \mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\}, \{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t} + A_2 + A_3 t$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{-t} + 6A_3 + 8A_2 + 8A_3 t = 2t + e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = -\frac{3}{16}, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-t}}{3} - \frac{3}{16} + \frac{t}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-4t}\right) + \left(\frac{e^{-t}}{3} - \frac{3}{16} + \frac{t}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + c_2 e^{-4t} + \frac{e^{-t}}{3} - \frac{3}{16} + \frac{t}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{7}{48} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2t} - 4c_2 \mathrm{e}^{-4t} - \frac{\mathrm{e}^{-t}}{3} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 4c_2 - \frac{1}{12} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{4}$$
$$c_2 = \frac{5}{48}$$

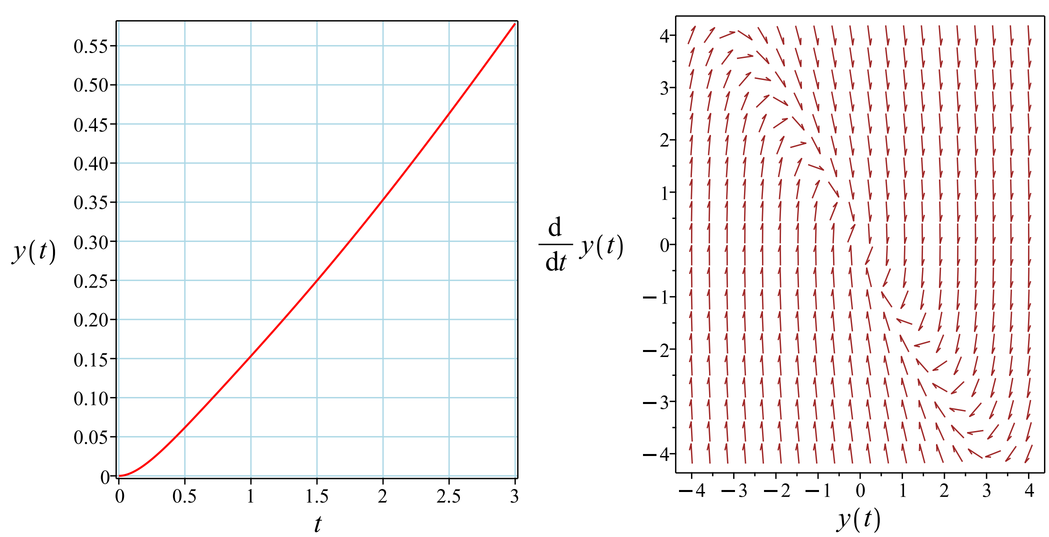
Substituting these values back in above solution results in

$$y = -\frac{3}{16} - \frac{\mathrm{e}^{-2t}}{4} + \frac{5\,\mathrm{e}^{-4t}}{48} + \frac{\mathrm{e}^{-t}}{3} + \frac{t}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{16} - \frac{\mathrm{e}^{-2t}}{4} + \frac{5\,\mathrm{e}^{-4t}}{48} + \frac{\mathrm{e}^{-t}}{3} + \frac{t}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{16} - \frac{\mathrm{e}^{-2t}}{4} + \frac{5\,\mathrm{e}^{-4t}}{48} + \frac{\mathrm{e}^{-t}}{3} + \frac{t}{4}$$

Verified OK.

16.37.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 483: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6}{1} \, dt}
\end{aligned}
$$

$$= z_1 e^{-3t}$$
$$= z_1 \left(e^{-3t}\right)$$

Which simplifies to

$$y_1 = e^{-4t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(y_1)^2}\, dt$$
$$= y_1 \int \frac{e^{-6t}}{(y_1)^2}\, dt$$
$$= y_1 \left(\frac{e^{2t}}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(e^{-4t}\right) + c_2 \left(e^{-4t} \left(\frac{e^{2t}}{2}\right)\right)$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2t + e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}, \{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-2t}}{2}, e^{-4t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t} + A_2 + A_3 t$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{-t} + 6A_3 + 8A_2 + 8A_3 t = 2t + e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = -\frac{3}{16}, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-t}}{3} - \frac{3}{16} + \frac{t}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2}\right) + \left(\frac{e^{-t}}{3} - \frac{3}{16} + \frac{t}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2} + \frac{e^{-t}}{3} - \frac{3}{16} + \frac{t}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} + \frac{7}{48} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 e^{-4t} - c_2 e^{-2t} - \frac{e^{-t}}{3} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_1 - c_2 - \frac{1}{12} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{5}{48}$$
$$c_2 = -\frac{1}{2}$$

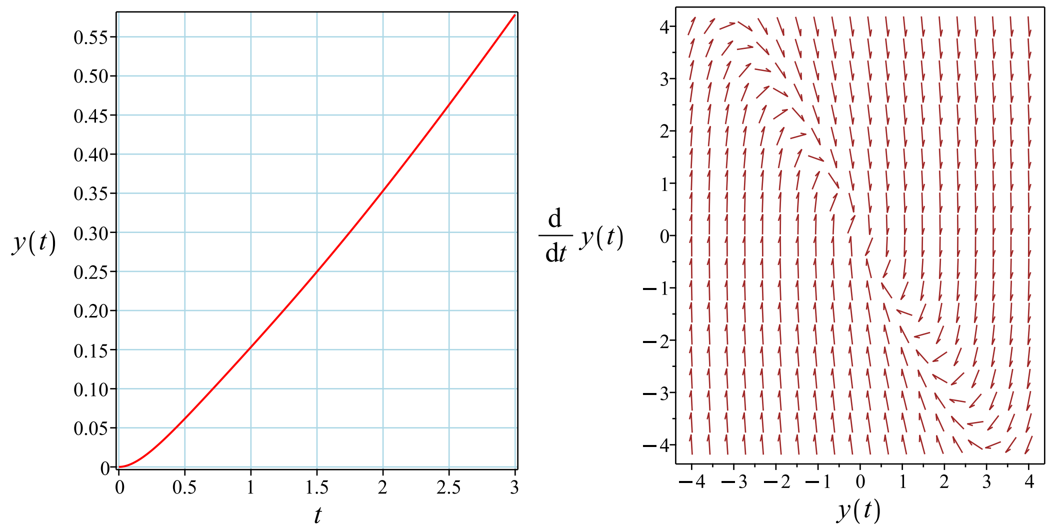
Substituting these values back in above solution results in

$$y = -\frac{3}{16} - \frac{e^{-2t}}{4} + \frac{5\,e^{-4t}}{48} + \frac{e^{-t}}{3} + \frac{t}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{16} - \frac{e^{-2t}}{4} + \frac{5\,e^{-4t}}{48} + \frac{e^{-t}}{3} + \frac{t}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{3}{16} - \frac{e^{-2t}}{4} + \frac{5\,e^{-4t}}{48} + \frac{e^{-t}}{3} + \frac{t}{4}$$

Verified OK.

16.37.4 Maple step by step solution

Let's solve

$\left[y'' + 6y' + 8y = 2t + e^{-t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 8 = 0$
- Factor the characteristic polynomial

  $(r + 4)(r + 2) = 0$
- Roots of the characteristic polynomial

  $r = (-4, -2)$
- 1st solution of the homogeneous ODE

$y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 2t + e^{-t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -4\,e^{-4t} & -2\,e^{-2t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-6t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^{-4t}\left(\int (2\,e^{4t}t + e^{3t})dt\right)}{2} + \frac{e^{-2t}\left(\int (2\,e^{2t}t + e^{t})dt\right)}{2}$

- ○ Compute integrals

  $y_p(t) = \frac{e^{-t}}{3} - \frac{3}{16} + \frac{t}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + \frac{e^{-t}}{3} - \frac{3}{16} + \frac{t}{4}$

☐ Check validity of solution $y = c_1 e^{-4t} + c_2 e^{-2t} + \frac{e^{-t}}{3} - \frac{3}{16} + \frac{t}{4}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 + \frac{7}{48}$

- ○ Compute derivative of the solution

  $y' = -4c_1 e^{-4t} - 2c_2 e^{-2t} - \frac{e^{-t}}{3} + \frac{1}{4}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -4c_1 - 2c_2 - \frac{1}{12}$

- Solve for c_1 and c_2

  $\{c_1 = \frac{5}{48}, c_2 = -\frac{1}{4}\}$
- Substitute constant values into general solution and simplify

  $y = -\frac{3}{16} - \frac{e^{-2t}}{4} + \frac{5\,e^{-4t}}{48} + \frac{e^{-t}}{3} + \frac{t}{4}$

- Solution to the IVP

  $y = -\frac{3}{16} - \frac{e^{-2t}}{4} + \frac{5\,e^{-4t}}{48} + \frac{e^{-t}}{3} + \frac{t}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+8*y(t)=2*t+exp(-t),y(0) = 0, D(y)(0) = 0],y(t), singso
```

$$y(t) = \frac{5\,e^{-4t}}{48} - \frac{3}{16} + \frac{t}{4} + \frac{e^{-t}}{3} - \frac{e^{-2t}}{4}$$

✓ Solution by Mathematica

Time used: 0.223 (sec). Leaf size: 42

```
DSolve[{y''[t]+6*y'[t]+8*y[t]==2*t+Exp[-t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutio
```

$$y(t) \to \frac{1}{48}e^{-4t}\left(3e^{4t}(4t-3) - 12e^{2t} + 16e^{3t} + 5\right)$$

16.38 problem 40

Internal problem ID [13198]
Internal file name [`OUTPUT/11853_Sunday_December_03_2023_07_19_56_PM_91969336/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 6y' + 8y = 2t + \mathrm{e}^{t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.38.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 8 \\ F &= 2t + \mathrm{e}^{t} \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 8y = 2t + e^t$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2t + e^t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.38.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 6, C = 8, f(t) = 2t + e^t$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 6, C = 8$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda e^{\lambda t} + 8 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(8)}\\ &= -3 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 1\\ \lambda_2 &= -3 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-2)t} + c_2 e^{(-4)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2t + \mathrm{e}^t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^t\}, \{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^t + A_2 + A_3 t$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$15A_1 e^t + 6A_3 + 8A_2 + 8A_3 t = 2t + e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{15}, A_2 = -\frac{3}{16}, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^t}{15} - \frac{3}{16} + \frac{t}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-4t}\right) + \left(\frac{e^t}{15} - \frac{3}{16} + \frac{t}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + c_2 e^{-4t} + \frac{e^t}{15} - \frac{3}{16} + \frac{t}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 - \frac{29}{240} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2t} - 4c_2 e^{-4t} + \frac{e^t}{15} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 4c_2 + \frac{19}{60} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{12}$$
$$c_2 = \frac{3}{80}$$

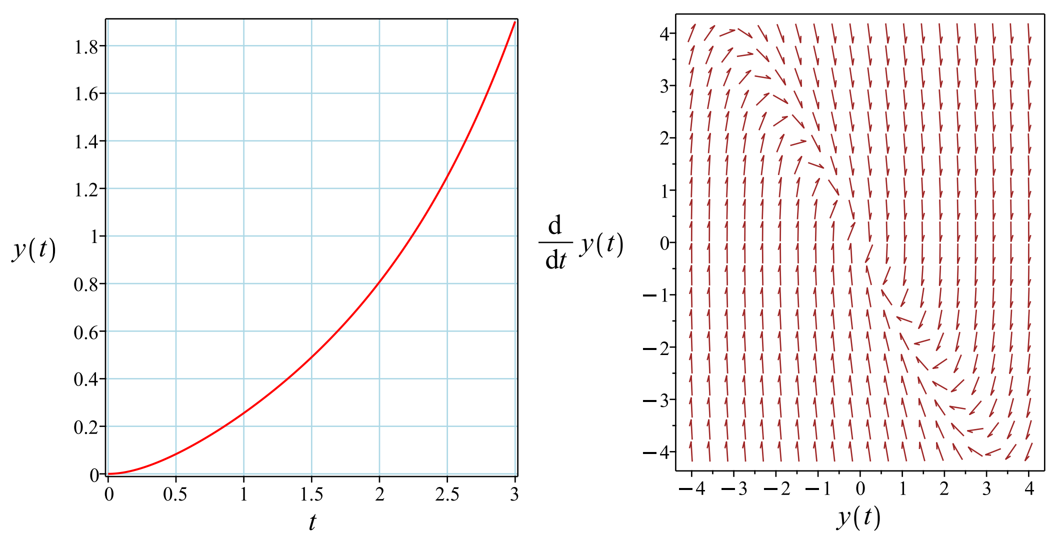
Substituting these values back in above solution results in

$$y = -\frac{3}{16} + \frac{e^{-2t}}{12} + \frac{3\,e^{-4t}}{80} + \frac{e^t}{15} + \frac{t}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{16} + \frac{e^{-2t}}{12} + \frac{3\,e^{-4t}}{80} + \frac{e^t}{15} + \frac{t}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{16} + \frac{e^{-2t}}{12} + \frac{3\,e^{-4t}}{80} + \frac{e^t}{15} + \frac{t}{4}$$

Verified OK.

16.38.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 485: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-3t} \\
&= z_1 \left(\mathrm{e}^{-3t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{\left(y_1\right)^2}\, dt \\
&= y_1 \int \frac{e^{-6t}}{\left(y_1\right)^2}\, dt \\
&= y_1 \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 6y' + 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2t + e^{t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{t}\}, \{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-2t}}{2}, e^{-4t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{t} + A_2 + A_3 t$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$15A_1 e^{t} + 6A_3 + 8A_2 + 8A_3 t = 2t + e^{t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{15}, A_2 = -\frac{3}{16}, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{t}}{15} - \frac{3}{16} + \frac{t}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2}\right) + \left(\frac{e^{t}}{15} - \frac{3}{16} + \frac{t}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2} + \frac{e^t}{15} - \frac{3}{16} + \frac{t}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} - \frac{29}{240} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 e^{-4t} - c_2 e^{-2t} + \frac{e^t}{15} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_1 - c_2 + \frac{19}{60} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3}{80}$$
$$c_2 = \frac{1}{6}$$

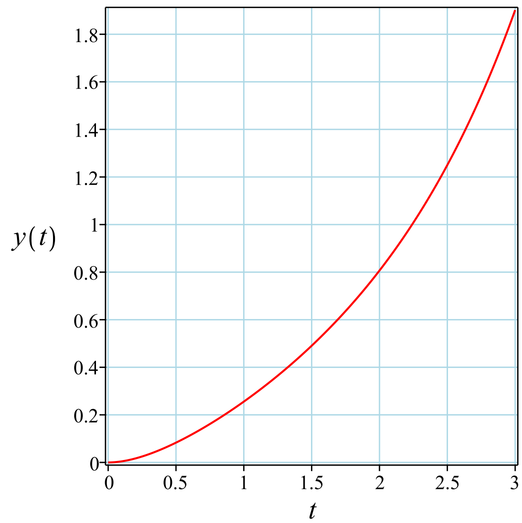
Substituting these values back in above solution results in

$$y = -\frac{3}{16} + \frac{e^{-2t}}{12} + \frac{3\,e^{-4t}}{80} + \frac{e^t}{15} + \frac{t}{4}$$

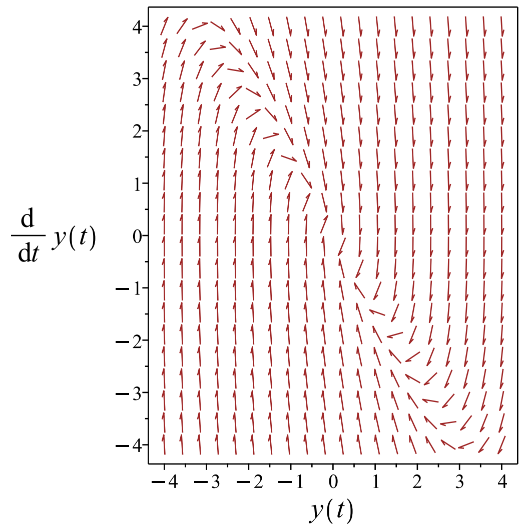
Summary

The solution(s) found are the following

$$y = -\frac{3}{16} + \frac{e^{-2t}}{12} + \frac{3\,e^{-4t}}{80} + \frac{e^t}{15} + \frac{t}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{16} + \frac{e^{-2t}}{12} + \frac{3\,e^{-4t}}{80} + \frac{e^{t}}{15} + \frac{t}{4}$$

Verified OK.

16.38.4 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 8y = 2t + e^{t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r + 4)(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -2)$

- 1st solution of the homogeneous ODE

$y_1(t) = \mathrm{e}^{-4t}$

- 2nd solution of the homogeneous ODE

$y_2(t) = \mathrm{e}^{-2t}$

- General solution of the ODE

$y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 2t + \mathrm{e}^t\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-4t} & \mathrm{e}^{-2t} \\ -4\,\mathrm{e}^{-4t} & -2\,\mathrm{e}^{-2t} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{-6t}$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\mathrm{e}^{-4t}\left(\int (2t+\mathrm{e}^t)\mathrm{e}^{4t}dt\right)}{2} + \frac{\mathrm{e}^{-2t}\left(\int (2t+\mathrm{e}^t)\mathrm{e}^{2t}dt\right)}{2}$

  ◦ Compute integrals

  $y_p(t) = \frac{\mathrm{e}^t}{15} - \frac{3}{16} + \frac{t}{4}$

- Substitute particular solution into general solution to ODE

$y = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-2t} + \frac{\mathrm{e}^t}{15} - \frac{3}{16} + \frac{t}{4}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-2t} + \frac{\mathrm{e}^t}{15} - \frac{3}{16} + \frac{t}{4}$

  ◦ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - \frac{29}{240}$

  ◦ Compute derivative of the solution

  $y' = -4c_1 \mathrm{e}^{-4t} - 2c_2 \mathrm{e}^{-2t} + \frac{\mathrm{e}^t}{15} + \frac{1}{4}$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -4c_1 - 2c_2 + \frac{19}{60}$

- Solve for c_1 and c_2

  $\{c_1 = \frac{3}{80}, c_2 = \frac{1}{12}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(16\,e^{5t}+60\,e^{4t}t-45\,e^{4t}+20\,e^{2t}+9)\,e^{-4t}}{240}$

- Solution to the IVP

  $y = \frac{(16\,e^{5t}+60\,e^{4t}t-45\,e^{4t}+20\,e^{2t}+9)\,e^{-4t}}{240}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+8*y(t)=2*t+exp(t),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \frac{(16\,e^{5t} + 60t\,e^{4t} - 45\,e^{4t} + 20\,e^{2t} + 9)\,e^{-4t}}{240}$$

✓ Solution by Mathematica

Time used: 0.2 (sec). Leaf size: 33

```
DSolve[{y''[t]+6*y'[t]+8*y[t]==2*t+Exp[t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{1}{240}\left(60t + 9e^{-4t} + 20e^{-2t} + 16e^t - 45\right)$$

16.39 problem 41

Internal problem ID [13199]
Internal file name [`OUTPUT/11854_Sunday_December_03_2023_07_19_59_PM_10470501/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__linear__constant__coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y = t + e^{-t}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.39.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}
p(t) &= 0 \\
q(t) &= 4 \\
F &= t + e^{-t}
\end{aligned}$$

Hence the ode is

$$y'' + 4y = t + e^{-t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t + e^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.39.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = t + e^{-t}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t + e^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\}, \{1, t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t} + A_2 + A_3 t$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^{-t} + 4A_2 + 4A_3 t = t + e^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = 0, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-t}}{5} + \frac{t}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2t) + c_2 \sin(2t)) + \left(\frac{e^{-t}}{5} + \frac{t}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{e^{-t}}{5} + \frac{t}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{5} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{e^{-t}}{5} + \frac{1}{4}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{20} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{5}$$
$$c_2 = -\frac{1}{40}$$

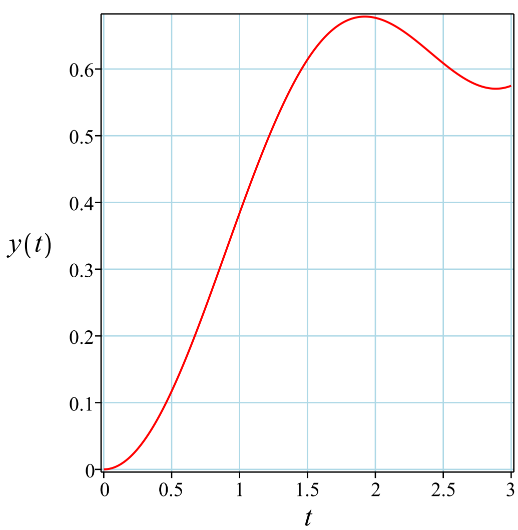
Substituting these values back in above solution results in

$$y = -\frac{\cos(2t)}{5} - \frac{\sin(2t)}{40} + \frac{e^{-t}}{5} + \frac{t}{4}$$

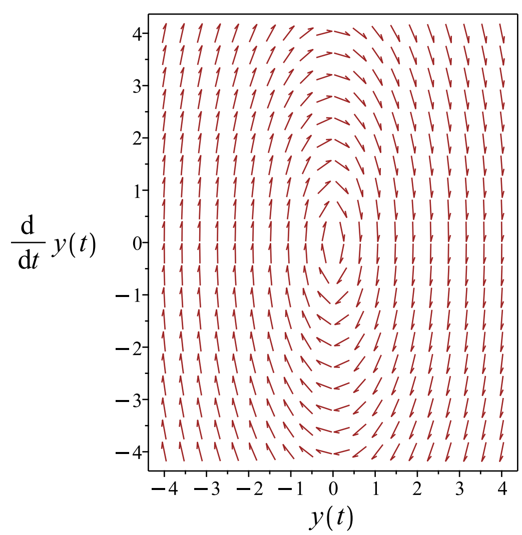
<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{\cos(2t)}{5} - \frac{\sin(2t)}{40} + \frac{e^{-t}}{5} + \frac{t}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y=-\frac{\cos(2t)}{5}-\frac{\sin(2t)}{40}+\frac{e^{-t}}{5}+\frac{t}{4}$$

Verified OK.

16.39.3 Solving using Kovacic algorithm

Writing the ode as

$$y''+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1\\B&=0\\C&=4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=-4\\t&=1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(t)=-4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 487: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= \cos(2t) \int \frac{1}{\cos(2t)^2}\,dt \\ &= \cos(2t)\left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\cos(2t)+\frac{c_2\sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t+\mathrm{e}^{-t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-t}\},\{1,t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2},\cos(2t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1\mathrm{e}^{-t}+A_2+A_3t$$

The unknowns $\{A_1,A_2,A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1\mathrm{e}^{-t}+4A_2+4A_3t=t+\mathrm{e}^{-t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{1}{5},A_2=0,A_3=\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=\frac{\mathrm{e}^{-t}}{5}+\frac{t}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + \left(\frac{e^{-t}}{5} + \frac{t}{4}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} + \frac{e^{-t}}{5} + \frac{t}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$
0 = \frac{1}{5} + c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \sin(2t) + c_2 \cos(2t) - \frac{e^{-t}}{5} + \frac{1}{4}
$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$
0 = \frac{1}{20} + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{5} \\
c_2 &= -\frac{1}{20}
\end{aligned}
$$

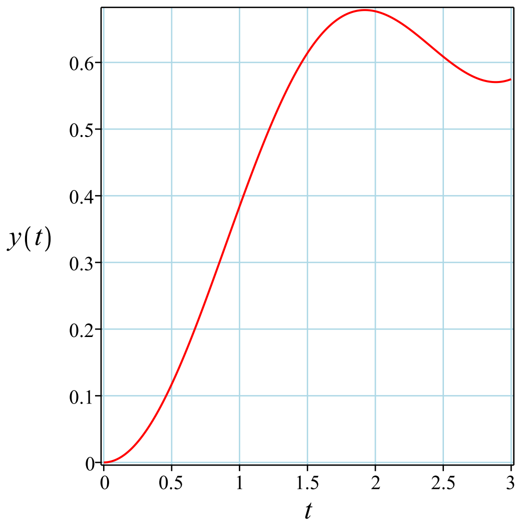
Substituting these values back in above solution results in

$$
y = -\frac{\cos(2t)}{5} - \frac{\sin(2t)}{40} + \frac{e^{-t}}{5} + \frac{t}{4}
$$

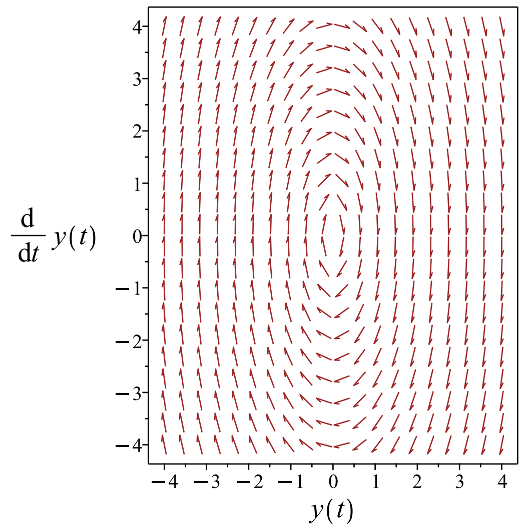
Summary

The solution(s) found are the following

$$
y = -\frac{\cos(2t)}{5} - \frac{\sin(2t)}{40} + \frac{e^{-t}}{5} + \frac{t}{4} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\cos(2t)}{5} - \frac{\sin(2t)}{40} + \frac{e^{-t}}{5} + \frac{t}{4}$$

Verified OK.

16.39.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y = t + e^{-t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = t + \mathrm{e}^{-t}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\cos(2t)\left(\int \sin(2t)(t+\mathrm{e}^{-t}) dt\right)}{2} + \frac{\sin(2t)\left(\int \cos(2t)(t+\mathrm{e}^{-t}) dt\right)}{2}$

  - Compute integrals

    $y_p(t) = \frac{t}{4} - \frac{\sin(2t)}{8} + \frac{\mathrm{e}^{-t}}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{t}{4} - \frac{\sin(2t)}{8} + \frac{\mathrm{e}^{-t}}{5}$

□ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{t}{4} - \frac{\sin(2t)}{8} + \frac{\mathrm{e}^{-t}}{5}$

  - Use initial condition $y(0) = 0$

    $0 = \frac{1}{5} + c_1$

  - Compute derivative of the solution

    $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{1}{4} - \frac{\cos(2t)}{4} - \frac{\mathrm{e}^{-t}}{5}$

  - Use the initial condition $y'\big|_{\{t=0\}} = 0$

    $0 = -\frac{1}{5} + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = -\frac{1}{5}, c_2 = \frac{1}{10}\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\cos(2t)}{5} - \frac{\sin(2t)}{40} + \frac{e^{-t}}{5} + \frac{t}{4}$

- Solution to the IVP

  $y = -\frac{\cos(2t)}{5} - \frac{\sin(2t)}{40} + \frac{e^{-t}}{5} + \frac{t}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve([diff(y(t),t$2)+4*y(t)=t+exp(-t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{\sin(2t)}{40} - \frac{\cos(2t)}{5} + \frac{t}{4} + \frac{e^{-t}}{5}$$

✓ Solution by Mathematica

Time used: 0.794 (sec). Leaf size: 32

```
DSolve[{y''[t]+4*y[t]==t+Exp[-t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to \frac{1}{40}\left(10t + 8e^{-t} - \sin(2t) - 8\cos(2t)\right)$$

16.40 problem 42

Internal problem ID [13200]
Internal file name [`OUTPUT/11855_Sunday_December_03_2023_07_20_04_PM_77441081/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.1 page 399
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 6 + t^2 + e^t}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.40.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= 6 + t^2 + e^t \end{aligned}$$

Hence the ode is

$$y'' + 4y = 6 + t^2 + e^t$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 6 + t^2 + e^t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.40.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = 6 + t^2 + e^t$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = +2i$$
$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6 + t^2 + e^t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^t\}, \{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^t + A_2 + A_3 t + A_4 t^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^t + 2A_4 + 4A_2 + 4A_3 t + 4A_4 t^2 = 6 + t^2 + e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = \frac{11}{8}, A_3 = 0, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^t}{5} + \frac{11}{8} + \frac{t^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2t) + c_2 \sin(2t)) + \left(\frac{e^t}{5} + \frac{11}{8} + \frac{t^2}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{e^t}{5} + \frac{11}{8} + \frac{t^2}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{63}{40} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{e^t}{5} + \frac{t}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{5} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{63}{40}$$
$$c_2 = -\frac{1}{10}$$

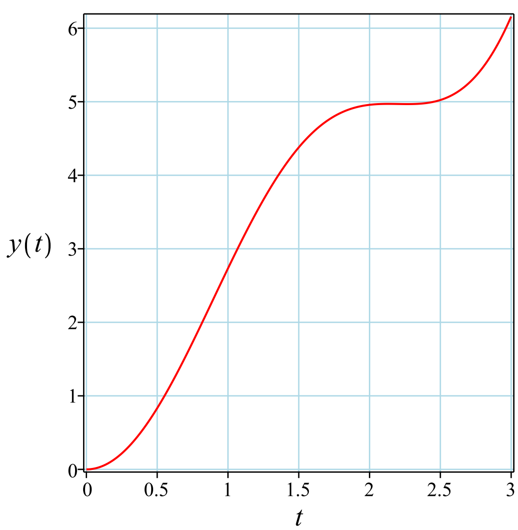
Substituting these values back in above solution results in

$$y = \frac{11}{8} - \frac{63\cos(2t)}{40} - \frac{\sin(2t)}{10} + \frac{e^t}{5} + \frac{t^2}{4}$$

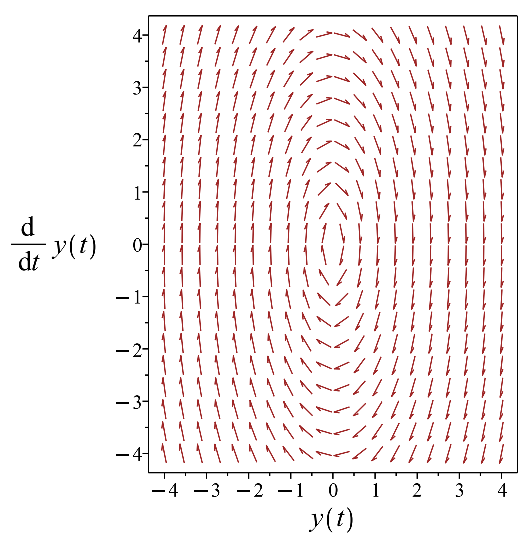
Summary

The solution(s) found are the following

$$y = \frac{11}{8} - \frac{63\cos(2t)}{40} - \frac{\sin(2t)}{10} + \frac{e^t}{5} + \frac{t^2}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{11}{8} - \frac{63 \cos (2t)}{40} - \frac{\sin (2t)}{10} + \frac{e^t}{5} + \frac{t^2}{4}$$

Verified OK.

16.40.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 489: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= \cos(2t) \int \frac{1}{\cos(2t)^2}\,dt \\ &= \cos(2t)\left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t)+By'(t)+Cy(t)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t)+By'(t)+Cy(t)=f(t)$. y_h is the solution to

$$y''+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\cos(2t)+\frac{c_2\sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6+t^2+e^t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^t\},\{1,t,t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2},\cos(2t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1e^t+A_2+A_3t+A_4t^2$$

The unknowns $\{A_1,A_2,A_3,A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1e^t+2A_4+4A_2+4A_3t+4A_4t^2=6+t^2+e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{1}{5},A_2=\frac{11}{8},A_3=0,A_4=\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^t}{5} + \frac{11}{8} + \frac{t^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + \left(\frac{e^t}{5} + \frac{11}{8} + \frac{t^2}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} + \frac{e^t}{5} + \frac{11}{8} + \frac{t^2}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{63}{40} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2t) + c_2 \cos(2t) + \frac{e^t}{5} + \frac{t}{2}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{1}{5} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{63}{40} \\ c_2 &= -\frac{1}{5} \end{aligned}$$

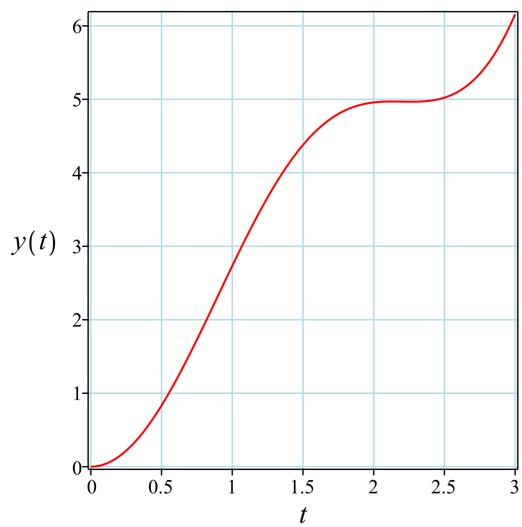
Substituting these values back in above solution results in

$$y = \frac{11}{8} - \frac{63\cos(2t)}{40} - \frac{\sin(2t)}{10} + \frac{e^t}{5} + \frac{t^2}{4}$$

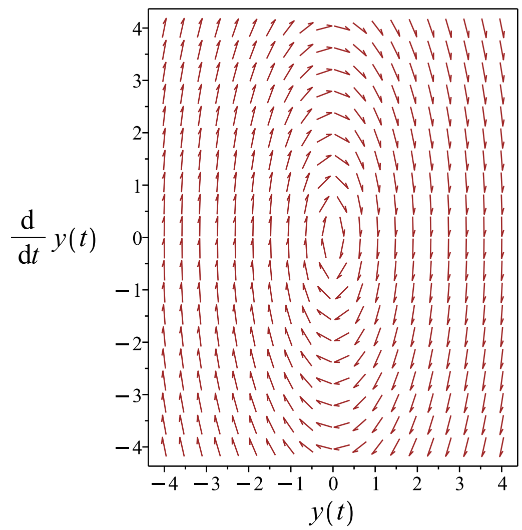
Summary

The solution(s) found are the following

$$y = \frac{11}{8} - \frac{63 \cos(2t)}{40} - \frac{\sin(2t)}{10} + \frac{e^{t}}{5} + \frac{t^{2}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{11}{8} - \frac{63 \cos(2t)}{40} - \frac{\sin(2t)}{10} + \frac{e^{t}}{5} + \frac{t^{2}}{4}$$

Verified OK.

16.40.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y = 6 + t^{2} + e^{t}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^{2} + 4 = 0$

- Use quadratic formula to solve for r

$r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 6 + t^2 + \mathrm{e}^t\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\cos(2t)\left(\int \sin(2t)(6+t^2+\mathrm{e}^t)dt\right)}{2} + \frac{\sin(2t)\left(\int \cos(2t)(6+t^2+\mathrm{e}^t)dt\right)}{2}$

  - Compute integrals

    $y_p(t) = \frac{13\cos(t)^2}{4} + \frac{t^2}{4} + \frac{\mathrm{e}^t}{5} - \frac{1}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{13\cos(t)^2}{4} + \frac{t^2}{4} + \frac{\mathrm{e}^t}{5} - \frac{1}{4}$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{13\cos(t)^2}{4} + \frac{t^2}{4} + \frac{\mathrm{e}^t}{5} - \frac{1}{4}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + \frac{16}{5}$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{13\cos(t)\sin(t)}{2} + \frac{t}{2} + \frac{e^t}{5}$
- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \frac{1}{5} + 2c_2$
- Solve for c_1 and c_2

  $\{c_1 = -\frac{16}{5}, c_2 = -\frac{1}{10}\}$
- Substitute constant values into general solution and simplify

  $y = \frac{11}{8} - \frac{63\cos(2t)}{40} - \frac{\sin(2t)}{10} + \frac{e^t}{5} + \frac{t^2}{4}$

- Solution to the IVP

  $y = \frac{11}{8} - \frac{63\cos(2t)}{40} - \frac{\sin(2t)}{10} + \frac{e^t}{5} + \frac{t^2}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve([diff(y(t),t$2)+4*y(t)=6+t^2+exp(t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -\frac{\sin(2t)}{10} - \frac{63\cos(2t)}{40} + \frac{11}{8} + \frac{t^2}{4} + \frac{e^t}{5}$$

✓ Solution by Mathematica

Time used: 0.352 (sec). Leaf size: 33

```
DSolve[{y''[t]+4*y[t]==6+t^2+Exp[t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> T
```

$$y(t) \to \frac{1}{40}\left(10t^2 + 8e^t - 4\sin(2t) - 63\cos(2t) + 55\right)$$

17 Chapter 4. Forcing and Resonance. Section 4.2 page 412

17.1 problem 1

Internal problem ID [13201]
Internal file name [`OUTPUT/11856_Sunday_December_03_2023_07_20_09_PM_36210487/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 3y' + 2y = \cos(t)}$$

17.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 3, C = 2, f(t) = \cos(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 3\lambda\, e^{\lambda t} + 2\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\
&= -\frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
y &= c_1 e^{(-1)t} + c_2 e^{(-2)t}
\end{aligned}$$

Or

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(t) + A_2 \sin(t) - 3A_1 \sin(t) + 3A_2 \cos(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = \frac{3}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(t)}{10} + \frac{3\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 e^{-2t}\right) + \left(\frac{\cos(t)}{10} + \frac{3\sin(t)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-t} + c_2 e^{-2t} + \frac{\cos(t)}{10} + \frac{3\sin(t)}{10} \tag{1}$$

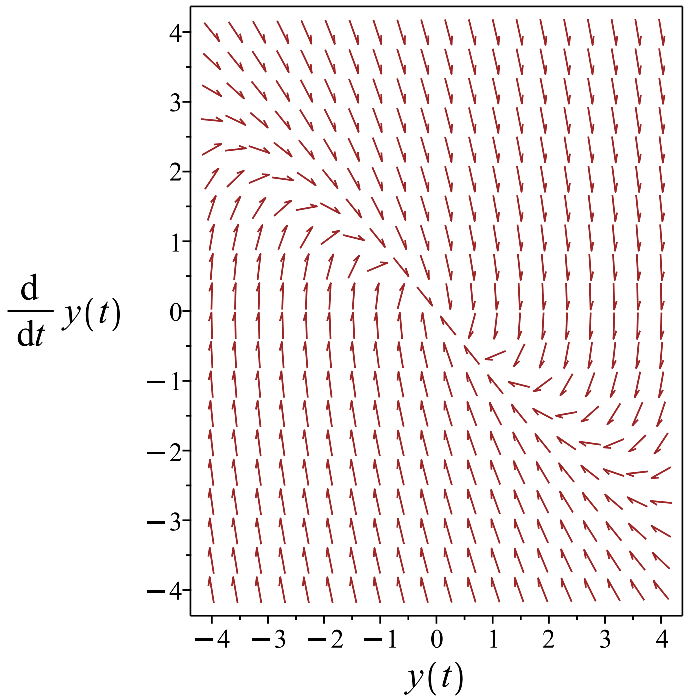


Figure 571: Slope field plot

Verification of solutions

$$y = c_1 e^{-t} + c_2 e^{-2t} + \frac{\cos(t)}{10} + \frac{3\sin(t)}{10}$$

Verified OK.

17.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 491: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt}\\ &= z_1 e^{-\frac{3t}{2}}\\ &= z_1\left(\mathrm{e}^{-\frac{3t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(y_1)^2}\,dt\\ &= y_1 \int \frac{e^{-3t}}{(y_1)^2}\,dt\\ &= y_1\left(\mathrm{e}^{t}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t} \left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(t), \sin(t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2t}, \mathrm{e}^{-t}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(t) + A_2 \sin(t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1 \cos(t) + A_2 \sin(t) - 3A_1 \sin(t) + 3A_2 \cos(t) = \cos(t)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = \frac{3}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(t)}{10} + \frac{3\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-t}\right) + \left(\frac{\cos(t)}{10} + \frac{3\sin(t)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2t} + c_2 e^{-t} + \frac{\cos(t)}{10} + \frac{3\sin(t)}{10} \tag{1}$$

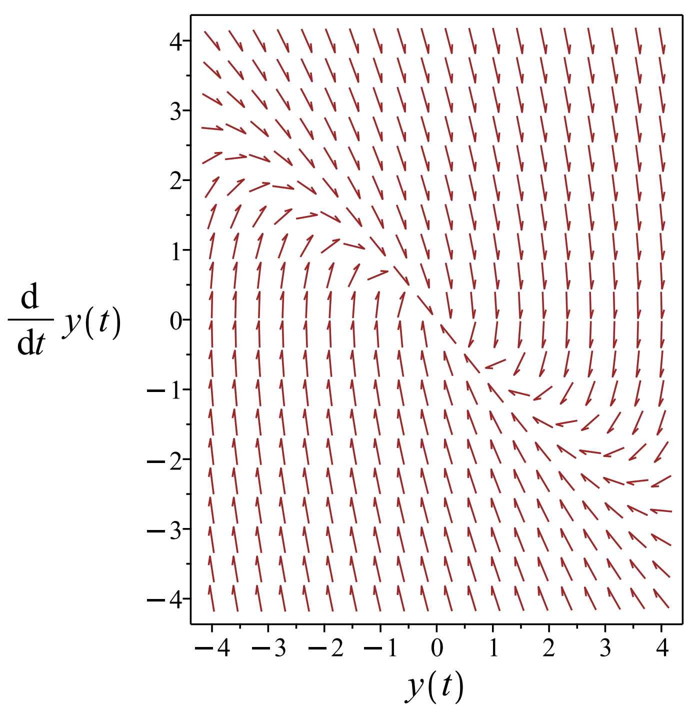


Figure 572: Slope field plot

Verification of solutions

$$y = c_1 e^{-2t} + c_2 e^{-t} + \frac{\cos(t)}{10} + \frac{3\sin(t)}{10}$$

Verified OK.

17.1.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = \cos(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \cos(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{-t} \\ -2\,e^{-2t} & -e^{-t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-3t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^{-2t}\left(\int e^{2t}\cos(t)\,dt\right) + e^{-t}\left(\int e^{t}\cos(t)\,dt\right)$

- Compute integrals

  $y_p(t) = \frac{\cos(t)}{10} + \frac{3\sin(t)}{10}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} + \frac{\cos(t)}{10} + \frac{3\sin(t)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=cos(t),y(t), singsol=all)
```

$$y(t) = -e^{-2t}c_1 + \frac{\cos(t)}{10} + \frac{3\sin(t)}{10} + c_2 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 32

```
DSolve[y''[t]+3*y'[t]+2*y[t]==Cos[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{10}\left(3\sin(t) + \cos(t) + 10e^{-2t}\left(c_2e^t + c_1\right)\right)$$

17.2 problem 2

Internal problem ID [13202]
Internal file name [`OUTPUT/11857_Sunday_December_03_2023_07_20_12_PM_2281889/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 3y' + 2y = 5\cos(t)}$$

17.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 3, C = 2, f(t) = 5\cos(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 3\lambda\, e^{\lambda t} + 2\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-1)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(t) + A_2\sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\cos(t) + A_2\sin(t) - 3A_1\sin(t) + 3A_2\cos(t) = 5\cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(t)}{2} + \frac{3\sin(t)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 e^{-2t}\right) + \left(\frac{\cos(t)}{2} + \frac{3\sin(t)}{2}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{-t} + c_2 e^{-2t} + \frac{\cos(t)}{2} + \frac{3\sin(t)}{2} \tag{1}$$

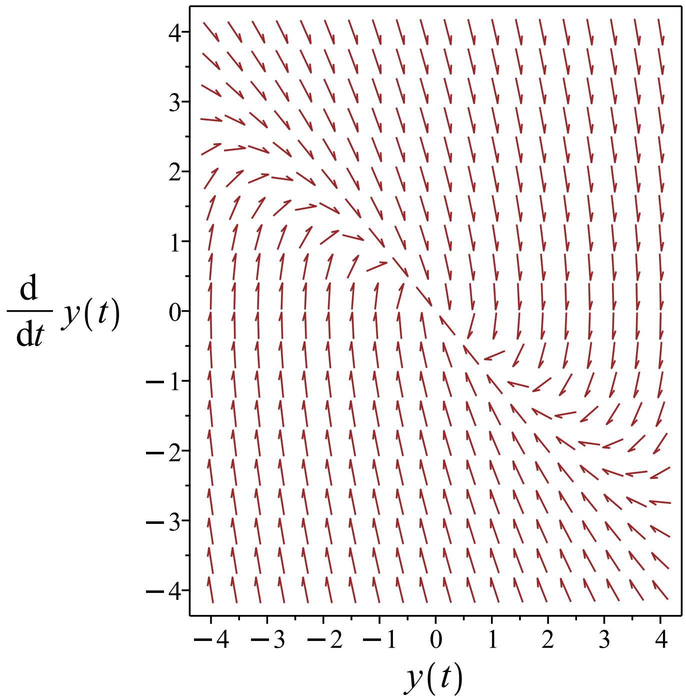


Figure 573: Slope field plot

Verification of solutions

$$
y = c_1 e^{-t} + c_2 e^{-2t} + \frac{\cos(t)}{2} + \frac{3\sin(t)}{2}
$$

Verified OK.

17.2.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' + 3y' + 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 3 \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = y e^{\int \frac{B}{2A} \, dt}
$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 493: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{t}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt} \\
&= z_1 e^{-\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-3t}}{(y_1)^2}\,dt \\
&= y_1 \left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2t}\right) + c_2\left(\mathrm{e}^{-2t}\left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
5\cos(t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(t), \sin(t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2t}, \mathrm{e}^{-t}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(t) + A_2 \sin(t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1 \cos(t) + A_2 \sin(t) - 3A_1 \sin(t) + 3A_2 \cos(t) = 5\cos(t)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(t)}{2} + \frac{3\sin(t)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-t}\right) + \left(\frac{\cos(t)}{2} + \frac{3\sin(t)}{2}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{-2t} + c_2 e^{-t} + \frac{\cos(t)}{2} + \frac{3\sin(t)}{2} \tag{1}$$

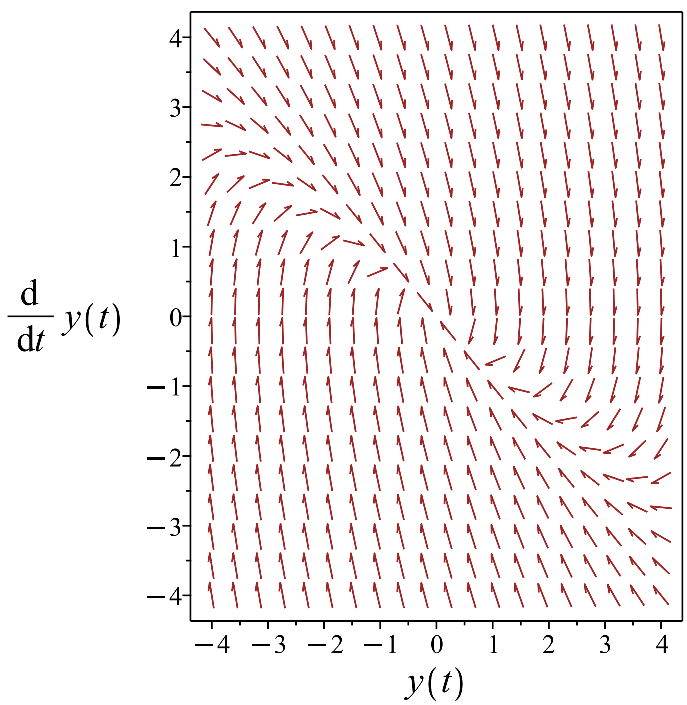


Figure 574: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + \frac{\cos(t)}{2} + \frac{3\sin(t)}{2}$$

Verified OK.

17.2.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = 5\cos(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 5\cos(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{-t} \\ -2\,e^{-2t} & -e^{-t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-3t}$
- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -5\,e^{-2t}\left(\int e^{2t}\cos(t)\,dt\right) + 5\,e^{-t}\left(\int e^{t}\cos(t)\,dt\right)$
- Compute integrals

  $y_p(t) = \frac{\cos(t)}{2} + \frac{3\sin(t)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} + \frac{\cos(t)}{2} + \frac{3\sin(t)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=5*cos(t),y(t), singsol=all)
```

$$y(t) = -e^{-2t}c_1 + \frac{\cos(t)}{2} + \frac{3\sin(t)}{2} + c_2 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 32

```
DSolve[y''[t]+3*y'[t]+2*y[t]==5*Cos[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}\left(3\sin(t) + \cos(t) + 2e^{-2t}\left(c_2e^t + c_1\right)\right)$$

17.3 problem 3

Internal problem ID [13203]
Internal file name [`OUTPUT/11858_Sunday_December_03_2023_07_20_15_PM_21715495/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 3y' + 2y = \sin(t)}$$

17.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 3, C = 2, f(t) = \sin(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 3\lambda\, e^{\lambda t} + 2\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-1)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(t) + A_2 \sin(t) - 3A_1 \sin(t) + 3A_2 \cos(t) = \sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 e^{-2t}\right) + \left(-\frac{3\cos(t)}{10} + \frac{\sin(t)}{10}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{-t} + c_2 e^{-2t} - \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$

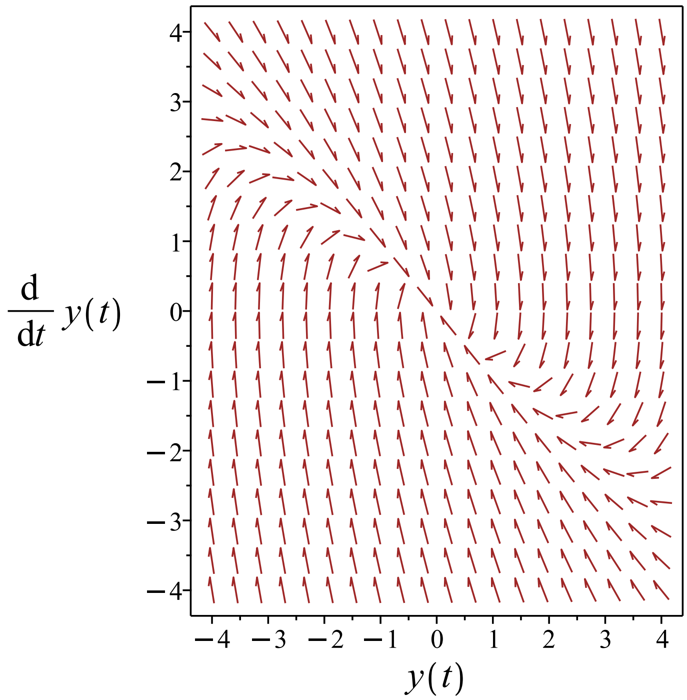


Figure 575: Slope field plot

Verification of solutions

$$y = c_1 e^{-t} + c_2 e^{-2t} - \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

Verified OK.

17.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 495: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt} \\ &= z_1 e^{-\frac{3t}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-3t}}{(y_1)^2}\,dt \\ &= y_1 \left(\mathrm{e}^{t}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2t}\right) + c_2\left(\mathrm{e}^{-2t}\left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(t), \sin(t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2t}, \mathrm{e}^{-t}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(t) + A_2 \sin(t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1 \cos(t) + A_2 \sin(t) - 3A_1 \sin(t) + 3A_2 \cos(t) = \sin(t)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t}\right) + \left(-\frac{3\cos(t)}{10} + \frac{\sin(t)}{10}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} - \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$

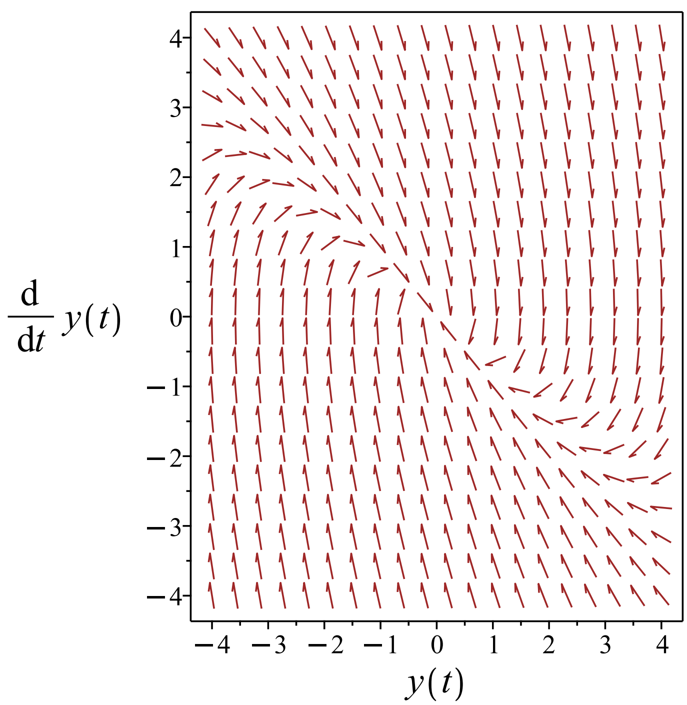


Figure 576: Slope field plot

Verification of solutions

$$y = c_1 e^{-2t} + c_2 e^{-t} - \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

Verified OK.

17.3.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = \sin(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \sin(t)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-3t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^{-2t}\left(\int e^{2t}\sin(t)\,dt\right) + e^{-t}\left(\int e^{t}\sin(t)\,dt\right)$

- Compute integrals

  $y_p(t) = -\frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$

• Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} - \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=sin(t),y(t), singsol=all)
```

$$y(t) = -e^{-2t}c_1 - \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} + c_2 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 32

```
DSolve[y''[t]+3*y'[t]+2*y[t]==Sin[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{10}\left(\sin(t) - 3\cos(t) + 10e^{-2t}\left(c_2 e^t + c_1\right)\right)$$

17.4 problem 4

Internal problem ID [13204]
Internal file name [`OUTPUT/11859_Sunday_December_03_2023_07_20_17_PM_94733438/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 3y' + 2y = 2\sin(t)}$$

17.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 3, C = 2, f(t) = 2\sin(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 3\lambda\, e^{\lambda t} + 2\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ y &= c_1 e^{(-1)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(t) + A_2 \sin(t) - 3A_1 \sin(t) + 3A_2 \cos(t) = 2\sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{5}, A_2 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(t)}{5} + \frac{\sin(t)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 e^{-2t}\right) + \left(-\frac{3\cos(t)}{5} + \frac{\sin(t)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-t} + c_2 e^{-2t} - \frac{3\cos(t)}{5} + \frac{\sin(t)}{5} \tag{1}$$

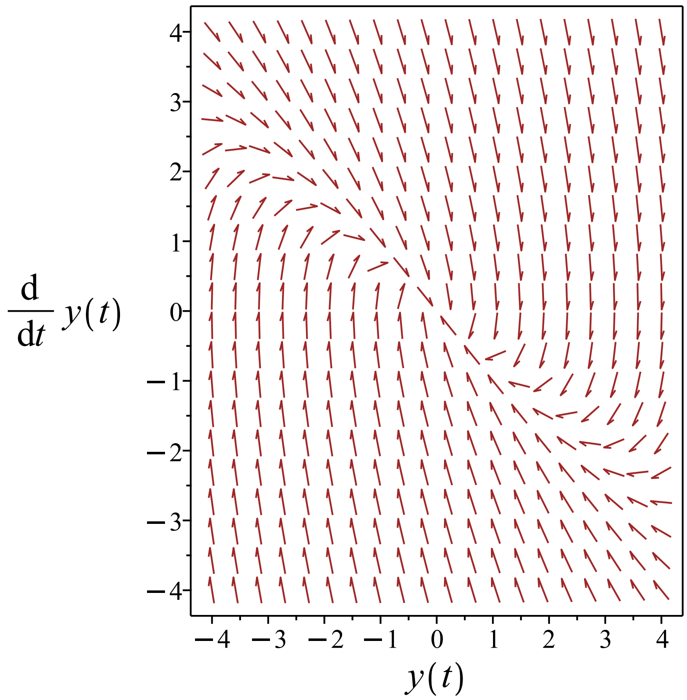


Figure 577: Slope field plot

Verification of solutions

$$y = c_1 e^{-t} + c_2 e^{-2t} - \frac{3\cos(t)}{5} + \frac{\sin(t)}{5}$$

Verified OK.

17.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 497: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = e^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\, dt} \\ &= z_1 e^{-\frac{3t}{2}} \\ &= z_1 \left(e^{-\frac{3t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-3t}}{(y_1)^2}\, dt \\ &= y_1 \left(e^{t}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2t}\right) + c_2\left(\mathrm{e}^{-2t}\left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2 \sin(t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(t), \sin(t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2t}, \mathrm{e}^{-t}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(t) + A_2 \sin(t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1 \cos(t) + A_2 \sin(t) - 3A_1 \sin(t) + 3A_2 \cos(t) = 2 \sin(t)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{5}, A_2 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(t)}{5} + \frac{\sin(t)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t}\right) + \left(-\frac{3\cos(t)}{5} + \frac{\sin(t)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} - \frac{3\cos(t)}{5} + \frac{\sin(t)}{5} \tag{1}$$

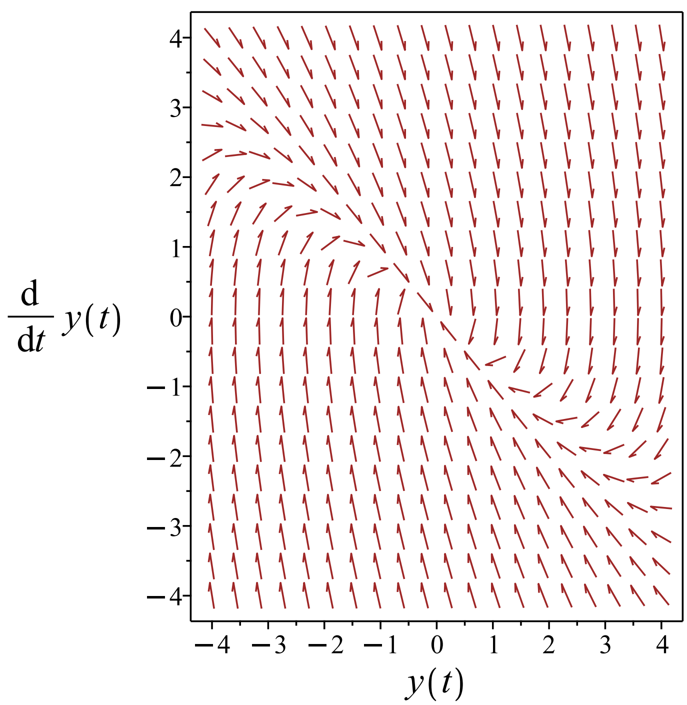


Figure 578: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} - \frac{3\cos(t)}{5} + \frac{\sin(t)}{5}$$

Verified OK.

17.4.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = 2\sin(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 2\sin(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-3t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -2e^{-2t}\left(\int e^{2t}\sin(t)\,dt\right) + 2e^{-t}\left(\int e^{t}\sin(t)\,dt\right)$

- Compute integrals

  $y_p(t) = -\frac{3\cos(t)}{5} + \frac{\sin(t)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-2t} + c_2e^{-t} - \frac{3\cos(t)}{5} + \frac{\sin(t)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=2*sin(t),y(t), singsol=all)
```

$$y(t) = -e^{-2t}c_1 - \frac{3\cos(t)}{5} + \frac{\sin(t)}{5} + c_2e^{-t}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 32

```
DSolve[y''[t]+3*y'[t]+2*y[t]==2*Sin[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{5}\left(\sin(t) - 3\cos(t) + 5e^{-2t}\left(c_2 e^t + c_1\right)\right)$$

17.5 problem 5

Internal problem ID [13205]
Internal file name [`OUTPUT/11860_Sunday_December_03_2023_07_20_20_PM_53181779/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 8y = \cos(t)}$$

17.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 6, C = 8, f(t) = \cos(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 6, C = 8$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda \, e^{\lambda t} + 8 \, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{6^2 - (4)\,(1)\,(8)} \\ &= -3 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 1 \\ \lambda_2 &= -3 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 \\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ y &= c_1 e^{(-2)t} + c_2 e^{(-4)t}\end{aligned}$$

Or

$$y = c_1 e^{-2t} + c_2 e^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2t} + c_2 e^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 \cos(t) + 7A_2 \sin(t) - 6A_1 \sin(t) + 6A_2 \cos(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{7}{85}, A_2 = \frac{6}{85}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-4t}\right) + \left(\frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2t} + c_2 e^{-4t} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85} \tag{1}$$

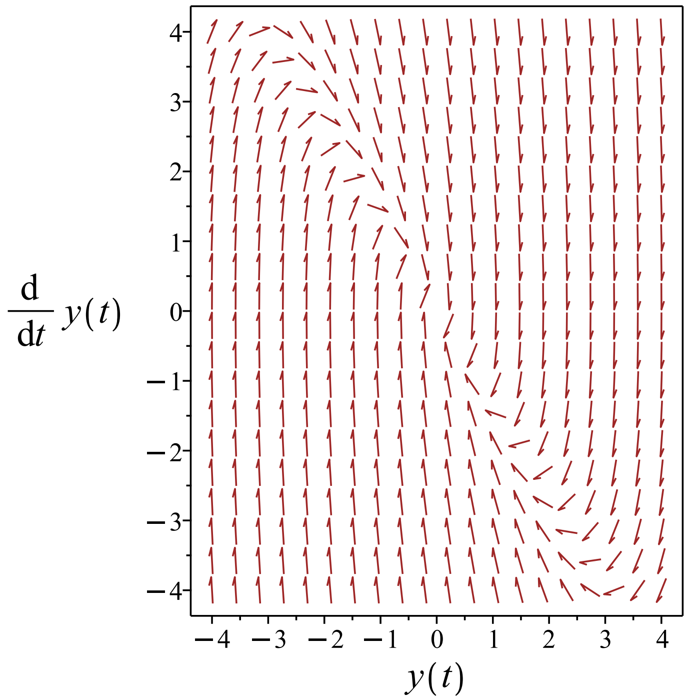


Figure 579: Slope field plot

Verification of solutions

$$y = c_1 e^{-2t} + c_2 e^{-4t} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$$

Verified OK.

17.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 499: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\, dt} \\
&= z_1 e^{-3t} \\
&= z_1 \left(\mathrm{e}^{-3t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-6t}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 6y' + 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(t), \sin(t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{-2t}}{2}, \mathrm{e}^{-4t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(t) + A_2 \sin(t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 \cos(t) + 7A_2 \sin(t) - 6A_1 \sin(t) + 6A_2 \cos(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{7}{85}, A_2 = \frac{6}{85}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2}\right) + \left(\frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85} \tag{1}$$

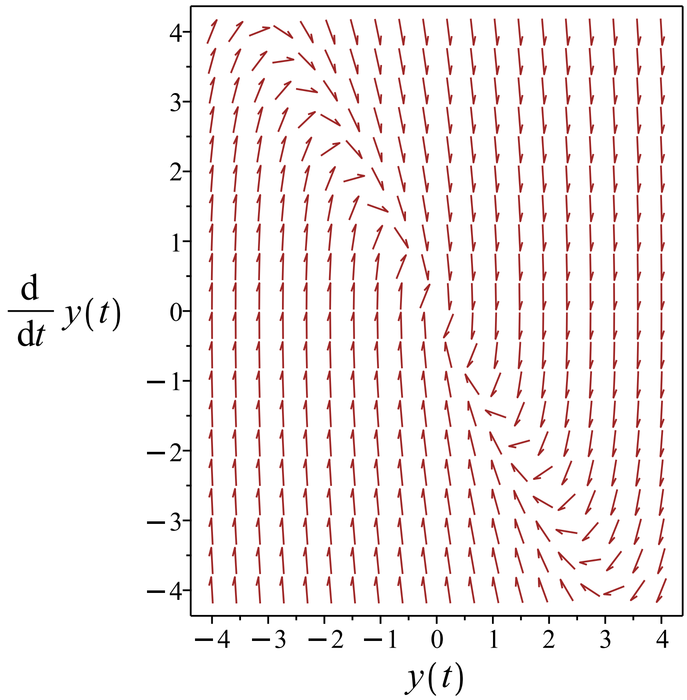


Figure 580: Slope field plot

Verification of solutions

$$y = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$$

Verified OK.

17.5.3 Maple step by step solution

Let's solve

$y'' + 6y' + 8y = \cos(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r+4)(r+2) = 0$

- Roots of the characteristic polynomial

$r = (-4, -2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \cos(t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -4\,e^{-4t} & -2\,e^{-2t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-6t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^{-4t}\left(\int \cos(t) e^{4t} dt\right)}{2} + \frac{e^{-2t}\left(\int e^{2t} \cos(t) dt\right)}{2}$

- ○ Compute integrals

  $y_p(t) = \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(t),t$2)+6*diff(y(t),t)+8*y(t)=cos(t),y(t), singsol=all)
```

$$y(t) = -\frac{e^{-4t}c_1}{2} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85} + c_2 e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 35

```
DSolve[y''[t]+6*y'[t]+8*y[t]==Cos[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{6\sin(t)}{85} + \frac{7\cos(t)}{85} + e^{-4t}(c_2 e^{2t} + c_1)$$

17.6 problem 6

Internal problem ID [13206]
Internal file name [`OUTPUT/11861_Sunday_December_03_2023_07_20_24_PM_42746866/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 8y = -4\cos(3t)}$$

17.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 6, C = 8, f(t) = -4\cos(3t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 6, C = 8$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda\, e^{\lambda t} + 8\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(8)} \\ &= -3 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 1 \\ \lambda_2 &= -3 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 \\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ y &= c_1 e^{(-2)t} + c_2 e^{(-4)t}\end{aligned}$$

Or

$$y = c_1 e^{-2t} + c_2 e^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2t} + c_2 e^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-4\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(3t) + A_2\sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\cos(3t) - A_2\sin(3t) - 18A_1\sin(3t) + 18A_2\cos(3t) = -4\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{325}, A_2 = -\frac{72}{325}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\cos(3t)}{325} - \frac{72\sin(3t)}{325}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-4t}\right) + \left(\frac{4\cos(3t)}{325} - \frac{72\sin(3t)}{325}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t} + \frac{4 \cos (3t)}{325} - \frac{72 \sin (3t)}{325} \tag{1}$$

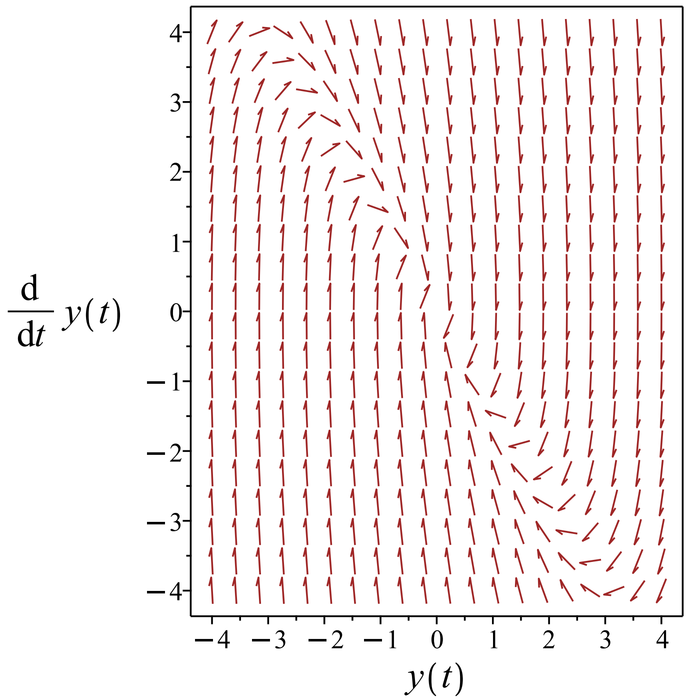


Figure 581: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t} + \frac{4 \cos (3t)}{325} - \frac{72 \sin (3t)}{325}$$

Verified OK.

17.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 6 \\
C &= 8
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = ye^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 501: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dt} \\
&= z_1 e^{-3t} \\
&= z_1\left(\mathrm{e}^{-3t}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-6t}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-4\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-2t}}{2}, e^{-4t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(3t) + A_2\sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\cos(3t) - A_2\sin(3t) - 18A_1\sin(3t) + 18A_2\cos(3t) = -4\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{325}, A_2 = -\frac{72}{325}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\cos(3t)}{325} - \frac{72\sin(3t)}{325}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2}\right) + \left(\frac{4\cos(3t)}{325} - \frac{72\sin(3t)}{325}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2} + \frac{4\cos(3t)}{325} - \frac{72\sin(3t)}{325} \tag{1}$$

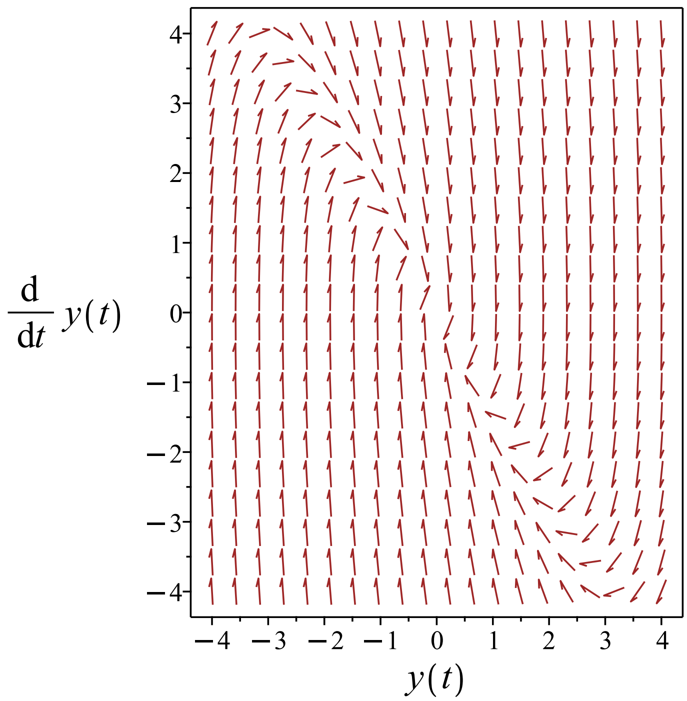


Figure 582: Slope field plot

Verification of solutions

$$y = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2} + \frac{4\cos(3t)}{325} - \frac{72\sin(3t)}{325}$$

Verified OK.

17.6.3 Maple step by step solution

Let's solve

$y'' + 6y' + 8y = -4\cos(3t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r+4)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = -4\cos(3t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -4\,e^{-4t} & -2\,e^{-2t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-6t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = 2\,e^{-4t}\left(\int \cos(3t)\, e^{4t} dt\right) - 2\,e^{-2t}\left(\int e^{2t} \cos(3t)\, dt\right)$

- ○ Compute integrals

  $y_p(t) = \frac{4\cos(3t)}{325} - \frac{72\sin(3t)}{325}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + \frac{4\cos(3t)}{325} - \frac{72\sin(3t)}{325}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(t),t$2)+6*diff(y(t),t)+8*y(t)=-4*cos(3*t),y(t), singsol=all)
```

$$y(t) = -\frac{e^{-4t}c_1}{2} + c_2e^{-2t} + \frac{4\cos(3t)}{325} - \frac{72\sin(3t)}{325}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 37

```
DSolve[y''[t]+6*y'[t]+8*y[t]==-4*Cos[3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1e^{-4t} + c_2e^{-2t} + \frac{4}{325}(\cos(3t) - 18\sin(3t))$$

17.7 problem 7

Internal problem ID [13207]
Internal file name [`OUTPUT/11862_Sunday_December_03_2023_07_20_28_PM_83117521/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 13y = 3\cos(2t)}$$

17.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 13, f(t) = 3\cos(2t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 13$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda \, e^{\lambda t} + 13 \, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(13)} \\ &= -2 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 3i \\ \lambda_2 &= -2 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 3i \\ \lambda_2 &= -2 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}\cos(3t), e^{-2t}\sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(2t) + A_2\sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1\cos(2t) + 9A_2\sin(2t) - 8A_1\sin(2t) + 8A_2\cos(2t) = 3\cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{27}{145}, A_2 = \frac{24}{145}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{27\cos(2t)}{145} + \frac{24\sin(2t)}{145}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}(c_1\cos(3t) + c_2\sin(3t))\right) + \left(\frac{27\cos(2t)}{145} + \frac{24\sin(2t)}{145}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t)) + \frac{27\cos(2t)}{145} + \frac{24\sin(2t)}{145} \tag{1}$$

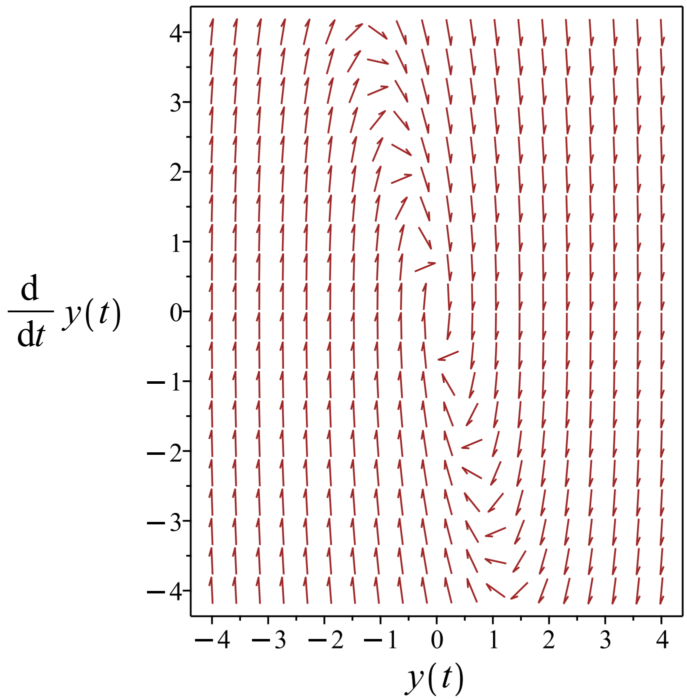


Figure 583: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t)) + \frac{27\cos(2t)}{145} + \frac{24\sin(2t)}{145}$$

Verified OK.

17.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 503: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(3t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1\left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t} \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-4t}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2t} \cos(3t)\right) + c_2 \left(\mathrm{e}^{-2t} \cos(3t) \left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 13y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-2t} \cos(3t)\, c_1 + \frac{\mathrm{e}^{-2t} \sin(3t)\, c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-2t}\cos(3t), \frac{e^{-2t}\sin(3t)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(2t) + A_2\sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1\cos(2t) + 9A_2\sin(2t) - 8A_1\sin(2t) + 8A_2\cos(2t) = 3\cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{27}{145}, A_2 = \frac{24}{145}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{27\cos(2t)}{145} + \frac{24\sin(2t)}{145}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}\cos(3t)c_1 + \frac{e^{-2t}\sin(3t)c_2}{3}\right) + \left(\frac{27\cos(2t)}{145} + \frac{24\sin(2t)}{145}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-2t}\cos(3t)c_1 + \frac{e^{-2t}\sin(3t)c_2}{3} + \frac{27\cos(2t)}{145} + \frac{24\sin(2t)}{145} \tag{1}$$

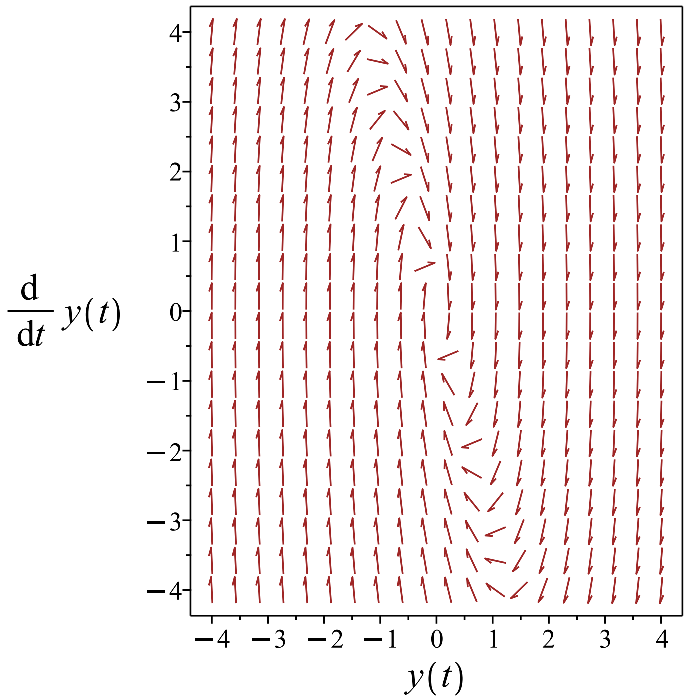


Figure 584: Slope field plot

Verification of solutions

$$y = e^{-2t} \cos(3t) c_1 + \frac{e^{-2t} \sin(3t) c_2}{3} + \frac{27 \cos(2t)}{145} + \frac{24 \sin(2t)}{145}$$

Verified OK.

17.7.3 Maple step by step solution

Let's solve

$y'' + 4y' + 13y = 3\cos(2t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-2 - 3\,\mathrm{I}, -2 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t} \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t} \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-2t} \cos(3t)\, c_1 + \mathrm{e}^{-2t} \sin(3t)\, c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 3\cos(2t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t}\cos(3t) & \mathrm{e}^{-2t}\sin(3t) \\ -2\,\mathrm{e}^{-2t}\cos(3t) - 3\,\mathrm{e}^{-2t}\sin(3t) & -2\,\mathrm{e}^{-2t}\sin(3t) + 3\,\mathrm{e}^{-2t}\cos(3t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,\mathrm{e}^{-4t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \mathrm{e}^{-2t}\left(-\cos(3t)\left(\int \sin(3t)\cos(2t)\,\mathrm{e}^{2t} dt\right) + \sin(3t)\left(\int \cos(3t)\cos(2t)\,\mathrm{e}^{2t} dt\right)\right)$

- ○ Compute integrals

  $y_p(t) = \frac{27\cos(2t)}{145} + \frac{24\sin(2t)}{145}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-2t}\sin(3t)\, c_2 + \mathrm{e}^{-2t}\cos(3t)\, c_1 + \frac{27\cos(2t)}{145} + \frac{24\sin(2t)}{145}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(t),t$2)+4*diff(y(t),t)+13*y(t)=3*cos(2*t),y(t), singsol=all)
```

$$y(t) = c_2 \mathrm{e}^{-2t} \sin(3t) + c_1 \mathrm{e}^{-2t} \cos(3t) + \frac{24 \sin(2t)}{145} + \frac{27 \cos(2t)}{145}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 47

```
DSolve[y''[t]+4*y'[t]+13*y[t]==3*Cos[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{3}{145}(8\sin(2t) + 9\cos(2t)) + c_2 e^{-2t}\cos(3t) + c_1 e^{-2t}\sin(3t)$$

17.8 problem 8

Internal problem ID [13208]
Internal file name [`OUTPUT/11863_Sunday_December_03_2023_07_20_34_PM_83144681/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 20y = -\cos(5t)}$$

17.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 20, f(t) = -\cos(5t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 20$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} + 20\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(20)}\\ &= -2 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 4i\\ \lambda_2 &= -2 - 4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 4i\\ \lambda_2 &= -2 - 4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\cos(5t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5t), \sin(5t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4t)\,\mathrm{e}^{-2t}, \sin(4t)\,\mathrm{e}^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(5t) + A_2 \sin(5t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(5t) - 5A_2 \sin(5t) - 20A_1 \sin(5t) + 20A_2 \cos(5t) = -\cos(5t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{85}, A_2 = -\frac{4}{85}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))\right) + \left(\frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85} \tag{1}$$

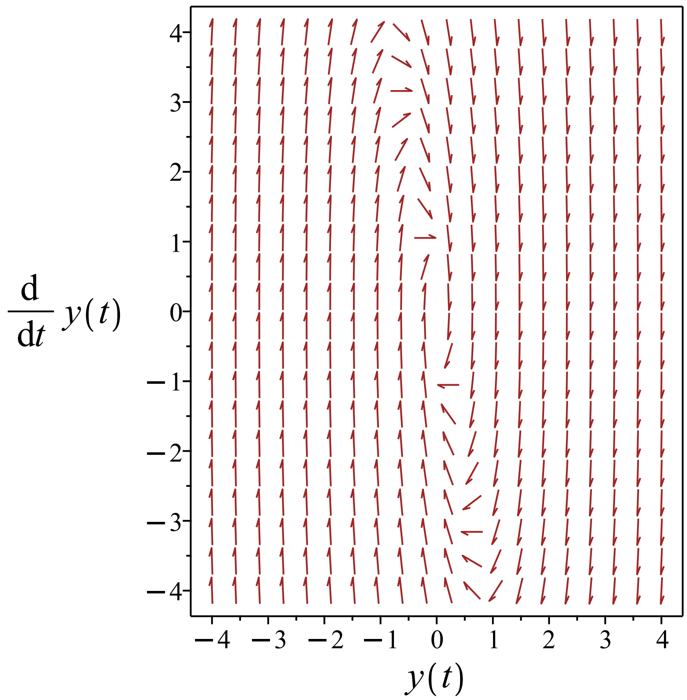


Figure 585: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85}$$

Verified OK.

17.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 20y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 4 \\
C &= 20
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = y e^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -16 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -16 z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 505: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -16$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(4t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1\left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(4t)\,\mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{\tan(4t)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(4t)\,\mathrm{e}^{-2t}\right) + c_2 \left(\cos(4t)\,\mathrm{e}^{-2t}\left(\frac{\tan(4t)}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \frac{\sin(4t)\,\mathrm{e}^{-2t} c_2}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\cos(5t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5t), \sin(5t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos(4t)\,\mathrm{e}^{-2t}, \frac{\sin(4t)\,\mathrm{e}^{-2t}}{4}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(5t) + A_2 \sin(5t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(5t) - 5A_2 \sin(5t) - 20A_1 \sin(5t) + 20A_2 \cos(5t) = -\cos(5t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{85}, A_2 = -\frac{4}{85}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(4t)\,\mathrm{e}^{-2t} c_1 + \frac{\sin(4t)\,\mathrm{e}^{-2t} c_2}{4}\right) + \left(\frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \frac{\sin(4t)\,\mathrm{e}^{-2t} c_2}{4} + \frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85} \tag{1}$$

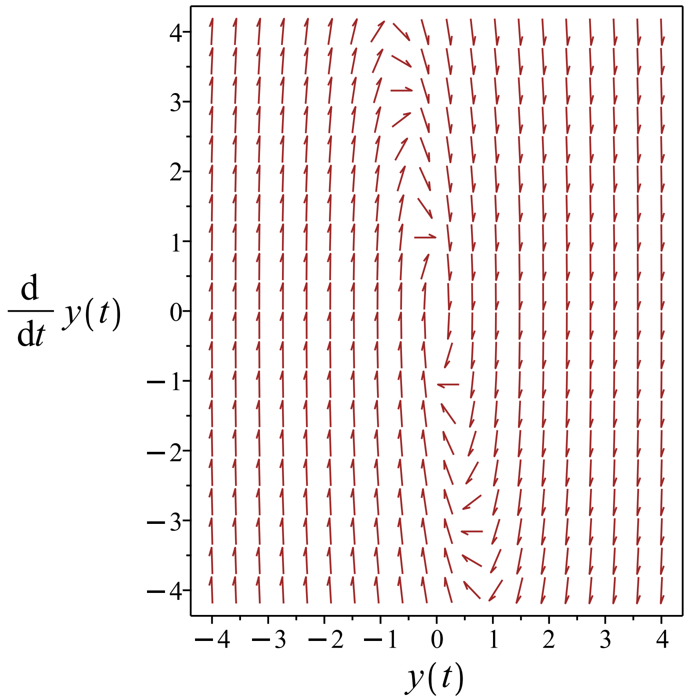


Figure 586: Slope field plot

<u>Verification of solutions</u>

$$y = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \frac{\sin(4t)\,\mathrm{e}^{-2t} c_2}{4} + \frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85}$$

Verified OK.

17.8.3 Maple step by step solution

Let's solve

$y'' + 4y' + 20y = -\cos(5t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 20 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

$r = (-2-4\,\mathrm{I}, -2+4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(4t)\,\mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(4t)\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \sin(4t)\,\mathrm{e}^{-2t} c_2 + y_p(t)$

- ☐ Find a particular solution $y_p(t)$ of the ODE
  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))} dt\right), f(t) = -\cos(5t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(4t)\,\mathrm{e}^{-2t} & \sin(4t)\,\mathrm{e}^{-2t} \\ -4\sin(4t)\,\mathrm{e}^{-2t} - 2\cos(4t)\,\mathrm{e}^{-2t} & 4\cos(4t)\,\mathrm{e}^{-2t} - 2\sin(4t)\,\mathrm{e}^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 4\,\mathrm{e}^{-4t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\mathrm{e}^{-2t}\left(\cos(4t)\left(\int(-\sin(9t)+\sin(t))\mathrm{e}^{2t}dt\right)+\sin(4t)\left(\int(\cos(t)+\cos(9t))\mathrm{e}^{2t}dt\right)\right)}{8}$

  - Compute integrals

    $y_p(t) = \frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85}$

- Substitute particular solution into general solution to ODE

  $y = \sin(4t)\,\mathrm{e}^{-2t} c_2 + \cos(4t)\,\mathrm{e}^{-2t} c_1 + \frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(t),t$2)+4*diff(y(t),t)+20*y(t)=-cos(5*t),y(t), singsol=all)
```

$$y(t) = \sin(4t)\,\mathrm{e}^{-2t}c_2 + \cos(4t)\,\mathrm{e}^{-2t}c_1 + \frac{\cos(5t)}{85} - \frac{4\sin(5t)}{85}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 45

```
DSolve[y''[t]+4*y'[t]+20*y[t]==-Cos[5*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{85}(\cos(5t) - 4\sin(5t)) + c_2e^{-2t}\cos(4t) + c_1e^{-2t}\sin(4t)$$

17.9 problem 9

Internal problem ID [13209]
Internal file name [`OUTPUT/11864_Sunday_December_03_2023_07_20_42_PM_30836973/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 20y = -3\sin(2t)}$$

17.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 20, f(t) = -3\sin(2t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 20$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} + 20\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(20)}\\ &= -2 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 4i\\ \lambda_2 &= -2 - 4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 4i\\ \lambda_2 &= -2 - 4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-3\sin(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4t)\,\mathrm{e}^{-2t}, \sin(4t)\,\mathrm{e}^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(2t) + A_2\sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1\cos(2t) + 16A_2\sin(2t) - 8A_1\sin(2t) + 8A_2\cos(2t) = -3\sin(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{40}, A_2 = -\frac{3}{20}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\cos(2t)}{40} - \frac{3\sin(2t)}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2t}(c_1\cos(4t) + c_2\sin(4t))\right) + \left(\frac{3\cos(2t)}{40} - \frac{3\sin(2t)}{20}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{3\cos(2t)}{40} - \frac{3\sin(2t)}{20} \tag{1}$$

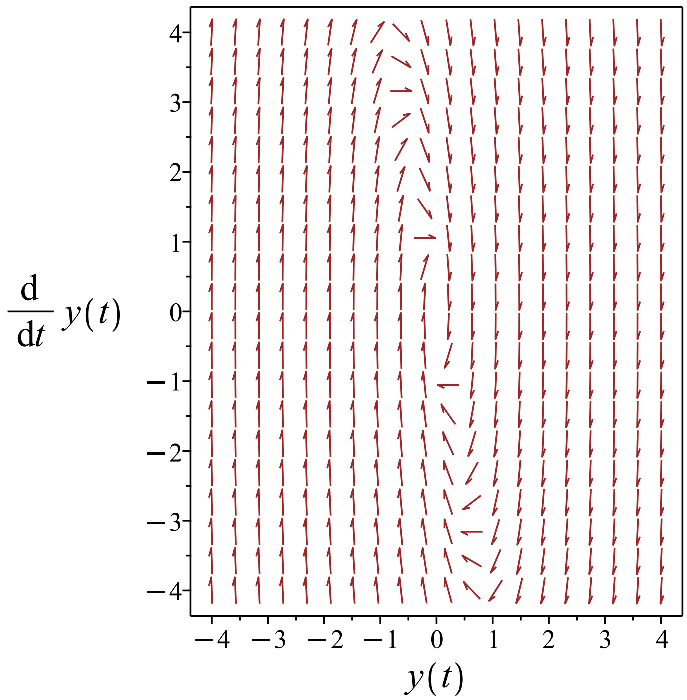


Figure 587: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{3\cos(2t)}{40} - \frac{3\sin(2t)}{20}$$

Verified OK.

17.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 20y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -16z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 507: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -16$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(4t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1\left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(4t)\,\mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\
&= y_1\left(\frac{\tan(4t)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\cos(4t)\,\mathrm{e}^{-2t}\right) + c_2\left(\cos(4t)\,\mathrm{e}^{-2t}\left(\frac{\tan(4t)}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \frac{\sin(4t)\,\mathrm{e}^{-2t} c_2}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-3\sin(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos (2t), \sin (2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos (4t)\, \mathrm{e}^{-2t}, \frac{\sin (4t)\, \mathrm{e}^{-2t}}{4}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos (2t) + A_2 \sin (2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1 \cos (2t) + 16A_2 \sin (2t) - 8A_1 \sin (2t) + 8A_2 \cos (2t) = -3 \sin (2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{40}, A_2 = -\frac{3}{20}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3 \cos (2t)}{40} - \frac{3 \sin (2t)}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos (4t)\, \mathrm{e}^{-2t} c_1 + \frac{\sin (4t)\, \mathrm{e}^{-2t} c_2}{4}\right) + \left(\frac{3 \cos (2t)}{40} - \frac{3 \sin (2t)}{20}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos (4t)\, \mathrm{e}^{-2t} c_1 + \frac{\sin (4t)\, \mathrm{e}^{-2t} c_2}{4} + \frac{3 \cos (2t)}{40} - \frac{3 \sin (2t)}{20} \tag{1}$$

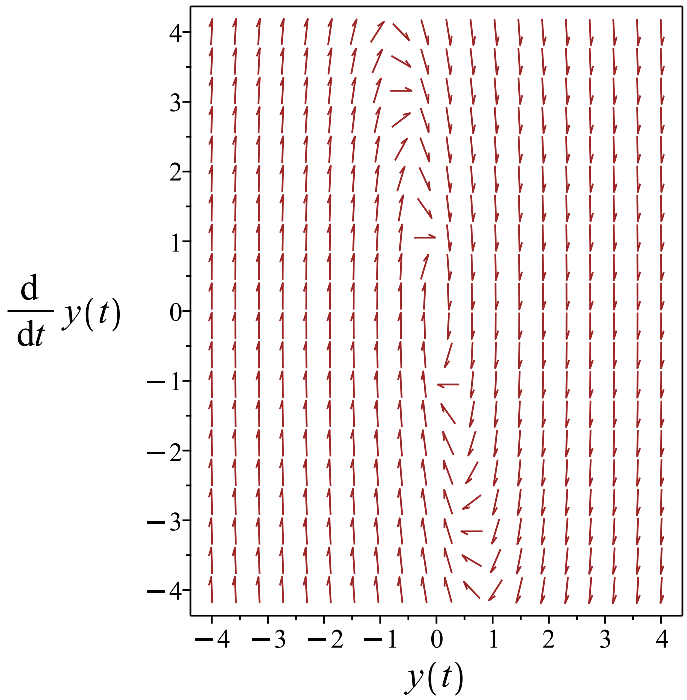


Figure 588: Slope field plot

Verification of solutions

$$y = \cos(4t)\,e^{-2t}c_1 + \frac{\sin(4t)\,e^{-2t}c_2}{4} + \frac{3\cos(2t)}{40} - \frac{3\sin(2t)}{20}$$

Verified OK.

17.9.3 Maple step by step solution

Let's solve

$y'' + 4y' + 20y = -3\sin(2t)$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4r + 20 = 0$

- Use quadratic formula to solve for r

$r = \frac{(-4)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

$r = (-2 - 4\,\mathrm{I}, -2 + 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(4t)\,\mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(4t)\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \sin(4t)\,\mathrm{e}^{-2t} c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = -3\sin(2t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(4t)\,\mathrm{e}^{-2t} & \sin(4t)\,\mathrm{e}^{-2t} \\ -4\sin(4t)\,\mathrm{e}^{-2t} - 2\cos(4t)\,\mathrm{e}^{-2t} & 4\cos(4t)\,\mathrm{e}^{-2t} - 2\sin(4t)\,\mathrm{e}^{-2t} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 4\,\mathrm{e}^{-4t}$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{3\,\mathrm{e}^{-2t}\left(\sin(4t)\left(\int \cos(4t)\sin(2t)\mathrm{e}^{2t}dt\right) - \cos(4t)\left(\int \sin(4t)\sin(2t)\mathrm{e}^{2t}dt\right)\right)}{4}$

  ◦ Compute integrals

  $y_p(t) = \frac{3\cos(2t)}{40} - \frac{3\sin(2t)}{20}$

- Substitute particular solution into general solution to ODE

  $y = \sin(4t)\,\mathrm{e}^{-2t} c_2 + \cos(4t)\,\mathrm{e}^{-2t} c_1 - \frac{3\sin(2t)}{20} + \frac{3\cos(2t)}{40}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(diff(y(t),t$2)+4*diff(y(t),t)+20*y(t)=-3*sin(2*t),y(t), singsol=all)
```

$$y(t) = \sin(4t)\,\mathrm{e}^{-2t} c_2 + \cos(4t)\,\mathrm{e}^{-2t} c_1 - \frac{3\sin(2t)}{20} + \frac{3\cos(2t)}{40}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 45

```
DSolve[y''[t]+4*y'[t]+20*y[t]==-3*Sin[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{3}{40}(\cos(2t) - 2\sin(2t)) + c_2 e^{-2t}\cos(4t) + c_1 e^{-2t}\sin(4t)$$

17.10 problem 10

Internal problem ID [13210]
Internal file name [`OUTPUT/11865_Sunday_December_03_2023_07_20_48_PM_49887873/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = \cos(3t)}$$

17.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 2, C = 1, f(t) = \cos(3t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 2\lambda\, \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\,\mathrm{e}^{-t}, \mathrm{e}^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \cos(3t) - 8A_2 \sin(3t) - 6A_1 \sin(3t) + 6A_2 \cos(3t) = \cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{25}, A_2 = \frac{3}{50}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t}\right) + \left(-\frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-t}(c_2 t + c_1) - \frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-t}(c_2 t + c_1) - \frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50} \tag{1}$$

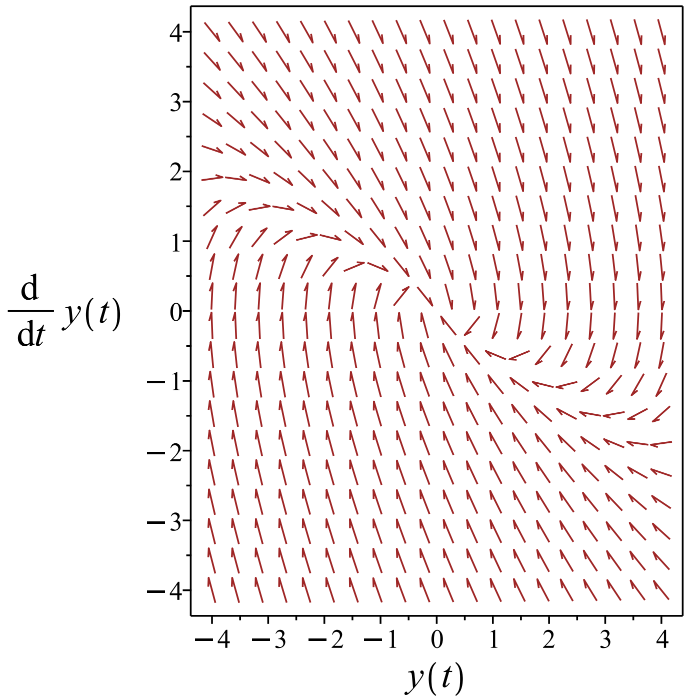


Figure 589: Slope field plot

Verification of solutions

$$y = e^{-t}(c_2 t + c_1) - \frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}$$

Verified OK.

17.10.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{\left(p(t)^2 + p'(t)\right) y}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2\, dx} \\ &= e^{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \mathrm{e}^{t}\cos(3t) \\
(\mathrm{e}^{t} y)'' &= \mathrm{e}^{t}\cos(3t)
\end{aligned}
$$

Integrating once gives

$$
(\mathrm{e}^{t} y)' = \frac{\mathrm{e}^{t}(\cos(3t) + 3\sin(3t))}{10} + c_1
$$

Integrating again gives

$$
(\mathrm{e}^{t} y) = -\frac{2\,\mathrm{e}^{t}\cos(3t)}{25} + \frac{3\,\mathrm{e}^{t}\sin(3t)}{50} + c_1 t + c_2
$$

Hence the solution is

$$
y = \frac{-\frac{2\,\mathrm{e}^{t}\cos(3t)}{25} + \frac{3\,\mathrm{e}^{t}\sin(3t)}{50} + c_1 t + c_2}{\mathrm{e}^{t}}
$$

Or

$$
y = -\frac{8\cos(t)^{3}}{25} + \frac{6\cos(t)^{2}\sin(t)}{25} + t\,\mathrm{e}^{-t} c_1 + \frac{6\cos(t)}{25} - \frac{3\sin(t)}{50} + c_2\mathrm{e}^{-t}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = -\frac{8\cos(t)^{3}}{25} + \frac{6\cos(t)^{2}\sin(t)}{25} + t\,\mathrm{e}^{-t} c_1 + \frac{6\cos(t)}{25} - \frac{3\sin(t)}{50} + c_2\mathrm{e}^{-t} \tag{1}
$$

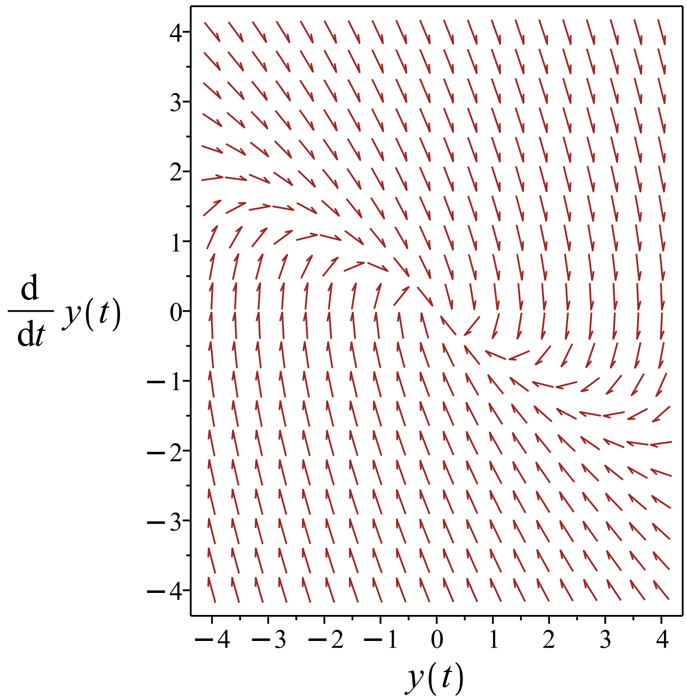


Figure 590: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{8\cos(t)^3}{25} + \frac{6\cos(t)^2\sin(t)}{25} + t\,e^{-t}c_1 + \frac{6\cos(t)}{25} - \frac{3\sin(t)}{50} + c_2e^{-t}$$

Verified OK.

17.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 509: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\ &= z_1 e^{-t} \\ &= z_1 \left(\mathrm{e}^{-t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt \\ &= y_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-t}\right) + c_2\left(\mathrm{e}^{-t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(3t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(3t), \sin(3t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{t\,\mathrm{e}^{-t}, \mathrm{e}^{-t}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(3t) + A_2 \sin(3t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-8A_1 \cos(3t) - 8A_2 \sin(3t) - 6A_1 \sin(3t) + 6A_2 \cos(3t) = \cos(3t)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{25}, A_2 = \frac{3}{50}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 t\, e^{-t}\right) + \left(-\frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-t}(c_2 t + c_1) - \frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}$$

Summary

The solution(s) found are the following

$$y = e^{-t}(c_2 t + c_1) - \frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50} \tag{1}$$

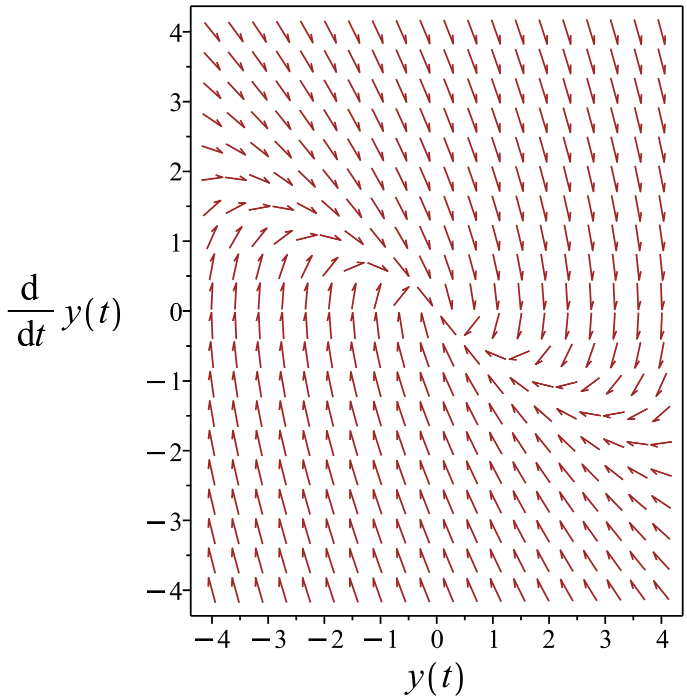


Figure 591: Slope field plot

Verification of solutions

$$y = e^{-t}(c_2 t + c_1) - \frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}$$

Verified OK.

17.10.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = \cos(3t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \cos(3t)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} \end{bmatrix}$$
- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$
- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \mathrm{e}^{-t}\left(-\left(\int t\cos(3t)\,\mathrm{e}^{t}dt\right) + t\left(\int \mathrm{e}^{t}\cos(3t)\,dt\right)\right)$
- ○ Compute integrals

  $y_p(t) = -\frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}$

- Substitute particular solution into general solution to ODE

  $y = c_2 t\,\mathrm{e}^{-t} + c_1 \mathrm{e}^{-t} - \frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(t),t$2)+2*diff(y(t),t)+y(t)=cos(3*t),y(t), singsol=all)
```

$$y(t) = (c_1 t + c_2)\,\mathrm{e}^{-t} - \frac{2\cos(3t)}{25} + \frac{3\sin(3t)}{50}$$

✓ Solution by Mathematica

Time used: 0.22 (sec). Leaf size: 35

```
DSolve[y''[t]+2*y'[t]+y[t]==Cos[3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{3}{50}\sin(3t) - \frac{2}{25}\cos(3t) + e^{-t}(c_2 t + c_1)$$

17.11 problem 11

Internal problem ID [13211]
Internal file name [`OUTPUT/11866_Sunday_December_03_2023_07_20_52_PM_32862123/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 8y = \cos(t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

17.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}p(t) &= 6\\ q(t) &= 8\\ F &= \cos(t)\end{aligned}$$

Hence the ode is

$$y'' + 6y' + 8y = \cos(t)$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

17.11.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 6, C = 8, f(t) = \cos(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 6, C = 8$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda e^{\lambda t} + 8 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(8)}\\ &= -3 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = -3 + 1$$
$$\lambda_2 = -3 - 1$$

Which simplifies to

$$\lambda_1 = -2$$
$$\lambda_2 = -4$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{(-2)t} + c_2 e^{(-4)t}$$

Or

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 \cos(t) + 7A_2 \sin(t) - 6A_1 \sin(t) + 6A_2 \cos(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{7}{85}, A_2 = \frac{6}{85}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-4t}\right) + \left(\frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + c_2 e^{-4t} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{7}{85} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2t} - 4c_2 e^{-4t} - \frac{7\sin(t)}{85} + \frac{6\cos(t)}{85}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 4c_2 + \frac{6}{85} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{5}$$
$$c_2 = \frac{2}{17}$$

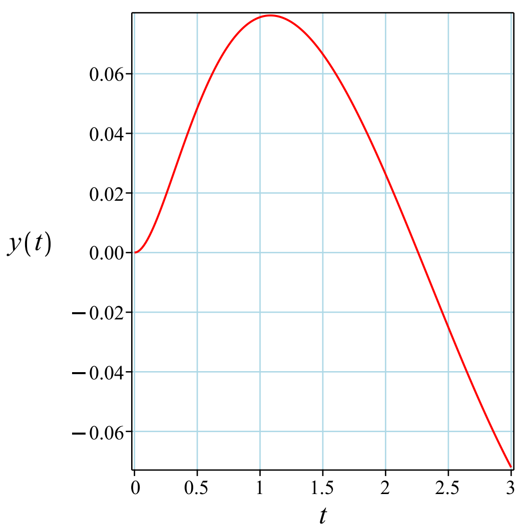
Substituting these values back in above solution results in

$$y = -\frac{e^{-2t}}{5} + \frac{2\,e^{-4t}}{17} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$$

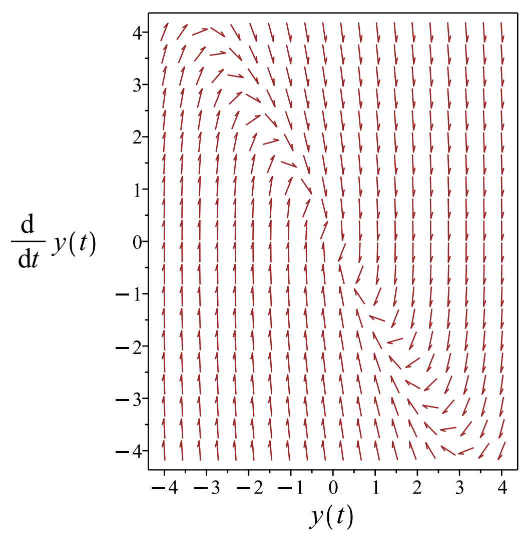
<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{e^{-2t}}{5} + \frac{2\,e^{-4t}}{17} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{e^{-2t}}{5} + \frac{2\,e^{-4t}}{17} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$$

Verified OK.

17.11.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 8y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 511: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dt}\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-3t} \\
&= z_1\left(\mathrm{e}^{-3t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-6t}}{(y_1)^2}\, dt \\
&= y_1\left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-4t}\right) + c_2\left(\mathrm{e}^{-4t}\left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 6y' + 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-2t}}{2}, e^{-4t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 \cos(t) + 7A_2 \sin(t) - 6A_1 \sin(t) + 6A_2 \cos(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{7}{85}, A_2 = \frac{6}{85}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2}\right) + \left(\frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2} + \frac{7 \cos(t)}{85} + \frac{6 \sin(t)}{85} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} + \frac{7}{85} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 e^{-4t} - c_2 e^{-2t} - \frac{7 \sin(t)}{85} + \frac{6 \cos(t)}{85}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_1 - c_2 + \frac{6}{85} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2}{17}$$
$$c_2 = -\frac{2}{5}$$

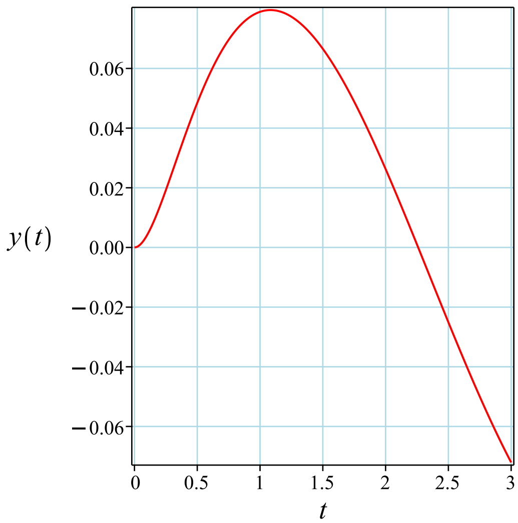
Substituting these values back in above solution results in

$$y = -\frac{e^{-2t}}{5} + \frac{2\,e^{-4t}}{17} + \frac{7 \cos(t)}{85} + \frac{6 \sin(t)}{85}$$

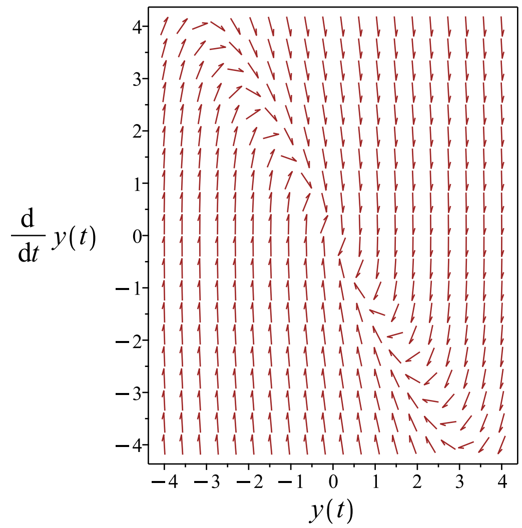
Summary

The solution(s) found are the following

$$y = -\frac{e^{-2t}}{5} + \frac{2\,e^{-4t}}{17} + \frac{7 \cos(t)}{85} + \frac{6 \sin(t)}{85} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-2t}}{5} + \frac{2\,e^{-4t}}{17} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$$

Verified OK.

17.11.4 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 8y = \cos(t), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 6r + 8 = 0$
- Factor the characteristic polynomial
  $(r+4)(r+2) = 0$
- Roots of the characteristic polynomial
  $r = (-4, -2)$
- 1st solution of the homogeneous ODE

$y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \cos(t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -4\,e^{-4t} & -2\,e^{-2t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-6t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^{-4t}\left(\int \cos(t) e^{4t} dt\right)}{2} + \frac{e^{-2t}\left(\int e^{2t}\cos(t) dt\right)}{2}$

- ○ Compute integrals

  $y_p(t) = \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$

☐ Check validity of solution $y = c_1 e^{-4t} + c_2 e^{-2t} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 + \frac{7}{85}$

- ○ Compute derivative of the solution

  $y' = -4c_1 e^{-4t} - 2c_2 e^{-2t} - \frac{7\sin(t)}{85} + \frac{6\cos(t)}{85}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -4c_1 - 2c_2 + \frac{6}{85}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{2}{17}, c_2 = -\frac{1}{5}\right\}$
- Substitute constant values into general solution and simplify

  $y = -\frac{e^{-2t}}{5} + \frac{2\,e^{-4t}}{17} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$

- Solution to the IVP

  $y = -\frac{e^{-2t}}{5} + \frac{2\,e^{-4t}}{17} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+8*y(t)=cos(t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all
```

$$y(t) = \frac{2\,e^{-4t}}{17} + \frac{7\cos(t)}{85} + \frac{6\sin(t)}{85} - \frac{e^{-2t}}{5}$$

✓ Solution by Mathematica

Time used: 2.147 (sec). Leaf size: 63

```
DSolve[{y''[t]+5*y'[t]+8*y[t]==Cos[t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to \frac{1}{518}\left(35\sin(t) - 45\sqrt{7}e^{-5t/2}\sin\left(\frac{\sqrt{7}t}{2}\right) + 49\cos(t) - 49e^{-5t/2}\cos\left(\frac{\sqrt{7}t}{2}\right)\right)$$

17.12 problem 12

Internal problem ID [13212]
Internal file name [`OUTPUT/11867_Sunday_December_03_2023_07_20_55_PM_55063930/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 8y = 2\cos(3t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

17.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 8 \\ F &= 2\cos(3t) \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 8y = 2\cos(3t)$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2\cos(3t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

17.12.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 6, C = 8, f(t) = 2\cos(3t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 6, C = 8$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 6\lambda\,\mathrm{e}^{\lambda t} + 8\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(8)} \\ &= -3 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 1 \\ \lambda_2 &= -3 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 \\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ y &= c_1 e^{(-2)t} + c_2 e^{(-4)t}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(3t) - A_2 \sin(3t) - 18A_1 \sin(3t) + 18A_2 \cos(3t) = 2\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{325}, A_2 = \frac{36}{325}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-4t}\right) + \left(-\frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2t} + c_2 e^{-4t} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 - \frac{2}{325} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2t} - 4c_2 e^{-4t} + \frac{6\sin(3t)}{325} + \frac{108\cos(3t)}{325}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 4c_2 + \frac{108}{325} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{2}{13}$$
$$c_2 = \frac{4}{25}$$

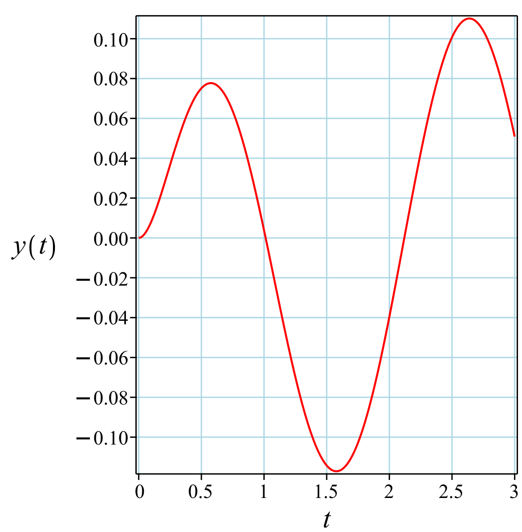
Substituting these values back in above solution results in

$$y = -\frac{2\,e^{-2t}}{13} + \frac{4\,e^{-4t}}{25} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$$

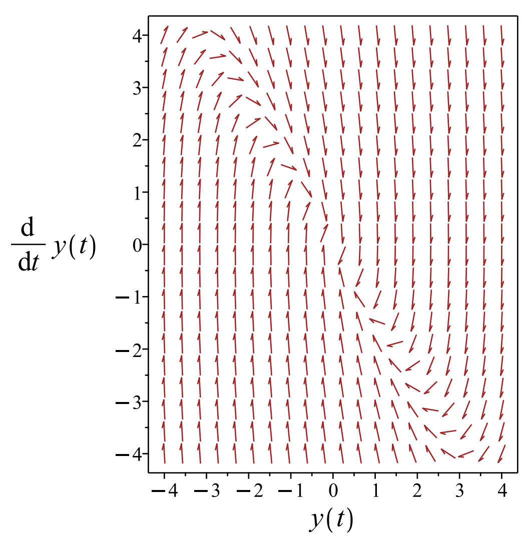
Summary

The solution(s) found are the following

$$y = -\frac{2\,e^{-2t}}{13} + \frac{4\,e^{-4t}}{25} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2\,e^{-2t}}{13} + \frac{4\,e^{-4t}}{25} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$$

Verified OK.

17.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 513: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dt} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-3t} \\
&= z_1 \left(\mathrm{e}^{-3t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-6t}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{\mathrm{e}^{2t}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 6y' + 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-2t}}{2}, e^{-4t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(3t) - A_2 \sin(3t) - 18A_1 \sin(3t) + 18A_2 \cos(3t) = 2\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{325}, A_2 = \frac{36}{325}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4t} + \frac{c_2 e^{-2t}}{2}\right) + \left(-\frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2} - \frac{2 \cos (3t)}{325} + \frac{36 \sin (3t)}{325} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} - \frac{2}{325} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 \mathrm{e}^{-4t} - c_2 \mathrm{e}^{-2t} + \frac{6 \sin (3t)}{325} + \frac{108 \cos (3t)}{325}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -4c_1 - c_2 + \frac{108}{325} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

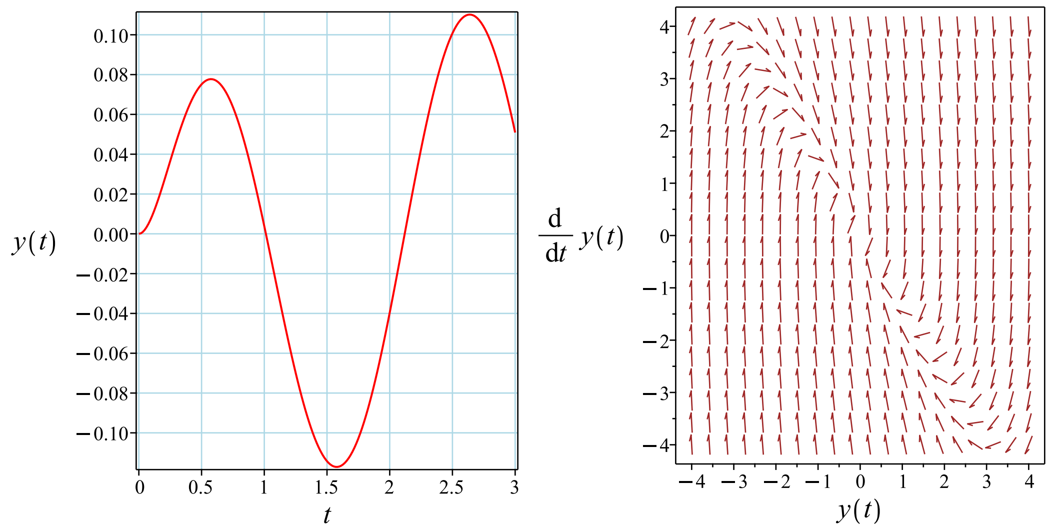
$$c_1 = \frac{4}{25}$$
$$c_2 = -\frac{4}{13}$$

Substituting these values back in above solution results in

$$y = -\frac{2\,\mathrm{e}^{-2t}}{13} + \frac{4\,\mathrm{e}^{-4t}}{25} - \frac{2 \cos (3t)}{325} + \frac{36 \sin (3t)}{325}$$

Summary
The solution(s) found are the following

$$y = -\frac{2\,\mathrm{e}^{-2t}}{13} + \frac{4\,\mathrm{e}^{-4t}}{25} - \frac{2 \cos (3t)}{325} + \frac{36 \sin (3t)}{325} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{2\,e^{-2t}}{13} + \frac{4\,e^{-4t}}{25} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$$

Verified OK.

17.12.4 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 8y = 2\cos(3t), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r+4)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -2)$

- 1st solution of the homogeneous ODE

$y_1(t) = e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 2\cos(3t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-4t} & e^{-2t} \\ -4\,e^{-4t} & -2\,e^{-2t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-6t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^{-4t}\left(\int \cos(3t)\,e^{4t}dt\right) + e^{-2t}\left(\int e^{2t}\cos(3t)\,dt\right)$

- ○ Compute integrals

  $y_p(t) = -\frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-4t} + c_2 e^{-2t} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$

□ Check validity of solution $y = c_1 e^{-4t} + c_2 e^{-2t} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - \frac{2}{325}$

- ○ Compute derivative of the solution

  $y' = -4c_1 e^{-4t} - 2c_2 e^{-2t} + \frac{6\sin(3t)}{325} + \frac{108\cos(3t)}{325}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -4c_1 - 2c_2 + \frac{108}{325}$

- Solve for c_1 and c_2

  $\{c_1 = \frac{4}{25}, c_2 = -\frac{2}{13}\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{2\,e^{-2t}}{13} + \frac{4\,e^{-4t}}{25} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$

- Solution to the IVP

  $y = -\frac{2\,e^{-2t}}{13} + \frac{4\,e^{-4t}}{25} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+8*y(t)=2*cos(3*t),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \frac{4\,e^{-4t}}{25} - \frac{2\,e^{-2t}}{13} - \frac{2\cos(3t)}{325} + \frac{36\sin(3t)}{325}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 74

```
DSolve[{y''[t]+5*y'[t]+8*y[t]==2*Cos[3*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{1}{791}e^{-5t/2}\left(105e^{5t/2}\sin(3t) - 85\sqrt{7}\sin\left(\frac{\sqrt{7}t}{2}\right) - 7e^{5t/2}\cos(3t) + 7\cos\left(\frac{\sqrt{7}t}{2}\right)\right)$$

17.13 problem 13

Internal problem ID [13213]
Internal file name [`OUTPUT/11868_Sunday_December_03_2023_07_21_00_PM_72067464/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 20y = -3\sin(2t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

17.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 20 \\ F &= -3\sin(2t) \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 20y = -3\sin(2t)$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 20$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -3\sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

17.13.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 6, C = 20, f(t) = -3\sin(2t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 6y' + 20y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 6, C = 20$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda\, e^{\lambda t} + 20\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(20)}\\ &= -3 \pm i\sqrt{11}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + i\sqrt{11}\\ \lambda_2 &= -3 - i\sqrt{11}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 + i\sqrt{11}\\ \lambda_2 &= -3 - i\sqrt{11}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = \sqrt{11}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-3t}\left(\cos\left(\sqrt{11}\,t\right) c_1 + \sin\left(\sqrt{11}\,t\right) c_2\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-3t}\left(\cos\left(\sqrt{11}\,t\right) c_1 + \sin\left(\sqrt{11}\,t\right) c_2\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-3\sin(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-3t} \cos\left(\sqrt{11}\, t\right), e^{-3t} \sin\left(\sqrt{11}\, t\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1 \cos(2t) + 16A_2 \sin(2t) - 12A_1 \sin(2t) + 12A_2 \cos(2t) = -3 \sin(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{9}{100}, A_2 = -\frac{3}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-3t}\left(\cos\left(\sqrt{11}\, t\right) c_1 + \sin\left(\sqrt{11}\, t\right) c_2\right)\right) + \left(\frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-3t}\left(\cos\left(\sqrt{11}\, t\right) c_1 + \sin\left(\sqrt{11}\, t\right) c_2\right) + \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{9}{100} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,e^{-3t}\left(\cos\left(\sqrt{11}\,t\right) c_1 + \sin\left(\sqrt{11}\,t\right) c_2\right) + e^{-3t}\left(-\sqrt{11}\,\sin\left(\sqrt{11}\,t\right) c_1 + \sqrt{11}\,\cos\left(\sqrt{11}\,t\right) c_2\right) - \frac{9\,\mathrm{s}}{}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 + \sqrt{11}\,c_2 - \frac{6}{25} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{9}{100} \\ c_2 &= -\frac{3\sqrt{11}}{1100} \end{aligned}$$

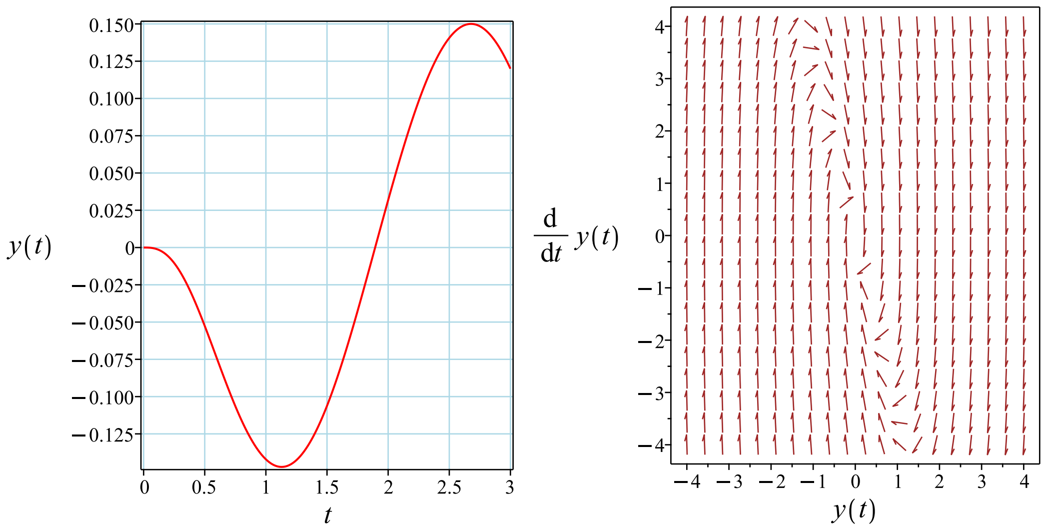
Substituting these values back in above solution results in

$$y = -\frac{9\,e^{-3t}\cos\left(\sqrt{11}\,t\right)}{100} - \frac{3\,e^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{1100} + \frac{9\cos\left(2t\right)}{100} - \frac{3\sin\left(2t\right)}{25}$$

Summary

The solution(s) found are the following

$$y = -\frac{9\,e^{-3t}\cos\left(\sqrt{11}\,t\right)}{100} - \frac{3\,e^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{1100} + \frac{9\cos\left(2t\right)}{100} - \frac{3\sin\left(2t\right)}{25} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{9\,e^{-3t}\cos\left(\sqrt{11}\,t\right)}{100} - \frac{3\,e^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{1100} + \frac{9\cos\left(2t\right)}{100} - \frac{3\sin\left(2t\right)}{25}$$

Verified OK.

17.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-11}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -11$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -11z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 515: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -11$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\sqrt{11}\, t\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\, dt} \\ &= z_1 e^{-3t} \\ &= z_1\left(\mathrm{e}^{-3t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3t} \cos\left(\sqrt{11}\, t\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-6t}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{\sqrt{11}\, \tan\left(\sqrt{11}\, t\right)}{11}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right)\right) + c_2\left(\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right)\left(\frac{\sqrt{11}\,\tan\left(\sqrt{11}\,t\right)}{11}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 6y' + 20y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right) + \frac{c_2 \mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
-3\sin(2t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(2t), \sin(2t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right), \frac{\mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{11}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\cos(2t) + A_2\sin(2t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1 \cos(2t) + 16A_2 \sin(2t) - 12A_1 \sin(2t) + 12A_2 \cos(2t) = -3\sin(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{9}{100}, A_2 = -\frac{3}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3t} \cos\left(\sqrt{11}\, t\right) + \frac{c_2 e^{-3t} \sin\left(\sqrt{11}\, t\right) \sqrt{11}}{11}\right) + \left(\frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-3t} \cos\left(\sqrt{11}\, t\right) + \frac{c_2 e^{-3t} \sin\left(\sqrt{11}\, t\right) \sqrt{11}}{11} + \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = c_1 + \frac{9}{100} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 e^{-3t} \cos\left(\sqrt{11}\, t\right) - c_1 e^{-3t} \sin\left(\sqrt{11}\, t\right) \sqrt{11} - \frac{3c_2 e^{-3t} \sin\left(\sqrt{11}\, t\right) \sqrt{11}}{11} + c_2 e^{-3t} \cos\left(\sqrt{11}\, t\right) - \frac{9\,\mathrm{s}}{}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{6}{25} - 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{9}{100} \\
c_2 &= -\frac{3}{100}
\end{aligned}
$$

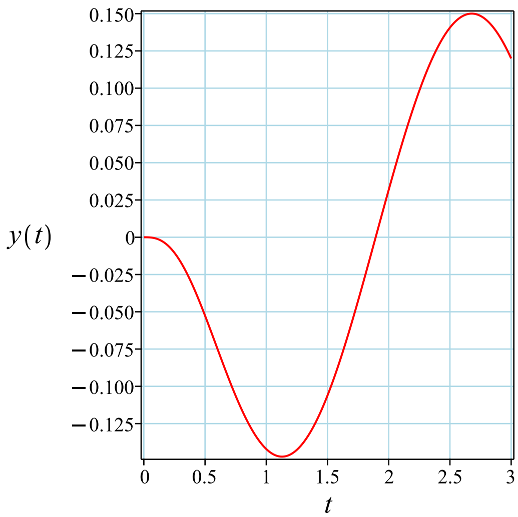
Substituting these values back in above solution results in

$$
y = -\frac{9\,e^{-3t}\cos\left(\sqrt{11}\,t\right)}{100} - \frac{3\,e^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{1100} + \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}
$$

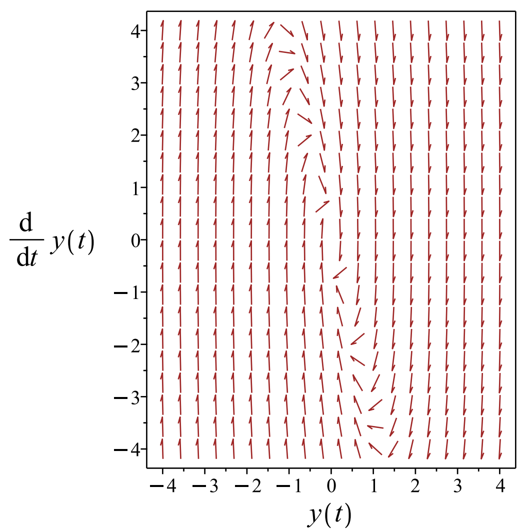
Summary

The solution(s) found are the following

$$
y = -\frac{9\,e^{-3t}\cos\left(\sqrt{11}\,t\right)}{100} - \frac{3\,e^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{1100} + \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = -\frac{9\,e^{-3t}\cos\left(\sqrt{11}\,t\right)}{100} - \frac{3\,e^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{1100} + \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}
$$

Verified OK.

17.13.4 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 20y = -3\sin(2t), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 20 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-44})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-3 - \mathrm{I}\sqrt{11}, -3 + \mathrm{I}\sqrt{11}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right) + \mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = -3\sin(2t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right) & \mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right) \\ -3\,\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right) - \mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11} & -3\,\mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right) + \mathrm{e}^{-3t}\sqrt{11} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \sqrt{11}\,\mathrm{e}^{-6t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = \frac{3\,\mathrm{e}^{-3t}\sqrt{11}\left(\cos\left(\sqrt{11}\,t\right)\left(\int \mathrm{e}^{3t}\sin(2t)\sin\left(\sqrt{11}\,t\right)dt\right)-\sin\left(\sqrt{11}\,t\right)\left(\int \mathrm{e}^{3t}\sin(2t)\cos\left(\sqrt{11}\,t\right)dt\right)\right)}{11}$$

- Compute integrals

$$y_p(t) = \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right) + \mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)c_2 + \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}$$

□ Check validity of solution $y = c_1\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right) + \mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)c_2 + \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}$

- Use initial condition $y(0) = 0$

$$0 = c_1 + \frac{9}{100}$$

- Compute derivative of the solution

$$y' = -3c_1\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right) - c_1\mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11} - 3\,\mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)c_2 + \mathrm{e}^{-3t}\sqrt{11}\,\cos\left(\sqrt{11}\,t\right)$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -3c_1 + \sqrt{11}\,c_2 - \frac{6}{25}$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{9}{100}, c_2 = -\frac{3\sqrt{11}}{1100}\right\}$$

- Substitute constant values into general solution and simplify

$$y = -\frac{9\,\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right)}{100} - \frac{3\,\mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{1100} + \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}$$

- Solution to the IVP

$$y = -\frac{9\,\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right)}{100} - \frac{3\,\mathrm{e}^{-3t}\sin\left(\sqrt{11}\,t\right)\sqrt{11}}{1100} + \frac{9\cos(2t)}{100} - \frac{3\sin(2t)}{25}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 44

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+20*y(t)=-3*sin(2*t),y(0) = 0, D(y)(0) = 0],y(t), sings
```

$$y(t) = -\frac{3\,\mathrm{e}^{-3t}\sqrt{11}\,\sin\left(\sqrt{11}\,t\right)}{1100} - \frac{9\,\mathrm{e}^{-3t}\cos\left(\sqrt{11}\,t\right)}{100} + \frac{9\cos\left(2t\right)}{100} - \frac{3\sin\left(2t\right)}{25}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 61

```
DSolve[{y''[t]+6*y'[t]+20*y[t]==-3*Sin[2*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to -\frac{3e^{-3t}\left(44e^{3t}\sin(2t) + \sqrt{11}\sin\left(\sqrt{11}t\right) - 33e^{3t}\cos(2t) + 33\cos\left(\sqrt{11}t\right)\right)}{1100}$$

17.14 problem 14

Internal problem ID [13214]
Internal file name [`OUTPUT/11869_Sunday_December_03_2023_07_22_11_PM_72682248/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = 2\cos(2t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

17.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= 2\cos(2t) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + y = 2\cos(2t)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2\cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

17.14.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 2, C = 1, f(t) = 2\cos(2t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-t} + c_2 t\, e^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\, e^{-t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2t) - 3A_2 \sin(2t) - 4A_1 \sin(2t) + 4A_2 \cos(2t) = 2\cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{6}{25}, A_2 = \frac{8}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t}\right) + \left(-\frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-t}(c_2 t + c_1) - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-t}(c_2 t + c_1) - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{6}{25} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\mathrm{e}^{-t}(c_2 t + c_1) + c_2 \mathrm{e}^{-t} + \frac{12\sin(2t)}{25} + \frac{16\cos(2t)}{25}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{16}{25} - c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{6}{25}$$
$$c_2 = -\frac{2}{5}$$

Substituting these values back in above solution results in

$$y = -\frac{2t\,\mathrm{e}^{-t}}{5} + \frac{6\,\mathrm{e}^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

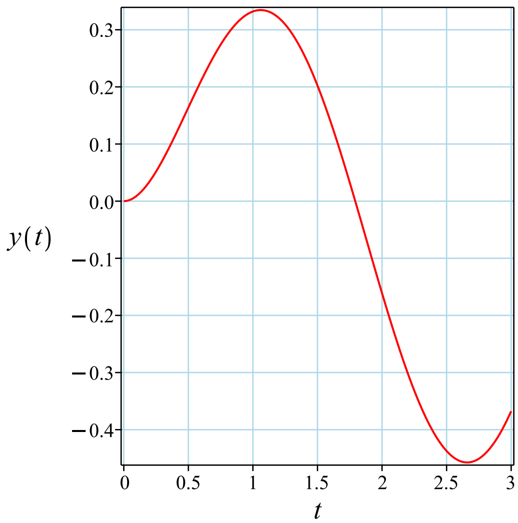
Which simplifies to

$$y = \frac{2(3-5t)\,\mathrm{e}^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

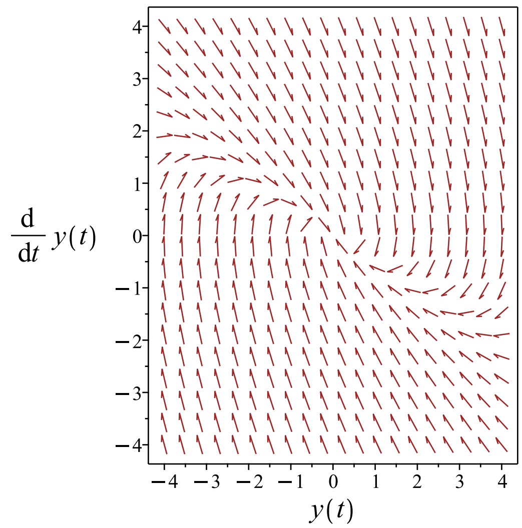
Summary

The solution(s) found are the following

$$y = \frac{2(3-5t)\,\mathrm{e}^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2(3-5t)\,e^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

Verified OK.

17.14.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\,y' + \frac{(p(t)^2 + p'(t))\,y}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 2\,e^{t}\cos(2t) \\ (e^{t}y)'' &= 2\,e^{t}\cos(2t) \end{aligned}$$

Integrating once gives

$$(e^t y)' = \frac{2\,e^t (2\sin(2t) + \cos(2t))}{5} + c_1$$

Integrating again gives

$$(e^t y) = -\frac{6\,e^t \cos(2t)}{25} + \frac{8\,e^t \sin(2t)}{25} + c_1 t + c_2$$

Hence the solution is

$$y = \frac{-\frac{6\,e^t \cos(2t)}{25} + \frac{8\,e^t \sin(2t)}{25} + c_1 t + c_2}{e^t}$$

Or

$$y = -\frac{12\cos(t)^2}{25} + \frac{16\cos(t)\sin(t)}{25} + t\,e^{-t} c_1 + c_2 e^{-t} + \frac{6}{25}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{12\cos(t)^2}{25} + \frac{16\cos(t)\sin(t)}{25} + t\,e^{-t} c_1 + c_2 e^{-t} + \frac{6}{25} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{6}{25} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{24\cos(t)\sin(t)}{25} - \frac{16\sin(t)^2}{25} + \frac{16\cos(t)^2}{25} + c_1 e^{-t} - t\,e^{-t} c_1 - c_2 e^{-t}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{16}{25} - c_2 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{2}{5}$$
$$c_2 = \frac{6}{25}$$

Substituting these values back in above solution results in

$$y = -\frac{2t\,e^{-t}}{5} + \frac{6\,e^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

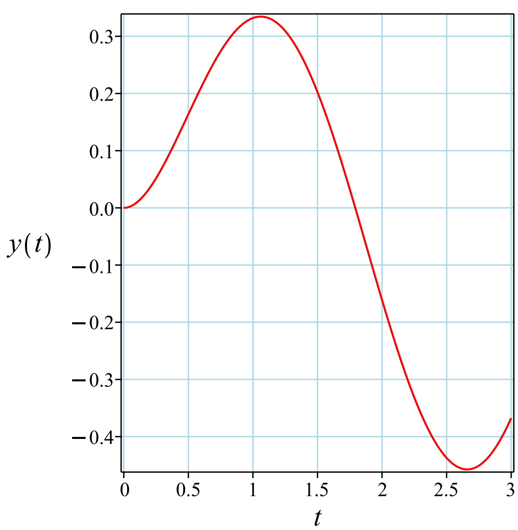
Which simplifies to

$$y = \frac{2(3-5t)\,e^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

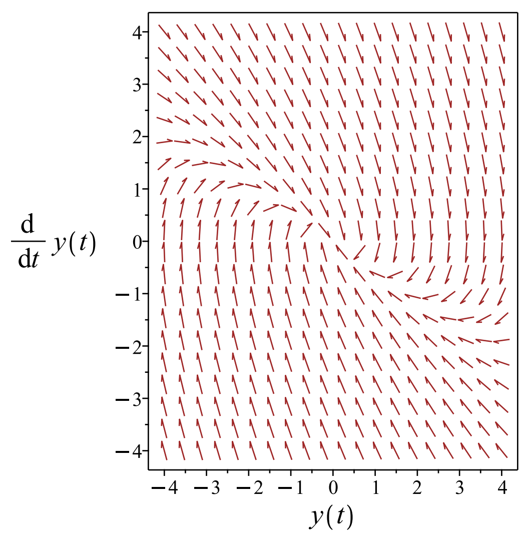
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{2(3-5t)\,e^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2(3-5t)\,e^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

Verified OK.

17.14.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 517: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-t} \\
&= z_1 \left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-2t}}{(y_1)^2}\, dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-t}\right) + c_2 \left(\mathrm{e}^{-t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\,e^{-t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2t) - 3A_2 \sin(2t) - 4A_1 \sin(2t) + 4A_2 \cos(2t) = 2\cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{6}{25}, A_2 = \frac{8}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 t\,e^{-t}\right) + \left(-\frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-t}(c_2 t + c_1) - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-t}(c_2 t + c_1) - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $t = 0$ in the above gives

$$0 = -\frac{6}{25} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\mathrm{e}^{-t}(c_2 t + c_1) + c_2 \mathrm{e}^{-t} + \frac{12\sin(2t)}{25} + \frac{16\cos(2t)}{25}$$

substituting $y' = 0$ and $t = 0$ in the above gives

$$0 = \frac{16}{25} - c_1 + c_2 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{6}{25}$$
$$c_2 = -\frac{2}{5}$$

Substituting these values back in above solution results in

$$y = -\frac{2t\,\mathrm{e}^{-t}}{5} + \frac{6\,\mathrm{e}^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

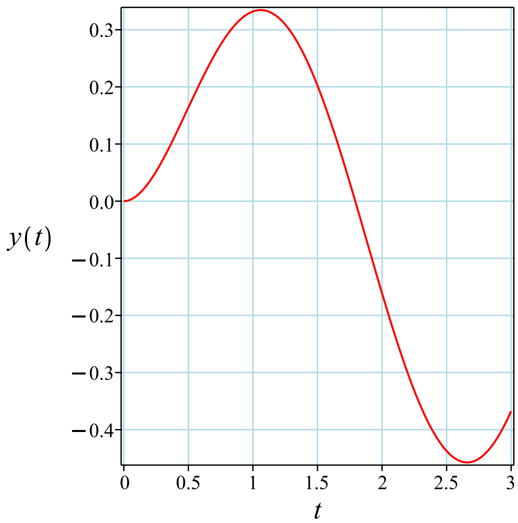
Which simplifies to

$$y = \frac{2(3-5t)\,\mathrm{e}^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

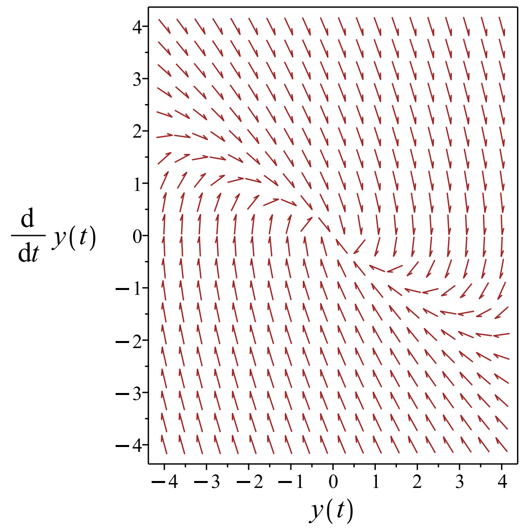
Summary

The solution(s) found are the following

$$y = \frac{2(3-5t)\,\mathrm{e}^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2(3-5t)\,e^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

Verified OK.

17.14.5 Maple step by step solution

Let's solve

$\left[y'' + 2y' + y = 2\cos(2t)\,, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the homogeneous ODE

$y_1(t) = e^{-t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-t} + c_2 t\,e^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 2\cos(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-t} & t\,e^{-t} \\ -e^{-t} & e^{-t} - t\,e^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = e^{-2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = 2\,e^{-t}\left(-\left(\int t\,e^{t}\cos(2t)\,dt\right) + t\left(\int e^{t}\cos(2t)\,dt\right)\right)$

  - Compute integrals

    $y_p(t) = -\frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-t} + c_2 t\,e^{-t} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$

☐ Check validity of solution $y = c_1 e^{-t} + c_2 t e^{-t} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$

  - Use initial condition $y(0) = 0$

    $0 = -\frac{6}{25} + c_1$

  - Compute derivative of the solution

    $y' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t\,e^{-t} + \frac{12\sin(2t)}{25} + \frac{16\cos(2t)}{25}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = \frac{16}{25} - c_1 + c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{6}{25}, c_2 = -\frac{2}{5}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{2(3-5t)e^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$

- Solution to the IVP

  $y = \frac{2(3-5t)e^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=2*cos(2*t),y(0) = 0, D(y)(0) = 0],y(t), singsol=a
```

$$y(t) = \frac{2(3-5t)\,e^{-t}}{25} - \frac{6\cos(2t)}{25} + \frac{8\sin(2t)}{25}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 37

```
DSolve[{y''[t]+2*y'[t]+y[t]==2*Cos[2*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to -\frac{2}{25}e^{-t}\left(5t - 4e^t\sin(2t) + 3e^t\cos(2t) - 3\right)$$

17.15 problem 15

Internal problem ID [13215]
Internal file name [`OUTPUT/11870_Sunday_December_03_2023_07_22_16_PM_16939157/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + y = \cos(3t)}$$

17.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 3, C = 1, f(t) = \cos(3t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 3y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 3, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 3\lambda\, \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(1)} \\
&= -\frac{3}{2} \pm \frac{\sqrt{5}}{2}
\end{aligned}$$

Hence

$$\lambda_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}$$

$$\lambda_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$$

Which simplifies to

$$\lambda_1 = \frac{\sqrt{5}}{2} - \frac{3}{2}$$

$$\lambda_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$y = c_1 e^{\left(\frac{\sqrt{5}}{2} - \frac{3}{2}\right)t} + c_2 e^{\left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t}$$

Or

$$y = c_1 \mathrm{e}^{\left(\frac{\sqrt{5}}{2} - \frac{3}{2}\right)t} + c_2 \mathrm{e}^{\left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} + c_2 \mathrm{e}^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}, \mathrm{e}^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \cos(3t) - 8A_2 \sin(3t) - 9A_1 \sin(3t) + 9A_2 \cos(3t) = \cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{8}{145}, A_2 = \frac{9}{145}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{8\cos(3t)}{145} + \frac{9\sin(3t)}{145}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} + c_2 \mathrm{e}^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}\right) + \left(-\frac{8\cos(3t)}{145} + \frac{9\sin(3t)}{145}\right) \end{aligned}$$

Which simplifies to

$$
y = c_1 \mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}} - \frac{8\cos\left(3t\right)}{145} + \frac{9\sin\left(3t\right)}{145}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}} - \frac{8\cos\left(3t\right)}{145} + \frac{9\sin\left(3t\right)}{145} \tag{1}
$$

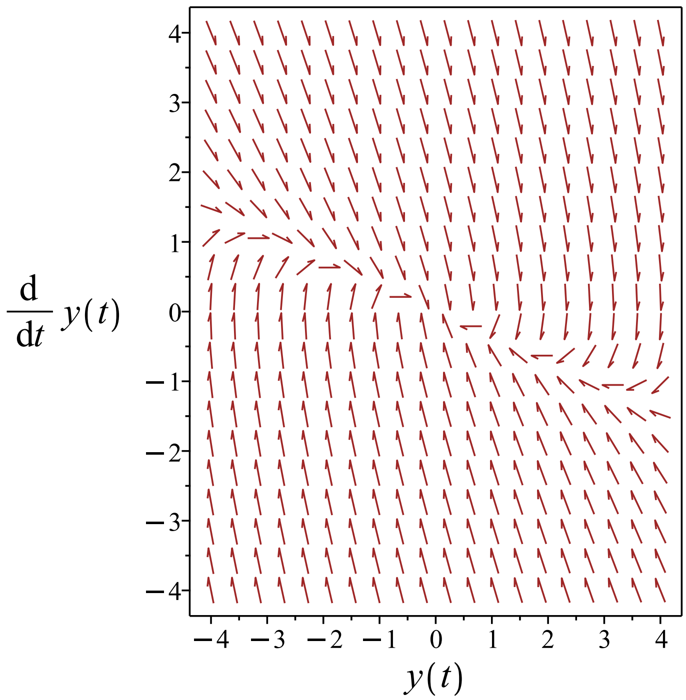


Figure 601: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}} + c_2 \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}} - \frac{8\cos\left(3t\right)}{145} + \frac{9\sin\left(3t\right)}{145}
$$

Verified OK.

17.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{5z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 519: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{5}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{t\sqrt{5}}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-3t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{\sqrt{5}\,\mathrm{e}^{t\sqrt{5}}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}} \left(\frac{\sqrt{5}\,\mathrm{e}^{t\sqrt{5}}}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 3y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}} + \frac{c_2 \mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}} \sqrt{5}}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{5}\,\mathrm{e}^{\frac{\left(\sqrt{5}-3\right)t}{2}}}{5}, \mathrm{e}^{-\frac{\left(3+\sqrt{5}\right)t}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \cos(3t) - 8A_2 \sin(3t) - 9A_1 \sin(3t) + 9A_2 \cos(3t) = \cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{8}{145}, A_2 = \frac{9}{145}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{8\cos(3t)}{145} + \frac{9\sin(3t)}{145}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}} + \frac{c_2 \mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}} \sqrt{5}}{5}\right) + \left(-\frac{8\cos(3t)}{145} + \frac{9\sin(3t)}{145}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}} + \frac{c_2 \mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}} \sqrt{5}}{5} - \frac{8\cos(3t)}{145} + \frac{9\sin(3t)}{145} \tag{1}
$$

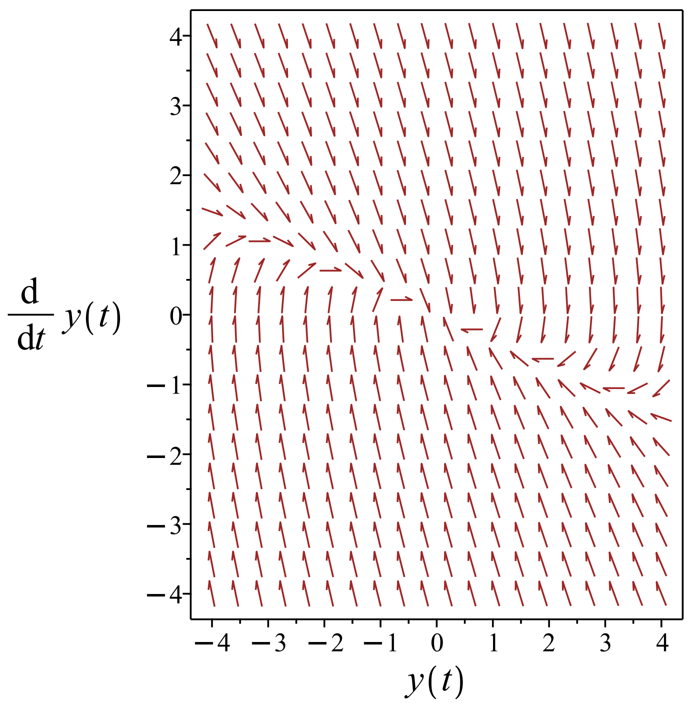


Figure 602: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}} + \frac{c_2 \mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}} \sqrt{5}}{5} - \frac{8\cos(3t)}{145} + \frac{9\sin(3t)}{145}
$$

Verified OK.

17.15.3 Maple step by step solution

Let's solve

$y'' + 3y' + y = \cos(3t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-3)\pm(\sqrt{5})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{2} - \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2} - \frac{3}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} + c_2 e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \cos(3t)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} & e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} \\ \left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} & \left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)e^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = \sqrt{5}\, e^{-3t}$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = \frac{\sqrt{5}\left(-\mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}\left(\int \cos(3t)\mathrm{e}^{\frac{(3+\sqrt{5})t}{2}}dt\right)+\mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}\left(\int \cos(3t)\mathrm{e}^{-\frac{(\sqrt{5}-3)t}{2}}dt\right)\right)}{5}$$

- Compute integrals

$$y_p(t) = -\frac{8\cos(3t)}{145} + \frac{9\sin(3t)}{145}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t} + c_2\mathrm{e}^{\left(\frac{\sqrt{5}}{2}-\frac{3}{2}\right)t} - \frac{8\cos(3t)}{145} + \frac{9\sin(3t)}{145}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(t),t$2)+3*diff(y(t),t)+y(t)=cos(3*t),y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{\frac{(\sqrt{5}-3)t}{2}}c_2 + \mathrm{e}^{-\frac{(3+\sqrt{5})t}{2}}c_1 - \frac{8\cos(3t)}{145} + \frac{9\sin(3t)}{145}$$

✓ Solution by Mathematica

Time used: 0.674 (sec). Leaf size: 52

```
DSolve[y''[t]+3*y'[t]+y[t]==Cos[3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{9}{145}\sin(3t) - \frac{8}{145}\cos(3t) + e^{-\frac{1}{2}(3+\sqrt{5})t}\left(c_2e^{\sqrt{5}t} + c_1\right)$$

17.16 problem 18

Internal problem ID [13216]
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Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 20y = 3 + 2\cos(2t)}$$

17.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 20, f(t) = 3 + 2\cos(2t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 20$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} + 20\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(20)} \\ &= -2 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 4i \\ \lambda_2 &= -2 - 4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 4i \\ \lambda_2 &= -2 - 4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3 + 2\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4t)\,e^{-2t}, \sin(4t)\,e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2\cos(2t) + A_3\sin(2t)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_2\cos(2t) + 16A_3\sin(2t) - 8A_2\sin(2t) + 8A_3\cos(2t) + 20A_1 = 3 + 2\cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{20}, A_2 = \frac{1}{10}, A_3 = \frac{1}{20}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3}{20} + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}(c_1\cos(4t) + c_2\sin(4t))\right) + \left(\frac{3}{20} + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{3}{20} + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20} \tag{1}$$

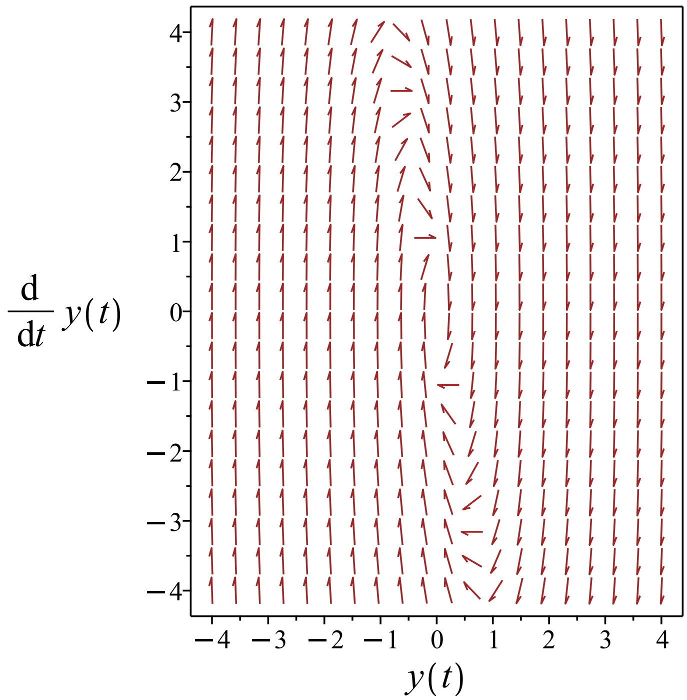

Figure 603: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{3}{20} + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20}$$

Verified OK.

17.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 20y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 4 \\
C &= 20
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = y e^{\int \frac{B}{2A} \, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -16 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -16 z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t) \, e^{-\int \frac{B}{2A} \, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 521: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -16$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(4t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4}{1} \, dt} \\
&= z_1 e^{-2t} \\
&= z_1 \left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(4t)\,\mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-4t}}{(y_1)^2}\,dt \\ &= y_1 \left(\frac{\tan(4t)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\cos(4t)\,\mathrm{e}^{-2t}\right) + c_2 \left(\cos(4t)\,\mathrm{e}^{-2t} \left(\frac{\tan(4t)}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \frac{\sin(4t)\,\mathrm{e}^{-2t} c_2}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3 + 2\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos(4t)\,e^{-2t}, \frac{\sin(4t)\,e^{-2t}}{4}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2\cos(2t) + A_3\sin(2t)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_2\cos(2t) + 16A_3\sin(2t) - 8A_2\sin(2t) + 8A_3\cos(2t) + 20A_1 = 3 + 2\cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{20}, A_2 = \frac{1}{10}, A_3 = \frac{1}{20}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3}{20} + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(4t)\,e^{-2t}c_1 + \frac{\sin(4t)\,e^{-2t}c_2}{4}\right) + \left(\frac{3}{20} + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(4t)\,e^{-2t}c_1 + \frac{\sin(4t)\,e^{-2t}c_2}{4} + \frac{3}{20} + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20} \tag{1}$$

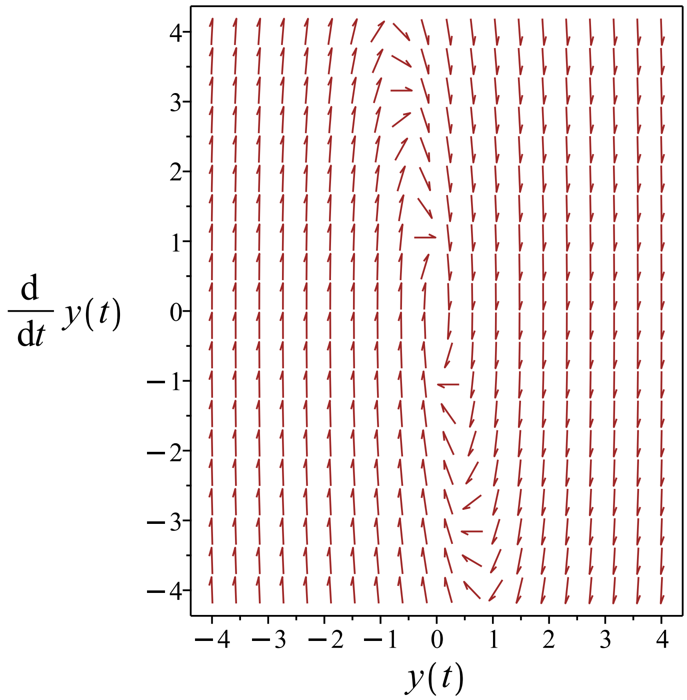

Figure 604: Slope field plot

Verification of solutions

$$y = \cos(4t)\, e^{-2t} c_1 + \frac{\sin(4t)\, e^{-2t} c_2}{4} + \frac{3}{20} + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20}$$

Verified OK.

17.16.3 Maple step by step solution

Let's solve

$y'' + 4y' + 20y = 3 + 2\cos(2t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 20 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4) \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

$r = (-2 - 4\,\mathrm{I}, -2 + 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(4t)\,\mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(4t)\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \sin(4t)\,\mathrm{e}^{-2t} c_2 + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 3 + 2\cos(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(4t)\,\mathrm{e}^{-2t} & \sin(4t)\,\mathrm{e}^{-2t} \\ -4\sin(4t)\,\mathrm{e}^{-2t} - 2\cos(4t)\,\mathrm{e}^{-2t} & 4\cos(4t)\,\mathrm{e}^{-2t} - 2\sin(4t)\,\mathrm{e}^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 4\,\mathrm{e}^{-4t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{\mathrm{e}^{-2t}\left(-\cos(4t)\left(\int \sin(4t)(3+2\cos(2t))\mathrm{e}^{2t}dt\right)+\sin(4t)\left(\int \cos(4t)(3+2\cos(2t))\mathrm{e}^{2t}dt\right)\right)}{4}$

  - Compute integrals

    $y_p(t) = \frac{3}{20} + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20}$

- Substitute particular solution into general solution to ODE

  $y = \sin(4t)\,\mathrm{e}^{-2t} c_2 + \cos(4t)\,\mathrm{e}^{-2t} c_1 + \frac{\cos(2t)}{10} + \frac{\sin(2t)}{20} + \frac{3}{20}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(t),t$2)+4*diff(y(t),t)+20*y(t)=3+2*cos(2*t),y(t), singsol=all)
```

$$y(t) = \sin(4t)\, \mathrm{e}^{-2t} c_2 + \cos(4t)\, \mathrm{e}^{-2t} c_1 + \frac{3}{20} + \frac{\sin(2t)}{20} + \frac{\cos(2t)}{10}$$

✓ Solution by Mathematica
Time used: 1.265 (sec). Leaf size: 47

```
DSolve[y''[t]+4*y'[t]+20*y[t]==3+2*Cos[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{20}\left(\sin(2t) + 2\cos(2t) + 20c_2 e^{-2t}\cos(4t) + 20c_1 e^{-2t}\sin(4t) + 3\right)$$

17.17 problem 19

Internal problem ID [13217]
Internal file name [`OUTPUT/11872_Sunday_December_03_2023_07_22_28_PM_67962004/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.2 page 412
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 20y = e^{-t} \cos(t)}$$

17.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 20, f(t) = e^{-t} \cos(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 20$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda\, e^{\lambda t} + 20\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(20)}\\ &= -2 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 4i\\ \lambda_2 &= -2 - 4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 4i\\ \lambda_2 &= -2 - 4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-t} \cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t} \cos(t), e^{-t} \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4t) e^{-2t}, \sin(4t) e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t} \cos(t) + A_2 e^{-t} \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{-t} \sin(t) + 2A_2 e^{-t} \cos(t) + 16A_1 e^{-t} \cos(t) + 16A_2 e^{-t} \sin(t) = e^{-t} \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{65}, A_2 = \frac{1}{130}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4 e^{-t} \cos(t)}{65} + \frac{e^{-t} \sin(t)}{130}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t))\right) + \left(\frac{4 e^{-t} \cos(t)}{65} + \frac{e^{-t} \sin(t)}{130}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{4 e^{-t} \cos(t)}{65} + \frac{e^{-t} \sin(t)}{130} \tag{1}$$

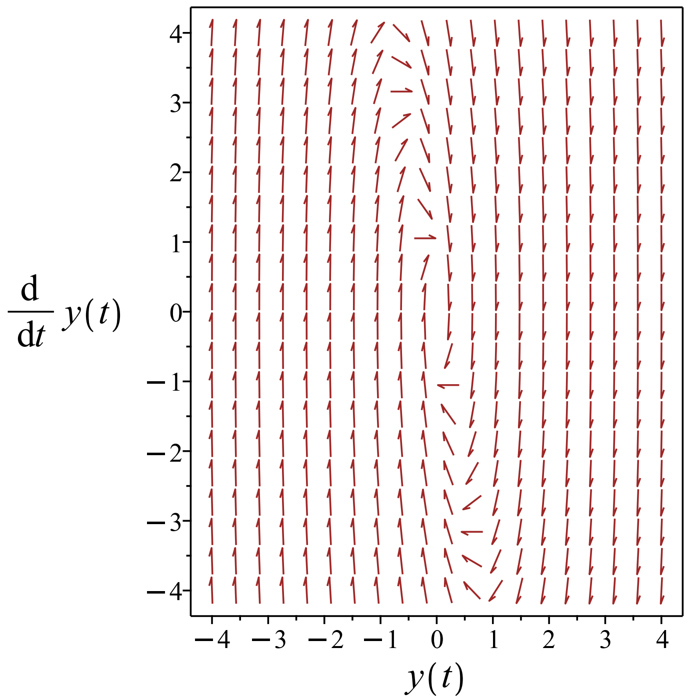

Figure 605: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_1 \cos(4t) + c_2 \sin(4t)) + \frac{4 e^{-t} \cos(t)}{65} + \frac{e^{-t} \sin(t)}{130}$$

Verified OK.

17.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 4 \\
C &= 20
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = ye^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -16 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -16z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 523: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(4t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1\left(\mathrm{e}^{-2t}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(4t)\,\mathrm{e}^{-2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-4t}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{\tan(4t)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(4t)\,\mathrm{e}^{-2t}\right) + c_2 \left(\cos(4t)\,\mathrm{e}^{-2t} \left(\frac{\tan(4t)}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 20y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \frac{\sin(4t)\,\mathrm{e}^{-2t} c_2}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-t} \cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-t}\cos(t), e^{-t}\sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos(4t)\,e^{-2t}, \frac{\sin(4t)\,e^{-2t}}{4}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-t}\cos(t) + A_2 e^{-t}\sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{-t}\sin(t) + 2A_2 e^{-t}\cos(t) + 16A_1 e^{-t}\cos(t) + 16A_2 e^{-t}\sin(t) = e^{-t}\cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{65}, A_2 = \frac{1}{130}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\,e^{-t}\cos(t)}{65} + \frac{e^{-t}\sin(t)}{130}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(4t)\,e^{-2t}c_1 + \frac{\sin(4t)\,e^{-2t}c_2}{4}\right) + \left(\frac{4\,e^{-t}\cos(t)}{65} + \frac{e^{-t}\sin(t)}{130}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(4t)\,e^{-2t}c_1 + \frac{\sin(4t)\,e^{-2t}c_2}{4} + \frac{4\,e^{-t}\cos(t)}{65} + \frac{e^{-t}\sin(t)}{130} \tag{1}$$

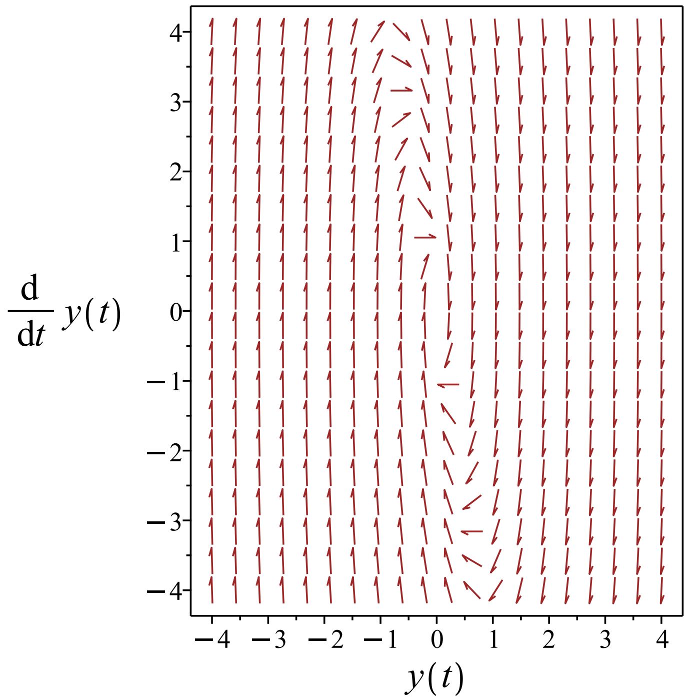

Figure 606: Slope field plot

Verification of solutions

$$y = \cos(4t)\, e^{-2t} c_1 + \frac{\sin(4t)\, e^{-2t} c_2}{4} + \frac{4\, e^{-t} \cos(t)}{65} + \frac{e^{-t} \sin(t)}{130}$$

Verified OK.

17.17.3 Maple step by step solution

Let's solve

$y'' + 4y' + 20y = e^{-t} \cos(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 20 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

$r = (-2 - 4\,\mathrm{I}, -2 + 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(4t)\,\mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(4t)\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \sin(4t)\,\mathrm{e}^{-2t} c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathrm{e}^{-t}\cos(t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(4t)\,\mathrm{e}^{-2t} & \sin(4t)\,\mathrm{e}^{-2t} \\ -4\sin(4t)\,\mathrm{e}^{-2t} - 2\cos(4t)\,\mathrm{e}^{-2t} & 4\cos(4t)\,\mathrm{e}^{-2t} - 2\sin(4t)\,\mathrm{e}^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 4\,\mathrm{e}^{-4t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\mathrm{e}^{-2t}\left(\cos(4t)\left(\int \sin(4t)\cos(t)\mathrm{e}^{t} dt\right) - \sin(4t)\left(\int \cos(4t)\cos(t)\mathrm{e}^{t} dt\right)\right)}{4}$

  - Compute integrals

    $y_p(t) = \frac{(8\cos(t) + \sin(t))\mathrm{e}^{-t}}{130}$

- Substitute particular solution into general solution to ODE

  $y = \cos(4t)\,\mathrm{e}^{-2t} c_1 + \sin(4t)\,\mathrm{e}^{-2t} c_2 + \frac{(8\cos(t) + \sin(t))\mathrm{e}^{-t}}{130}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(t),t$2)+4*diff(y(t),t)+20*y(t)=exp(-t)*cos(t),y(t), singsol=all)
```

$$y(t) = (c_1 \cos(4t) + c_2 \sin(4t))\, e^{-2t} + \frac{4\left(\cos(t) + \frac{\sin(t)}{8}\right) e^{-t}}{65}$$

✓ Solution by Mathematica

Time used: 0.457 (sec). Leaf size: 44

```
DSolve[y''[t]+4*y'[t]+20*y[t]==Exp[-t]*Cos[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{130} e^{-2t} \left(e^t \sin(t) + 8 e^t \cos(t) + 130 c_2 \cos(4t) + 130 c_1 \sin(4t)\right)$$

18 Chapter 4. Forcing and Resonance. Section 4.3 page 424

18.1 problem 1

Internal problem ID [13218]
Internal file name [`OUTPUT/11873_Sunday_December_03_2023_07_22_35_PM_20933045/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.3 page 424
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = \cos(t)}$$

18.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 9, f(t) = \cos(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 9\,e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(3t) + c_2 \sin(3t))$$

Or

$$y = c_1 \cos(3t) + c_2 \sin(3t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3t) + c_2 \sin(3t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3t), \sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(t) + 8A_2 \sin(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(t)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3t) + c_2 \sin(3t)) + \left(\frac{\cos(t)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\cos(t)}{8} \tag{1}$$

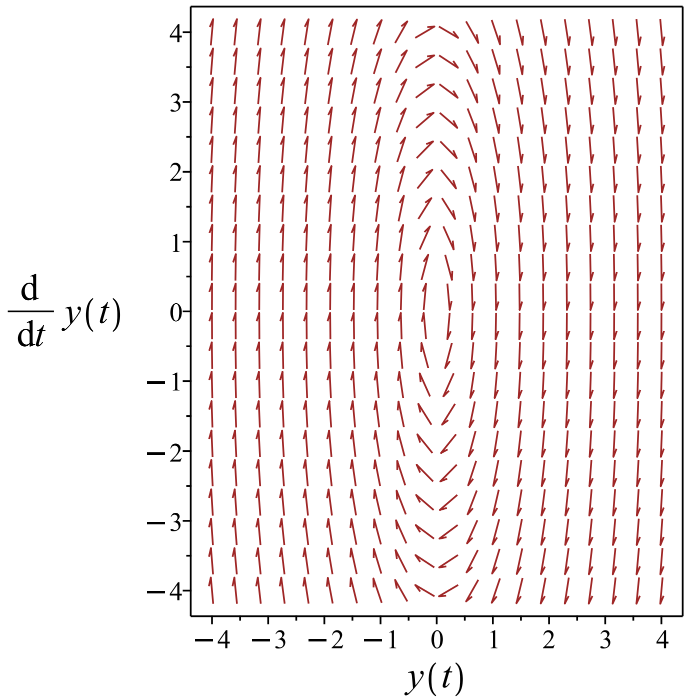

Figure 607: Slope field plot

Verification of solutions

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\cos(t)}{8}$$

Verified OK.

18.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 9
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = y e^{\int \frac{B}{2A} \, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -9 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -9z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t) \, e^{-\int \frac{B}{2A} \, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 525: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\ &= \cos(3t) \int \frac{1}{\cos(3t)^2}\, dt \\ &= \cos(3t) \left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3t)) + c_2\left(\cos(3t)\left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3t)}{3}, \cos(3t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(t) + 8A_2 \sin(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(t)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}\right) + \left(\frac{\cos(t)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} + \frac{\cos(t)}{8} \tag{1}$$

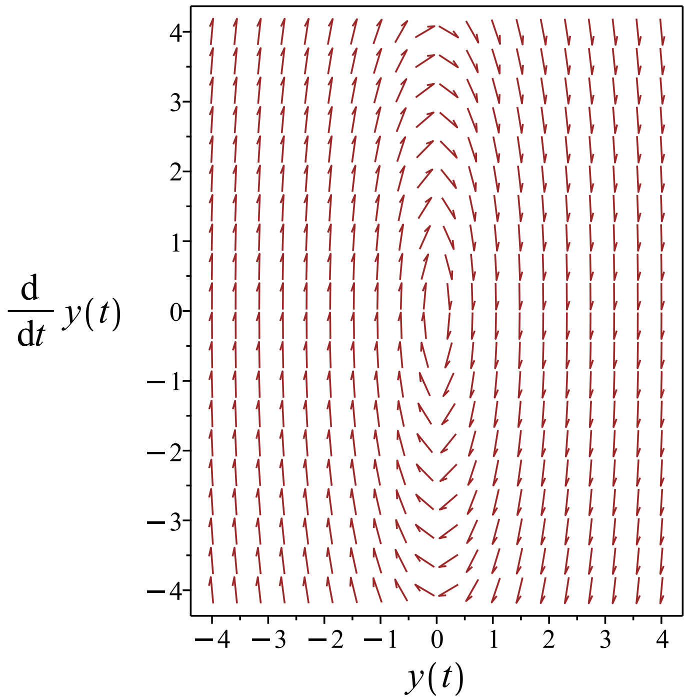

Figure 608: Slope field plot

Verification of solutions

$$y = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} + \frac{\cos(t)}{8}$$

Verified OK.

18.1.3 Maple step by step solution

Let's solve

$y'' + 9y = \cos(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t),y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t),y_2(t))} dt\right), f(t) = \cos(t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 3$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\cos(3t)\left(\int \sin(3t)\cos(t) dt\right)}{3} + \frac{\sin(3t)\left(\int \cos(3t)\cos(t) dt\right)}{3}$

  - Compute integrals

    $y_p(t) = \frac{\cos(t)}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\cos(t)}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(t),t$2)+9*y(t)=cos(t),y(t), singsol=all)
```

$$y(t) = c_2 \sin(3t) + c_1 \cos(3t) + \frac{\cos(t)}{8}$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 30

```
DSolve[y''[t]+9*y[t]==Cos[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\cos(t)}{8} + \left(\frac{1}{12} + c_1\right)\cos(3t) + c_2 \sin(3t)$$

18.2 problem 2

Internal problem ID [13219]
Internal file name [`OUTPUT/11874_Sunday_December_03_2023_07_22_38_PM_28158158/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.3 page 424
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 9y = 5\sin(2t)}$$

18.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 9, f(t) = 5\sin(2t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 9\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos{(3t)} + c_2 \sin{(3t)})$$

Or

$$y = c_1 \cos{(3t)} + c_2 \sin{(3t)}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3t) + c_2 \sin(3t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5 \sin(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3t), \sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(2t) + 5A_2 \sin(2t) = 5 \sin(2t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(2t)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3t) + c_2 \sin(3t)) + (\sin(2t)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \sin(2t) \tag{1}$$

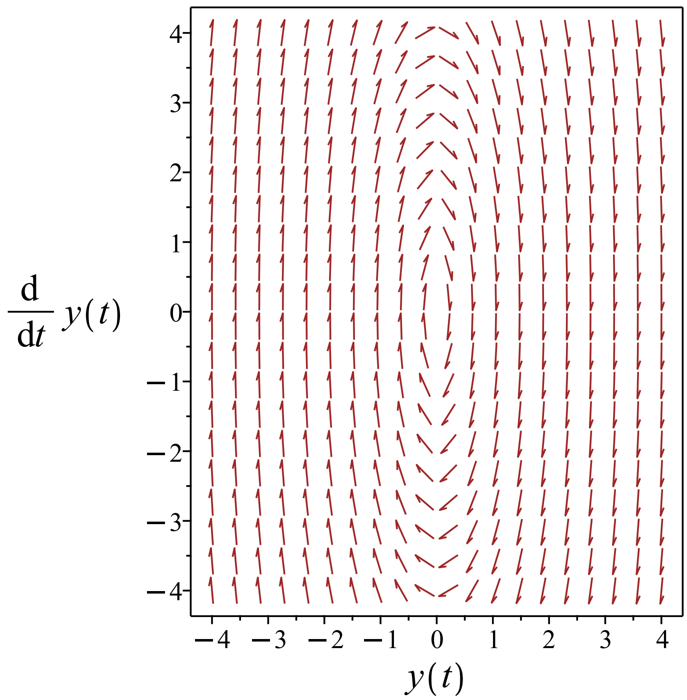

Figure 609: Slope field plot

Verification of solutions

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \sin(2t)$$

Verified OK.

18.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 527: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(3t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(3t)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\ &= \cos(3t) \int \frac{1}{\cos(3t)^2}\, dt \\ &= \cos(3t) \left(\frac{\tan(3t)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3t)) + c_2\left(\cos(3t) \left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5 \sin(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3t)}{3}, \cos(3t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(2t) + 5A_2 \sin(2t) = 5 \sin(2t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(2t)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}\right) + (\sin(2t)) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} + \sin(2t) \tag{1}$$

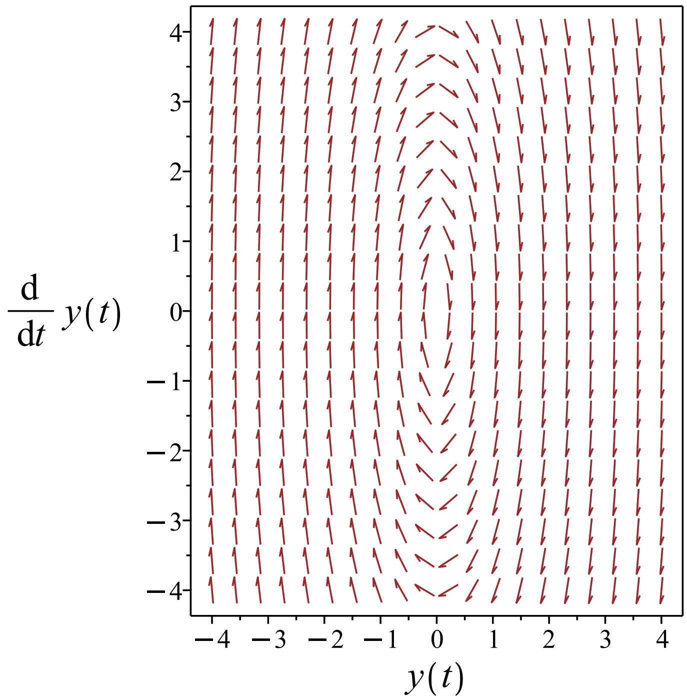

Figure 610: Slope field plot

Verification of solutions

$$y = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} + \sin(2t)$$

Verified OK.

18.2.3 Maple step by step solution

Let's solve

$y'' + 9y = 5\sin(2t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 5\sin(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 3$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{5\cos(3t)\left(\int(\cos(t) - \cos(5t))dt\right)}{6} + \frac{5\sin(3t)\left(\int(\sin(5t) - \sin(t))dt\right)}{6}$

  - Compute integrals

    $y_p(t) = \sin(2t)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + \sin(2t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(t),t$2)+9*y(t)=5*sin(2*t),y(t), singsol=all)
```

$$y(t) = c_2 \sin(3t) + c_1 \cos(3t) + \sin(2t)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 24

```
DSolve[y''[t]+9*y[t]==5*Sin[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \sin(2t) + c_1 \cos(3t) + c_2 \sin(3t)$$

18.3 problem 3

Internal problem ID [13220]
Internal file name [`OUTPUT/11875_Sunday_December_03_2023_07_22_40_PM_34995297/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.3 page 424
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y = -\cos\left(\frac{t}{2}\right)}$$

18.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = -\cos\left(\frac{t}{2}\right)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\,e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\cos\left(\frac{t}{2}\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\cos\left(\frac{t}{2}\right), \sin\left(\frac{t}{2}\right)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos\left(\frac{t}{2}\right) + A_2 \sin\left(\frac{t}{2}\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\frac{15A_1 \cos\left(\frac{t}{2}\right)}{4} + \frac{15A_2 \sin\left(\frac{t}{2}\right)}{4} = -\cos\left(\frac{t}{2}\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{15}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4\cos\left(\frac{t}{2}\right)}{15}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2t) + c_2 \sin(2t)) + \left(-\frac{4 \cos\left(\frac{t}{2}\right)}{15}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{4 \cos\left(\frac{t}{2}\right)}{15} \tag{1}
$$

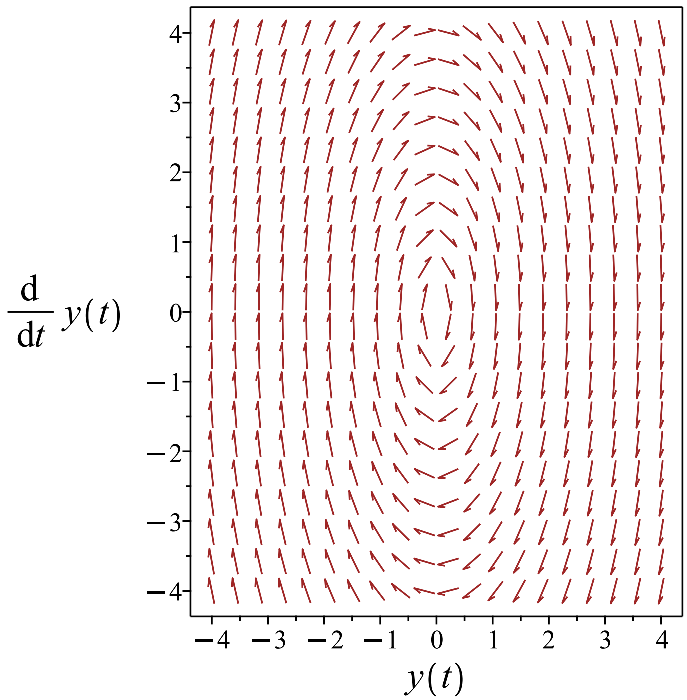

Figure 611: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{4 \cos\left(\frac{t}{2}\right)}{15}
$$

Verified OK.

18.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 529: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dt}}{y_1^2} dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dt \\ &= \cos(2t) \int \frac{1}{\cos(2t)^2} dt \\ &= \cos(2t) \left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2t)) + c_2\left(\cos(2t) \left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\cos\left(\frac{t}{2}\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\cos\left(\frac{t}{2}\right), \sin\left(\frac{t}{2}\right)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2}, \cos(2t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos\left(\frac{t}{2}\right) + A_2 \sin\left(\frac{t}{2}\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\frac{15A_1 \cos\left(\frac{t}{2}\right)}{4} + \frac{15A_2 \sin\left(\frac{t}{2}\right)}{4} = -\cos\left(\frac{t}{2}\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{15}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4\cos\left(\frac{t}{2}\right)}{15}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + \left(-\frac{4\cos\left(\frac{t}{2}\right)}{15}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} - \frac{4\cos\left(\frac{t}{2}\right)}{15} \tag{1}
$$

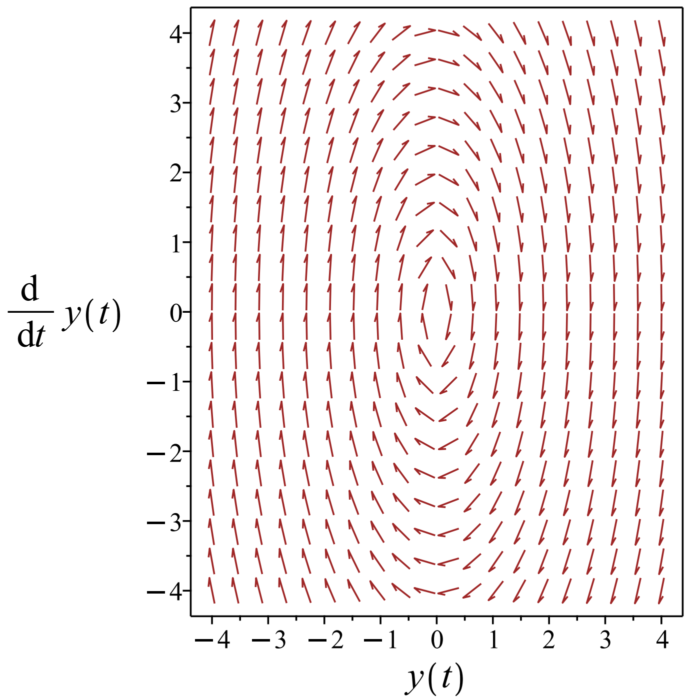

Figure 612: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} - \frac{4\cos\left(\frac{t}{2}\right)}{15}
$$

Verified OK.

18.3.3 Maple step by step solution

Let's solve

$y'' + 4y = -\cos\left(\frac{t}{2}\right)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = -\cos\left(\frac{t}{2}\right)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2$

  - Substitute functions into equation for $y_p(t)$

$$y_p(t) = \frac{\cos(2t)\left(\int \sin(2t)\cos\left(\frac{t}{2}\right)dt\right)}{2} - \frac{\sin(2t)\left(\int \cos(2t)\cos\left(\frac{t}{2}\right)dt\right)}{2}$$

- Compute integrals

$$y_p(t) = -\frac{4\cos\left(\frac{t}{2}\right)}{15}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{4\cos\left(\frac{t}{2}\right)}{15}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(t),t$2)+4*y(t)=-cos(t/2),y(t), singsol=all)
```

$$y(t) = \sin(2t) c_2 + \cos(2t) c_1 - \frac{4\cos\left(\frac{t}{2}\right)}{15}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 30

```
DSolve[y''[t]+4*y[t]==-Cos[t/2],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{4}{15}\cos\left(\frac{t}{2}\right) + c_1\cos(2t) + c_2\sin(2t)$$

18.4 problem 4

Internal problem ID [13221]
Internal file name [`OUTPUT/11876_Sunday_December_03_2023_07_22_44_PM_30650530/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.3 page 424
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 3\cos(2t)}$$

18.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = 3\cos(2t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since $\cos(2t)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{\cos(2t)\,t, \sin(2t)\,t\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(2t)\,t + A_2 \sin(2t)\,t$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2t) + 4A_2 \cos(2t) = 3\cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\sin(2t)\,t}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2t) + c_2 \sin(2t)) + \left(\frac{3 \sin(2t)\, t}{4}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{3 \sin(2t)\, t}{4} \tag{1}
$$

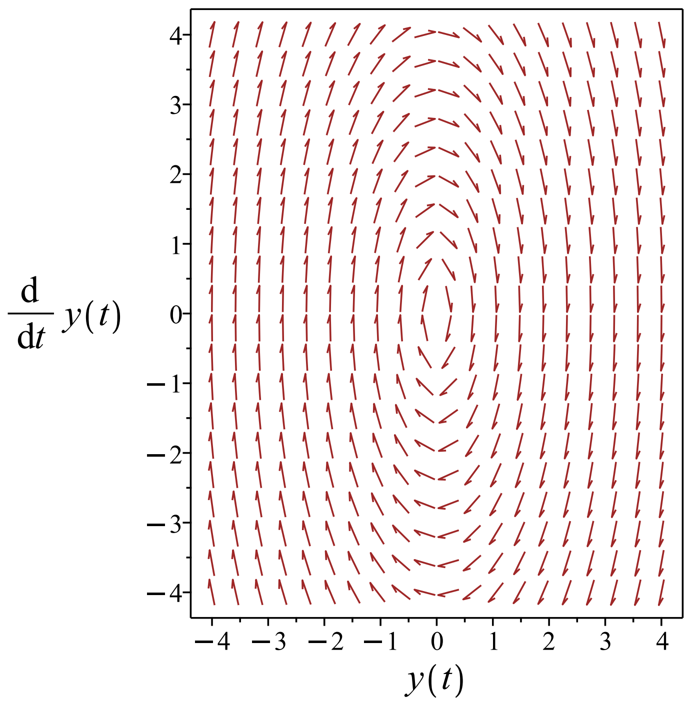

Figure 613: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{3 \sin(2t)\, t}{4}
$$

Verified OK.

18.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 531: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dt}}{y_1^2} dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dt \\ &= \cos(2t) \int \frac{1}{\cos(2t)^2} dt \\ &= \cos(2t) \left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3 \cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2}, \cos(2t)\right\}$$

Since $\cos(2t)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{\cos(2t)\, t, \sin(2t)\, t\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(2t)\, t + A_2 \sin(2t)\, t$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2t) + 4A_2 \cos(2t) = 3 \cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3 \sin(2t)\, t}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + \left(\frac{3 \sin(2t)\, t}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} + \frac{3 \sin(2t)\, t}{4} \tag{1}
$$

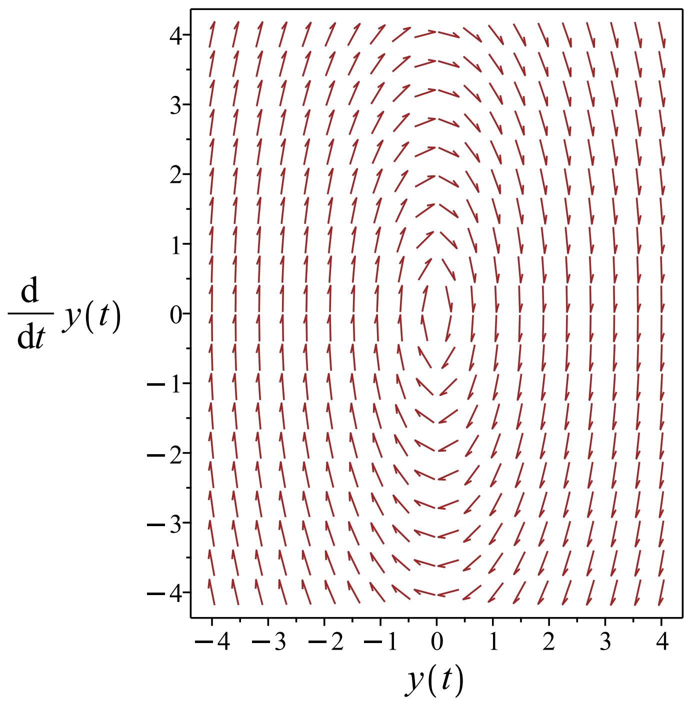

Figure 614: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} + \frac{3 \sin(2t)\, t}{4}
$$

Verified OK.

18.4.3 Maple step by step solution

Let's solve

$y'' + 4y = 3\cos(2t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 3\cos(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2$

  - Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{3\cos(2t)\left(\int \sin(4t)dt\right)}{4} + \frac{3\sin(2t)\left(\int \cos(2t)^2 dt\right)}{2}$$

- Compute integrals

$$y_p(t) = \frac{3\cos(2t)}{16} + \frac{3\sin(2t)t}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{3\cos(2t)}{16} + \frac{3\sin(2t)t}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(t),t$2)+4*y(t)=3*cos(2*t),y(t), singsol=all)
```

$$y(t) = \frac{(6t + 8c_2)\sin(2t)}{8} + \frac{(8c_1 + 3)\cos(2t)}{8}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 33

```
DSolve[y''[t]+4*y[t]==3*Cos[2*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \left(\frac{3}{16} + c_1\right)\cos(2t) + \frac{1}{4}(3t + 4c_2)\sin(2t)$$

18.5 problem 5

Internal problem ID [13222]
Internal file name [`OUTPUT/11877_Sunday_December_03_2023_07_22_47_PM_72174500/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 4. Forcing and Resonance. Section 4.3 page 424
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 2\cos(3t)}$$

18.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 9, f(t) = 2\cos(3t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 9\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^0(c_1 \cos(3t) + c_2 \sin(3t))$$

Or

$$y = c_1 \cos(3t) + c_2 \sin(3t)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3t) + c_2 \sin(3t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3t), \sin(3t)\}$$

Since $\cos(3t)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\cos(3t), t\sin(3t)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t \cos(3t) + A_2 t \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \sin(3t) + 6A_2 \cos(3t) = 2\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{t \sin(3t)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3t) + c_2 \sin(3t)) + \left(\frac{t \sin(3t)}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{t \sin(3t)}{3} \tag{1}$$

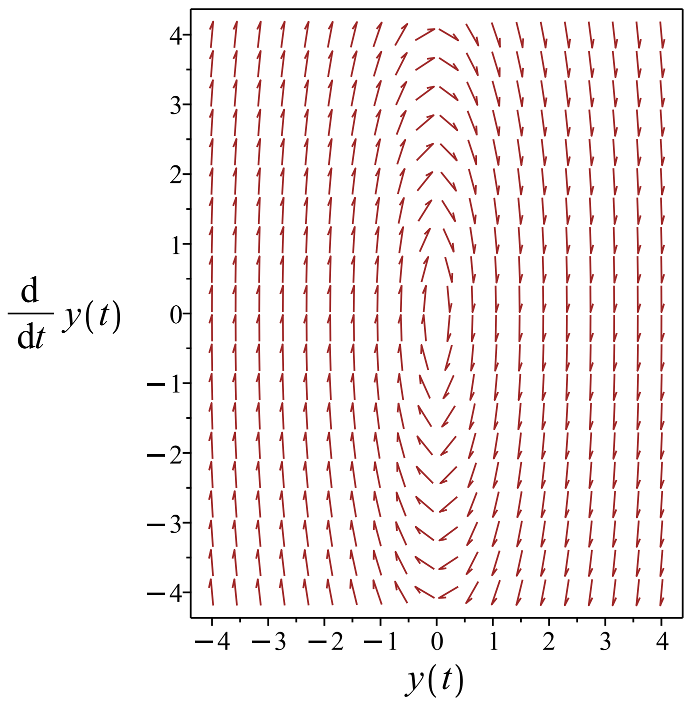

Figure 615: Slope field plot

Verification of solutions

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{t \sin(3t)}{3}$$

Verified OK.

18.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 533: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(3t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3t) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dt}}{y_1^2} dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dt \\ &= \cos(3t) \int \frac{1}{\cos(3t)^2} dt \\ &= \cos(3t) \left(\frac{\tan(3t)}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3t)) + c_2\left(\cos(3t)\left(\frac{\tan(3t)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3t)}{3}, \cos(3t)\right\}$$

Since $\cos(3t)$ is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\cos(3t), t\sin(3t)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t\cos(3t) + A_2 t\sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \sin(3t) + 6A_2 \cos(3t) = 2\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{t\sin(3t)}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}\right) + \left(\frac{t \sin(3t)}{3}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} + \frac{t \sin(3t)}{3} \tag{1}$$

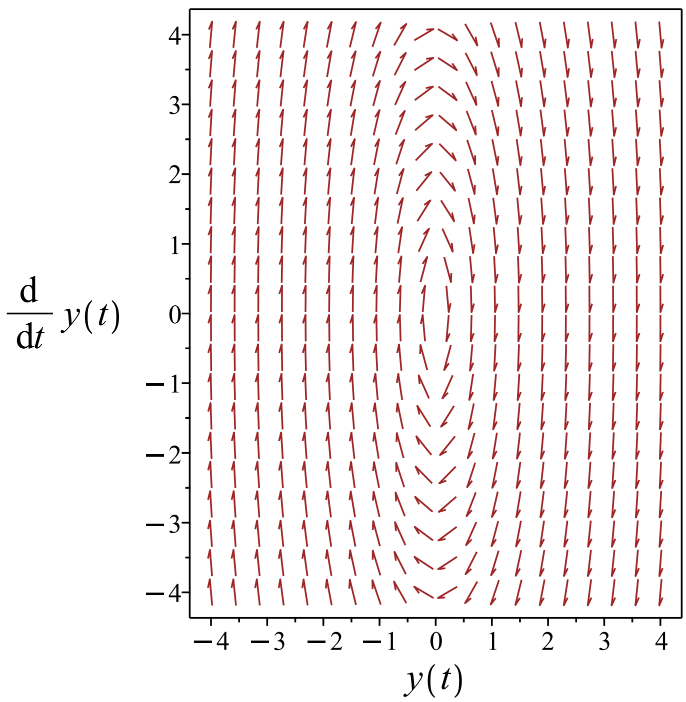

Figure 616: Slope field plot

Verification of solutions

$$y = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} + \frac{t \sin(3t)}{3}$$

Verified OK.

18.5.3 Maple step by step solution

Let's solve

$y'' + 9y = 2\cos(3t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 2\cos(3t)\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$

- ∘ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{\cos(3t)\left(\int \sin(6t)dt\right)}{3} + \frac{2\sin(3t)\left(\int \cos(3t)^2 dt\right)}{3}$$

- Compute integrals

$$y_p(t) = \frac{\cos(3t)}{18} + \frac{t\sin(3t)}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(3t) + c_2 \sin(3t) + \frac{\cos(3t)}{18} + \frac{t\sin(3t)}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(t),t$2)+9*y(t)=2*cos(3*t),y(t), singsol=all)
```

$$y(t) = \frac{(9c_1 + 1)\cos(3t)}{9} + \frac{(t + 3c_2)\sin(3t)}{3}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 31

```
DSolve[y''[t]+9*y[t]==2*Cos[3*t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \left(\frac{1}{18} + c_1\right)\cos(3t) + \frac{1}{3}(t + 3c_2)\sin(3t)$$

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19.1 problem 27

Internal problem ID [13223]
Internal file name [OUTPUT/11878_Sunday_December_03_2023_07_22_50_PM_75951210/index.tex]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.3 page 600
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y = 8}$$

With initial conditions

$$[y(0) = 11, y'(0) = 5]$$

19.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= 8 \end{aligned}$$

Hence the ode is

$$y'' + 4y = 8$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{8}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 11\\ y'(0) &= 5\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 5 - 11s + 4Y(s) = \frac{8}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{11s^2 + 5s + 8}{s\,(s^2 + 4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{9}{2}-\frac{5i}{4}}{s-2i}+\frac{\frac{9}{2}+\frac{5i}{4}}{s+2i}+\frac{2}{s}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{9}{2}-\frac{5i}{4}}{s-2i}\right) = \left(\frac{9}{2}-\frac{5i}{4}\right)\mathrm{e}^{2it}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{9}{2}+\frac{5i}{4}}{s+2i}\right) = \left(\frac{9}{2}+\frac{5i}{4}\right)\mathrm{e}^{-2it}$$
$$\mathcal{L}^{-1}\left(\frac{2}{s}\right) = 2$$

Adding the above results and simplifying gives

$$y = 9\cos(2t)+\frac{5\sin(2t)}{2}+2$$

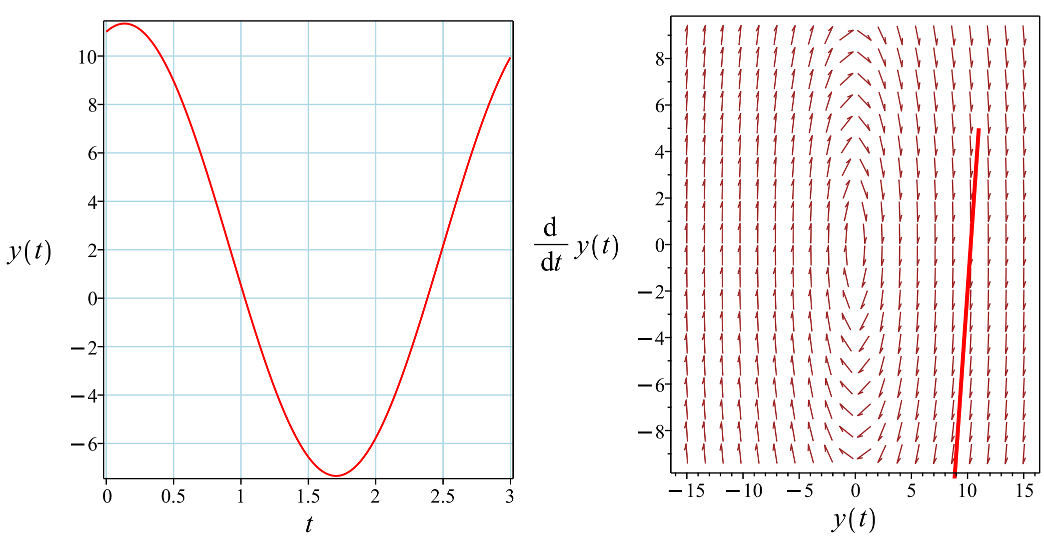
Simplifying the solution gives

$$y = 9\cos(2t)+\frac{5\sin(2t)}{2}+2$$

Summary

The solution(s) found are the following

$$y = 9\cos(2t)+\frac{5\sin(2t)}{2}+2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 9\cos(2t) + \frac{5\sin(2t)}{2} + 2$$

Verified OK.

19.1.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = 8, y(0) = 11, y'\Big|_{\{t=0\}} = 5\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1\cos(2t) + c_2\sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 8\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -4\cos(2t)\left(\int \sin(2t)\, dt\right) + 4\sin(2t)\left(\int \cos(2t)\, dt\right)$

- Compute integrals

  $y_p(t) = 2$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + 2$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + 2$

- Use initial condition $y(0) = 11$

  $11 = c_1 + 2$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 5$

  $5 = 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 9, c_2 = \frac{5}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = 9\cos(2t) + \frac{5\sin(2t)}{2} + 2$

- Solution to the IVP

  $y = 9\cos(2t) + \frac{5\sin(2t)}{2} + 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.797 (sec). Leaf size: 18

```
dsolve([diff(y(t),t$2)+4*y(t)=8,y(0) = 11, D(y)(0) = 5],y(t), singsol=all)
```

$$y(t) = 9\cos(2t) + \frac{5\sin(2t)}{2} + 2$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 19

```
DSolve[{y''[t]+4*y[t]==8,{y[0]==11,y'[0]==5}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 9\cos(2t) + 5\sin(t)\cos(t) + 2$$

19.2 problem 28

Internal problem ID [13224]
Internal file name [`OUTPUT/11879_Tuesday_December_05_2023_12_12_38_PM_64086885/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.3 page 600
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y = \mathrm{e}^{2t}}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

19.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -4 \\ F &= \mathrm{e}^{2t} \end{aligned}$$

Hence the ode is

$$y'' - 4y = \mathrm{e}^{2t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4Y(s) = \frac{1}{s-2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= -1
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - s - 4Y(s) = \frac{1}{s-2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 - 3s + 3}{(s-2)(s^2-4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{4(s-2)^2} + \frac{13}{16(s+2)} + \frac{3}{16(s-2)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{4(s-2)^2}\right) = \frac{e^{2t}t}{4}$$
$$\mathcal{L}^{-1}\left(\frac{13}{16(s+2)}\right) = \frac{13\,e^{-2t}}{16}$$
$$\mathcal{L}^{-1}\left(\frac{3}{16(s-2)}\right) = \frac{3\,e^{2t}}{16}$$

Adding the above results and simplifying gives

$$y = \frac{13\,e^{-2t}}{16} + \frac{e^{2t}(3+4t)}{16}$$

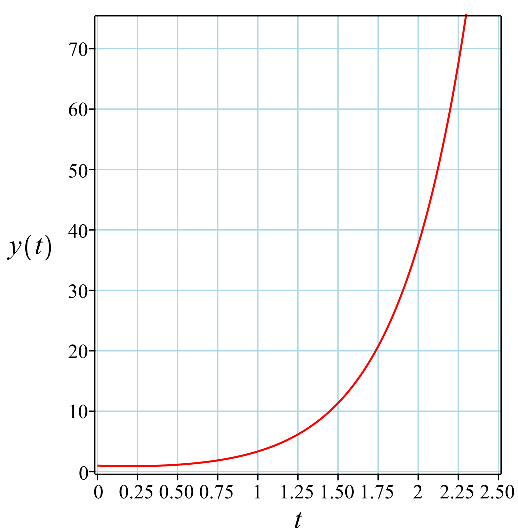
Simplifying the solution gives

$$y = \frac{13\,e^{-2t}}{16} + \frac{e^{2t}(3+4t)}{16}$$

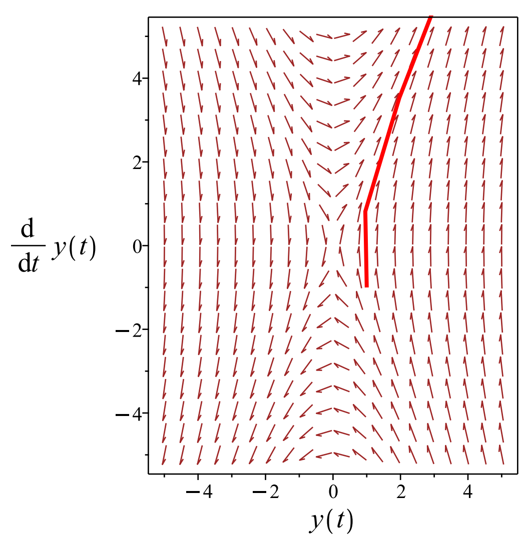
Summary

The solution(s) found are the following

$$y = \frac{13\,e^{-2t}}{16} + \frac{e^{2t}(3+4t)}{16} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{13\,\mathrm{e}^{-2t}}{16} + \frac{\mathrm{e}^{2t}(3+4t)}{16}$$

Verified OK.

19.2.2 Maple step by step solution

Let's solve

$\left[y'' - 4y = \mathrm{e}^{2t}, y(0) = 1, y'\Big|_{\{t=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathrm{e}^{2t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{2t} \\ -2\,e^{-2t} & 2\,e^{2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 4$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^{-2t}(\int e^{4t}dt)}{4} + \frac{e^{2t}(\int 1dt)}{4}$

- Compute integrals

  $y_p(t) = \frac{e^{2t}(-1+4t)}{16}$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-2t} + c_2e^{2t} + \frac{e^{2t}(-1+4t)}{16}$

☐ Check validity of solution $y = c_1e^{-2t} + c_2e^{2t} + \frac{e^{2t}(-1+4t)}{16}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 - \frac{1}{16}$

- Compute derivative of the solution

  $y' = -2c_1e^{-2t} + 2c_2e^{2t} + \frac{e^{2t}(-1+4t)}{8} + \frac{e^{2t}}{4}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = -2c_1 + 2c_2 + \frac{1}{8}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{13}{16}, c_2 = \frac{1}{4}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{13\,e^{-2t}}{16} + \frac{e^{2t}(3+4t)}{16}$

- Solution to the IVP

  $y = \frac{13\,e^{-2t}}{16} + \frac{e^{2t}(3+4t)}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.0 (sec). Leaf size: 22

```
dsolve([diff(y(t),t$2)-4*y(t)=exp(2*t),y(0) = 1, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = \frac{13\,e^{-2t}}{16} + \frac{e^{2t}(4t+3)}{16}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 27

```
DSolve[{y''[t]-4*y[t]==Exp[2*t],{y[0]==1,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to \frac{1}{16} e^{-2t} \left(e^{4t}(4t+3)+13\right)$$

19.3 problem 29

Internal problem ID [13225]
Internal file name [`OUTPUT/11880_Tuesday_December_05_2023_12_12_41_PM_3851012/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.3 page 600
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 5y = 2\,e^{t}}$$

With initial conditions

$$[y(0) = 3, y'(0) = 1]$$

19.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 5 \\ F &= 2\,e^{t} \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 5y = 2\,e^{t}$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2\,\mathrm{e}^t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) + 5Y(s) = \frac{2}{s-1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 3 \\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 11 - 3s - 4sY(s) + 5Y(s) = \frac{2}{s-1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3s^2 - 14s + 13}{(s-1)(s^2 - 4s + 5)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-1} + \frac{1+2i}{s-2-i} + \frac{1-2i}{s-2+i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = \mathrm{e}^{t}$$
$$\mathcal{L}^{-1}\left(\frac{1+2i}{s-2-i}\right) = (1+2i)\,\mathrm{e}^{(2+i)t}$$
$$\mathcal{L}^{-1}\left(\frac{1-2i}{s-2+i}\right) = (1-2i)\,\mathrm{e}^{(2-i)t}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{t} + 2\,\mathrm{e}^{2t}(\cos(t) - 2\sin(t))$$

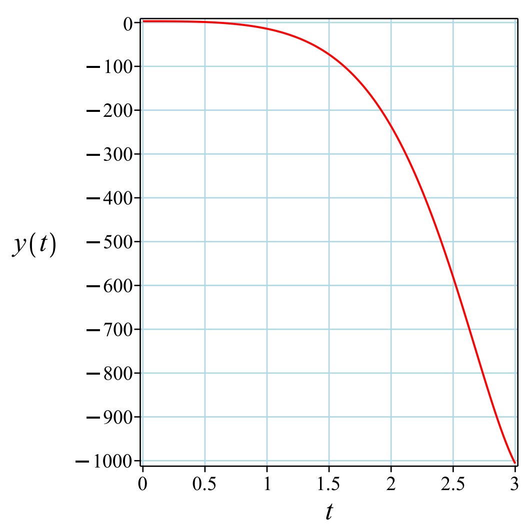
Simplifying the solution gives

$$y = \mathrm{e}^{t} + (2\cos(t) - 4\sin(t))\,\mathrm{e}^{2t}$$

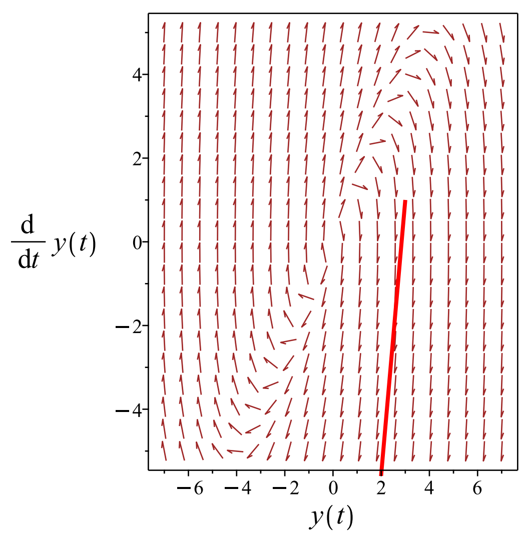
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{t} + (2\cos(t) - 4\sin(t))\,\mathrm{e}^{2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{t} + (2\cos(t) - 4\sin(t))\,\mathrm{e}^{2t}$$

Verified OK.

19.3.2 Maple step by step solution

Let's solve

$\left[y'' - 4y' + 5y = 2\,\mathrm{e}^{t}, y(0) = 3, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - \mathrm{I}, 2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{2t} \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{2t} \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{2t} \cos(t) + c_2 \mathrm{e}^{2t} \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 2\,\mathrm{e}^{t}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{2t} \cos(t) & \mathrm{e}^{2t} \sin(t) \\ 2\,\mathrm{e}^{2t} \cos(t) - \mathrm{e}^{2t} \sin(t) & 2\,\mathrm{e}^{2t} \sin(t) + \mathrm{e}^{2t} \cos(t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{4t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -2\,\mathrm{e}^{2t}\left(\cos(t)\left(\int \mathrm{e}^{-t}\sin(t)\,dt\right) - \sin(t)\left(\int \mathrm{e}^{-t}\cos(t)\,dt\right)\right)$$

- ◦ Compute integrals

  $$y_p(t) = \mathrm{e}^{t}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\mathrm{e}^{2t}\cos(t) + c_2\mathrm{e}^{2t}\sin(t) + \mathrm{e}^{t}$$

- □ Check validity of solution $y = c_1\mathrm{e}^{2t}\cos(t) + c_2\mathrm{e}^{2t}\sin(t) + \mathrm{e}^{t}$
  - ◦ Use initial condition $y(0) = 3$

    $$3 = 1 + c_1$$
  - ◦ Compute derivative of the solution

    $$y' = 2c_1\mathrm{e}^{2t}\cos(t) - c_1\mathrm{e}^{2t}\sin(t) + 2c_2\mathrm{e}^{2t}\sin(t) + c_2\mathrm{e}^{2t}\cos(t) + \mathrm{e}^{t}$$
  - ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $$1 = 2c_1 + 1 + c_2$$
  - ◦ Solve for c_1 and c_2

    $$\{c_1 = 2, c_2 = -4\}$$
  - ◦ Substitute constant values into general solution and simplify

    $$y = \mathrm{e}^{t} + (2\cos(t) - 4\sin(t))\,\mathrm{e}^{2t}$$

- Solution to the IVP

  $$y = \mathrm{e}^{t} + (2\cos(t) - 4\sin(t))\,\mathrm{e}^{2t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.5 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)+5*y(t)=2*exp(t),y(0) = 3, D(y)(0) = 1],y(t), singsol=a
```

$$y(t) = e^{t} + (2\cos(t) - 4\sin(t))\, e^{2t}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 25

```
DSolve[{y''[t]-4*y'[t]+5*y[t]==2*Exp[t],{y[0]==3,y'[0]==1}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to e^t \left(-4e^t \sin(t) + 2e^t \cos(t) + 1\right)$$

19.4 problem 30

Internal problem ID [13226]
Internal file name [`OUTPUT/11881_Tuesday_December_05_2023_12_12_41_PM_10656506/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.3 page 600
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 13y = 13\,\text{Heaviside}\,(t-4)}$$

With initial conditions

$$[y(0) = 3, y'(0) = 1]$$

19.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 13 \\ F &= 13\,\text{Heaviside}\,(t-4) \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 13y = 13\,\text{Heaviside}\,(t-4)$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 13\,\mathrm{Heaviside}\,(t - 4)$ is

$$\{t < 4 \vee 4 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 6sY(s) - 6y(0) + 13Y(s) = \frac{13\,\mathrm{e}^{-4s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 3\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 19 - 3s + 6sY(s) + 13Y(s) = \frac{13\,\mathrm{e}^{-4s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3s^2 + 13\,\mathrm{e}^{-4s} + 19s}{s\,(s^2 + 6s + 13)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{3s^2 + 13\,\mathrm{e}^{-4s} + 19s}{s\,(s^2 + 6s + 13)}\right) \\
&= \mathrm{e}^{-3t}(3\cos(2t) + 5\sin(2t)) + \left(\frac{1}{26} + \frac{3i}{52}\right)\left(8 - 12i - 13\,\mathrm{e}^{(-3-2i)(t-4)} + (5 + 12i)\,\mathrm{e}^{(-3+2i)(t-4)}\right) \text{Heavisid}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= \mathrm{e}^{-3t}(3\cos(2t) + 5\sin(2t)) \\
&\quad + \left(\frac{1}{26} + \frac{3i}{52}\right)\left(8 - 12i - 13\,\mathrm{e}^{(-3-2i)(t-4)} + (5 + 12i)\,\mathrm{e}^{(-3+2i)(t-4)}\right) \text{Heaviside}(t-4)
\end{aligned}
$$

Simplifying the solution gives

$$
\begin{aligned}
y &= \left(-\frac{1}{2} - \frac{3i}{4}\right) \text{Heaviside}(t-4)\,\mathrm{e}^{(-3-2i)(t-4)} + \left(-\frac{1}{2} + \frac{3i}{4}\right) \text{Heaviside}(t-4)\,\mathrm{e}^{(-3+2i)(t-4)} \\
&\quad + \text{Heaviside}(t-4) + \mathrm{e}^{-3t}(3\cos(2t) + 5\sin(2t))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \left(-\frac{1}{2} - \frac{3i}{4}\right) \text{Heaviside}(t-4)\,\mathrm{e}^{(-3-2i)(t-4)} \\
&\quad + \left(-\frac{1}{2} + \frac{3i}{4}\right) \text{Heaviside}(t-4)\,\mathrm{e}^{(-3+2i)(t-4)} \\
&\quad + \text{Heaviside}(t-4) + \mathrm{e}^{-3t}(3\cos(2t) + 5\sin(2t))
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \left(-\frac{1}{2} - \frac{3i}{4}\right) \text{Heaviside}(t-4)\,\mathrm{e}^{(-3-2i)(t-4)} + \left(-\frac{1}{2} + \frac{3i}{4}\right) \text{Heaviside}(t-4)\,\mathrm{e}^{(-3+2i)(t-4)} \\
&\quad + \text{Heaviside}(t-4) + \mathrm{e}^{-3t}(3\cos(2t) + 5\sin(2t))
\end{aligned}
$$

Verified OK.

19.4.2 Maple step by step solution

Let's solve

$$\left[y''+6y'+13y=13 Heaviside(t-4), y(0)=3, y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+6r+13=0$

- Use quadratic formula to solve for r

  $r=\frac{(-6)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(-3-2\,\mathrm{I}, -3+2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-3t}\cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-3t}\sin(2t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-3t}\cos(2t)+c_2\mathrm{e}^{-3t}\sin(2t)+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=13 Heaviside(t-4)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t))=\begin{bmatrix}\mathrm{e}^{-3t}\cos(2t) & \mathrm{e}^{-3t}\sin(2t)\\ -3\,\mathrm{e}^{-3t}\cos(2t)-2\,\mathrm{e}^{-3t}\sin(2t) & -3\,\mathrm{e}^{-3t}\sin(2t)+2\,\mathrm{e}^{-3t}\cos(2t)\end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t))=2\,\mathrm{e}^{-6t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{13\,\mathrm{e}^{-3t}\left(\cos(2t)\left(\int \sin(2t)\, Heaviside(t-4)\mathrm{e}^{3t}dt\right)-\sin(2t)\left(\int \cos(2t)\, Heaviside(t-4)\mathrm{e}^{3t}dt\right)\right)}{2}$$

- Compute integrals

$$y_p(t) = -\left(-1+\left(\left(\cos(8)-\frac{3\sin(8)}{2}\right)\cos(2t)+\frac{3\sin(2t)\left(\cos(8)+\frac{2\sin(8)}{3}\right)}{2}\right)\mathrm{e}^{-3t+12}\right) Heaviside(t-4)$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-3t}\cos(2t)+c_2\mathrm{e}^{-3t}\sin(2t)-\left(-1+\left(\left(\cos(8)-\frac{3\sin(8)}{2}\right)\cos(2t)+\frac{3\sin(2t)\left(\cos(8)+\frac{2\sin(8)}{3}\right)}{2}\right)\right.$$

□ Check validity of solution $y = c_1\mathrm{e}^{-3t}\cos(2t)+c_2\mathrm{e}^{-3t}\sin(2t)-\left(-1+\left(\left(\cos(8)-\frac{3\sin(8)}{2}\right)\cos(\right.\right.$

- Use initial condition $y(0)=3$

$$3 = c_1$$

- Compute derivative of the solution

$$y' = -3c_1\mathrm{e}^{-3t}\cos(2t)-2c_1\mathrm{e}^{-3t}\sin(2t)-3c_2\mathrm{e}^{-3t}\sin(2t)+2c_2\mathrm{e}^{-3t}\cos(2t)-\left(\left(-2\left(\cos(8)-\frac{3\,\mathrm{s}}{}\right.\right.\right.$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

$$1 = -3c_1+2c_2$$

- Solve for c_1 and c_2

$$\{c_1 = 3, c_2 = 5\}$$

- Substitute constant values into general solution and simplify

$$y = -\left(\left(\cos(8)-\frac{3\sin(8)}{2}\right)\cos(2t)+\frac{3\sin(2t)\left(\cos(8)+\frac{2\sin(8)}{3}\right)}{2}\right) Heaviside(t-4)\,\mathrm{e}^{-3t+12}+3\,\mathrm{e}^{-3t}\cos(2$$

- Solution to the IVP

$$y = -\left(\left(\cos(8)-\frac{3\sin(8)}{2}\right)\cos(2t)+\frac{3\sin(2t)\left(\cos(8)+\frac{2\sin(8)}{3}\right)}{2}\right) Heaviside(t-4)\,\mathrm{e}^{-3t+12}+3\,\mathrm{e}^{-3t}\cos(2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 7.156 (sec). Leaf size: 57

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+13*y(t)=13*Heaviside(t-4),y(0) = 3, D(y)(0) = 1],y(t),
```

$$y(t) = \left(-\frac{1}{2} - \frac{3i}{4}\right) \text{Heaviside}\,(t-4)\, \mathrm{e}^{(-3-2i)(t-4)} + \left(-\frac{1}{2} + \frac{3i}{4}\right) \text{Heaviside}\,(t-4)\, \mathrm{e}^{(-3+2i)(t-4)} + \text{Heaviside}\,(t-4) + \mathrm{e}^{-3t}(3\cos(2t) + 5\sin(2t))$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 82

```
DSolve[{y''[t]-4*y'[t]+5*y[t]==UnitStep[t-4],{y[0]==3,y'[0]==1}},y[t],t,IncludeSingularSolut
```

$$y(t) \to \begin{cases} e^{2t}(3\cos(t) - 5\sin(t)) & t \leq 4 \\ -\frac{1}{5}e^{2t-8}\cos(4-t) + 3e^{2t}\cos(t) - \frac{2}{5}e^{2t-8}\sin(4-t) - 5e^{2t}\sin(t) + \frac{1}{5} & \text{True} \end{cases}$$

19.5 problem 31

Internal problem ID [13227]
Internal file name [`OUTPUT/11882_Tuesday_December_05_2023_12_12_42_PM_12105076/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.3 page 600
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \cos(2t)}$$

With initial conditions

$$[y(0) = -2, y'(0) = 0]$$

19.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= \cos(2t) \end{aligned}$$

Hence the ode is

$$y'' + 4y = \cos(2t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{s}{s^2+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -2\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2s + 4Y(s) = \frac{s}{s^2+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s(2s^2+7)}{(s^2+4)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{8(s-2i)^2} + \frac{i}{8(s+2i)^2} - \frac{1}{s-2i} - \frac{1}{s+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{i}{8\left(s-2i\right)^{2}}\right) = -\frac{it\,\mathrm{e}^{2it}}{8}$$

$$\mathcal{L}^{-1}\left(\frac{i}{8\left(s+2i\right)^{2}}\right) = \frac{it\,\mathrm{e}^{-2it}}{8}$$

$$\mathcal{L}^{-1}\left(-\frac{1}{s-2i}\right) = -\mathrm{e}^{2it}$$

$$\mathcal{L}^{-1}\left(-\frac{1}{s+2i}\right) = -\mathrm{e}^{-2it}$$

Adding the above results and simplifying gives

$$y = -2\cos\left(2t\right)+\frac{\sin\left(2t\right)t}{4}$$

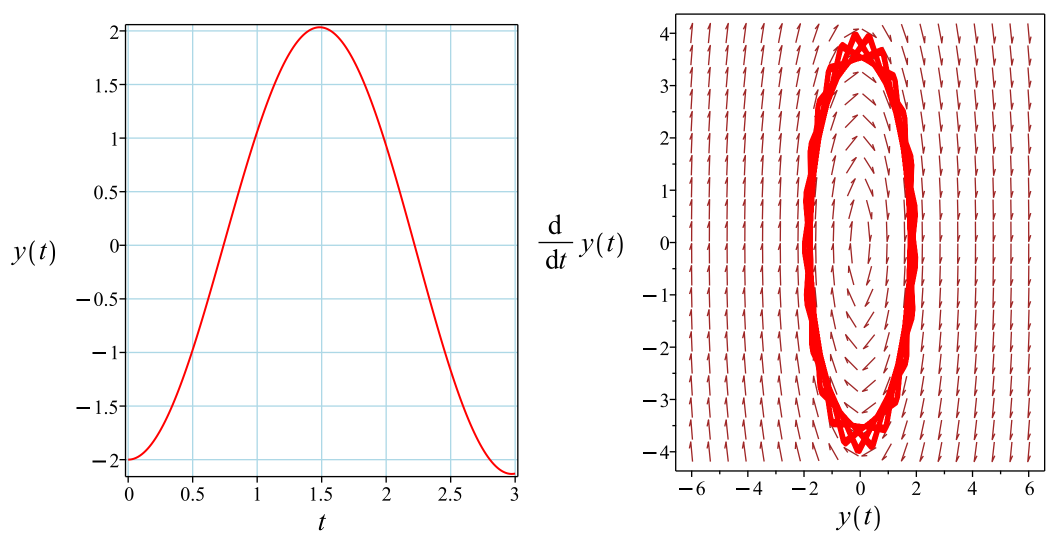
Simplifying the solution gives

$$y = -2\cos\left(2t\right)+\frac{\sin\left(2t\right)t}{4}$$

Summary

The solution(s) found are the following

$$y = -2\cos\left(2t\right)+\frac{\sin\left(2t\right)t}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2\cos(2t) + \frac{\sin(2t)\, t}{4}$$

Verified OK.

19.5.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = \cos(2t), y(0) = -2, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \cos(2t)\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(2t)\left(\int \sin(4t)dt\right)}{4} + \frac{\sin(2t)\left(\int \cos(2t)^2 dt\right)}{2}$

- Compute integrals

  $y_p(t) = \frac{\cos(2t)}{16} + \frac{\sin(2t)t}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\cos(2t)}{16} + \frac{\sin(2t)t}{4}$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\cos(2t)}{16} + \frac{\sin(2t)t}{4}$

- Use initial condition $y(0) = -2$

  $-2 = c_1 + \frac{1}{16}$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{\sin(2t)}{8} + \frac{\cos(2t)t}{2}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{33}{16}, c_2 = 0\right\}$

- Substitute constant values into general solution and simplify

  $y = -2\cos(2t) + \frac{\sin(2t)t}{4}$

- Solution to the IVP

  $y = -2\cos(2t) + \frac{\sin(2t)t}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.438 (sec). Leaf size: 18

```
dsolve([diff(y(t),t$2)+4*y(t)=cos(2*t),y(0) = -2, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = -2\cos(2t) + \frac{t\sin(2t)}{4}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 21

```
DSolve[{y''[t]+4*y[t]==Cos[2*t],{y[0]==-2,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to \frac{1}{4}t\sin(2t) - 2\cos(2t)$$

19.6 problem 32

Internal problem ID [13228]
Internal file name [`OUTPUT/11883_Tuesday_December_05_2023_12_12_43_PM_58150107/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.3 page 600
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y = \text{Heaviside}\left(t-4\right)\cos\left(5t-20\right)}$$

With initial conditions

$$[y(0) = 0, y'(0) = -2]$$

19.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 3 \\ F &= \text{Heaviside}\left(t-4\right)\cos\left(5t-20\right) \end{aligned}$$

Hence the ode is

$$y'' + 3y = \text{Heaviside}\left(t-4\right)\cos\left(5t-20\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}\,(t-4)\cos(5t-20)$ is

$$\{t < 4 \lor 4 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3Y(s) = \frac{e^{-4s}s}{s^2+25} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= -2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 + 3Y(s) = \frac{e^{-4s}s}{s^2+25}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-4s}s - 2s^2 - 50}{(s^2+25)(s^2+3)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-4s}s-2s^{2}-50}{(s^{2}+25)\,(s^{2}+3)}\right) \\
&= -\frac{2\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3}+\frac{\text{Heaviside}\left(t-4\right)\left(\cos\left(\sqrt{3}\,(t-4)\right)-\cos\left(5t-20\right)\right)}{22}
\end{aligned}
$$

Hence the final solution is

$$
y=-\frac{2\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3}+\frac{\text{Heaviside}\left(t-4\right)\left(\cos\left(\sqrt{3}\,(t-4)\right)-\cos\left(5t-20\right)\right)}{22}
$$

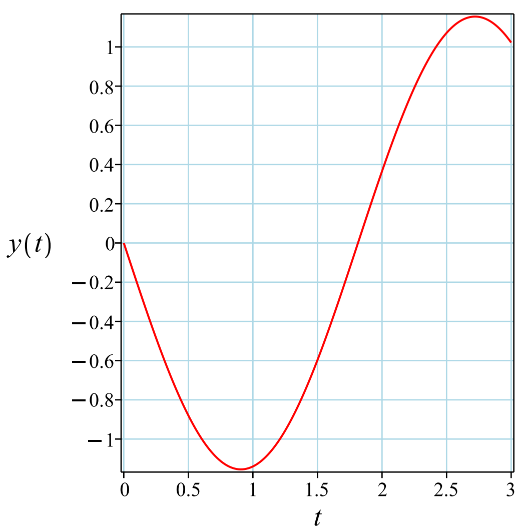
Simplifying the solution gives

$$
y=-\frac{\text{Heaviside}\left(t-4\right)\cos\left(5t-20\right)}{22}+\frac{\text{Heaviside}\left(t-4\right)\cos\left(\sqrt{3}\,(t-4)\right)}{22}-\frac{2\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3}
$$

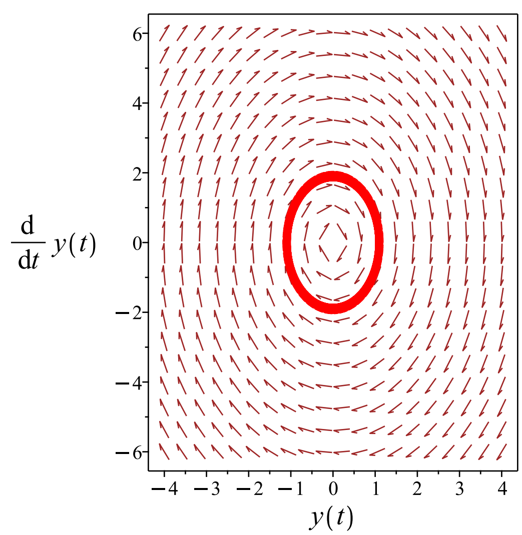
Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &-\frac{\text{Heaviside}\left(t-4\right)\cos\left(5t-20\right)}{22} \\
&+\frac{\text{Heaviside}\left(t-4\right)\cos\left(\sqrt{3}\,(t-4)\right)}{22}-\frac{2\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3}
\end{aligned}
\tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\text{Heaviside}(t-4)\cos(5t-20)}{22} + \frac{\text{Heaviside}(t-4)\cos\left(\sqrt{3}\,(t-4)\right)}{22} - \frac{2\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3}$$

Verified OK.

19.6.2 Maple step by step solution

Let's solve

$$\left[y'' + 3y = \mathit{Heaviside}(t-4)\cos(5t-20)\,, y(0) = 0, y'\Big|_{\{t=0\}} = -2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-12})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{3}, \mathrm{I}\sqrt{3}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos\left(\sqrt{3}\,t\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin\left(\sqrt{3}\,t\right)$

- General solution of the ODE

  $y = c_1y_1(t) + c_2y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\sqrt{3}\,t\right)c_1 + \sin\left(\sqrt{3}\,t\right)c_2 + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathit{Heaviside}(t-4)\cos(5t-2$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos\left(\sqrt{3}\,t\right) & \sin\left(\sqrt{3}\,t\right) \\ -\sin\left(\sqrt{3}\,t\right)\sqrt{3} & \cos\left(\sqrt{3}\,t\right)\sqrt{3} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(t), y_2(t)) = \sqrt{3}$$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\frac{\sqrt{3}\left(\cos\left(\sqrt{3}\,t\right)\left(\int \sin\left(\sqrt{3}\,t\right) \mathit{Heaviside}(t-4)\cos(5t-20)dt\right) - \sin\left(\sqrt{3}\,t\right)\left(\int \cos\left(\sqrt{3}\,t\right) \mathit{Heaviside}(t-4)\cos(5t-20)dt\right)\right)}{3}$$

- Compute integrals

$$y_p(t) = -\frac{\mathit{Heaviside}(t-4)\left(-\cos\left(\sqrt{3}\,(t-4)\right)+\cos(5t-20)\right)}{22}$$

- Substitute particular solution into general solution to ODE

$$y = \cos\left(\sqrt{3}\,t\right) c_1 + \sin\left(\sqrt{3}\,t\right) c_2 - \frac{\mathit{Heaviside}(t-4)\left(-\cos\left(\sqrt{3}\,(t-4)\right)+\cos(5t-20)\right)}{22}$$

□ Check validity of solution $y = \cos\left(\sqrt{3}\,t\right) c_1 + \sin\left(\sqrt{3}\,t\right) c_2 - \frac{\mathit{Heaviside}(t-4)\left(-\cos\left(\sqrt{3}\,(t-4)\right)+\cos(5t-20)\right)}{22}$

- Use initial condition $y(0) = 0$

$$0 = c_1$$

- Compute derivative of the solution

$$y' = -\sqrt{3}\sin\left(\sqrt{3}\,t\right) c_1 + \sqrt{3}\cos\left(\sqrt{3}\,t\right) c_2 - \frac{\mathit{Dirac}(t-4)\left(-\cos\left(\sqrt{3}\,(t-4)\right)+\cos(5t-20)\right)}{22} - \frac{\mathit{Heaviside}(t-4)\left(\sqrt{3}\right.}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = -2$

$$-2 = c_2\sqrt{3}$$

- Solve for c_1 and c_2

$$\left\{c_1 = 0, c_2 = -\frac{2\sqrt{3}}{3}\right\}$$

- Substitute constant values into general solution and simplify

$$y = -\frac{\mathit{Heaviside}(t-4)\cos(5t-20)}{22} + \frac{\mathit{Heaviside}(t-4)\cos\left(\sqrt{3}\,(t-4)\right)}{22} - \frac{2\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3}$$

- Solution to the IVP

$$y = -\frac{\mathit{Heaviside}(t-4)\cos(5t-20)}{22} + \frac{\mathit{Heaviside}(t-4)\cos\left(\sqrt{3}\,(t-4)\right)}{22} - \frac{2\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.578 (sec). Leaf size: 39

```
dsolve([diff(y(t),t$2)+3*y(t)=Heaviside(t-4)*cos(5*(t-4)),y(0) = 0, D(y)(0) = -2],y(t), sing
```

$$y(t) = -\frac{2\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)}{3} - \frac{\text{Heaviside}\left(t-4\right)\cos\left(5t-20\right)}{22} + \frac{\text{Heaviside}\left(t-4\right)\cos\left(\sqrt{3}\,(t-4)\right)}{22}$$

✓ Solution by Mathematica

Time used: 0.797 (sec). Leaf size: 66

```
DSolve[{y''[t]+3*y[t]==UnitStep[t-4]*Cos[5*(t-4)],{y[0]==0,y'[0]==-2}},y[t],t,IncludeSingula
```

$$y(t) \to \begin{cases} -\frac{2\sin\left(\sqrt{3}t\right)}{\sqrt{3}} & t \leq 4 \\ \frac{1}{66}\left(-3\cos(5(t-4)) + 3\cos\left(\sqrt{3}(t-4)\right) - 44\sqrt{3}\sin\left(\sqrt{3}t\right)\right) & \text{True} \end{cases}$$

19.7 problem 33

Internal problem ID [13229]
Internal file name [`OUTPUT/11884_Tuesday_December_05_2023_12_12_43_PM_19494184/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.3 page 600
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 9y = 20 \operatorname{Heaviside}(-2+t) \sin(-2+t)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

19.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 9 \\ F &= 20 \operatorname{Heaviside}(-2+t) \sin(-2+t) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 9y = 20 \operatorname{Heaviside}(-2+t) \sin(-2+t)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 20\,\mathrm{Heaviside}\,(-2+t)\sin(-2+t)$ is

$$\{t < 2 \lor 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 9Y(s) = \frac{20\,\mathrm{e}^{-2s}}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 6 - s + 4sY(s) + 9Y(s) = \frac{20\,\mathrm{e}^{-2s}}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 + 6s^2 + 20\,\mathrm{e}^{-2s} + s + 6}{(s^2+1)(s^2+4s+9)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{s^3 + 6s^2 + 20\,\mathrm{e}^{-2s} + s + 6}{(s^2+1)\,(s^2+4s+9)}\right) \\
&= \frac{\mathrm{e}^{-2t}\left(4\sqrt{5}\,\sin\left(t\sqrt{5}\right) + 5\cos\left(t\sqrt{5}\right)\right)}{5} + \left(\mathrm{e}^{4-2t}\cos\left(\sqrt{5}\,(-2+t)\right) - \cos(-2+t) + 2\sin(-2+t)\right) \mathrm{Hea}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y = {} & \frac{\mathrm{e}^{-2t}\left(4\sqrt{5}\,\sin\left(t\sqrt{5}\right) + 5\cos\left(t\sqrt{5}\right)\right)}{5} \\
& + \left(\mathrm{e}^{4-2t}\cos\left(\sqrt{5}\,(-2+t)\right) - \cos(-2+t) + 2\sin(-2+t)\right) \mathrm{Heaviside}(-2+t)
\end{aligned}
$$

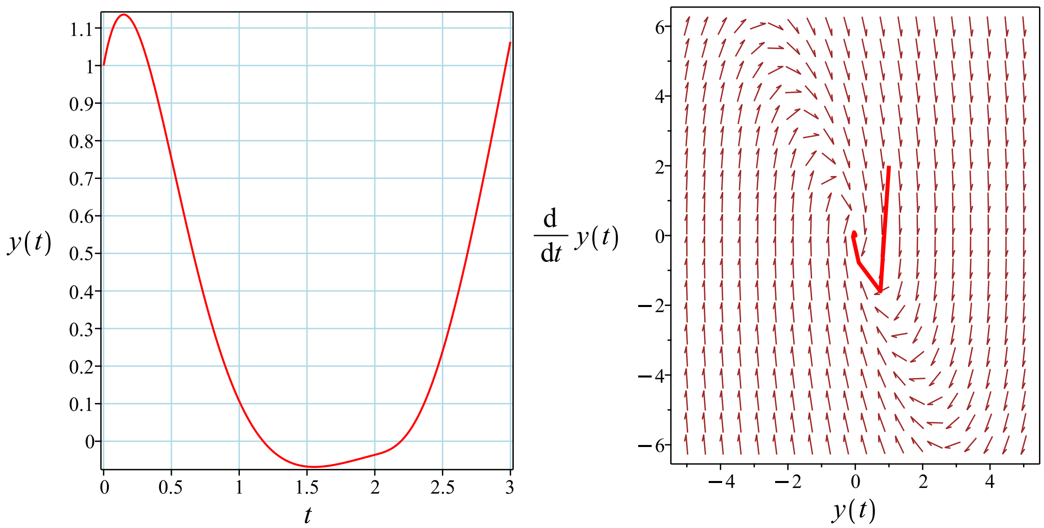
Simplifying the solution gives

$$
\begin{aligned}
y = {} & \mathrm{Heaviside}(-2+t)\cos\left(\sqrt{5}\,(-2+t)\right)\mathrm{e}^{4-2t} + \cos\left(t\sqrt{5}\right)\mathrm{e}^{-2t} \\
& + \frac{4\sqrt{5}\,\sin\left(t\sqrt{5}\right)\mathrm{e}^{-2t}}{5} - \mathrm{Heaviside}(-2+t)\left(\cos(-2+t) - 2\sin(-2+t)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \mathrm{Heaviside}(-2+t)\cos\left(\sqrt{5}\,(-2+t)\right)\mathrm{e}^{4-2t} + \cos\left(t\sqrt{5}\right)\mathrm{e}^{-2t} \\
& + \frac{4\sqrt{5}\,\sin\left(t\sqrt{5}\right)\mathrm{e}^{-2t}}{5} - \mathrm{Heaviside}(-2+t)\left(\cos(-2+t) - 2\sin(-2+t)\right)
\end{aligned} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \text{Heaviside}\,(-2+t)\cos\left(\sqrt{5}\,(-2+t)\right)\mathrm{e}^{4-2t} + \cos\left(t\sqrt{5}\right)\mathrm{e}^{-2t} + \frac{4\sqrt{5}\,\sin\left(t\sqrt{5}\right)\mathrm{e}^{-2t}}{5} - \text{Heaviside}\,(-2+t)\left(\cos\,(-2+t) - 2\sin\,(-2+t)\right)$$

Verified OK.

19.7.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 9y = 20\mathit{Heaviside}(-2+t)\sin\,(-2+t)\,, y(0) = 1, y'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-20})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-2 - \mathrm{I}\sqrt{5}, -2 + \mathrm{I}\sqrt{5}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos\left(t\sqrt{5}\right) e^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin\left(t\sqrt{5}\right) e^{-2t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos\left(t\sqrt{5}\right) e^{-2t} + c_2 \sin\left(t\sqrt{5}\right) e^{-2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 20\mathit{Heaviside}(-2+t)\sin(-\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos\left(t\sqrt{5}\right) e^{-2t} & \sin\left(t\sqrt{5}\right) e^{-2t} \\ -\sqrt{5}\sin\left(t\sqrt{5}\right) e^{-2t} - 2\cos\left(t\sqrt{5}\right) e^{-2t} & \sqrt{5}\cos\left(t\sqrt{5}\right) e^{-2t} - 2\sin\left(t\sqrt{5}\right) e^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \sqrt{5}\, e^{-4t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -4\sqrt{5}\, e^{-2t}\left(\cos\left(t\sqrt{5}\right)\left(\int e^{2t}\sin(-2+t)\sin\left(t\sqrt{5}\right)\mathit{Heaviside}(-2+t)\,dt\right) - \sin\left(t\sqrt{5}\right)\left(\int e\right.\right.$

  - Compute integrals

    $y_p(t) = -\left(-e^{4-2t}\cos\left(\sqrt{5}\,(-2+t)\right) + \cos(-2+t) - 2\sin(-2+t)\right)\mathit{Heaviside}(-2+t)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos\left(t\sqrt{5}\right) e^{-2t} + c_2 \sin\left(t\sqrt{5}\right) e^{-2t} - \left(-e^{4-2t}\cos\left(\sqrt{5}\,(-2+t)\right) + \cos(-2+t) - 2\sin(-2\right.$

□ Check validity of solution $y = c_1 \cos\left(t\sqrt{5}\right) e^{-2t} + c_2 \sin\left(t\sqrt{5}\right) e^{-2t} - \left(-e^{4-2t}\cos\left(\sqrt{5}\,(-2+t)\right)\right.$

  - Use initial condition $y(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $y' = -c_1\sqrt{5}\sin\left(t\sqrt{5}\right) e^{-2t} - 2c_1\cos\left(t\sqrt{5}\right) e^{-2t} + c_2\sqrt{5}\cos\left(t\sqrt{5}\right) e^{-2t} - 2c_2\sin\left(t\sqrt{5}\right) e^{-2t} - (2e$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 2$

$$2 = -2c_1 + c_2\sqrt{5}$$

- Solve for c_1 and c_2

$$\left\{c_1 = 1, c_2 = \frac{4\sqrt{5}}{5}\right\}$$

- Substitute constant values into general solution and simplify

$$y = \mathit{Heaviside}(-2+t)\cos\left(\sqrt{5}\,(-2+t)\right)\mathrm{e}^{4-2t} + \cos\left(t\sqrt{5}\right)\mathrm{e}^{-2t} + \frac{4\sqrt{5}\sin\left(t\sqrt{5}\right)\mathrm{e}^{-2t}}{5} - \mathit{Heaviside}(-2$$

- Solution to the IVP

$$y = \mathit{Heaviside}(-2+t)\cos\left(\sqrt{5}\,(-2+t)\right)\mathrm{e}^{4-2t} + \cos\left(t\sqrt{5}\right)\mathrm{e}^{-2t} + \frac{4\sqrt{5}\sin\left(t\sqrt{5}\right)\mathrm{e}^{-2t}}{5} - \mathit{Heaviside}(-2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.953 (sec). Leaf size: 64

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+9*y(t)=20*Heaviside(t-2)*sin(t-2),y(0) = 1, D(y)(0) =
```

$$y(t) = \cos\left(\sqrt{5}\,(t-2)\right)\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{-2t+4} + \mathrm{e}^{-2t}\cos\left(t\sqrt{5}\right) + \frac{4\,\mathrm{e}^{-2t}\sqrt{5}\,\sin\left(t\sqrt{5}\right)}{5} - \mathrm{Heaviside}\,(t-2)\left(\cos\left(t-2\right) - 2\sin\left(t-2\right)\right)$$

✓ Solution by Mathematica

Time used: 2.391 (sec). Leaf size: 115

```
DSolve[{y''[t]+4*y'[t]+9*y[t]==20*UnitStep[t-2]*Sin[t-2],{y[0]==1,y'[0]==2}},y[t],t,IncludeS
```

$y(t)$

$$\rightarrow \begin{cases} -\cos(2-t) + e^{4-2t}\cos\left(\sqrt{5}(t-2)\right) + e^{-2t}\cos\left(\sqrt{5}t\right) - 2\sin(2-t) + \frac{4e^{-2t}\sin\left(\sqrt{5}t\right)}{\sqrt{5}} & t > 2 \\ \frac{1}{5}e^{-2t}\left(5\cos\left(\sqrt{5}t\right) + 4\sqrt{5}\sin\left(\sqrt{5}t\right)\right) & \text{True} \end{cases}$$

19.8 problem 34

Internal problem ID [13230]
Internal file name [`OUTPUT/11885_Tuesday_December_05_2023_12_12_44_PM_54095803/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.3 page 600
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y = \begin{cases} t & 0 \leq t < 1 \\ 1 & 1 \leq t \end{cases}}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

19.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 0$$
$$q(t) = 3$$
$$F = \begin{cases} 0 & t < 0 \\ t & t < 1 \\ 1 & 1 \leq t \end{cases}$$

Hence the ode is

$$y'' + 3y = \begin{cases} 0 & t < 0 \\ t & t < 1 \\ 1 & 1 \leq t \end{cases}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ t & t < 1 \\ 1 & 1 \leq t \end{cases}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3Y(s) = \frac{-e^{-s} + 1}{s^2} \tag{1}$$

But the initial conditions are

$$y(0) = 2$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2s + 3Y(s) = \frac{-e^{-s} + 1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-2s^3 + e^{-s} - 1}{s^2(s^2 + 3)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-2s^3 + e^{-s} - 1}{s^2(s^2 + 3)}\right) \\
&= 2\cos\left(\sqrt{3}\,t\right) - \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{9} + \frac{t}{3} - \frac{\text{Heaviside}(t-1)\sqrt{3}\left(\sqrt{3}\,t - \sqrt{3} - \sin\left(\sqrt{3}\,(t-1)\right)\right)}{9}
\end{aligned}$$

Hence the final solution is

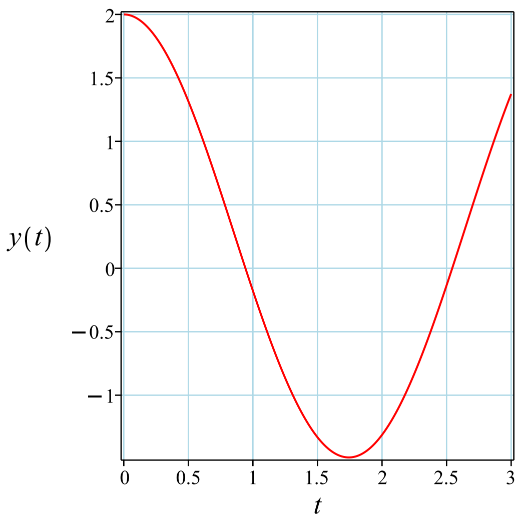
$$y = 2\cos\left(\sqrt{3}\,t\right) - \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{9} + \frac{t}{3} - \frac{\text{Heaviside}(t-1)\sqrt{3}\left(\sqrt{3}\,t - \sqrt{3} - \sin\left(\sqrt{3}\,(t-1)\right)\right)}{9}$$

Simplifying the solution gives

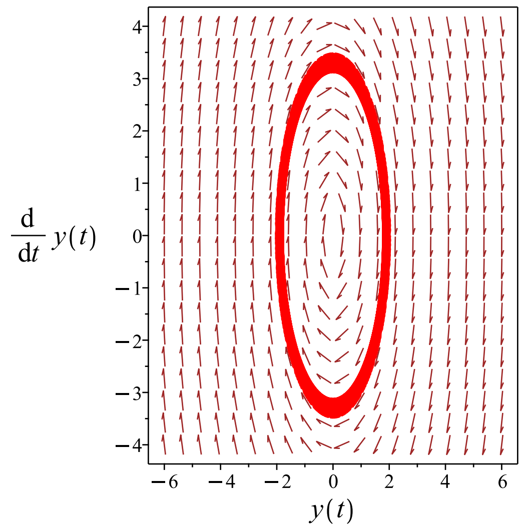
$$y = \frac{\sin\left(\sqrt{3}\,(t-1)\right)\sqrt{3}\ \text{Heaviside}(t-1)}{9} + 2\cos\left(\sqrt{3}\,t\right) - \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{9} + \frac{(-3t+3)\ \text{Heaviside}(t-1)}{9} + \frac{t}{3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sin\left(\sqrt{3}\,(t-1)\right)\sqrt{3}\ \text{Heaviside}(t-1)}{9} + 2\cos\left(\sqrt{3}\,t\right) - \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{9} + \frac{(-3t+3)\ \text{Heaviside}(t-1)}{9} + \frac{t}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sin\left(\sqrt{3}\,(t-1)\right)\sqrt{3}\ \text{Heaviside}\,(t-1)}{9} + 2\cos\left(\sqrt{3}\,t\right) - \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{9} + \frac{(-3t+3)\ \text{Heaviside}\,(t-1)}{9} + \frac{t}{3}$$

Verified OK.

19.8.2 Maple step by step solution

Let's solve

$$\left[y''+3y=\begin{cases} 0 & t<0 \\ t & t<1 \\ 1 & 1\le t \end{cases}, y(0)=2, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2+3=0$
- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-12})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{3}, \mathrm{I}\sqrt{3}\right)$
- 1st solution of the homogeneous ODE

  $y_1(t) = \cos\left(\sqrt{3}\, t\right)$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin\left(\sqrt{3}\, t\right)$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\sqrt{3}\, t\right) c_1 + \sin\left(\sqrt{3}\, t\right) c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \left\{\begin{array}{cc} 0 & t < 0 \\ t & t < 1 \\ 1 & 1 \leq t \end{array}\right.\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \cos\left(\sqrt{3}\, t\right) & \sin\left(\sqrt{3}\, t\right) \\ -\sin\left(\sqrt{3}\, t\right)\sqrt{3} & \cos\left(\sqrt{3}\, t\right)\sqrt{3} \end{array}\right]$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \sqrt{3}$
- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = \frac{\sqrt{3}\left(-\cos\left(\sqrt{3}\, t\right)\left(\int \sin\left(\sqrt{3}\, t\right)\left(\left\{\begin{array}{cc} 0 & t < 0 \\ t & t < 1 \\ 1 & 1 \leq t \end{array}\right.\right) dt\right) + \sin\left(\sqrt{3}\, t\right)\left(\int \cos\left(\sqrt{3}\, t\right)\left(\left\{\begin{array}{cc} 0 & t < 0 \\ t & t < 1 \\ 1 & 1 \leq t \end{array}\right.\right) dt\right)\right)}{3}$$

- ○ Compute integrals

$$y_p(t) = \frac{\left(\begin{cases} 0 & t \le 0 \\ 3t - \sin\left(\sqrt{3}\, t\right)\sqrt{3} & t \le 1 \\ 3 + \sqrt{3}\left(-1+\cos\left(\sqrt{3}\right)\right)\sin\left(\sqrt{3}\, t\right) - \sqrt{3}\cos\left(\sqrt{3}\, t\right)\sin\left(\sqrt{3}\right) & 1 < t \end{cases}\right)}{9}$$

- Substitute particular solution into general solution to ODE

$$y = \cos\left(\sqrt{3}\, t\right) c_1 + \sin\left(\sqrt{3}\, t\right) c_2 + \frac{\left(\begin{cases} 0 \\ 3t - \sin\left(\sqrt{3}\, t\right)\sqrt{3} \\ 3 + \sqrt{3}\left(-1+\cos\left(\sqrt{3}\right)\right)\sin\left(\sqrt{3}\, t\right) - \sqrt{3}\cos\left(\sqrt{3}\, t\right)\sin(\end{cases}\right.}{9}$$

☐ Check validity of solution $y = \cos\left(\sqrt{3}\, t\right) c_1 + \sin\left(\sqrt{3}\, t\right) c_2 + \frac{\left(\begin{cases} 0 \\ 3t - \sin(\\ 3 + \sqrt{3}\left(-1+\cos\left(\sqrt{3}\right)\right)\sin(\end{cases}\right.}{}$

- Use initial condition $y(0) = 2$

  $2 = c_1$

- Compute derivative of the solution

$$y' = -\sqrt{3}\sin\left(\sqrt{3}\, t\right) c_1 + \sqrt{3}\cos\left(\sqrt{3}\, t\right) c_2 + \frac{\left(\begin{cases} 0 \\ 3 - 3\cos\left(\sqrt{3}\, t\right) \\ 3\left(-1+\cos\left(\sqrt{3}\right)\right)\cos\left(\sqrt{3}\, t\right) + 3\sin\left(\sqrt{3}\, t\right)\sin \end{cases}\right.}{9}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_2\sqrt{3}$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 0\}$

- Substitute constant values into general solution and simplify

$$y = 2\cos\left(\sqrt{3}\, t\right) + \frac{\left(\begin{cases} 0 & t \le 0 \\ 3t - \sin\left(\sqrt{3}\, t\right)\sqrt{3} & t \le 1 \\ 3 + \sqrt{3}\left(-1+\cos\left(\sqrt{3}\right)\right)\sin\left(\sqrt{3}\, t\right) - \sqrt{3}\cos\left(\sqrt{3}\, t\right)\sin\left(\sqrt{3}\right) & 1 < t \end{cases}\right)}{9}$$

- Solution to the IVP

$$y = 2\cos\left(\sqrt{3}\,t\right) + \frac{\left(\begin{cases} 0 & t \leq 0 \\ 3t - \sin\left(\sqrt{3}\,t\right)\sqrt{3} & t \leq 1 \\ 3 + \sqrt{3}\left(-1 + \cos\left(\sqrt{3}\right)\right)\sin\left(\sqrt{3}\,t\right) - \sqrt{3}\,\cos\left(\sqrt{3}\,t\right)\sin\left(\sqrt{3}\right) & 1 < t \end{cases}\right)}{9}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 7.829 (sec). Leaf size: 83

```
dsolve([diff(y(t),t$2)+3*y(t)=piecewise(0<=t and t<1,t,t>=1,1),y(0) = 2, D(y)(0) = 0],y(t),
```

$$y(t) = 2\cos\left(\sqrt{3}\,t\right) - \frac{\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)}{9} + \frac{\left(\begin{cases} t & t < 1 \\ 1 + \frac{\sqrt{3}\,\sin\left(\sqrt{3}\,(t-1)\right)}{3} & 1 \leq t \end{cases}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 108

```
DSolve[{y''[t]+3*y[t]==Piecewise[{{t,0<=t<1},{1,t>=1}}],{y[0]==2,y'[0]==0}},y[t],t,IncludeSi
```

$$y(t) \to \begin{cases} 2\cos\left(\sqrt{3}t\right) & t \leq 0 \\ \frac{1}{9}\left(3t + 18\cos\left(\sqrt{3}t\right) - \sqrt{3}\sin\left(\sqrt{3}t\right)\right) & 0 < t \leq 1 \\ \frac{1}{9}\left(18\cos\left(\sqrt{3}t\right) + \sqrt{3}\sin\left(\sqrt{3}(t-1)\right) - \sqrt{3}\sin\left(\sqrt{3}t\right) + 3\right) & \text{True} \end{cases}$$

20 Chapter 6. Laplace transform. Section 6.4. page 608

20.1 problem 2

Internal problem ID [13231]
Internal file name [`OUTPUT/11886_Tuesday_December_05_2023_12_12_45_PM_51119002/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.4. page 608
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y = 5\delta(-2 + t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

20.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 3 \\ F &= 5\delta(-2 + t) \end{aligned}$$

Hence the ode is

$$y'' + 3y = 5\delta(-2 + t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 5\delta(-2 + t)$ is

$$\{t < 2 \vee 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3Y(s) = 5\,\mathrm{e}^{-2s} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3Y(s) = 5\,\mathrm{e}^{-2s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{5\,\mathrm{e}^{-2s}}{s^2 + 3}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{5\,\mathrm{e}^{-2s}}{s^2+3}\right) \\
&= \frac{5\,\mathrm{Heaviside}\left(-2+t\right)\sin\left(\sqrt{3}\left(-2+t\right)\right)\sqrt{3}}{3}
\end{aligned}$$

Hence the final solution is

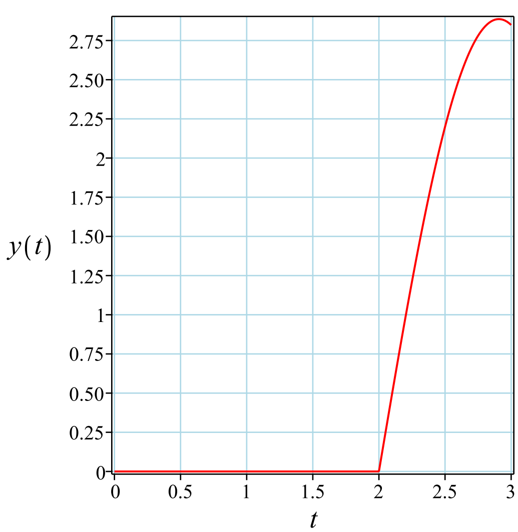
$$y = \frac{5\,\mathrm{Heaviside}\left(-2+t\right)\sin\left(\sqrt{3}\left(-2+t\right)\right)\sqrt{3}}{3}$$

Simplifying the solution gives

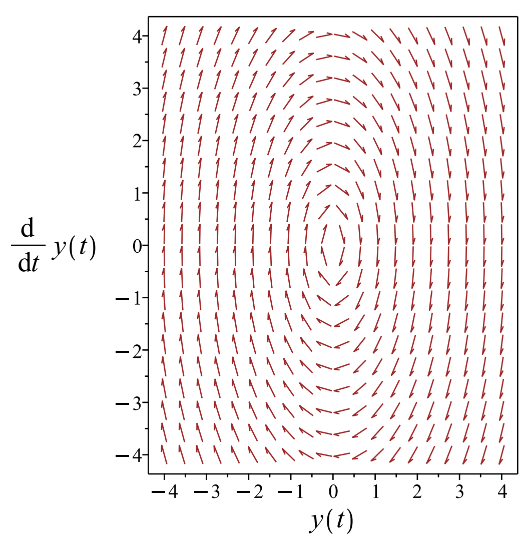
$$y = \frac{5\,\mathrm{Heaviside}\left(-2+t\right)\sin\left(\sqrt{3}\left(-2+t\right)\right)\sqrt{3}}{3}$$

Summary
The solution(s) found are the following

$$y = \frac{5\,\mathrm{Heaviside}\left(-2+t\right)\sin\left(\sqrt{3}\left(-2+t\right)\right)\sqrt{3}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5\,\mathrm{Heaviside}\,(-2+t)\sin\left(\sqrt{3}\,(-2+t)\right)\sqrt{3}}{3}$$

Verified OK.

20.1.2 Maple step by step solution

Let's solve

$\left[y''+3y=5\mathit{Dirac}(-2+t)\,,y(0)=0,y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+3=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-12})}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\mathrm{I}\sqrt{3},\mathrm{I}\sqrt{3}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos\left(\sqrt{3}\,t\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin\left(\sqrt{3}\,t\right)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=\cos\left(\sqrt{3}\,t\right)c_1+\sin\left(\sqrt{3}\,t\right)c_2+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=5\mathit{Dirac}(-2+t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(\sqrt{3}\,t) & \sin(\sqrt{3}\,t) \\ -\sin(\sqrt{3}\,t)\sqrt{3} & \cos(\sqrt{3}\,t)\sqrt{3} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(t), y_2(t)) = \sqrt{3}$$

- Substitute functions into equation for $y_p(t)$

  $$y_p(t) = \frac{5\sqrt{3}\left(\int \mathit{Dirac}(-2+t)dt\right)\left(\sin\left(\sqrt{3}\,t\right)\cos\left(2\sqrt{3}\right)-\cos\left(\sqrt{3}\,t\right)\sin\left(2\sqrt{3}\right)\right)}{3}$$

- Compute integrals

  $$y_p(t) = \frac{5\sqrt{3}\,\mathit{Heaviside}(-2+t)\left(\sin\left(\sqrt{3}\,t\right)\cos\left(2\sqrt{3}\right)-\cos\left(\sqrt{3}\,t\right)\sin\left(2\sqrt{3}\right)\right)}{3}$$

• Substitute particular solution into general solution to ODE

$$y = \cos\left(\sqrt{3}\,t\right)c_1 + \sin\left(\sqrt{3}\,t\right)c_2 + \frac{5\sqrt{3}\,\mathit{Heaviside}(-2+t)\left(\sin\left(\sqrt{3}\,t\right)\cos\left(2\sqrt{3}\right)-\cos\left(\sqrt{3}\,t\right)\sin\left(2\sqrt{3}\right)\right)}{3}$$

□ Check validity of solution $y = \cos\left(\sqrt{3}\,t\right)c_1 + \sin\left(\sqrt{3}\,t\right)c_2 + \frac{5\sqrt{3}\,\mathit{Heaviside}(-2+t)\left(\sin\left(\sqrt{3}\,t\right)\cos\left(2\sqrt{3}\right)-\mathrm{co}\right.}{3}$

- Use initial condition $y(0) = 0$

  $$0 = c_1$$

- Compute derivative of the solution

  $$y' = -\sqrt{3}\sin\left(\sqrt{3}\,t\right)c_1 + \sqrt{3}\cos\left(\sqrt{3}\,t\right)c_2 + \frac{5\sqrt{3}\,\mathit{Dirac}(-2+t)\left(\sin\left(\sqrt{3}\,t\right)\cos\left(2\sqrt{3}\right)-\cos\left(\sqrt{3}\,t\right)\sin\left(2\sqrt{3}\right)\right)}{3} + \frac{5}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $$0 = c_2\sqrt{3}$$

- Solve for c_1 and c_2

  $$\{c_1 = 0, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

  $$y = \frac{5\sqrt{3}\,\mathit{Heaviside}(-2+t)\left(\sin\left(\sqrt{3}\,t\right)\cos\left(2\sqrt{3}\right)-\cos\left(\sqrt{3}\,t\right)\sin\left(2\sqrt{3}\right)\right)}{3}$$

• Solution to the IVP

$$y = \frac{5\sqrt{3}\,\mathit{Heaviside}(-2+t)\left(\sin\left(\sqrt{3}\,t\right)\cos\left(2\sqrt{3}\right)-\cos\left(\sqrt{3}\,t\right)\sin\left(2\sqrt{3}\right)\right)}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 15.422 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)+3*y(t)=5*Dirac(t-2),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{5\sqrt{3}\ \mathrm{Heaviside}\,(t-2)\sin\left(\sqrt{3}\,(t-2)\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.288 (sec). Leaf size: 36

```
DSolve[{y''[t]+3*y[t]==DiracDelta[t-2],{y[0]==2,y'[0]==0}},y[t],t,IncludeSingularSolutions -
```

$$y(t) \to \frac{\theta(t-2)\sin\left(\sqrt{3}(t-2)\right)}{\sqrt{3}} + 2\cos\left(\sqrt{3}t\right)$$

20.2 problem 3

Internal problem ID [13232]
Internal file name [`OUTPUT/11887_Tuesday_December_05_2023_12_12_45_PM_35239012/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.4. page 608
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = \delta(-3+t)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

20.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= \delta(-3+t) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = \delta(-3+t)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(-3 + t)$ is

$$\{t < 3 \lor 3 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 5Y(s) = \mathrm{e}^{-3s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1 \\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - s + 2sY(s) + 5Y(s) = \mathrm{e}^{-3s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-3s} + s + 3}{s^2 + 2s + 5}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-3s} + s + 3}{s^2 + 2s + 5}\right) \\
&= e^{-t}(\sin(2t) + \cos(2t)) + \frac{\text{Heaviside}(-3+t)\, e^{3-t} \sin(-6+2t)}{2}
\end{aligned}
$$

Hence the final solution is

$$
y = e^{-t}(\sin(2t) + \cos(2t)) + \frac{\text{Heaviside}(-3+t)\, e^{3-t} \sin(-6+2t)}{2}
$$

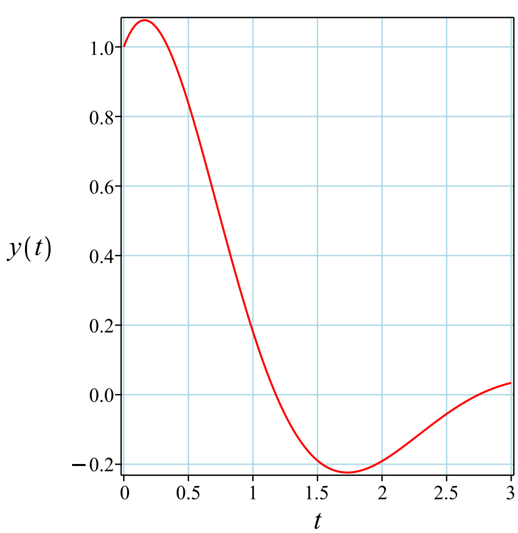
Simplifying the solution gives

$$
y = e^{-t}(\sin(2t) + \cos(2t)) + \frac{\text{Heaviside}(-3+t)\, e^{3-t} \sin(-6+2t)}{2}
$$

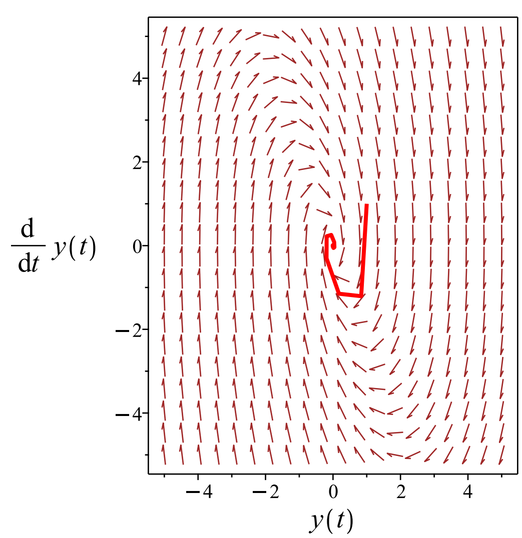
Summary

The solution(s) found are the following

$$
y = e^{-t}(\sin(2t) + \cos(2t)) + \frac{\text{Heaviside}(-3+t)\, e^{3-t} \sin(-6+2t)}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-t}(\sin(2t) + \cos(2t)) + \frac{\text{Heaviside}(-3+t)\, e^{3-t} \sin(-6+2t)}{2}$$

Verified OK.

20.2.2 Maple step by step solution

Let's solve

$\left[y'' + 2y' + 5y = Dirac(-3+t), y(0) = 1, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-t}\cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}\sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Dirac(-3+t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t}\cos(2t) & \mathrm{e}^{-t}\sin(2t) \\ -\mathrm{e}^{-t}\cos(2t) - 2\,\mathrm{e}^{-t}\sin(2t) & -\mathrm{e}^{-t}\sin(2t) + 2\,\mathrm{e}^{-t}\cos(2t) \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{-2t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{\left(\int \mathit{Dirac}(-3+t)dt\right)(\sin(2t)\cos(6)-\cos(2t)\sin(6))\mathrm{e}^{3-t}}{2}$

- ○ Compute integrals

  $y_p(t) = \frac{\mathit{Heaviside}(-3+t)\mathrm{e}^{3-t}(\sin(2t)\cos(6)-\cos(2t)\sin(6))}{2}$

- • Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-t}\cos(2t) + c_2\mathrm{e}^{-t}\sin(2t) + \frac{\mathit{Heaviside}(-3+t)\mathrm{e}^{3-t}(\sin(2t)\cos(6)-\cos(2t)\sin(6))}{2}$

- □ Check validity of solution $y = c_1\mathrm{e}^{-t}\cos(2t) + c_2\mathrm{e}^{-t}\sin(2t) + \frac{\mathit{Heaviside}(-3+t)\mathrm{e}^{3-t}(\sin(2t)\cos(6)-\cos(2t)\,\mathrm{si}}{2}$

- ○ Use initial condition $y(0) = 1$

  $1 = c_1$

- ○ Compute derivative of the solution

  $y' = -c_1\mathrm{e}^{-t}\cos(2t) - 2c_1\mathrm{e}^{-t}\sin(2t) - c_2\mathrm{e}^{-t}\sin(2t) + 2c_2\mathrm{e}^{-t}\cos(2t) + \frac{\mathit{Dirac}(-3+t)\mathrm{e}^{3-t}(\sin(2t)\cos(6)-}{2}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = -c_1 + 2c_2$

- ○ Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 1\}$

- ○ Substitute constant values into general solution and simplify

  $y = \mathrm{e}^{-t}(\sin(2t) + \cos(2t)) + \frac{\mathit{Heaviside}(-3+t)\mathrm{e}^{3-t}(\sin(2t)\cos(6)-\cos(2t)\sin(6))}{2}$

- • Solution to the IVP

  $y = \mathrm{e}^{-t}(\sin(2t) + \cos(2t)) + \frac{\mathit{Heaviside}(-3+t)\mathrm{e}^{3-t}(\sin(2t)\cos(6)-\cos(2t)\sin(6))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.718 (sec). Leaf size: 37

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+5*y(t)=Dirac(t-3),y(0) = 1, D(y)(0) = 1],y(t), singsol
```

$$y(t) = e^{-t}(\cos(2t) + \sin(2t)) + \frac{e^{-t+3} \operatorname{Heaviside}(t-3) \sin(2t-6)}{2}$$

✓ Solution by Mathematica

Time used: 0.179 (sec). Leaf size: 41

```
DSolve[{y''[t]+2*y'[t]+5*y[t]==DiracDelta[t-3],{y[0]==1,y'[0]==1}},y[t],t,IncludeSingularSol
```

$$y(t) \to \frac{1}{2} e^{-t} \left(2(\sin(2t) + \cos(2t)) - e^3 \theta(t-3) \sin(6-2t)\right)$$

20.3 problem 4

Internal problem ID [13233]
Internal file name [`OUTPUT/11888_Tuesday_December_05_2023_12_12_46_PM_25479048/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.4. page 608
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = -2\delta(-2+t)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

20.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 2 \\ F &= -2\delta(-2+t) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 2y = -2\delta(-2+t)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -2\delta(-2+t)$ is

$$\{t < 2 \vee 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 2Y(s) = -2\,\mathrm{e}^{-2s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4 - 2s + 2sY(s) + 2Y(s) = -2\,\mathrm{e}^{-2s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{2(\mathrm{e}^{-2s} - s - 2)}{s^2 + 2s + 2}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(-\frac{2(\mathrm{e}^{-2s} - s - 2)}{s^2 + 2s + 2}\right)\\ &= -2\,\mathrm{Heaviside}\,(-2+t)\sin\,(-2+t)\,\mathrm{e}^{-t+2} + 2(\cos\,(t) + \sin\,(t))\,\mathrm{e}^{-t}\end{aligned}$$

Hence the final solution is

$$y = -2\,\mathrm{Heaviside}(-2+t)\sin(-2+t)\,\mathrm{e}^{-t+2} + 2(\cos(t)+\sin(t))\,\mathrm{e}^{-t}$$

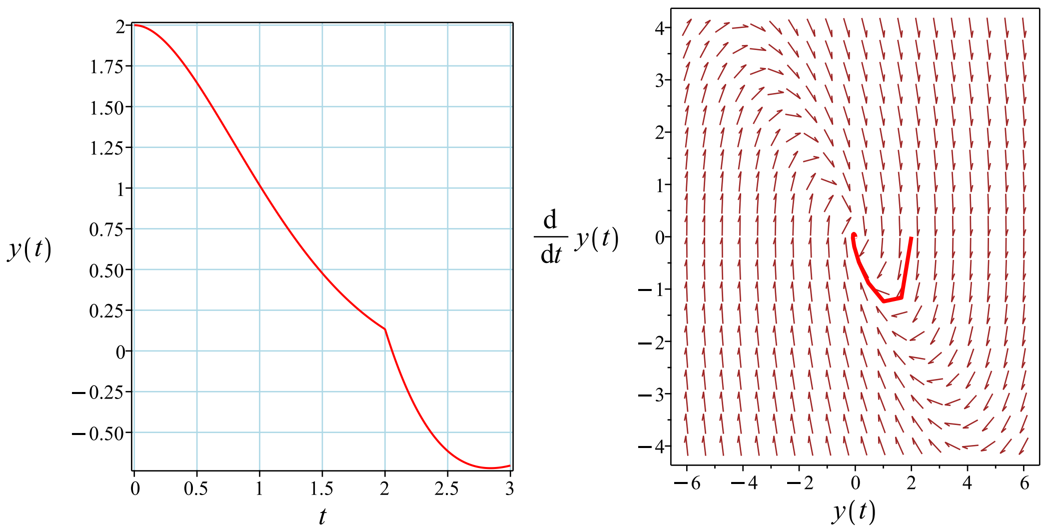
Simplifying the solution gives

$$y = -2\,\mathrm{Heaviside}(-2+t)\sin(-2+t)\,\mathrm{e}^{-t+2} + 2(\cos(t)+\sin(t))\,\mathrm{e}^{-t}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -2\,\mathrm{Heaviside}(-2+t)\sin(-2+t)\,\mathrm{e}^{-t+2} + 2(\cos(t)+\sin(t))\,\mathrm{e}^{-t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -2\,\mathrm{Heaviside}(-2+t)\sin(-2+t)\,\mathrm{e}^{-t+2} + 2(\cos(t)+\sin(t))\,\mathrm{e}^{-t}$$

Verified OK.

20.3.2 Maple step by step solution

Let's solve

$\left[y''+2y'+2y=-2 Dirac(-2+t), y(0)=2, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+2r+2=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-1-\mathrm{I}, -1+\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-t}\cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{-t}\sin(t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-t}\cos(t)+c_2\mathrm{e}^{-t}\sin(t)+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=-2 Dirac(-2+t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t))=\left[\begin{array}{cc}\mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t) \\ -\mathrm{e}^{-t}\cos(t)-\mathrm{e}^{-t}\sin(t) & -\mathrm{e}^{-t}\sin(t)+\mathrm{e}^{-t}\cos(t)\end{array}\right]$

  - Compute Wronskian

    $W(y_1(t), y_2(t))=\mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = 2\left(\int Dirac(-2+t)\,dt\right)(\sin(2)\cos(t) - \cos(2)\sin(t))\,\mathrm{e}^{-t+2}$

- Compute integrals

  $y_p(t) = 2Heaviside(-2+t)(\sin(2)\cos(t) - \cos(2)\sin(t))\,\mathrm{e}^{-t+2}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t) + 2Heaviside(-2+t)(\sin(2)\cos(t) - \cos(2)\sin(t))\,\mathrm{e}^{-t+2}$

□ Check validity of solution $y = c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t) + 2Heaviside(-2+t)(\sin(2)\cos(t) -$

- Use initial condition $y(0) = 2$

  $2 = c_1$

- Compute derivative of the solution

  $y' = -c_1\mathrm{e}^{-t}\cos(t) - c_1\mathrm{e}^{-t}\sin(t) - c_2\mathrm{e}^{-t}\sin(t) + c_2\mathrm{e}^{-t}\cos(t) + 2Dirac(-2+t)(\sin(2)\cos(t) -$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = -2Heaviside(-2+t)(-\sin(2)\cos(t) + \cos(2)\sin(t))\,\mathrm{e}^{-t+2} + 2(\cos(t) + \sin(t))\,\mathrm{e}^{-t}$

- Solution to the IVP

  $y = -2Heaviside(-2+t)(-\sin(2)\cos(t) + \cos(2)\sin(t))\,\mathrm{e}^{-t+2} + 2(\cos(t) + \sin(t))\,\mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.25 (sec). Leaf size: 32

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=-2*Dirac(t-2),y(0) = 2, D(y)(0) = 0],y(t), sing
```

$$y(t) = -2\,\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{2-t}\sin\,(t-2) + 2\,\mathrm{e}^{-t}(\sin\,(t) + \cos\,(t))$$

✓ Solution by Mathematica

Time used: 0.3 (sec). Leaf size: 31

```
DSolve[{y''[t]+2*y'[t]+2*y[t]==-2*DiracDelta[t-2],{y[0]==2,y'[0]==0}},y[t],t,IncludeSingular
```

$$y(t) \to 2e^{-t}\left(e^2\theta(t-2)\sin(2-t) + \sin(t) + \cos(t)\right)$$

20.4 problem 5

Internal problem ID [13234]
Internal file name [`OUTPUT/11889_Tuesday_December_05_2023_12_12_46_PM_81295945/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.4. page 608
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 3y = \delta(t-1) - 3\delta(t-4)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

20.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 3 \\ F &= \delta(t-1) - 3\delta(t-4) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 3y = \delta(t-1) - 3\delta(t-4)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t-1) - 3\delta(t-4)$ is

$$\{1 \leq t \leq 4, 4 \leq t \leq \infty, -\infty \leq t \leq 1\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 3Y(s) = \mathrm{e}^{-s} - 3\,\mathrm{e}^{-4s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2sY(s) + 3Y(s) = \mathrm{e}^{-s} - 3\,\mathrm{e}^{-4s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s} - 3\,\mathrm{e}^{-4s}}{s^2 + 2s + 3}$$

Taking the inverse Laplace transform gives

$$y = \mathcal{L}^{-1}(Y(s))$$
$$= \mathcal{L}^{-1}\left(\frac{e^{-s} - 3\,e^{-4s}}{s^2 + 2s + 3}\right)$$
$$= \frac{\text{Heaviside}(t-1)\sqrt{2}\,e^{1-t}\sin\left(\sqrt{2}\,(t-1)\right)}{2} - \frac{3\,\text{Heaviside}(t-4)\sqrt{2}\,e^{-t+4}\sin\left(\sqrt{2}\,(t-4)\right)}{2}$$

Hence the final solution is

$$y = \frac{\text{Heaviside}(t-1)\sqrt{2}\,e^{1-t}\sin\left(\sqrt{2}\,(t-1)\right)}{2} - \frac{3\,\text{Heaviside}(t-4)\sqrt{2}\,e^{-t+4}\sin\left(\sqrt{2}\,(t-4)\right)}{2}$$

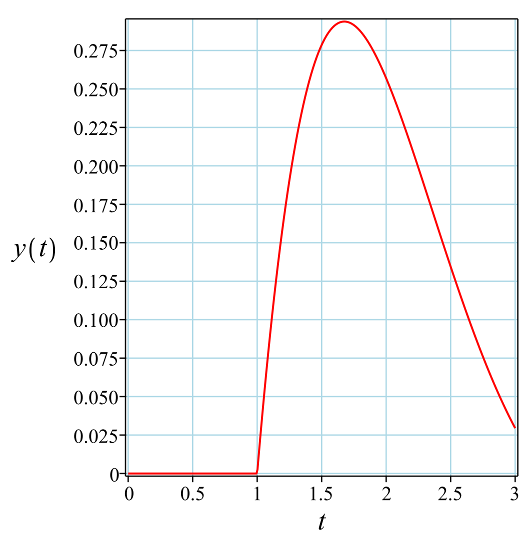
Simplifying the solution gives

$$y = \frac{\sqrt{2}\left(\text{Heaviside}(t-1)\,e^{1-t}\sin\left(\sqrt{2}\,(t-1)\right) - 3\,\text{Heaviside}(t-4)\,e^{-t+4}\sin\left(\sqrt{2}\,(t-4)\right)\right)}{2}$$

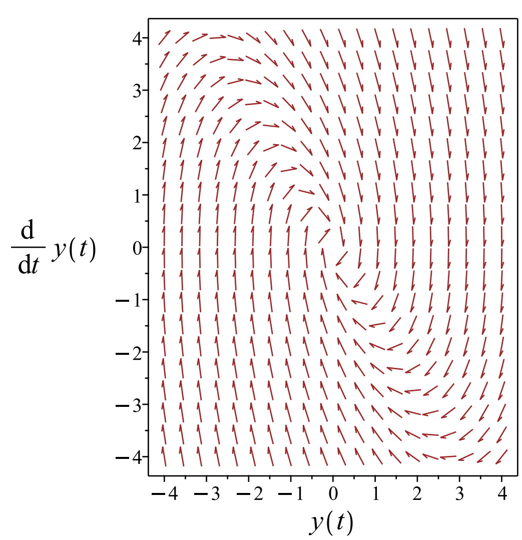
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\left(\text{Heaviside}(t-1)\,e^{1-t}\sin\left(\sqrt{2}\,(t-1)\right) - 3\,\text{Heaviside}(t-4)\,e^{-t+4}\sin\left(\sqrt{2}\,(t-4)\right)\right)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2}\left(\mathrm{Heaviside}\left(t-1\right) \mathrm{e}^{1-t} \sin\left(\sqrt{2}\left(t-1\right)\right)-3\,\mathrm{Heaviside}\left(t-4\right) \mathrm{e}^{-t+4} \sin\left(\sqrt{2}\left(t-4\right)\right)\right)}{2}$$

Verified OK.

20.4.2 Maple step by step solution

Let's solve

$\left[y''+2y'+3y=Dirac(t-1)-3Dirac(t-4), y(0)=0, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+2r+3=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm\left(\sqrt{-8}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-1-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}-1\right)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos\left(\sqrt{2}\,t\right)\mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin\left(\sqrt{2}\,t\right)\mathrm{e}^{-t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos\left(\sqrt{2}\,t\right)\mathrm{e}^{-t}+c_2\sin\left(\sqrt{2}\,t\right)\mathrm{e}^{-t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=Dirac(t-1)-3Dirac(t-\right.$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(\sqrt{2}\,t)\,e^{-t} & \sin(\sqrt{2}\,t)\,e^{-t} \\ -\sin(\sqrt{2}\,t)\,\sqrt{2}\,e^{-t} - \cos(\sqrt{2}\,t)\,e^{-t} & \sqrt{2}\,e^{-t}\cos(\sqrt{2}\,t) - \sin(\sqrt{2}\,t)\,e^{-t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = \sqrt{2}\,e^{-2t}$

- Substitute functions into equation for $y_p(t)$

  $$y_p(t) = \frac{\sqrt{2}\,e^{1-t}\left(\cos(\sqrt{2}\,t)\left(\int\left(3Dirac(t-4)e^3\sin(4\sqrt{2}) - Dirac(t-1)\sin(\sqrt{2})\right)dt\right) - \sin(\sqrt{2}\,t)\left(\int\left(3Dirac(t-4)e^3\cos(4\sqrt{2}) - D\right.\right.\right.}{2}$$

- Compute integrals

  $$y_p(t) = \frac{\left(3\sin(4\sqrt{2})\,Heaviside(t-4)e^3\cos(\sqrt{2}\,t) - 3Heaviside(t-4)e^3\cos(4\sqrt{2})\sin(\sqrt{2}\,t) + \cos(\sqrt{2})\,Heaviside(t-1)\sin(\sqrt{2}\,t) - \right.}{2}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\cos(\sqrt{2}\,t)\,e^{-t} + c_2\sin(\sqrt{2}\,t)\,e^{-t} + \frac{\left(3\sin(4\sqrt{2})\,Heaviside(t-4)e^3\cos(\sqrt{2}\,t) - 3Heaviside(t-4)e^3\cos(4\sqrt{2})\,\mathrm{si}\right.}{}$$

☐ Check validity of solution $y = c_1\cos(\sqrt{2}\,t)\,e^{-t} + c_2\sin(\sqrt{2}\,t)\,e^{-t} + \frac{\left(3\sin(4\sqrt{2})\,Heaviside(t-4)e^3\cos(\sqrt{}\right.}{}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $$y' = -c_1\sqrt{2}\,\sin(\sqrt{2}\,t)\,e^{-t} - c_1\cos(\sqrt{2}\,t)\,e^{-t} + c_2\sqrt{2}\,\cos(\sqrt{2}\,t)\,e^{-t} - c_2\sin(\sqrt{2}\,t)\,e^{-t} + \frac{\left(3\sin(4\sqrt{}\right.}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -c_1 + \sqrt{2}\,c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $$y = \frac{\left(3\sin(4\sqrt{2})\,Heaviside(t-4)e^3\cos(\sqrt{2}\,t) - 3Heaviside(t-4)e^3\cos(4\sqrt{2})\sin(\sqrt{2}\,t) + \cos(\sqrt{2})\,Heaviside(t-1)\sin(\sqrt{2}\,t) - \sin(\right.}{2}$$

- Solution to the IVP

  $$y = \frac{\left(3\sin(4\sqrt{2})\,Heaviside(t-4)e^3\cos(\sqrt{2}\,t) - 3Heaviside(t-4)e^3\cos(4\sqrt{2})\sin(\sqrt{2}\,t) + \cos(\sqrt{2})\,Heaviside(t-1)\sin(\sqrt{2}\,t) - \sin(\right.}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.562 (sec). Leaf size: 51

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+3*y(t)=Dirac(t-1)-3*Dirac(t-4),y(0) = 0, D(y)(0) = 0],
```

$$y(t) = -\frac{3\sqrt{2}\left(\operatorname{Heaviside}(t-4)\,\mathrm{e}^{4-t}\sin\left(\sqrt{2}\,(t-4)\right)-\frac{\operatorname{Heaviside}(t-1)\mathrm{e}^{-t+1}\sin\left(\sqrt{2}\,(t-1)\right)}{3}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.371 (sec). Leaf size: 53

```
DSolve[{y''[t]+2*y'[t]+3*y[t]==DiracDelta[t-1]-3*DiracDelta[t-4],{y[0]==0,y'[0]==0}},y[t],t,
```

$$y(t) \to \frac{e^{1-t}\left(\theta(t-1)\sin\left(\sqrt{2}(t-1)\right)-3e^{3}\theta(t-4)\sin\left(\sqrt{2}(t-4)\right)\right)}{\sqrt{2}}$$

21 Chapter 6. Laplace transform. Section 6.6. page 624

21.1 problem 1

Internal problem ID [13235]
Internal file name [`OUTPUT/11890_Tuesday_December_05_2023_12_12_47_PM_52077065/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.6. page 624
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = \sin(4t)\,e^{-2t}}$$

With initial conditions

$$[y(0) = 2, y'(0) = -2]$$

21.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 2 \\ F &= \sin(4t)\,e^{-2t} \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 2y = \sin(4t)\,e^{-2t}$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(4t)\,\mathrm{e}^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 2Y(s) = \frac{4}{(s+2)^2 + 16} \tag{1}$$

But the initial conditions are

$$y(0) = 2$$
$$y'(0) = -2$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2 - 2s + 2sY(s) + 2Y(s) = \frac{4}{(s+2)^2 + 16}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s^3 + 10s^2 + 48s + 44}{(s^2 + 4s + 20)(s^2 + 2s + 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{1}{65}+\frac{7i}{260}}{s+2-4i}+\frac{\frac{1}{65}-\frac{7i}{260}}{s+2+4i}+\frac{\frac{64}{65}-\frac{8i}{65}}{s+1-i}+\frac{\frac{64}{65}+\frac{8i}{65}}{s+1+i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{65}+\frac{7i}{260}}{s+2-4i}\right) = \left(\frac{1}{65}+\frac{7i}{260}\right) e^{(-2+4i)t}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{1}{65}-\frac{7i}{260}}{s+2+4i}\right) = \left(\frac{1}{65}-\frac{7i}{260}\right) e^{(-2-4i)t}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{64}{65}-\frac{8i}{65}}{s+1-i}\right) = \left(\frac{64}{65}-\frac{8i}{65}\right) e^{(-1+i)t}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{64}{65}+\frac{8i}{65}}{s+1+i}\right) = \left(\frac{64}{65}+\frac{8i}{65}\right) e^{(-1-i)t}$$

Adding the above results and simplifying gives

$$y = \frac{e^{-2t}(4\cos(4t)-7\sin(4t))}{130}+\frac{16(8\cos(t)+\sin(t))\, e^{-t}}{65}$$

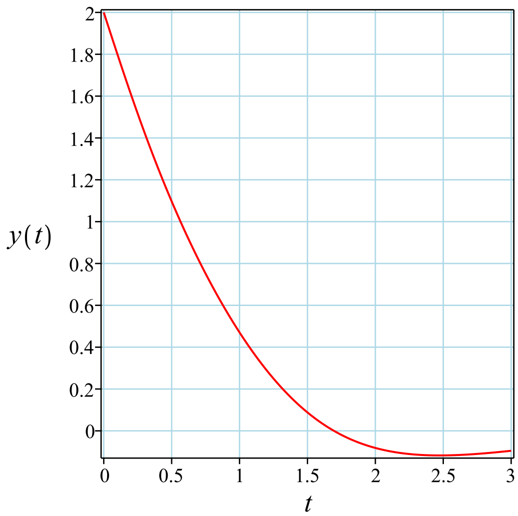
Simplifying the solution gives

$$y = \frac{e^{-2t}(4\cos(4t)-7\sin(4t))}{130}+\frac{128\, e^{-t}\left(\cos(t)+\frac{\sin(t)}{8}\right)}{65}$$

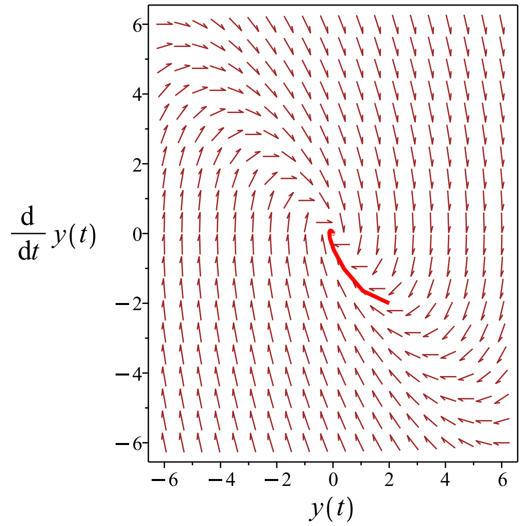
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^{-2t}(4\cos(4t)-7\sin(4t))}{130}+\frac{128\, e^{-t}\left(\cos(t)+\frac{\sin(t)}{8}\right)}{65} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{-2t}(4\cos(4t) - 7\sin(4t))}{130} + \frac{128\,e^{-t}\left(\cos(t) + \frac{\sin(t)}{8}\right)}{65}$$

Verified OK.

21.1.2 Maple step by step solution

Let's solve

$\left[y'' + 2y' + 2y = \sin(4t)\,e^{-2t}, y(0) = 2, y'\Big|_{\{t=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - I, -1 + I)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t} \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t} \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} \cos(t) + c_2 \mathrm{e}^{-t} \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \sin(4t) \mathrm{e}^{-2t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} \cos(t) & \mathrm{e}^{-t} \sin(t) \\ -\mathrm{e}^{-t} \cos(t) - \mathrm{e}^{-t} \sin(t) & -\mathrm{e}^{-t} \sin(t) + \mathrm{e}^{-t} \cos(t) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$

  ◦ Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\mathrm{e}^{-t}\left(\cos(t) \left(\int \mathrm{e}^{-t} \sin(t) \sin(4t) dt\right) - \sin(t) \left(\int \mathrm{e}^{-t} \cos(t) \sin(4t) dt\right)\right)$

  ◦ Compute integrals

    $y_p(t) = -\frac{\mathrm{e}^{-2t}(-4\cos(4t)+7\sin(4t))}{130}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-t} \cos(t) + c_2 \mathrm{e}^{-t} \sin(t) - \frac{\mathrm{e}^{-2t}(-4\cos(4t)+7\sin(4t))}{130}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-t} \cos(t) + c_2 \mathrm{e}^{-t} \sin(t) - \frac{\mathrm{e}^{-2t}(-4\cos(4t)+7\sin(4t))}{130}$

  ◦ Use initial condition $y(0) = 2$

    $2 = c_1 + \frac{2}{65}$

  ◦ Compute derivative of the solution

    $y' = -c_1 \mathrm{e}^{-t} \cos(t) - c_1 \mathrm{e}^{-t} \sin(t) - c_2 \mathrm{e}^{-t} \sin(t) + c_2 \mathrm{e}^{-t} \cos(t) + \frac{\mathrm{e}^{-2t}(-4\cos(4t)+7\sin(4t))}{65} - \frac{\mathrm{e}^{-2t}(16\sin(}{}$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = -2$

$-2 = -c_1 - \frac{18}{65} + c_2$

- Solve for c_1 and c_2

  $\{c_1 = \frac{128}{65}, c_2 = \frac{16}{65}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^{-2t}(4\cos(4t) - 7\sin(4t))}{130} + \frac{128\, e^{-t}\left(\cos(t) + \frac{\sin(t)}{8}\right)}{65}$

- Solution to the IVP

  $y = \frac{e^{-2t}(4\cos(4t) - 7\sin(4t))}{130} + \frac{128\, e^{-t}\left(\cos(t) + \frac{\sin(t)}{8}\right)}{65}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.156 (sec). Leaf size: 37

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=exp(-2*t)*sin(4*t),y(0) = 2, D(y)(0) = -2],y(t)
```

$$y(t) = \frac{e^{-2t}(-7\sin(4t) + 4\cos(4t))}{130} + \frac{128\left(\cos(t) + \frac{\sin(t)}{8}\right) e^{-t}}{65}$$

✓ Solution by Mathematica

Time used: 0.379 (sec). Leaf size: 41

```
DSolve[{y''[t]+2*y'[t]+2*y[t]==Exp[-2*t]*Sin[4*t],{y[0]==2,y'[0]==-2}},y[t],t,IncludeSingula
```

$$y(t) \to \frac{1}{130}e^{-2t}(32e^t\sin(t) - 7\sin(4t) + 256e^t\cos(t) + 4\cos(4t))$$

21.2 problem 2

Internal problem ID [13236]
Internal file name [`OUTPUT/11891_Tuesday_December_05_2023_12_12_48_PM_64669449/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.6. page 624
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + 5y = \text{Heaviside}\left(-2 + t\right) \sin\left(-8 + 4t\right)}$$

With initial conditions

$$[y(0) = -2, y'(0) = 0]$$

21.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 5 \\ F &= \text{Heaviside}\left(-2 + t\right) \sin\left(-8 + 4t\right) \end{aligned}$$

Hence the ode is

$$y'' + y' + 5y = \text{Heaviside}\left(-2 + t\right) \sin\left(-8 + 4t\right)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}\left(-2 + t\right) \sin\left(-8 + 4t\right)$ is

$$\{t < 2 \vee 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) + 5Y(s) = \frac{4\,\mathrm{e}^{-2s}}{s^2 + 16} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -2\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 + 2s + sY(s) + 5Y(s) = \frac{4\,\mathrm{e}^{-2s}}{s^2 + 16}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{-2s^3 - 2s^2 + 4\,\mathrm{e}^{-2s} - 32s - 32}{(s^2 + 16)\,(s^2 + s + 5)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{-2s^3 - 2s^2 + 4\,\mathrm{e}^{-2s} - 32s - 32}{(s^2+16)\,(s^2+s+5)}\right) \\
&= -\frac{2\,\mathrm{e}^{-\frac{t}{2}}\left(\sqrt{19}\,\sin\left(\frac{\sqrt{19}\,t}{2}\right) + 19\cos\left(\frac{\sqrt{19}\,t}{2}\right)\right)}{19} + \frac{\left(-76\cos(-8+4t) - 209\sin(-8+4t) + 4\,\mathrm{e}^{-\frac{t}{2}+1}\left(23\sqrt{\right.\right.}{}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= -\frac{2\,\mathrm{e}^{-\frac{t}{2}}\left(\sqrt{19}\,\sin\left(\frac{\sqrt{19}\,t}{2}\right) + 19\cos\left(\frac{\sqrt{19}\,t}{2}\right)\right)}{19} \\
&\quad + \frac{\left(-76\cos(-8+4t) - 209\sin(-8+4t) + 4\,\mathrm{e}^{-\frac{t}{2}+1}\left(23\sqrt{19}\,\sin\left(\frac{\sqrt{19}\,(-2+t)}{2}\right) + 19\cos\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)\right)\right)\,\mathrm{He}}{2603}
\end{aligned}
$$

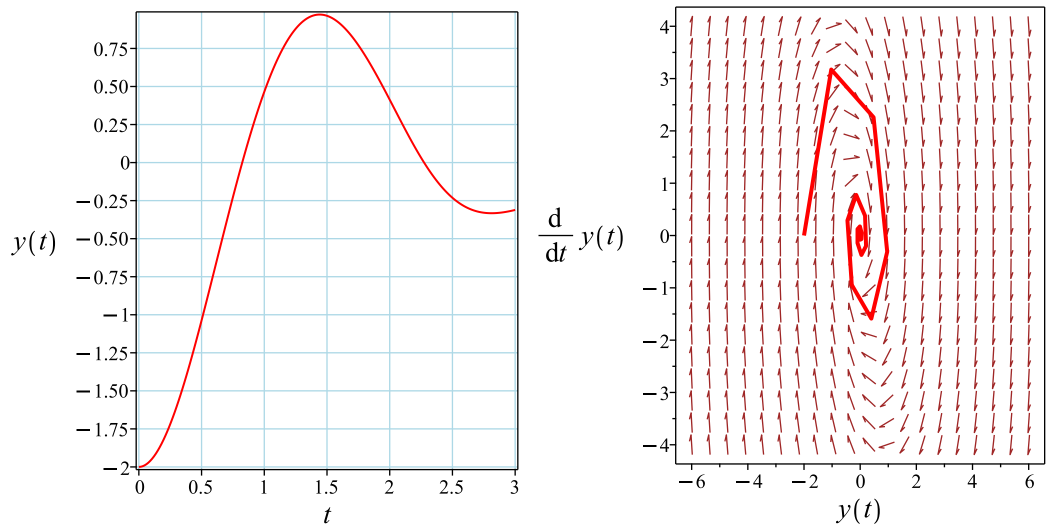
Simplifying the solution gives

$$
\begin{aligned}
y = {} & \frac{4\,\mathrm{Heaviside}(-2+t)\,\mathrm{e}^{-\frac{t}{2}+1}\cos\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{137} \\
& + \frac{92\,\mathrm{Heaviside}(-2+t)\,\sqrt{19}\,\mathrm{e}^{-\frac{t}{2}+1}\sin\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{2603} - 2\cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}} \\
& - \frac{2\sqrt{19}\,\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right)}{19} - \frac{4\,\mathrm{Heaviside}(-2+t)\left(\cos(-8+4t) + \frac{11\sin(-8+4t)}{4}\right)}{137}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{4\,\mathrm{Heaviside}(-2+t)\,\mathrm{e}^{-\frac{t}{2}+1}\cos\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{137} \\
& + \frac{92\,\mathrm{Heaviside}(-2+t)\,\sqrt{19}\,\mathrm{e}^{-\frac{t}{2}+1}\sin\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{2603} - 2\cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}} \\
& - \frac{2\sqrt{19}\,\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right)}{19} - \frac{4\,\mathrm{Heaviside}(-2+t)\left(\cos(-8+4t) + \frac{11\sin(-8+4t)}{4}\right)}{137}
\end{aligned} \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{4\,\mathrm{Heaviside}(-2+t)\,e^{-\frac{t}{2}+1}\cos\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{137} + \frac{92\,\mathrm{Heaviside}(-2+t)\sqrt{19}\,e^{-\frac{t}{2}+1}\sin\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{2603} - 2\cos\left(\frac{\sqrt{19}\,t}{2}\right)e^{-\frac{t}{2}} - \frac{2\sqrt{19}\,e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right)}{19} - \frac{4\,\mathrm{Heaviside}(-2+t)\left(\cos(-8+4t)+\frac{11\sin(-8+4t)}{4}\right)}{137}$$

Verified OK.

21.2.2 Maple step by step solution

Let's solve

$$\left[y''+y'+5y = \mathit{Heaviside}(-2+t)\sin(-8+4t),\, y(0)=-2,\, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+r+5=0$

- Use quadratic formula to solve for r

$r = \frac{(-1)\pm(\sqrt{-19})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{19}}{2}, -\frac{1}{2}+\frac{\mathrm{I}\sqrt{19}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}} + c_2\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathit{Heaviside}(-2+t)\sin(-8\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}} & \mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right) \\ -\frac{\sqrt{19}\,\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right)}{2} - \frac{\cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2} & -\frac{\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{t}{2}}\sqrt{19}\cos\left(\frac{\sqrt{19}\,t}{2}\right)}{2} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \frac{\sqrt{19}\,\mathrm{e}^{-t}}{2}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{2\sqrt{19}\,\mathrm{e}^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{19}\,t}{2}\right)\left(\int \mathrm{e}^{\frac{t}{2}}\sin(-8+4t)\mathit{Heaviside}(-2+t)\sin\left(\frac{\sqrt{19}\,t}{2}\right)dt\right) - \sin\left(\frac{\sqrt{19}\,t}{2}\right)\left(\int \mathrm{e}^{\frac{t}{2}}\sin(-8+4t)\mathit{Heaviside}(-2\right.\right.}{19}$

  - Compute integrals

    $y_p(t) = -\frac{\mathit{Heaviside}(-2+t)\left(-92\,\mathrm{e}^{-\frac{t}{2}+1}\sqrt{19}\sin\left(\frac{\sqrt{19}\,(-2+t)}{2}\right) + 209\sin(-8+4t) + 76\cos(-8+4t) - 76\,\mathrm{e}^{-\frac{t}{2}+1}\cos\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)\right)}{2603}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}} + c_2\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right) - \frac{\mathit{Heaviside}(-2+t)\left(-92\,\mathrm{e}^{-\frac{t}{2}+1}\sqrt{19}\sin\left(\frac{\sqrt{19}\,(-2+t)}{2}\right) + 209\sin(-8+4t) + \right.}{2603}$

- $\square$ Check validity of solution $y = c_1 \cos\left(\frac{\sqrt{19}\,t}{2}\right) \mathrm{e}^{-\frac{t}{2}} + c_2 \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{19}\,t}{2}\right) - \frac{\mathit{Heaviside}(-2+t)\left(-92\mathrm{e}^{-\frac{t}{2}+1}\sqrt{19}\right.}{}$
  - Use initial condition $y(0) = -2$

    $-2 = c_1$
  - Compute derivative of the solution

    $y' = -\frac{c_1\sqrt{19}\,\sin\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2} - \frac{c_1\cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2} - \frac{c_2\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{19}\,t}{2}\right)}{2} + \frac{c_2\mathrm{e}^{-\frac{t}{2}}\sqrt{19}\,\cos\left(\frac{\sqrt{19}\,t}{2}\right)}{2} - \frac{\mathit{Dirac}(-2+t)\left(-92\,\mathrm{e}^{-}\right.}{}$
  - Use the initial condition $y'\big|_{\{t=0\}} = 0$

    $0 = -\frac{c_1}{2} + \frac{c_2\sqrt{19}}{2}$
  - Solve for c_1 and c_2

    $\left\{c_1 = -2, c_2 = -\frac{2\sqrt{19}}{19}\right\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{4\mathit{Heaviside}(-2+t)\mathrm{e}^{-\frac{t}{2}+1}\cos\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{137} + \frac{92\mathit{Heaviside}(-2+t)\sqrt{19}\,\mathrm{e}^{-\frac{t}{2}+1}\sin\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{2603} - 2\cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}} -$
- Solution to the IVP

  $y = \frac{4\mathit{Heaviside}(-2+t)\mathrm{e}^{-\frac{t}{2}+1}\cos\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{137} + \frac{92\mathit{Heaviside}(-2+t)\sqrt{19}\,\mathrm{e}^{-\frac{t}{2}+1}\sin\left(\frac{\sqrt{19}\,(-2+t)}{2}\right)}{2603} - 2\cos\left(\frac{\sqrt{19}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}} -$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.703 (sec). Leaf size: 89

```
dsolve([diff(y(t),t$2)+diff(y(t),t)+5*y(t)=Heaviside(t-2)*sin(4*(t-2)),y(0) = -2, D(y)(0) =
```

$$y(t) = \frac{4\cos\left(\frac{\sqrt{19}\,(t-2)}{2}\right)\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{1-\frac{t}{2}}}{137} + \frac{92\sin\left(\frac{\sqrt{19}\,(t-2)}{2}\right)\mathrm{Heaviside}\,(t-2)\sqrt{19}\,\mathrm{e}^{1-\frac{t}{2}}}{2603} - 2\,\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{19}\,t}{2}\right) - \frac{2\,\mathrm{e}^{-\frac{t}{2}}\sqrt{19}\,\sin\left(\frac{\sqrt{19}\,t}{2}\right)}{19} - \frac{4\left(\cos(4t-8)+\frac{11\sin(4t-8)}{4}\right)\mathrm{Heaviside}\,(t-2)}{137}$$

✓ Solution by Mathematica

Time used: 6.103 (sec). Leaf size: 163

```
DSolve[{y''[t]+y'[t]+5*y[t]==UnitStep[t-2]*Sin[4*(t-2)],{y[0]==-2,y'[0]==0}},y[t],t,IncludeS
```

$y(t)$

$$\rightarrow \left\{ \begin{array}{c} -\frac{2}{19}e^{-t/2}\left(19\cos\left(\frac{\sqrt{19}t}{2}\right)+\sqrt{19}\sin\left(\frac{\sqrt{19}t}{2}\right)\right) \\ \frac{e^{-t/2}\left(-76e^{t/2}\cos(8-4t)+76e\cos\left(\frac{1}{2}\sqrt{19}(t-2)\right)-5206\cos\left(\frac{\sqrt{19}t}{2}\right)+209e^{t/2}\sin(8-4t)+92\sqrt{19}e\sin\left(\frac{1}{2}\sqrt{19}(t-2)\right)-274\sqrt{19}\sin\left(\frac{\sqrt{19}t}{2}\right)\right.}{2603} \end{array} \right.$$

21.3 problem 3

Internal problem ID [13237]
Internal file name [`OUTPUT/11892_Tuesday_December_05_2023_12_12_49_PM_88058161/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.6. page 624
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + 8y = (1 - \text{Heaviside}\,(t - 4)) \cos\,(t - 4)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

21.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 8 \\ F &= (1 - \text{Heaviside}\,(t - 4)) \cos\,(t - 4) \end{aligned}$$

Hence the ode is

$$y'' + y' + 8y = (1 - \text{Heaviside}\,(t - 4)) \cos\,(t - 4)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = (1 - \text{Heaviside}(t-4)) \cos(t-4)$ is

$$\{t < 4 \lor 4 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) + 8Y(s) = -\frac{-\sin(4) + s(e^{-4s} - \cos(4))}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + sY(s) + 8Y(s) = -\frac{-\sin(4) + s(e^{-4s} - \cos(4))}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s\cos(4) - s\,e^{-4s} + \sin(4)}{(s^2+1)(s^2+s+8)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{s\cos(4) - s\,\mathrm{e}^{-4s} + \sin(4)}{(s^2+1)(s^2+s+8)}\right) \\
&= \frac{\cos(t)\,(7\cos(4)-\sin(4))}{50} + \frac{\left(-31\cos\left(\frac{\sqrt{31}\,t}{2}\right)(7\cos(4)-\sin(4)) + \sin\left(\frac{\sqrt{31}\,t}{2}\right)\sqrt{31}\,(-9\cos(4) - 13\right.}{1550}
\end{aligned}
$$

Hence the final solution is

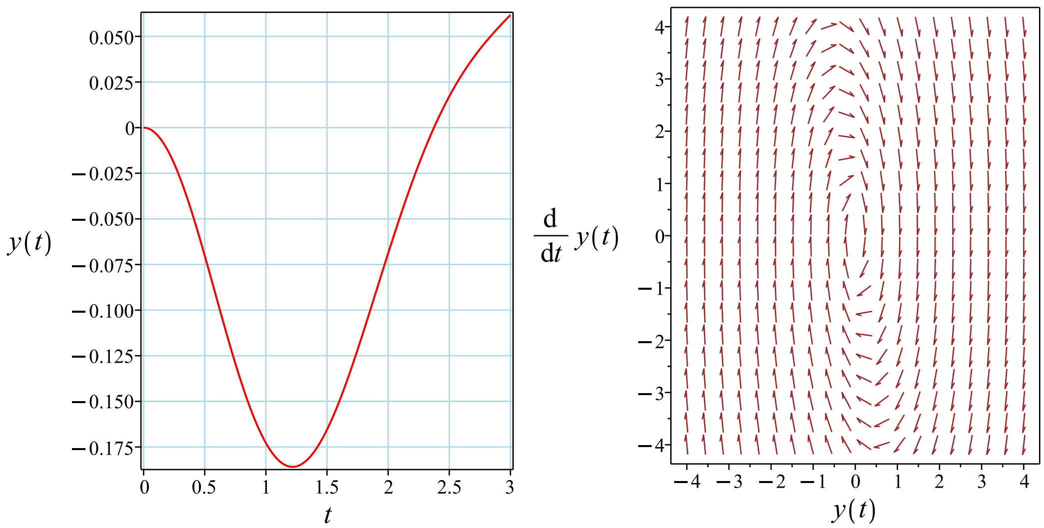
$$
\begin{aligned}
y = {} & \frac{\cos(t)\,(7\cos(4)-\sin(4))}{50} \\
& + \frac{\left(-31\cos\left(\frac{\sqrt{31}\,t}{2}\right)(7\cos(4)-\sin(4)) + \sin\left(\frac{\sqrt{31}\,t}{2}\right)\sqrt{31}\,(-9\cos(4) - 13\sin(4))\right)\mathrm{e}^{-\frac{t}{2}}}{1550} \\
& - \frac{\sin(t)\,(-\cos(4) - 7\sin(4))}{50} \\
& + \frac{\left(-217\cos(t-4) - 31\sin(t-4) + \mathrm{e}^{-\frac{t}{2}+2}\left(9\sqrt{31}\,\sin\left(\frac{\sqrt{31}\,(t-4)}{2}\right) + 217\cos\left(\frac{\sqrt{31}\,(t-4)}{2}\right)\right)\right)\mathrm{Heaviside}\,(t}{1550}
\end{aligned}
$$

Simplifying the solution gives

$$
\begin{aligned}
y = {} & \\
& - \frac{9\left(\left(\sqrt{31}\,\sin(2\sqrt{31}) - \frac{217\cos(2\sqrt{31})}{9}\right)\cos\left(\frac{\sqrt{31}\,t}{2}\right) - \frac{217\sin\left(\frac{\sqrt{31}\,t}{2}\right)\left(\frac{9\sqrt{31}\,\cos(2\sqrt{31})}{217} + \sin(2\sqrt{31})\right)}{9}\right)\mathrm{Heaviside}\,(t}{1550} \\
& - \frac{7\left(\cos(4) - \frac{\sin(4)}{7}\right)\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{31}\,t}{2}\right)}{50} - \frac{9\left(\cos(4) + \frac{13\sin(4)}{9}\right)\sqrt{31}\,\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{31}\,t}{2}\right)}{1550} \\
& - \frac{7(-1+\mathrm{Heaviside}\,(t-4))\left(\left(\cos(t) + \frac{\sin(t)}{7}\right)\cos(4) - \frac{\sin(4)(\cos(t)-7\sin(t))}{7}\right)}{50}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & -\frac{9\left(\left(\sqrt{31}\,\sin\left(2\sqrt{31}\right)-\frac{217\cos\left(2\sqrt{31}\right)}{9}\right)\cos\left(\frac{\sqrt{31}\,t}{2}\right)-\frac{217\sin\left(\frac{\sqrt{31}\,t}{2}\right)\left(\frac{9\sqrt{31}\,\cos\left(2\sqrt{31}\right)}{217}+\sin\left(2\sqrt{31}\right)\right)}{9}\right)\mathrm{Heaviside}\,(t}{1550} \\
& -\frac{7\left(\cos(4)-\frac{\sin(4)}{7}\right)\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{31}\,t}{2}\right)}{50}-\frac{9\left(\cos(4)+\frac{13\sin(4)}{9}\right)\sqrt{31}\,\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{31}\,t}{2}\right)}{1550} \\
& -\frac{7(-1+\mathrm{Heaviside}\,(t-4))\left(\left(\cos(t)+\frac{\sin(t)}{7}\right)\cos(4)-\frac{\sin(4)(\cos(t)-7\sin(t))}{7}\right)}{50}
\end{aligned}
\tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{9\left(\left(\sqrt{31}\,\sin\left(2\sqrt{31}\right)-\frac{217\cos\left(2\sqrt{31}\right)}{9}\right)\cos\left(\frac{\sqrt{31}\,t}{2}\right)-\frac{217\sin\left(\frac{\sqrt{31}\,t}{2}\right)\left(\frac{9\sqrt{31}\,\cos\left(2\sqrt{31}\right)}{217}+\sin\left(2\sqrt{31}\right)\right)}{9}\right)\text{Heaviside}\,(t}{1550}$$
$$-\frac{7\left(\cos(4)-\frac{\sin(4)}{7}\right)\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{31}\,t}{2}\right)}{50}-\frac{9\left(\cos(4)+\frac{13\sin(4)}{9}\right)\sqrt{31}\,\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{31}\,t}{2}\right)}{1550}$$
$$-\frac{7(-1+\text{Heaviside}\,(t-4))\left(\left(\cos(t)+\frac{\sin(t)}{7}\right)\cos(4)-\frac{\sin(4)(\cos(t)-7\sin(t))}{7}\right)}{50}$$

Verified OK.

21.3.2 Maple step by step solution

Let's solve

$$\left[y''+y'+8y=(1-\mathit{Heaviside}(t-4))\cos(t-4)\,,y(0)=0,y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\cos(t-4)\,\mathit{Heaviside}(t-4)+\cos(t-4)-8y-y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+y'+8y=-(-1+\mathit{Heaviside}(t-4))\cos(t-4)$

- Characteristic polynomial of homogeneous ODE

  $r^2+r+8=0$

- Use quadratic formula to solve for r

  $r=\frac{(-1)\pm\left(\sqrt{-31}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2},-\frac{1}{2}+\frac{\mathrm{I}\sqrt{31}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}$

- 2nd solution of the homogeneous ODE

$$y_2(t)=\sin\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}$$

- General solution of the ODE

$$y=c_1y_1(t)+c_2y_2(t)+y_p(t)$$

- Substitute in solutions of the homogeneous ODE

$$y=c_1\cos\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}+c_2\sin\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}+y_p(t)$$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=-(-1+\mathit{Heaviside}(t-4))\,c\right.$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t),y_2(t))=\begin{bmatrix}\cos\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}} & \sin\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}\\ -\frac{\sqrt{31}\sin\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{\cos\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2} & \frac{\sqrt{31}\cos\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{\sin\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2}\end{bmatrix}$$

- ○ Compute Wronskian

$$W(y_1(t),y_2(t))=\frac{\sqrt{31}\,\mathrm{e}^{-t}}{2}$$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t)=\frac{2\sqrt{31}\,\mathrm{e}^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{31}\,t}{2}\right)\left(\int\mathrm{e}^{\frac{t}{2}}\cos(t-4)\sin\left(\frac{\sqrt{31}\,t}{2}\right)(-1+\mathit{Heaviside}(t-4))dt\right)-\sin\left(\frac{\sqrt{31}\,t}{2}\right)\left(\int\mathrm{e}^{\frac{t}{2}}\cos(t-4)\cos\left(\frac{\sqrt{31}\,t}{2}\right)(-1+\right.\right.}{31}$$

- ○ Compute integrals

$$y_p(t)=\frac{7\mathit{Heaviside}(t-4)\mathrm{e}^{-\frac{t}{2}+2}\cos\left(\frac{\sqrt{31}\,(t-4)}{2}\right)}{50}+\frac{9\mathit{Heaviside}(t-4)\sqrt{31}\,\mathrm{e}^{-\frac{t}{2}+2}\sin\left(\frac{\sqrt{31}\,(t-4)}{2}\right)}{1550}-\frac{7(-1+\mathit{Heaviside}(t-4))\left(\cos(\right.}{50}$$

- Substitute particular solution into general solution to ODE

$$y=c_1\cos\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}+c_2\sin\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}+\frac{7\mathit{Heaviside}(t-4)\mathrm{e}^{-\frac{t}{2}+2}\cos\left(\frac{\sqrt{31}\,(t-4)}{2}\right)}{50}+\frac{9\mathit{Heaviside}(t-4)\sqrt{31}\,\mathrm{e}^{-\frac{t}{2}+}}{1550}$$

☐ Check validity of solution $y=c_1\cos\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}+c_2\sin\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}+\frac{7\mathit{Heaviside}(t-4)\mathrm{e}^{-\frac{t}{2}+2}\cos\left(\frac{\sqrt{31}\,(}{2}\right.}{50}$

- ○ Use initial condition $y(0)=0$

$$0=c_1+\frac{7\cos(4)}{50}-\frac{\sin(4)}{50}$$

- ○ Compute derivative of the solution

$$y'=-\frac{c_1\sqrt{31}\sin\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{c_1\cos\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2}+\frac{c_2\sqrt{31}\cos\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2}-\frac{c_2\sin\left(\frac{\sqrt{31}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{2}+\frac{7\mathit{Dirac}(t-4)\mathrm{e}^{-\frac{t}{2}+2}\mathrm{co}}{50}$$

- ○ Use the initial condition $y'\Big|_{\{t=0\}}=0$

$$0=-\frac{c_1}{2}+\frac{c_2\sqrt{31}}{2}+\frac{\cos(4)}{50}+\frac{7\sin(4)}{50}$$

- Solve for c_1 and c_2

$$\left\{c_1=-\frac{7\cos(4)}{50}+\frac{\sin(4)}{50}, c_2=-\frac{\sqrt{31}\,(9\cos(4)+13\sin(4))}{1550}\right\}$$

- Substitute constant values into general solution and simplify

$$y=-\frac{9\left(\left(\sqrt{31}\,\sin\left(2\sqrt{31}\right)-\frac{217\cos\left(2\sqrt{31}\right)}{9}\right)\cos\left(\frac{\sqrt{31}\,t}{2}\right)-\frac{217\sin\left(\frac{\sqrt{31}\,t}{2}\right)\left(\frac{9\sqrt{31}\,\cos\left(2\sqrt{31}\right)}{217}+\sin\left(2\sqrt{31}\right)\right)}{9}\right)\mathit{Heaviside}(t-4)\,\mathrm{e}^{-\frac{t}{2}+2}}{1550}-$$

- Solution to the IVP

$$y=-\frac{9\left(\left(\sqrt{31}\,\sin\left(2\sqrt{31}\right)-\frac{217\cos\left(2\sqrt{31}\right)}{9}\right)\cos\left(\frac{\sqrt{31}\,t}{2}\right)-\frac{217\sin\left(\frac{\sqrt{31}\,t}{2}\right)\left(\frac{9\sqrt{31}\,\cos\left(2\sqrt{31}\right)}{217}+\sin\left(2\sqrt{31}\right)\right)}{9}\right)\mathit{Heaviside}(t-4)\,\mathrm{e}^{-\frac{t}{2}+2}}{1550}-$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.797 (sec). Leaf size: 128

```
dsolve([diff(y(t),t$2)+diff(y(t),t)+8*y(t)=(1-Heaviside(t-4))*cos(t-4),y(0) = 0, D(y)(0) = 0
```

$$y(t) = -\frac{9\,\mathrm{Heaviside}(t-4)\left(\left(\sin\left(2\sqrt{31}\right)\sqrt{31}-\frac{217\cos\left(2\sqrt{31}\right)}{9}\right)\cos\left(\frac{\sqrt{31}\,t}{2}\right)-\frac{217\sin\left(\frac{\sqrt{31}\,t}{2}\right)\left(\frac{9\sqrt{31}\,\cos\left(2\sqrt{31}\right)}{217}+\sin\left(2\sqrt{3\right.}\right.}{9}\right.}{1550}$$

$$-\frac{7\,\mathrm{e}^{-\frac{t}{2}}\left(\cos(4)-\frac{\sin(4)}{7}\right)\cos\left(\frac{\sqrt{31}\,t}{2}\right)}{50}-\frac{9\left(\cos(4)+\frac{13\sin(4)}{9}\right)\sqrt{31}\,\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{31}\,t}{2}\right)}{1550}$$

$$-\frac{7\left(\left(\cos(t)+\frac{\sin(t)}{7}\right)\cos(4)-\frac{\sin(4)(-7\sin(t)+\cos(t))}{7}\right)(-1+\mathrm{Heaviside}(t-4))}{50}$$

✓ Solution by Mathematica

Time used: 4.688 (sec). Leaf size: 207

```
DSolve[{y''[t]+y'[t]+8*y[t]==(1-UnitStep[t-4])*Cos[t-4],{y[0]==0,y'[0]==0}},y[t],t,IncludeSi
```

$$y(t) \to \frac{e^{-t/2}\left(\theta(4-t)\left(-31e^{t/2}\sin(4-t)-9\sqrt{31}e^2\sin\left(\frac{1}{2}\sqrt{31}(t-4)\right)+217e^{t/2}\cos(4-t)-217e^2\cos\left(\frac{1}{2}\sqrt{31}(\right.\right.\right.}{}$$

21.4 problem 4

Internal problem ID [13238]
Internal file name [`OUTPUT/11893_Tuesday_December_05_2023_12_12_50_PM_50415031/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.6. page 624
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + y' + 3y = (1 - \text{Heaviside}\,(-2+t))\,\mathrm{e}^{\frac{1}{5}-\frac{t}{10}} \sin\,(-2+t)$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

21.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 3 \\ F &= -(-1 + \text{Heaviside}\,(-2+t))\,\mathrm{e}^{\frac{1}{5}-\frac{t}{10}} \sin\,(-2+t) \end{aligned}$$

Hence the ode is

$$y'' + y' + 3y = -(-1 + \text{Heaviside}\,(-2+t))\,\mathrm{e}^{\frac{1}{5}-\frac{t}{10}} \sin\,(-2+t)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -(-1 + \text{Heaviside}(-2 + t))\, e^{\frac{1}{5} - \frac{t}{10}} \sin($ is

$$\{t < 2 \vee 2 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) + 3Y(s) = 5i\left(\frac{e^{-2s} - e^{\frac{1}{5}-2i}}{10s + 1 - 10i} + \frac{-e^{-2s} + e^{\frac{1}{5}+2i}}{10s + 1 + 10i}\right) \tag{1}$$

But the initial conditions are

$$y(0) = 1$$
$$y'(0) = 2$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - s + sY(s) + 3Y(s) = 5i\left(\frac{e^{-2s} - e^{\frac{1}{5}-2i}}{10s + 1 - 10i} + \frac{-e^{-2s} + e^{\frac{1}{5}+2i}}{10s + 1 + 10i}\right)$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{50ie^{\frac{1}{5}-2i}s - 50ie^{\frac{1}{5}+2i}s - 100s^3 + 5ie^{\frac{1}{5}-2i} - 5ie^{\frac{1}{5}+2i} - 320s^2 + 100\,e^{-2s} - 50\,e^{\frac{1}{5}-2i} - 50\,e^{\frac{1}{5}+2i} - 161}{(-10s - 1 + 10i)\,(10s + 1 + 10i)\,(s^2 + s + 3)}$$

Taking the inverse Laplace transform gives

$$y = \mathcal{L}^{-1}(Y(s))$$
$$= \mathcal{L}^{-1}\left(\frac{50i\mathrm{e}^{\frac{1}{5}-2i}s - 50i\mathrm{e}^{\frac{1}{5}+2i}s - 100s^3 + 5i\mathrm{e}^{\frac{1}{5}-2i} - 5i\mathrm{e}^{\frac{1}{5}+2i} - 320s^2 + 100\,\mathrm{e}^{-2s} - 50\,\mathrm{e}^{\frac{1}{5}-2i} - 50\,\mathrm{e}^{\frac{1}{5}+2i} - 1}{(-10s-1+10i)(10s+1+10i)(s^2+s+3)}\right.$$
$$= \left(\frac{80}{1838780161} - \frac{191i}{1838780161}\right)\left(2144050\,\mathrm{e}^{\frac{1}{5}+2i-\frac{t}{2}} + (3430480+8190271i)\,\mathrm{e}^{-\frac{t}{2}} + (-1504050 + 152800\right.$$

Hence the final solution is

$$y = \left(\frac{80}{1838780161} - \frac{191i}{1838780161}\right)\left(2144050\,\mathrm{e}^{\frac{1}{5}+2i-\frac{t}{2}} + (3430480+8190271i)\,\mathrm{e}^{-\frac{t}{2}}\right.$$
$$\left. + (-1504050+1528000i)\,\mathrm{e}^{\frac{1}{5}-2i-\frac{t}{2}}\right)\cos\left(\frac{\sqrt{11}\,t}{2}\right)$$
$$+ \left(-\frac{4000}{42881} + \frac{9550i}{42881}\right)\mathrm{e}^{\left(-\frac{1}{10}-i\right)(-2+t)} + \left(-\frac{4000}{42881} - \frac{9550i}{42881}\right)\mathrm{e}^{\left(-\frac{1}{10}+i\right)(-2+t)}$$
$$+ \left(\frac{3975}{586570871359} + \frac{3910i}{586570871359}\right)\sqrt{11}\left(-4974196\,\mathrm{e}^{\frac{1}{5}+2i-\frac{t}{2}}\right.$$
$$\left. + (34090395-33532942i)\,\mathrm{e}^{-\frac{t}{2}} + (-82004+4973520i)\,\mathrm{e}^{\frac{1}{5}-2i-\frac{t}{2}}\right)\sin\left(\frac{\sqrt{11}\,t}{2}\right)$$
$$+ \frac{100\left(2\,\mathrm{e}^{-\frac{t}{2}+1}\left(159\sqrt{11}\,\sin\left(\frac{\sqrt{11}\,(-2+t)}{2}\right) - 440\cos\left(\frac{\sqrt{11}\,(-2+t)}{2}\right)\right) + 11(-191\sin(-2+t) + 80\cos(-2+\right.}{471691}$$

Simplifying the solution gives

$$y$$
$$= \frac{8000\,\text{Heaviside}(-2+t)\left(\left(\cos(t) - \frac{191\sin(t)}{80}\right)\cos(2) + \frac{191\left(\cos(t)+\frac{80\sin(t)}{191}\right)\sin(2)}{80}\right)\mathrm{e}^{\frac{1}{5}-\frac{t}{10}}}{42881}$$
$$+ \frac{100\left(11(191\sin(2)+80\cos(2))\cos\left(\frac{\sqrt{11}\,t}{2}\right) - 318\sqrt{11}\left(\cos(2) - \frac{782\sin(2)}{795}\right)\sin\left(\frac{\sqrt{11}\,t}{2}\right)\right)\mathrm{e}^{\frac{1}{5}-\frac{t}{2}}}{471691}$$
$$+ \left(-\frac{4000}{42881} + \frac{9550i}{42881}\right)\mathrm{e}^{\left(-\frac{1}{10}-i\right)(-2+t)} + \left(-\frac{4000}{42881} - \frac{9550i}{42881}\right)\mathrm{e}^{\left(-\frac{1}{10}+i\right)(-2+t)}$$
$$+ \frac{200\,\text{Heaviside}(-2+t)\left(\left(-159\sqrt{11}\,\sin\left(\sqrt{11}\right) - 440\cos\left(\sqrt{11}\right)\right)\cos\left(\frac{\sqrt{11}\,t}{2}\right) + \left(159\sqrt{11}\,\cos\left(\sqrt{11}\right) - \right.\right.}{471691}$$
$$+ \frac{5\sqrt{11}\,\sin\left(\frac{\sqrt{11}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}}{11} + \cos\left(\frac{\sqrt{11}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{8000 \,\mathrm{Heaviside}\left(-2+t\right) \left(\left(\cos\left(t\right)-\frac{191 \sin\left(t\right)}{80}\right) \cos\left(2\right)+\frac{191 \left(\cos\left(t\right)+\frac{80 \sin\left(t\right)}{191}\right) \sin\left(2\right)}{80}\right) \mathrm{e}^{\frac{1}{5}-\frac{t}{10}}}{42881} \qquad (1) \\
& + \frac{100 \left(11 \left(191 \sin\left(2\right)+80 \cos\left(2\right)\right) \cos\left(\frac{\sqrt{11}\, t}{2}\right)-318 \sqrt{11}\, \left(\cos\left(2\right)-\frac{782 \sin\left(2\right)}{795}\right) \sin\left(\frac{\sqrt{11}\, t}{2}\right)\right) \mathrm{e}^{\frac{1}{5}-\frac{t}{2}}}{471691} \\
& + \left(-\frac{4000}{42881}+\frac{9550 i}{42881}\right) \mathrm{e}^{\left(-\frac{1}{10}-i\right) \left(-2+t\right)}+\left(-\frac{4000}{42881}-\frac{9550 i}{42881}\right) \mathrm{e}^{\left(-\frac{1}{10}+i\right) \left(-2+t\right)} \\
& + \frac{200 \,\mathrm{Heaviside}\left(-2+t\right) \left(\left(-159 \sqrt{11}\, \sin\left(\sqrt{11}\right)-440 \cos\left(\sqrt{11}\right)\right) \cos\left(\frac{\sqrt{11}\, t}{2}\right)+\left(159 \sqrt{11}\, \cos\left(\sqrt{11}\right)-\right.}{471691} \\
& + \frac{5 \sqrt{11}\, \sin\left(\frac{\sqrt{11}\, t}{2}\right) \mathrm{e}^{-\frac{t}{2}}}{11}+\cos\left(\frac{\sqrt{11}\, t}{2}\right) \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
$$

Verification of solutions

$$
\begin{aligned}
& y \\
& = \frac{8000 \,\mathrm{Heaviside}\left(-2+t\right) \left(\left(\cos\left(t\right)-\frac{191 \sin\left(t\right)}{80}\right) \cos\left(2\right)+\frac{191 \left(\cos\left(t\right)+\frac{80 \sin\left(t\right)}{191}\right) \sin\left(2\right)}{80}\right) \mathrm{e}^{\frac{1}{5}-\frac{t}{10}}}{42881} \\
& + \frac{100 \left(11 \left(191 \sin\left(2\right)+80 \cos\left(2\right)\right) \cos\left(\frac{\sqrt{11}\, t}{2}\right)-318 \sqrt{11}\, \left(\cos\left(2\right)-\frac{782 \sin\left(2\right)}{795}\right) \sin\left(\frac{\sqrt{11}\, t}{2}\right)\right) \mathrm{e}^{\frac{1}{5}-\frac{t}{2}}}{471691} \\
& + \left(-\frac{4000}{42881}+\frac{9550 i}{42881}\right) \mathrm{e}^{\left(-\frac{1}{10}-i\right) \left(-2+t\right)}+\left(-\frac{4000}{42881}-\frac{9550 i}{42881}\right) \mathrm{e}^{\left(-\frac{1}{10}+i\right) \left(-2+t\right)} \\
& + \frac{200 \,\mathrm{Heaviside}\left(-2+t\right) \left(\left(-159 \sqrt{11}\, \sin\left(\sqrt{11}\right)-440 \cos\left(\sqrt{11}\right)\right) \cos\left(\frac{\sqrt{11}\, t}{2}\right)+\left(159 \sqrt{11}\, \cos\left(\sqrt{11}\right)-\right.}{471691} \\
& + \frac{5 \sqrt{11}\, \sin\left(\frac{\sqrt{11}\, t}{2}\right) \mathrm{e}^{-\frac{t}{2}}}{11}+\cos\left(\frac{\sqrt{11}\, t}{2}\right) \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
$$

Verified OK.

21.4.2 Maple step by step solution

Let's solve

$$\left[y''+y'+3y=(1-\mathit{Heaviside}(-2+t))\,\mathrm{e}^{\frac{1}{5}-\frac{t}{10}}\sin(-2+t)\,,y(0)=1,y'\Big|_{\{t=0\}}=2\right]$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y''=-3y-\mathrm{e}^{\frac{1}{5}-\frac{t}{10}}\sin(-2+t)\,\mathit{Heaviside}(-2+t)+\mathrm{e}^{\frac{1}{5}-\frac{t}{10}}\sin(-2+t)-y'$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y''+y'+3y=-(-1+\mathit{Heaviside}(-2+t))\,\mathrm{e}^{\frac{1}{5}-\frac{t}{10}}\sin(-2+t)$$

- Characteristic polynomial of homogeneous ODE

$$r^2+r+3=0$$

- Use quadratic formula to solve for r

$$r=\frac{(-1)\pm(\sqrt{-11})}{2}$$

- Roots of the characteristic polynomial

$$r=\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{11}}{2},-\frac{1}{2}+\frac{\mathrm{I}\sqrt{11}}{2}\right)$$

- 1st solution of the homogeneous ODE

$$y_1(t)=\cos\left(\frac{\sqrt{11}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}$$

- 2nd solution of the homogeneous ODE

$$y_2(t)=\sin\left(\frac{\sqrt{11}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}$$

- General solution of the ODE

$$y=c_1y_1(t)+c_2y_2(t)+y_p(t)$$

- Substitute in solutions of the homogeneous ODE

$$y=c_1\cos\left(\frac{\sqrt{11}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}+c_2\sin\left(\frac{\sqrt{11}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}+y_p(t)$$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

$$\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right),f(t)=-(-1+\mathit{Heaviside}(-2+t)\right.$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}} & \sin\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}} \\ -\frac{\sqrt{11}\,\sin\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}}}{2} - \frac{\cos\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}}}{2} & \frac{\sqrt{11}\,\cos\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}}}{2} - \frac{\sin\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}}}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = \frac{\sqrt{11}\,e^{-t}}{2}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{2\sqrt{11}\,e^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{11}\,t}{2}\right)\left(\int e^{\frac{2t}{5}+\frac{1}{5}} \sin(-2+t) \sin\left(\frac{\sqrt{11}\,t}{2}\right)(-1+Heaviside(-2+t))dt\right) - \sin\left(\frac{\sqrt{11}\,t}{2}\right)\left(\int e^{\frac{2t}{5}+\frac{1}{5}} \sin(-2+t) \cos\left(\right.\right.\right.}{11}$

- Compute integrals

  $y_p(t) = \frac{31800\,e^{-\frac{t}{2}}\left(-\frac{440 Heaviside(-2+t) e \cos\left(\frac{\sqrt{11}\,(-2+t)}{2}\right)}{159} + Heaviside(-2+t)\sqrt{11}\,\sin\left(\frac{\sqrt{11}\,(-2+t)}{2}\right) e + \frac{440\,e^{\frac{2t}{5}+\frac{1}{5}}(-1+Heaviside(-2\right.}{471691}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}} + c_2 \sin\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}} + \frac{31800\,e^{-\frac{t}{2}}\left(-\frac{440 Heaviside(-2+t) e \cos\left(\frac{\sqrt{11}\,(-2+t)}{2}\right)}{159} + Heaviside(-2+t)\right.}{}$

☐ Check validity of solution $y = c_1 \cos\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}} + c_2 \sin\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}} + \frac{31800 e^{-\frac{t}{2}}\left(-\frac{440 Heaviside(-2+t) e}{159}\right.}{}$

- Use initial condition $y(0) = 1$

  $1 = c_1 - \frac{8000\,e^{\frac{1}{5}}\left(\cos(2) + \frac{191\sin(2)}{80}\right)}{42881}$

- Compute derivative of the solution

  $y' = -\frac{c_1 \sin\left(\frac{\sqrt{11}\,t}{2}\right)\sqrt{11}\,e^{-\frac{t}{2}}}{2} - \frac{c_1 \cos\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}}}{2} + \frac{c_2\sqrt{11}\,\cos\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}}}{2} - \frac{c_2 \sin\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{t}{2}}}{2} - \frac{15900\,e^{-\frac{t}{2}}\left(-\frac{440 Heav}{}\right.}{}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = -\frac{c_1}{2} + \frac{\sqrt{11}\,c_2}{2} + \frac{800\,e^{\frac{1}{5}}\left(\cos(2) + \frac{191\sin(2)}{80}\right)}{42881} - \frac{8000\,e^{\frac{1}{5}}\left(-\frac{191\cos(2)}{80} + \sin(2)\right)}{42881}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{19100\,e^{\frac{1}{5}}\sin(2)}{42881} + \frac{8000\,e^{\frac{1}{5}}\cos(2)}{42881} + 1, c_2 = \frac{5\left(6256\,e^{\frac{1}{5}}\sin(2) - 6360\,e^{\frac{1}{5}}\cos(2) + 42881\right)\sqrt{11}}{471691}\right\}$

- Substitute constant values into general solution and simplify

$$y = -\frac{8000\,e^{-\frac{t}{2}}\left(-\left(-1+Heaviside(-2+t)\right)\left(\left(\left(\cos(t)-\frac{191\sin(t)}{80}\right)\cos(2)+\frac{191\left(\cos(t)+\frac{80\sin(t)}{191}\right)\sin(2)}{80}\right)e^{\frac{2t}{5}+\frac{1}{5}}+\left(e\left(\frac{159\sqrt{11}\,\sin\left(\sqrt{11}\right.}{440}\right.\right.\right.\right.}{}$$

- Solution to the IVP

$$y = -\frac{8000\,e^{-\frac{t}{2}}\left(-\left(-1+Heaviside(-2+t)\right)\left(\left(\left(\cos(t)-\frac{191\sin(t)}{80}\right)\cos(2)+\frac{191\left(\cos(t)+\frac{80\sin(t)}{191}\right)\sin(2)}{80}\right)e^{\frac{2t}{5}+\frac{1}{5}}+\left(e\left(\frac{159\sqrt{11}\,\sin\left(\sqrt{11}\right.}{440}\right.\right.\right.\right.}{}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.812 (sec). Leaf size: 178

```
dsolve([diff(y(t),t$2)+diff(y(t),t)+3*y(t)=(1-Heaviside(t-2))*exp(-(t-2)/10)*sin(t-2),y(0) =
```

$$y(t) = \frac{8000\left(\left(\cos(t)-\frac{191\sin(t)}{80}\right)\cos(2)+\frac{191\sin(2)\left(\cos(t)+\frac{80\sin(t)}{191}\right)}{80}\right)\text{Heaviside}(t-2)\,e^{-\frac{t}{10}+\frac{1}{5}}}{42881}$$
$$+\frac{100\left(11(80\cos(2)+191\sin(2))\cos\left(\frac{\sqrt{11}\,t}{2}\right)-318\left(\cos(2)-\frac{782\sin(2)}{795}\right)\sin\left(\frac{\sqrt{11}\,t}{2}\right)\sqrt{11}\right)e^{\frac{1}{5}-\frac{t}{2}}}{471691}$$
$$+\left(-\frac{4000}{42881}+\frac{9550i}{42881}\right)e^{\left(-\frac{1}{10}-i\right)(t-2)}+\left(-\frac{4000}{42881}-\frac{9550i}{42881}\right)e^{\left(-\frac{1}{10}+i\right)(t-2)}$$
$$+\frac{200\,\text{Heaviside}(t-2)\left(\left(-159\sqrt{11}\sin\left(\sqrt{11}\right)-440\cos\left(\sqrt{11}\right)\right)\cos\left(\frac{\sqrt{11}\,t}{2}\right)+\left(159\cos\left(\sqrt{11}\right)\sqrt{11}-4\right.}{471691}$$
$$+\frac{5\,e^{-\frac{t}{2}}\sqrt{11}\sin\left(\frac{\sqrt{11}\,t}{2}\right)}{11}+e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right)$$

✓ Solution by Mathematica

Time used: 6.103 (sec). Leaf size: 243

```
DSolve[{y''[t]+y'[t]+8*y[t]==(1-UnitStep[t-2])*Exp[-(t-2)/10]*Sin[t-2],{y[0]==1,y'[0]==2}},y
```

$$y(t)$$
$$\to \left\{ \begin{array}{l} \frac{e^{-t/2}\left(-248000e^{\frac{2t}{5}+\frac{1}{5}}\cos(2-t)+5\left(\sqrt{31}\left(483881-8\sqrt[5]{e}(3295\cos(2)-1782\sin(2))\right)\sin\left(\frac{\sqrt{31}t}{2}\right)-428420e^{\frac{2t}{5}+\frac{1}{5}}\sin(2-t)\right)+31\cos\left(\right.\right.}{15000311} \\ \frac{e^{-t/2}\left(-248000e\cos\left(\frac{1}{2}\sqrt{31}(t-2)\right)+5\sqrt{31}\left(26360e\sin\left(\frac{1}{2}\sqrt{31}(t-2)\right)+\left(483881-8\sqrt[5]{e}(3295\cos(2)-1782\sin(2))\right)\sin\left(\frac{\sqrt{31}t}{2}\right)\right)+31\cos\right.}{15000311} \end{array}\right.$$

21.5 problem 5

Internal problem ID [13239]
Internal file name [`OUTPUT/11894_Tuesday_December_05_2023_12_12_51_PM_18318295/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.6. page 624
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 16y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

21.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 16 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 16y = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 16$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 16Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s + 16Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s+1}{s^2+16}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{1}{2} - \frac{i}{8}}{s-4i} + \frac{\frac{1}{2} + \frac{i}{8}}{s+4i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2}-\frac{i}{8}}{s-4i}\right) = \left(\frac{1}{2}-\frac{i}{8}\right) e^{4it}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2}+\frac{i}{8}}{s+4i}\right) = \left(\frac{1}{2}+\frac{i}{8}\right) e^{-4it}$$

Adding the above results and simplifying gives

$$y = \cos(4t) + \frac{\sin(4t)}{4}$$

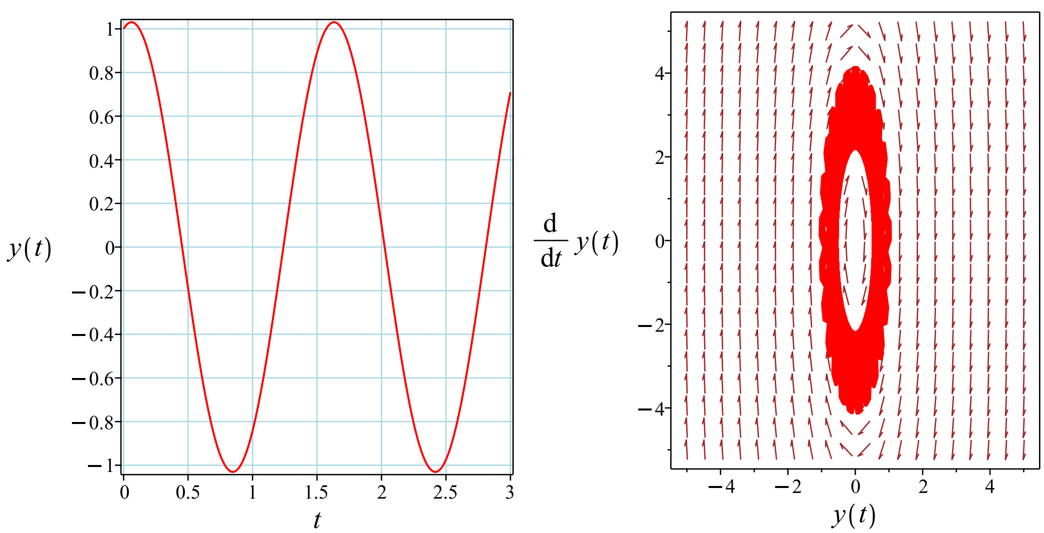
Simplifying the solution gives

$$y = \cos(4t) + \frac{\sin(4t)}{4}$$

Summary

The solution(s) found are the following

$$y = \cos(4t) + \frac{\sin(4t)}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \cos(4t) + \frac{\sin(4t)}{4}$$

Verified OK.

21.5.2 Maple step by step solution

Let's solve

$\left[y'' + 16y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-4\,\mathrm{I}, 4\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \cos(4t)$

- 2nd solution of the ODE

  $y_2(t) = \sin(4t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \cos(4t) + c_2 \sin(4t)$

□ Check validity of solution $y = c_1 \cos(4t) + c_2 \sin(4t)$

  - Use initial condition $y(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $y' = -4c_1 \sin(4t) + 4c_2 \cos(4t)$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = 4c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = 1, c_2 = \frac{1}{4}\right\}$

- Substitute constant values into general solution and simplify

  $y = \cos(4t) + \frac{\sin(4t)}{4}$

- Solution to the IVP

  $y = \cos(4t) + \frac{\sin(4t)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.594 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)+16*y(t)=0,y(0) = 1, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \cos(4t) + \frac{\sin(4t)}{4}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 18

```
DSolve[{y''[t]+16*y[t]==0,{y[0]==1,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \frac{1}{4}\sin(4t) + \cos(4t)$$

21.6 problem 6

Internal problem ID [13240]
Internal file name [`OUTPUT/11895_Tuesday_December_05_2023_12_12_51_PM_74600233/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.6. page 624
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sin(2t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

21.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= \sin(2t) \end{aligned}$$

Hence the ode is

$$y'' + 4y = \sin(2t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{2}{s^2+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4Y(s) = \frac{2}{s^2+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2}{(s^2+4)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{8(s-2i)^2} - \frac{1}{8(s+2i)^2} - \frac{i}{16(s-2i)} + \frac{i}{16s+32i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{8\left(s-2i\right)^{2}}\right) = -\frac{t\,\mathrm{e}^{2it}}{8}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{8\left(s+2i\right)^{2}}\right) = -\frac{t\,\mathrm{e}^{-2it}}{8}$$
$$\mathcal{L}^{-1}\left(-\frac{i}{16\left(s-2i\right)}\right) = -\frac{i\mathrm{e}^{2it}}{16}$$
$$\mathcal{L}^{-1}\left(\frac{i}{16s+32i}\right) = \frac{i\mathrm{e}^{-2it}}{16}$$

Adding the above results and simplifying gives

$$y = \frac{\sin\left(2t\right)}{8}-\frac{\cos\left(2t\right)t}{4}$$

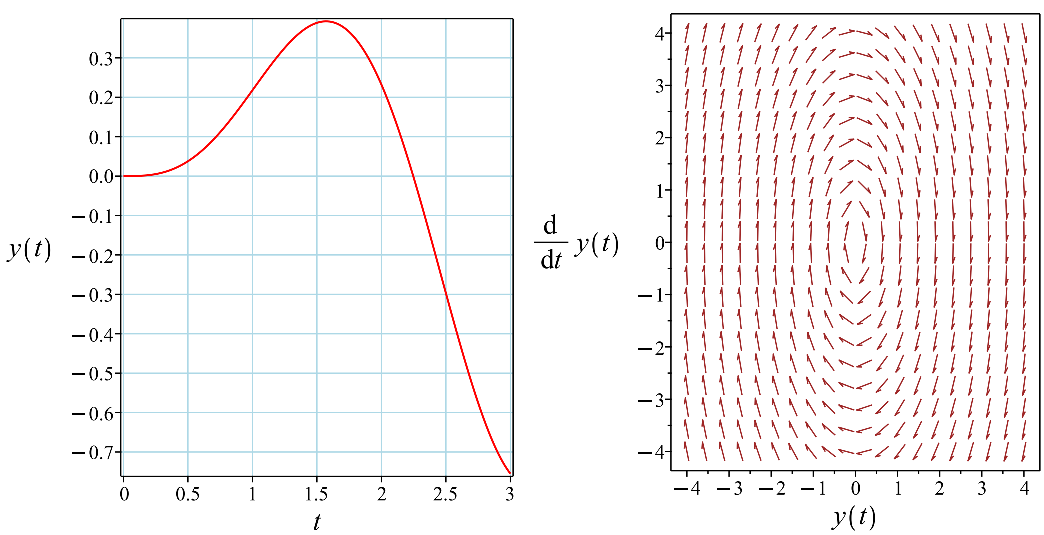
Simplifying the solution gives

$$y = \frac{\sin\left(2t\right)}{8}-\frac{\cos\left(2t\right)t}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin\left(2t\right)}{8}-\frac{\cos\left(2t\right)t}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\sin(2t)}{8} - \frac{\cos(2t)\,t}{4}$$

Verified OK.

21.6.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = \sin(2t), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \sin(2t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(2t)\left(\int \sin(2t)^2 dt\right)}{2} + \frac{\sin(2t)\left(\int \sin(4t) dt\right)}{4}$

- Compute integrals

  $y_p(t) = \frac{\sin(2t)}{16} - \frac{\cos(2t)t}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\sin(2t)}{16} - \frac{\cos(2t)t}{4}$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\sin(2t)}{16} - \frac{\cos(2t)t}{4}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{\cos(2t)}{8} + \frac{\sin(2t)t}{2}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -\frac{1}{8} + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 0, c_2 = \frac{1}{16}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{\sin(2t)}{8} - \frac{\cos(2t)t}{4}$

- Solution to the IVP

  $y = \frac{\sin(2t)}{8} - \frac{\cos(2t)t}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.359 (sec). Leaf size: 18

```
dsolve([diff(y(t),t$2)+4*y(t)=sin(2*t),y(0) = 0, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = \frac{\sin(2t)}{8} - \frac{t\cos(2t)}{4}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 21

```
DSolve[{y''[t]+4*y[t]==Sin[2*t],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{8}(\sin(2t) - 2t\cos(2t))$$

21.7 problem 7

Internal problem ID [13241]
Internal file name [`OUTPUT/11896_Tuesday_December_05_2023_12_12_51_PM_53853751/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.6. page 624
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

21.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + Y(s) = 0 \tag{1}$$

But the initial conditions are

$$y(0) = 1$$
$$y'(0) = 2$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4 - s + 2sY(s) + Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s+4}{s^2 + 2s + 1}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{3}{(s+1)^2} + \frac{1}{s+1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{3}{(s+1)^2}\right) = 3t\,\mathrm{e}^{-t}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = \mathrm{e}^{-t}$$

Adding the above results and simplifying gives

$$y = (3t+1)\,\mathrm{e}^{-t}$$

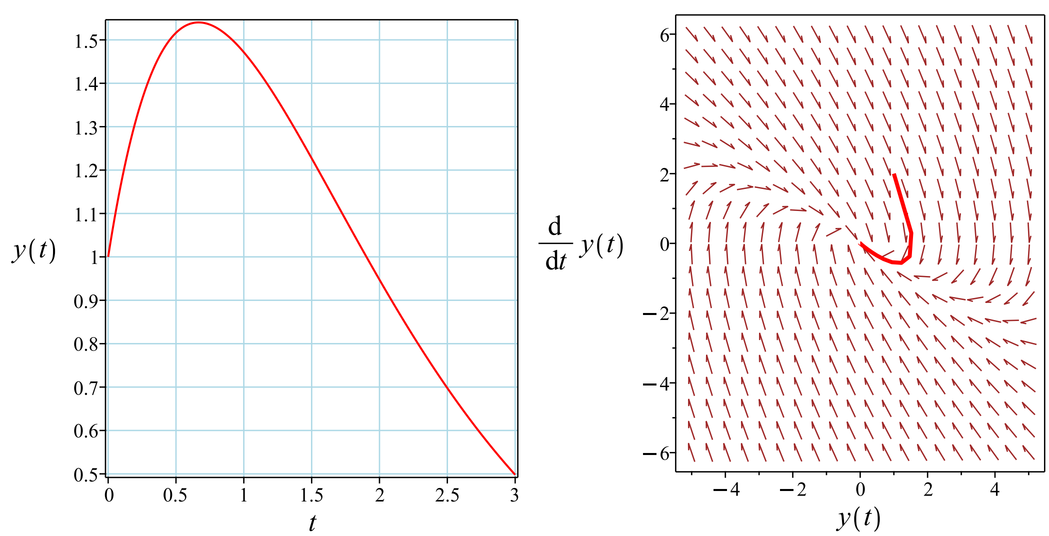
Simplifying the solution gives

$$y = (3t+1)\,\mathrm{e}^{-t}$$

Summary

The solution(s) found are the following

$$y = (3t+1)\,\mathrm{e}^{-t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (3t+1)\,\mathrm{e}^{-t}$$

Verified OK.

21.7.2 Maple step by step solution

Let's solve

$\left[y''+2y'+y=0, y(0)=1, y'\Big|_{\{t=0\}}=2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+2r+1=0$

- Factor the characteristic polynomial

  $(r+1)^2=0$

- Root of the characteristic polynomial

  $r=-1$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t)=t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-t}+c_2t\,\mathrm{e}^{-t}$

□ Check validity of solution $y=c_1\mathrm{e}^{-t}+c_2t\mathrm{e}^{-t}$

  - Use initial condition $y(0)=1$

    $1=c_1$

  - Compute derivative of the solution

    $y'=-c_1\mathrm{e}^{-t}+c_2\mathrm{e}^{-t}-c_2t\,\mathrm{e}^{-t}$

  - Use the initial condition $y'\Big|_{\{t=0\}}=2$

    $2=-c_1+c_2$

  - Solve for c_1 and c_2

    $\{c_1=1, c_2=3\}$

- Substitute constant values into general solution and simplify

  $$y = (3t+1)\,\mathrm{e}^{-t}$$

- Solution to the IVP

  $$y = (3t+1)\,\mathrm{e}^{-t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.343 (sec). Leaf size: 14

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=0,y(0) = 1, D(y)(0) = 2],y(t), singsol=all)
```

$$y(t) = (3t+1)\,\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 16

```
DSolve[{y''[t]+2*y'[t]+y[t]==0,{y[0]==1,y'[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}(3t+1)$$

21.8 problem 8

Internal problem ID [13242]
Internal file name [`OUTPUT/11897_Tuesday_December_05_2023_12_12_52_PM_9883834/index.tex`]

Book: DIFFERENTIAL EQUATIONS by Paul Blanchard, Robert L. Devaney, Glen R. Hall. 4th edition. Brooks/Cole. Boston, USA. 2012
Section: Chapter 6. Laplace transform. Section 6.6. page 624
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 16y = t}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

21.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 16 \\ F &= t \end{aligned}$$

Hence the ode is

$$y'' + 16y = t$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 16$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 16Y(s) = \frac{1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s + 16Y(s) = \frac{1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 + s^2 + 1}{s^2 (s^2 + 16)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{1}{2} - \frac{15i}{128}}{s - 4i} + \frac{\frac{1}{2} + \frac{15i}{128}}{s + 4i} + \frac{1}{16s^2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2}-\frac{15i}{128}}{s-4i}\right) = \left(\frac{1}{2}-\frac{15i}{128}\right)\mathrm{e}^{4it}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2}+\frac{15i}{128}}{s+4i}\right) = \left(\frac{1}{2}+\frac{15i}{128}\right)\mathrm{e}^{-4it}$$
$$\mathcal{L}^{-1}\left(\frac{1}{16s^2}\right) = \frac{t}{16}$$

Adding the above results and simplifying gives

$$y = \cos(4t) + \frac{15\sin(4t)}{64} + \frac{t}{16}$$

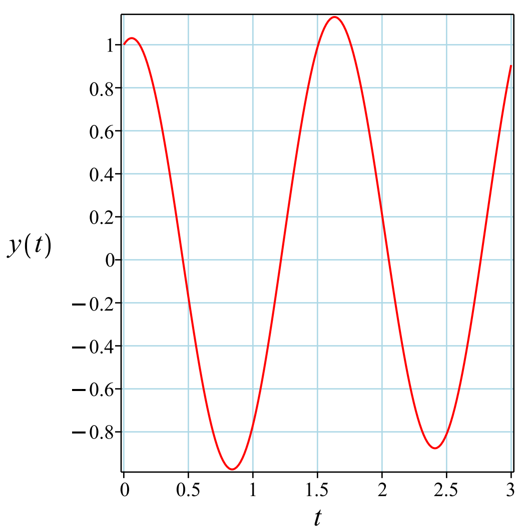
Simplifying the solution gives

$$y = \cos(4t) + \frac{15\sin(4t)}{64} + \frac{t}{16}$$

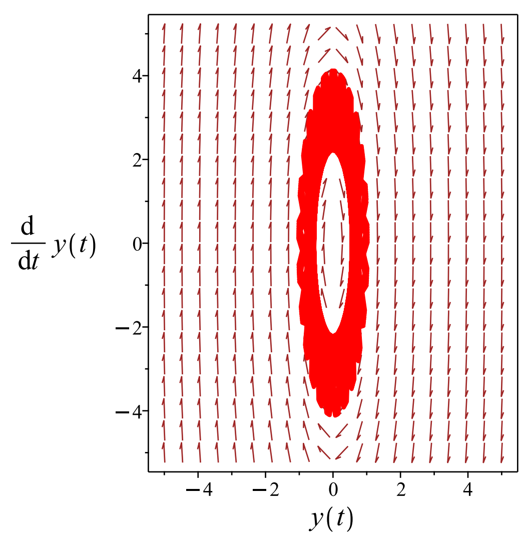
Summary

The solution(s) found are the following

$$y = \cos(4t) + \frac{15\sin(4t)}{64} + \frac{t}{16} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos(4t) + \frac{15\sin(4t)}{64} + \frac{t}{16}$$

Verified OK.

21.8.2 Maple step by step solution

Let's solve

$\left[y'' + 16y = t, y(0) = 1, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-4\,\mathrm{I}, 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(4t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(4t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4t) + c_2 \sin(4t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = t\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 4$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(4t)\left(\int t\sin(4t)dt\right)}{4} + \frac{\sin(4t)\left(\int \cos(4t)tdt\right)}{4}$

- Compute integrals

  $y_p(t) = \frac{t}{16}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(4t) + c_2\sin(4t) + \frac{t}{16}$

☐ Check validity of solution $y = c_1\cos(4t) + c_2\sin(4t) + \frac{t}{16}$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $y' = -4c_1\sin(4t) + 4c_2\cos(4t) + \frac{1}{16}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = \frac{1}{16} + 4c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 1, c_2 = \frac{15}{64}\right\}$

- Substitute constant values into general solution and simplify

  $y = \cos(4t) + \frac{15\sin(4t)}{64} + \frac{t}{16}$

- Solution to the IVP

  $y = \cos(4t) + \frac{15\sin(4t)}{64} + \frac{t}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.344 (sec). Leaf size: 18

```
dsolve([diff(y(t),t$2)+16*y(t)=t,y(0) = 1, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \cos(4t) + \frac{15\sin(4t)}{64} + \frac{t}{16}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 24

```
DSolve[{y''[t]+16*y[t]==t,{y[0]==1,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{64}(4t + 15\sin(4t)) + \cos(4t)$$