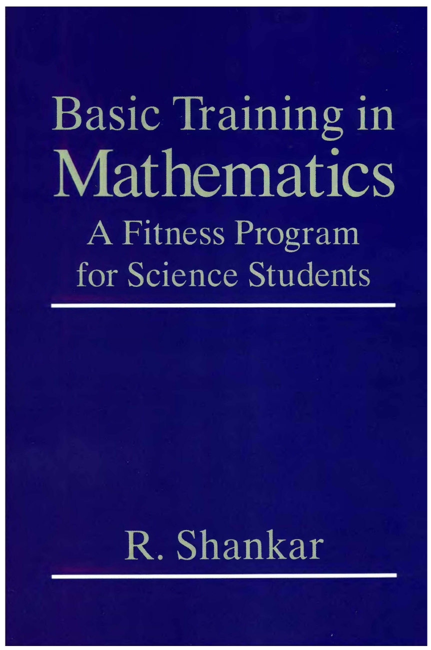
A Solution Manual For

Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995



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May 15, 2024

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1.1 problem 10.2.4

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Internal problem ID [5045]
Internal file name [`OUTPUT/4538_Sunday_June_05_2022_03_00_32_PM_16368218/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - \omega^2 x = 0}$$

1.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -\omega^2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - \omega^2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - \omega^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -\omega^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-\omega^2)}\\ &= \pm\sqrt{\omega^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{\omega^2}\\ \lambda_2 &= -\sqrt{\omega^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{\omega^2}\\ \lambda_2 &= -\sqrt{\omega^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{\left(\sqrt{\omega^2}\right)t} + c_2 e^{\left(-\sqrt{\omega^2}\right)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{\sqrt{\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{\omega^2}\, t}$$

<u>Summary</u>

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{\sqrt{\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{\omega^2}\, t} \tag{1}$$

<u>Verification of solutions</u>

$$x = c_1 \mathrm{e}^{\sqrt{\omega^2}\, t} + c_2 \mathrm{e}^{-\sqrt{\omega^2}\, t}$$

Verified OK.

1.1.2 Solving as second order ode can be made integrable ode

Multiplying the ode by x' gives

$$x'x'' - \omega^2 x' x = 0$$

Integrating the above w.r.t t gives

$$\int \left(x'x'' - \omega^2 x' x\right) dt = 0$$
$$\frac{x'^2}{2} - \frac{\omega^2 x^2}{2} = c_2$$

Which is now solved for x. Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \sqrt{\omega^2 x^2 + 2c_1} \tag{1}$$
$$x' = -\sqrt{\omega^2 x^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{\omega^2 x^2 + 2c_1}} dx = \int dt$$
$$\frac{\ln\left(\frac{\omega^2 x}{\sqrt{\omega^2}} + \sqrt{\omega^2 x^2 + 2c_1}\right)}{\sqrt{\omega^2}} = t + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln\left(\frac{\omega^2 x}{\sqrt{\omega^2}} + \sqrt{\omega^2 x^2 + 2c_1}\right)}{\sqrt{\omega^2}}} = \mathrm{e}^{t + c_2}$$

Which simplifies to

$$\left(\omega x \operatorname{csgn}(\omega) + \sqrt{\omega^2 x^2 + 2c_1}\right)^{\frac{1}{\sqrt{\omega^2}}} = c_3 \mathrm{e}^t$$

Simplifying the solution $x = \frac{\operatorname{csgn}(\omega)\left(\left(c_3 \mathrm{e}^t\right)^{\operatorname{csgn}(\omega)\omega} - 2\left(c_3 \mathrm{e}^t\right)^{-\operatorname{csgn}(\omega)\omega} c_1\right)}{2\omega}$ to $x = \frac{\left(c_3 \mathrm{e}^t\right)^{\omega} - 2\left(c_3 \mathrm{e}^t\right)^{-\omega} c_1}{2\omega}$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{\omega^2 x^2 + 2c_1}} dx = \int dt$$
$$-\frac{\ln\left(\frac{\omega^2 x}{\sqrt{\omega^2}} + \sqrt{\omega^2 x^2 + 2c_1}\right)}{\sqrt{\omega^2}} = t + c_4$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(\frac{\omega^2 x}{\sqrt{\omega^2}}+\sqrt{\omega^2 x^2+2c_1}\right)}{\sqrt{\omega^2}}} = e^{t+c_4}$$

Which simplifies to

$$\left(\omega x\ \mathrm{csgn}\left(\omega\right)+\sqrt{\omega^2 x^2+2c_1}\right)^{-\frac{\mathrm{csgn}(\omega)}{\omega}} = c_5 e^{t}$$

Simplifying the solution $x = -\frac{\mathrm{csgn}(\omega)\left(2\left(c_5 e^t\right)^{\mathrm{csgn}(\omega)\omega} c_1-\left(c_5 e^t\right)^{-\mathrm{csgn}(\omega)\omega}\right)}{2\omega}$ to $x = -\frac{2\left(c_5 e^t\right)^{\omega} c_1-\left(c_5 e^t\right)^{-\omega}}{2\omega}$

Summary

The solution(s) found are the following

$$x = \frac{\left(c_3 e^t\right)^{\omega}-2\left(c_3 e^t\right)^{-\omega} c_1}{2\omega} \tag{1}$$

$$x = -\frac{2\left(c_5 e^t\right)^{\omega} c_1-\left(c_5 e^t\right)^{-\omega}}{2\omega} \tag{2}$$

Verification of solutions

$$x = \frac{\left(c_3 e^t\right)^{\omega}-2\left(c_3 e^t\right)^{-\omega} c_1}{2\omega}$$

Verified OK.

$$x = -\frac{2\left(c_5 e^t\right)^{\omega} c_1-\left(c_5 e^t\right)^{-\omega}}{2\omega}$$

Verified OK.

1.1.3 Solving using Kovacic algorithm

Writing the ode as

$$x''-\omega^2 x = 0 \tag{1}$$

$$Ax''+Bx'+Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\omega^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{\omega^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= \omega^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\omega^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \omega^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\sqrt{\omega^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \mathrm{e}^{\sqrt{\omega^2}\, t}
\end{aligned}
$$

Which simplifies to

$$x_1 = \mathrm{e}^{\sqrt{\omega^2}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\
&= \mathrm{e}^{\sqrt{\omega^2}\, t} \int \frac{1}{\mathrm{e}^{2\sqrt{\omega^2}\, t}}\, dt \\
&= \mathrm{e}^{\sqrt{\omega^2}\, t} \left(-\frac{\operatorname{csgn}(\omega)\, \mathrm{e}^{-2\operatorname{csgn}(\omega)\omega t}}{2\omega} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(\mathrm{e}^{\sqrt{\omega^2}\, t}\right) + c_2\left(\mathrm{e}^{\sqrt{\omega^2}\, t}\left(-\frac{\mathrm{csgn}\left(\omega\right)\mathrm{e}^{-2\,\mathrm{csgn}(\omega)\omega t}}{2\omega}\right)\right)
\end{aligned}
$$

Simplifying the solution $x = c_1\mathrm{e}^{\sqrt{\omega^2}\, t} - \frac{c_2\,\mathrm{csgn}(\omega)\mathrm{e}^{-\,\mathrm{csgn}(\omega)\omega t}}{2\omega}$ to $x = c_1\mathrm{e}^{\sqrt{\omega^2}\, t} - \frac{c_2\mathrm{e}^{-\omega t}}{2\omega}$

Summary

The solution(s) found are the following

$$
x = c_1\mathrm{e}^{\sqrt{\omega^2}\, t} - \frac{c_2\mathrm{e}^{-\omega t}}{2\omega} \tag{1}
$$

Verification of solutions

$$
x = c_1\mathrm{e}^{\sqrt{\omega^2}\, t} - \frac{c_2\mathrm{e}^{-\omega t}}{2\omega}
$$

Verified OK.

1.1.4 Maple step by step solution

Let's solve

$x'' - \omega^2x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $-\omega^2 + r^2 = 0$

- Factor the characteristic polynomial

  $-(\omega - r)(\omega + r) = 0$

- Roots of the characteristic polynomial

  $r = (\omega, -\omega)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{\omega t}$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-\omega t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{\omega t} + c_2 e^{-\omega t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(x(t),t$2)-omega^2*x(t)=0,x(t), singsol=all)
```

$$x(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[x''[t]-\[Omega]^2*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \rightarrow c_1 e^{t\omega} + c_2 e^{-t\omega}$$

1.2 problem 10.2.5

1.2.1 Maple step by step solution 12

Internal problem ID [5046]
Internal file name [`OUTPUT/4539_Sunday_June_05_2022_03_00_33_PM_16585502/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.5.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{x''' - x'' + x' - x = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$x_h(t) = c_1 e^t + e^{-it} c_2 + e^{it} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = e^t$$
$$x_2 = e^{-it}$$
$$x_3 = e^{it}$$

Summary
The solution(s) found are the following

$$x = c_1 e^t + e^{-it} c_2 + e^{it} c_3 \tag{1}$$

Verification of solutions

$$x = c_1 e^t + e^{-it} c_2 + e^{it} c_3$$

Verified OK.

1.2.1 Maple step by step solution

Let's solve

$x''' - x'' + x' - x = 0$

- Highest derivative means the order of the ODE is 3

  x'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $x_1(t)$

    $x_1(t) = x$

  - Define new variable $x_2(t)$

    $x_2(t) = x'$

  - Define new variable $x_3(t)$

    $x_3(t) = x''$

  - Isolate for $x_3'(t)$ using original ODE

    $x_3'(t) = x_3(t) - x_2(t) + x_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[x_2(t) = x_1'(t), x_3(t) = x_2'(t), x_3'(t) = x_3(t) - x_2(t) + x_1(t)]$

- Define vector

  $$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-It} \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - I\sin(t)) \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(t) + I\sin(t) \\ I(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = \begin{bmatrix} -\cos(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_3(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -c_2\cos(t) + c_3\sin(t) \\ c_2\sin(t) + c_3\cos(t) \\ c_2\cos(t) - c_3\sin(t) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$x = c_1 e^t + c_3\sin(t) - c_2\cos(t)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(x(t),t$3)-diff(x(t),t$2)+diff(x(t),t)-x(t)=0,x(t), singsol=all)
```

$$x(t) = c_1 \mathrm{e}^t + c_2 \sin(t) + c_3 \cos(t)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 22

```
DSolve[x'''[t]-x''[t]+x'[t]-x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_3 e^t + c_1 \cos(t) + c_2 \sin(t)$$

1.3 problem 10.2.8 part(1)

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1.3.3 Solving using Kovacic algorithm 20
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Internal problem ID [5047]
Internal file name [`OUTPUT/4540_Sunday_June_05_2022_03_00_33_PM_66076883/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.8 part(1).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + 42x' + x = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

1.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 42 \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 42x' + x = 0$$

The domain of $p(t) = 42$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.3.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 42, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 42\lambda\, \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 42\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 42, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-42}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{42^2 - (4)\,(1)\,(1)}\\ &= -21 \pm 2\sqrt{110}\end{aligned}$$

Hence

$$\lambda_1 = -21 + 2\sqrt{110}$$

$$\lambda_2 = -21 - 2\sqrt{110}$$

Which simplifies to

$$\lambda_1 = -21 + 2\sqrt{110}$$
$$\lambda_2 = -21 - 2\sqrt{110}$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{\left(-21+2\sqrt{110}\right)t} + c_2 e^{\left(-21-2\sqrt{110}\right)t}$$

Or

$$x = c_1 \mathrm{e}^{\left(-21+2\sqrt{110}\right)t} + c_2 \mathrm{e}^{\left(-21-2\sqrt{110}\right)t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{\left(-21+2\sqrt{110}\right)t} + c_2 \mathrm{e}^{\left(-21-2\sqrt{110}\right)t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 \left(-21 + 2\sqrt{110}\right) \mathrm{e}^{\left(-21+2\sqrt{110}\right)t} + c_2 \left(-21 - 2\sqrt{110}\right) \mathrm{e}^{\left(-21-2\sqrt{110}\right)t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = (2c_1 - 2c_2)\sqrt{110} - 21c_1 - 21c_2 \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2} + \frac{21\sqrt{110}}{440}$$
$$c_2 = \frac{1}{2} - \frac{21\sqrt{110}}{440}$$

Substituting these values back in above solution results in

$$x = \frac{e^{\left(-21+2\sqrt{110}\right)t}}{2} + \frac{21\,e^{\left(-21+2\sqrt{110}\right)t}\sqrt{110}}{440} + \frac{e^{-\left(21+2\sqrt{110}\right)t}}{2} - \frac{21\,e^{-\left(21+2\sqrt{110}\right)t}\sqrt{110}}{440}$$

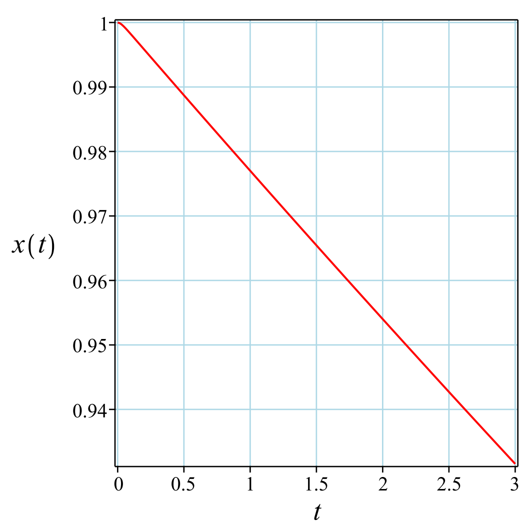
Which simplifies to

$$x = \frac{\left(220+21\sqrt{110}\right)e^{\left(-21+2\sqrt{110}\right)t}}{440} + \frac{\left(220-21\sqrt{110}\right)e^{\left(-21-2\sqrt{110}\right)t}}{440}$$

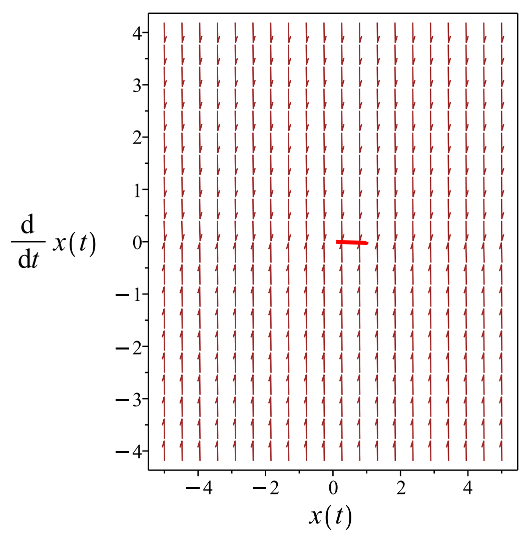
Summary

The solution(s) found are the following

$$x = \frac{\left(220+21\sqrt{110}\right)e^{\left(-21+2\sqrt{110}\right)t}}{440} + \frac{\left(220-21\sqrt{110}\right)e^{\left(-21-2\sqrt{110}\right)t}}{440} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{\left(220+21\sqrt{110}\right)e^{\left(-21+2\sqrt{110}\right)t}}{440} + \frac{\left(220-21\sqrt{110}\right)e^{\left(-21-2\sqrt{110}\right)t}}{440}$$

Verified OK.

1.3.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 42x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 42 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{440}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 440 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 440z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 4: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 440$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-2t\sqrt{110}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{42}{1}\,dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-21t} \\
&= z_1 \left(\mathrm{e}^{-21t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{\left(-21-2\sqrt{110}\right)t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{42}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-42t}}{(x_1)^2}\, dt \\
&= x_1 \left(\frac{\sqrt{110}\,\mathrm{e}^{4t\sqrt{110}}}{440}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{\left(-21-2\sqrt{110}\right)t}\right) + c_2 \left(\mathrm{e}^{\left(-21-2\sqrt{110}\right)t} \left(\frac{\sqrt{110}\,\mathrm{e}^{4t\sqrt{110}}}{440}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{\left(-21-2\sqrt{110}\right)t} + \frac{c_2 \mathrm{e}^{\left(-21+2\sqrt{110}\right)t}\sqrt{110}}{440} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 1$ and $t = 0$ in the above gives

$$
1 = c_1 + \frac{c_2\sqrt{110}}{440} \tag{1A}
$$

Taking derivative of the solution gives

$$x' = c_1\left(-21-2\sqrt{110}\right) e^{\left(-21-2\sqrt{110}\right)t} + \frac{c_2\left(-21+2\sqrt{110}\right) e^{\left(-21+2\sqrt{110}\right)t}\sqrt{110}}{440}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = \frac{(-880c_1 - 21c_2)\sqrt{110}}{440} - 21c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2} - \frac{21\sqrt{110}}{440}$$
$$c_2 = 21 + 2\sqrt{110}$$

Substituting these values back in above solution results in

$$x = \frac{e^{\left(-21+2\sqrt{110}\right)t}}{2} + \frac{21\, e^{\left(-21+2\sqrt{110}\right)t}\sqrt{110}}{440} + \frac{e^{-\left(21+2\sqrt{110}\right)t}}{2} - \frac{21\, e^{-\left(21+2\sqrt{110}\right)t}\sqrt{110}}{440}$$

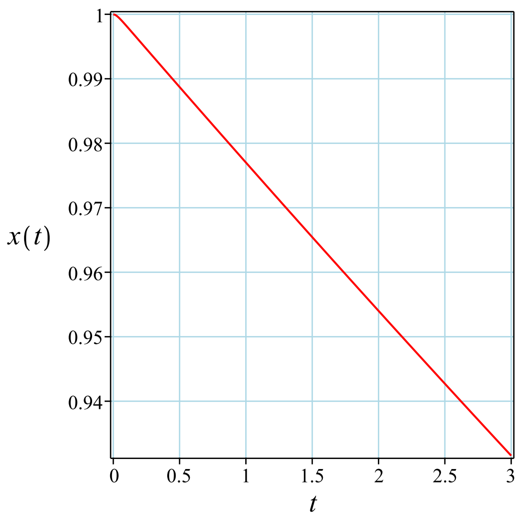
Which simplifies to

$$x = \frac{\left(220+21\sqrt{110}\right) e^{\left(-21+2\sqrt{110}\right)t}}{440} + \frac{\left(220-21\sqrt{110}\right) e^{\left(-21-2\sqrt{110}\right)t}}{440}$$

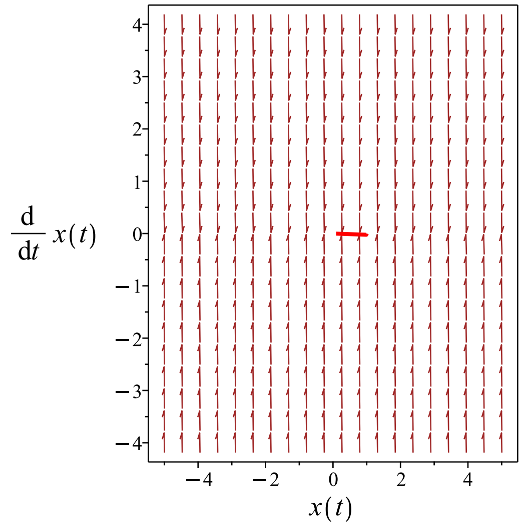
Summary

The solution(s) found are the following

$$x = \frac{\left(220+21\sqrt{110}\right) e^{\left(-21+2\sqrt{110}\right)t}}{440} + \frac{\left(220-21\sqrt{110}\right) e^{\left(-21-2\sqrt{110}\right)t}}{440} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{(220 + 21\sqrt{110})\, e^{(-21+2\sqrt{110})t}}{440} + \frac{(220 - 21\sqrt{110})\, e^{(-21-2\sqrt{110})t}}{440}$$

Verified OK.

1.3.4 Maple step by step solution

Let's solve

$$\left[x'' + 42x' + x = 0, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + 42r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-42)\pm\left(\sqrt{1760}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-21 - 2\sqrt{110}, -21 + 2\sqrt{110}\right)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{\left(-21-2\sqrt{110}\right)t}$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{\left(-21+2\sqrt{110}\right)t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{\left(-21-2\sqrt{110}\right)t} + c_2 \mathrm{e}^{\left(-21+2\sqrt{110}\right)t}$

☐ Check validity of solution $x = c_1 \mathrm{e}^{\left(-21-2\sqrt{110}\right)t} + c_2 \mathrm{e}^{\left(-21+2\sqrt{110}\right)t}$

  - Use initial condition $x(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = c_1\left(-21-2\sqrt{110}\right)\mathrm{e}^{\left(-21-2\sqrt{110}\right)t} + c_2\left(-21+2\sqrt{110}\right)\mathrm{e}^{\left(-21+2\sqrt{110}\right)t}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = c_1\left(-21-2\sqrt{110}\right) + \left(-21+2\sqrt{110}\right)c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{1}{2} - \frac{21\sqrt{110}}{440}, c_2 = \frac{1}{2} + \frac{21\sqrt{110}}{440}\right\}$

  - Substitute constant values into general solution and simplify

    $x = \frac{\left(220+21\sqrt{110}\right)\mathrm{e}^{\left(-21+2\sqrt{110}\right)t}}{440} + \frac{\left(220-21\sqrt{110}\right)\mathrm{e}^{\left(-21-2\sqrt{110}\right)t}}{440}$

- Solution to the IVP

  $x = \frac{\left(220+21\sqrt{110}\right)\mathrm{e}^{\left(-21+2\sqrt{110}\right)t}}{440} + \frac{\left(220-21\sqrt{110}\right)\mathrm{e}^{\left(-21-2\sqrt{110}\right)t}}{440}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 43

```
dsolve([diff(x(t),t$2)+42*diff(x(t),t)+x(t)=0,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{(220 + 21\sqrt{110})\,\mathrm{e}^{\left(-21+2\sqrt{110}\right)t}}{440} + \frac{(220 - 21\sqrt{110})\,\mathrm{e}^{\left(-21-2\sqrt{110}\right)t}}{440}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 53

```
DSolve[{x''[t]+42*x'[t]+x[t]==0,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{e^{-\left(\left(21+2\sqrt{110}\right)t\right)}\left(\left(881 + 84\sqrt{110}\right)e^{4\sqrt{110}t} - 1\right)}{880 + 84\sqrt{110}}$$

1.4 problem 10.2.8 part(2)

1.4.1 Maple step by step solution 28

Internal problem ID [5048]
Internal file name [OUTPUT/4541_Sunday_June_05_2022_03_00_34_PM_44292396/index.tex]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.8 part(2).
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{x'''' + x = 0}$$

The characteristic equation is

$$\lambda^4 + 1 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \\
\lambda_2 &= -\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \\
\lambda_3 &= -\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2} \\
\lambda_4 &= \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}
\end{aligned}$$

Therefore the homogeneous solution is

$$x_h(t) = e^{\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)t} c_1 + e^{\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)t} c_2 + e^{\left(-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)t} c_3 + e^{\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)t} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = e^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)t}$$
$$x_2 = e^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)t}$$
$$x_3 = e^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)t}$$
$$x_4 = e^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)t}$$

Summary

The solution(s) found are the following

$$x = e^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)t}c_1 + e^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)t}c_2 + e^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)t}c_3 + e^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)t}c_4 \tag{1}$$

Verification of solutions

$$x = e^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)t}c_1 + e^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)t}c_2 + e^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)t}c_3 + e^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)t}c_4$$

Verified OK.

1.4.1 Maple step by step solution

Let's solve

$x'''' + x = 0$

- Highest derivative means the order of the ODE is 4

  x''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $x_1(t)$

    $x_1(t) = x$

  - Define new variable $x_2(t)$

    $x_2(t) = x'$

  - Define new variable $x_3(t)$

    $x_3(t) = x''$

  - Define new variable $x_4(t)$

    $x_4(t) = x'''$

  - Isolate for $x_4'(t)$ using original ODE

    $x_4'(t) = -x_1(t)$

Convert linear ODE into a system of first order ODEs

$[x_2(t) = x_1'(t), x_3(t) = x_2'(t), x_4(t) = x_3'(t), x_4'(t) = -x_1(t)]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}} \\ 1 \end{array}\right]\right], \left[-\frac{\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}} \\ 1 \end{array}\right]\right], \left[\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-\frac{t\sqrt{2}}{2}} \cdot \left(\cos\left(\frac{t\sqrt{2}}{2}\right) - I\sin\left(\frac{t\sqrt{2}}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{t\sqrt{2}}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)-I\sin\left(\frac{t\sqrt{2}}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^3} \\ \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)-I\sin\left(\frac{t\sqrt{2}}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^2} \\ \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)-I\sin\left(\frac{t\sqrt{2}}{2}\right)}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ \cos\left(\frac{t\sqrt{2}}{2}\right) - I\sin\left(\frac{t\sqrt{2}}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \mathrm{e}^{-\frac{t\sqrt{2}}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{t\sqrt{2}}{2}\right) \\ -\frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{t\sqrt{2}}{2}\right) \end{array}\right], \vec{x}_2(t) = \mathrm{e}^{-\frac{t\sqrt{2}}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} - \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ -\cos\left(\frac{t\sqrt{2}}{2}\right) \\ \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{t\sqrt{2}}{2}\right) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{\frac{t\sqrt{2}}{2}} \cdot \left(\cos\left(\frac{t\sqrt{2}}{2}\right) - \mathrm{I}\sin\left(\frac{t\sqrt{2}}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\frac{t\sqrt{2}}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{t\sqrt{2}}{2}\right) - I\sin\left(\frac{t\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^3} \\ \frac{\cos\left(\frac{t\sqrt{2}}{2}\right) - I\sin\left(\frac{t\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^2} \\ \frac{\cos\left(\frac{t\sqrt{2}}{2}\right) - I\sin\left(\frac{t\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}} \\ \cos\left(\frac{t\sqrt{2}}{2}\right) - I\sin\left(\frac{t\sqrt{2}}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_3(t) = e^{\frac{t\sqrt{2}}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ \sin\left(\frac{t\sqrt{2}}{2}\right) \\ \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{t\sqrt{2}}{2}\right) \end{bmatrix}, \vec{x}_4(t) = e^{\frac{t\sqrt{2}}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{t\sqrt{2}}{2}\right) \\ \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} - \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{t\sqrt{2}}{2}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-\frac{t\sqrt{2}}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{t\sqrt{2}}{2}\right) \\ -\frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{t\sqrt{2}}{2}\right) \end{bmatrix} + c_2 e^{-\frac{t\sqrt{2}}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} - \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ -\cos\left(\frac{t\sqrt{2}}{2}\right) \\ \frac{\cos\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{t\sqrt{2}}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{t\sqrt{2}}{2}\right) \end{bmatrix} + c_3 e^{\frac{t\sqrt{2}}{2}} \cdot \Bigg[$$

- First component of the vector is the solution to the ODE

$$x = \frac{\left(\left((c_1+c_2)e^{-\frac{t\sqrt{2}}{2}} - e^{\frac{t\sqrt{2}}{2}}(c_3-c_4)\right)\cos\left(\frac{t\sqrt{2}}{2}\right) + \sin\left(\frac{t\sqrt{2}}{2}\right)\left((c_1-c_2)e^{-\frac{t\sqrt{2}}{2}} + e^{\frac{t\sqrt{2}}{2}}(c_3+c_4)\right)\right)\sqrt{2}}{2}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 61

```
dsolve(diff(x(t),t$4)+x(t)=0,x(t), singsol=all)
```

$$x(t) = \left(-c_1 \mathrm{e}^{-\frac{\sqrt{2}\,t}{2}} - c_2 \mathrm{e}^{\frac{\sqrt{2}\,t}{2}}\right) \sin\left(\frac{\sqrt{2}\,t}{2}\right) + \left(c_3 \mathrm{e}^{-\frac{\sqrt{2}\,t}{2}} + c_4 \mathrm{e}^{\frac{\sqrt{2}\,t}{2}}\right) \cos\left(\frac{\sqrt{2}\,t}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 65

```
DSolve[x''''[t]+x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-\frac{t}{\sqrt{2}}} \left(\left(c_1 e^{\sqrt{2}t} + c_2\right) \cos\left(\frac{t}{\sqrt{2}}\right) + \left(c_4 e^{\sqrt{2}t} + c_3\right) \sin\left(\frac{t}{\sqrt{2}}\right)\right)$$

1.5 problem 10.2.8 part(3)

1.5.1 Maple step by step solution 35

Internal problem ID [5049]
Internal file name [`OUTPUT/4542_Sunday_June_05_2022_03_00_35_PM_42275176/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.8 part(3).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{x''' - 3x'' - 9x' - 5x = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$$

The roots of the above equation are

$$\lambda_1 = 5$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$x_h(t) = \mathrm{e}^{-t} c_1 + t\,\mathrm{e}^{-t} c_2 + \mathrm{e}^{5t} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$x_1 = \mathrm{e}^{-t}$$
$$x_2 = t\,\mathrm{e}^{-t}$$
$$x_3 = \mathrm{e}^{5t}$$

Summary
The solution(s) found are the following

$$x = e^{-t}c_1 + t\,e^{-t}c_2 + e^{5t}c_3 \tag{1}$$

Verification of solutions

$$x = e^{-t}c_1 + t\,e^{-t}c_2 + e^{5t}c_3$$

Verified OK.

1.5.1 Maple step by step solution

Let's solve

$x''' - 3x'' - 9x' - 5x = 0$

- Highest derivative means the order of the ODE is 3

  x'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $x_1(t)$

    $x_1(t) = x$

  ◦ Define new variable $x_2(t)$

    $x_2(t) = x'$

  ◦ Define new variable $x_3(t)$

    $x_3(t) = x''$

  ◦ Isolate for $x_3'(t)$ using original ODE

    $x_3'(t) = 3x_3(t) + 9x_2(t) + 5x_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[x_2(t) = x_1'(t), x_3(t) = x_2'(t), x_3'(t) = 3x_3(t) + 9x_2(t) + 5x_1(t)]$

- Define vector

  $$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 9 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 9 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[5, \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\vec{x}_1(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\vec{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obta
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

  $$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 9 & 3 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

  $$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

  $$\vec{x}_2(t) = \mathrm{e}^{-t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

  $$\left[5, \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_3 = \mathrm{e}^{5t} \cdot \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t) + c_3 \overrightarrow{x}_3$

- Substitute solutions into the general solution

$$\overrightarrow{x} = \mathrm{e}^{-t} c_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + \mathrm{e}^{5t} c_3 \cdot \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $x = ((t+1)\, c_2 + c_1)\, \mathrm{e}^{-t} + \frac{\mathrm{e}^{5t} c_3}{25}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(x(t),t$3)-3*diff(x(t),t$2)-9*diff(x(t),t)-5*x(t)=0,x(t), singsol=all)
```

$$x(t) = (c_3 t + c_2)\, \mathrm{e}^{-t} + c_1 \mathrm{e}^{5t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[x'''[t]-3*x''[t]-9*x'[t]-5*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t}\left(c_2 t + c_3 e^{6t} + c_1\right)$$

1.6 problem 10.2.10

1.6.1 Solving as second order linear constant coeff ode 39
1.6.2 Solving using Kovacic algorithm 42
1.6.3 Maple step by step solution 47

Internal problem ID [5050]
Internal file name [`OUTPUT/4543_Sunday_June_05_2022_03_00_36_PM_76937184/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + 2\gamma x' + \omega_0 x = F\cos(\omega t)}$$

1.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 2\gamma, C = \omega_0, f(t) = F\cos(\omega t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2\gamma x' + \omega_0 x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A=1, B=2\gamma, C=\omega_0$. Let the solution be $x=e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 2\gamma\lambda\, \mathrm{e}^{\lambda t} + \omega_0 \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$2\gamma\lambda + \lambda^2 + \omega_0 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=2\gamma, C=\omega_0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2\gamma}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2\gamma^2 - (4)(1)(\omega_0)}\\ &= -\gamma \pm \sqrt{\gamma^2 - \omega_0}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\gamma + \sqrt{\gamma^2 - \omega_0}\\ \lambda_2 &= -\gamma - \sqrt{\gamma^2 - \omega_0}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\gamma + \sqrt{\gamma^2 - \omega_0}\\ \lambda_2 &= -\gamma - \sqrt{\gamma^2 - \omega_0}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{\left(-\gamma + \sqrt{\gamma^2 - \omega_0}\right)t} + c_2 e^{\left(-\gamma - \sqrt{\gamma^2 - \omega_0}\right)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{\left(-\gamma + \sqrt{\gamma^2 - \omega_0}\right)t} + c_2 \mathrm{e}^{\left(-\gamma - \sqrt{\gamma^2 - \omega_0}\right)t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{\left(-\gamma + \sqrt{\gamma^2 - \omega_0}\right)t} + c_2 \mathrm{e}^{\left(-\gamma - \sqrt{\gamma^2 - \omega_0}\right)t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$F\cos(\omega t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\omega t), \sin(\omega t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\left(-\gamma-\sqrt{\gamma^2-\omega_0}\right)t}, e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(\omega t) + A_2\sin(\omega t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-A_1\omega^2\cos(\omega t) - A_2\omega^2\sin(\omega t) + 2\gamma(-A_1\omega\sin(\omega t) + A_2\omega\cos(\omega t))\\ &+ \omega_0(A_1\cos(\omega t) + A_2\sin(\omega t)) = F\cos(\omega t)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{(\omega^2-\omega_0)F}{\omega^4+(4\gamma^2-2\omega_0)\omega^2+\omega_0^2}, A_2 = \frac{2\gamma F\omega}{\omega^4+(4\gamma^2-2\omega_0)\omega^2+\omega_0^2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{(\omega^2-\omega_0)F\cos(\omega t)}{\omega^4+(4\gamma^2-2\omega_0)\omega^2+\omega_0^2} + \frac{2\gamma F\omega\sin(\omega t)}{\omega^4+(4\gamma^2-2\omega_0)\omega^2+\omega_0^2}$$

Therefore the general solution is

$$\begin{aligned}x &= x_h + x_p\\ &= \left(c_1e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} + c_2e^{\left(-\gamma-\sqrt{\gamma^2-\omega_0}\right)t}\right)\\ &\quad + \left(-\frac{(\omega^2-\omega_0)F\cos(\omega t)}{\omega^4+(4\gamma^2-2\omega_0)\omega^2+\omega_0^2} + \frac{2\gamma F\omega\sin(\omega t)}{\omega^4+(4\gamma^2-2\omega_0)\omega^2+\omega_0^2}\right)\end{aligned}$$

Which simplifies to

$$x = c_1 \mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} + c_2 \mathrm{e}^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t} - \frac{(\omega^2-\omega_0) F \cos(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} + \frac{2\gamma F \omega \sin(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2}$$

Summary

The solution(s) found are the following

$$x = c_1 \mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} + c_2 \mathrm{e}^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t} - \frac{(\omega^2-\omega_0) F \cos(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} + \frac{2\gamma F \omega \sin(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} \tag{1}$$

Verification of solutions

$$x = c_1 \mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} + c_2 \mathrm{e}^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t} - \frac{(\omega^2-\omega_0) F \cos(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} + \frac{2\gamma F \omega \sin(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2}$$

Verified OK.

1.6.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2\gamma x' + \omega_0 x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2\gamma \\ C &= \omega_0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{\gamma^2 - \omega_0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= \gamma^2 - \omega_0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\gamma^2 - \omega_0\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 8: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \gamma^2 - \omega_0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{t\sqrt{\gamma^2-\omega_0}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2\gamma}{1}\,dt} \\ &= z_1 e^{-t\gamma} \\ &= z_1\left(\mathrm{e}^{-t\gamma}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{2\gamma}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-2t\gamma}}{(x_1)^2}\,dt \\ &= x_1\left(-\frac{\mathrm{e}^{-2t\sqrt{\gamma^2-\omega_0}}}{2\sqrt{\gamma^2-\omega_0}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(\mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t}\right) + c_2\left(\mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t}\left(-\frac{\mathrm{e}^{-2t\sqrt{\gamma^2-\omega_0}}}{2\sqrt{\gamma^2-\omega_0}}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 2\gamma x' + \omega_0 x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1\mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} - \frac{c_2\mathrm{e}^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t}}{2\sqrt{\gamma^2-\omega_0}}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
F\cos(\omega t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(\omega t), \sin(\omega t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{-\frac{\mathrm{e}^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t}}{2\sqrt{\gamma^2-\omega_0}}, \mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1\cos(\omega t) + A_2\sin(\omega t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\omega^2 \cos(\omega t) - A_2\omega^2 \sin(\omega t) + 2\gamma(-A_1\omega \sin(\omega t) + A_2\omega \cos(\omega t)) \\ + \omega_0(A_1 \cos(\omega t) + A_2 \sin(\omega t)) = F \cos(\omega t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{(\omega^2 - \omega_0) F}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2}, A_2 = \frac{2\gamma F\omega}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{(\omega^2 - \omega_0) F \cos(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} + \frac{2\gamma F\omega \sin(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} - \frac{c_2 e^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t}}{2\sqrt{\gamma^2 - \omega_0}}\right) \\ &\quad + \left(-\frac{(\omega^2 - \omega_0) F \cos(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} + \frac{2\gamma F\omega \sin(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned} x = c_1 e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} - \frac{c_2 e^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t}}{2\sqrt{\gamma^2 - \omega_0}} \\ - \frac{(\omega^2 - \omega_0) F \cos(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} + \frac{2\gamma F\omega \sin(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} \end{aligned} \tag{1}$$

<u>Verification of solutions</u>

$$\begin{aligned} x = c_1 e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} - \frac{c_2 e^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t}}{2\sqrt{\gamma^2 - \omega_0}} \\ - \frac{(\omega^2 - \omega_0) F \cos(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} + \frac{2\gamma F\omega \sin(\omega t)}{\omega^4 + (4\gamma^2 - 2\omega_0)\omega^2 + \omega_0^2} \end{aligned}$$

Verified OK.

1.6.3 Maple step by step solution

Let's solve

$x'' + 2\gamma x' + \omega_0 x = F\cos(\omega t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $2\gamma r + r^2 + \omega_0 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2\gamma)\pm\left(\sqrt{4\gamma^2-4\omega_0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\gamma - \sqrt{\gamma^2-\omega_0}, -\gamma + \sqrt{\gamma^2-\omega_0}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{\left(-\gamma-\sqrt{\gamma^2-\omega_0}\right)t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{\left(-\gamma-\sqrt{\gamma^2-\omega_0}\right)t} + c_2 e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = F\cos(\omega t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{\left(-\gamma-\sqrt{\gamma^2-\omega_0}\right)t} & e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} \\ \left(-\gamma-\sqrt{\gamma^2-\omega_0}\right)e^{\left(-\gamma-\sqrt{\gamma^2-\omega_0}\right)t} & \left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 2\sqrt{\gamma^2-\omega_0}\, e^{-2t\gamma}$

- Substitute functions into equation for $x_p(t)$

$$x_p(t) = \frac{F\left(\mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t}\left(\int \cos(\omega t)\mathrm{e}^{\left(\gamma-\sqrt{\gamma^2-\omega_0}\right)t}dt\right)-\mathrm{e}^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t}\left(\int \cos(\omega t)\mathrm{e}^{\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t}dt\right)\right)}{2\sqrt{\gamma^2-\omega_0}}$$

- Compute integrals

$$x_p(t) = \frac{F((-\omega^2+\omega_0)\cos(\omega t)+2\gamma\omega\sin(\omega t))}{\omega^4+2(2\gamma^2-\omega_0)\omega^2+\omega_0^2}$$

- Substitute particular solution into general solution to ODE

$$x = c_1\mathrm{e}^{\left(-\gamma-\sqrt{\gamma^2-\omega_0}\right)t} + c_2\mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right)t} + \frac{F((-\omega^2+\omega_0)\cos(\omega t)+2\gamma\omega\sin(\omega t))}{\omega^4+2(2\gamma^2-\omega_0)\omega^2+\omega_0^2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 111

```
dsolve(diff(x(t),t$2)+2*gamma*diff(x(t),t)+omega__0*x(t)=F*cos(omega*t),x(t), singsol=all)
```

$x(t)$

$$= \frac{-F(\omega^2-\omega_0)\cos(\omega t)+2F\sin(\omega t)\gamma\omega+4\left(\frac{\omega^4}{4}+\left(\gamma^2-\frac{\omega_0}{2}\right)\omega^2+\frac{\omega_0^2}{4}\right)\left(\mathrm{e}^{-\left(\gamma+\sqrt{\gamma^2-\omega_0}\right)t}c_1+\mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega_0}\right.}\right.}{\omega^4+(4\gamma^2-2\omega_0)\,\omega^2+\omega_0^2}$$

✓ Solution by Mathematica

Time used: 0.509 (sec). Leaf size: 108

```
DSolve[x''[t]+2*\[Gamma]*x'[t]+Subscript[\[Omega],0]*x[t]==F*Cos[\[Omega]*t],x[t],t,IncludeS
```

$$x(t) \to \frac{F(\omega(2\gamma\sin(t\omega) - \omega\cos(t\omega)) + \omega_0\cos(t\omega))}{4\gamma^2\omega^2 + \omega^4 - 2\omega_0\omega^2 + \omega_0^2} + c_1 e^{-t\left(\sqrt{\gamma^2-\omega_0}+\gamma\right)} + c_2 e^{t\left(\sqrt{\gamma^2-\omega_0}-\gamma\right)}$$

1.7 problem 10.2.11 (i)

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Internal problem ID [5051]
Internal file name [`OUTPUT/4544_Sunday_June_05_2022_03_00_37_PM_98654921/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.11 (i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 2y = e^{2x}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

1.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -2 \\ F &= e^{2x} \end{aligned}$$

Hence the ode is

$$y'' - y' - 2y = e^{2x}$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^{2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.7.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)} \\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{3}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x} x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-x}\right) + \left(\frac{e^{2x} x}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-x} + \frac{e^{2x} x}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - c_2 e^{-x} + \frac{2\,e^{2x} x}{3} + \frac{e^{2x}}{3}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 2c_1 - c_2 + \frac{1}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2}{9}$$
$$c_2 = \frac{7}{9}$$

Substituting these values back in above solution results in

$$y = \frac{2\,e^{2x}}{9} + \frac{7\,e^{-x}}{9} + \frac{e^{2x} x}{3}$$

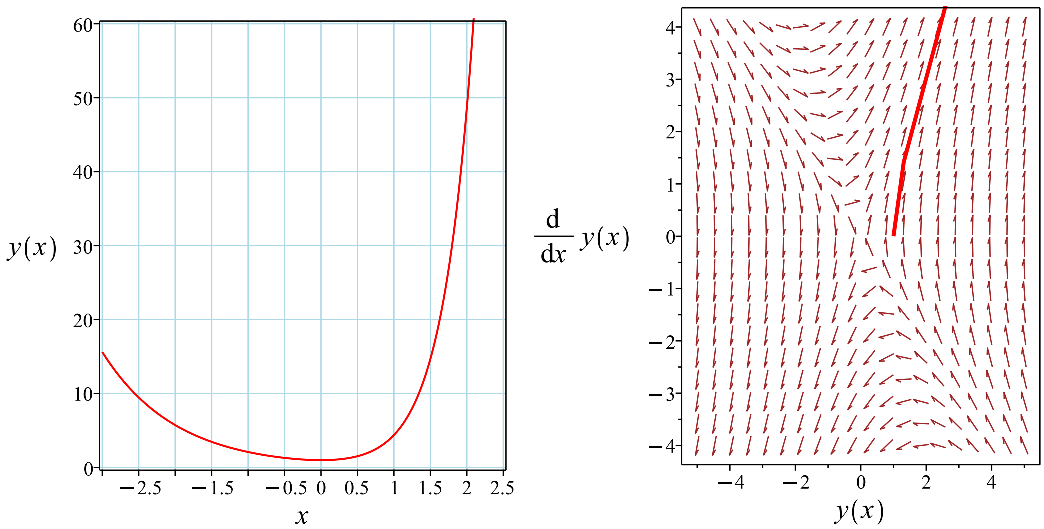
Which simplifies to

$$y = \frac{(2 + 3x)\,e^{2x}}{9} + \frac{7\,e^{-x}}{9}$$

Summary

The solution(s) found are the following

$$y = \frac{(2 + 3x)\,e^{2x}}{9} + \frac{7\,e^{-x}}{9} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{(2+3x)\,\mathrm{e}^{2x}}{9} + \frac{7\,\mathrm{e}^{-x}}{9}$$

Verified OK.

1.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 10: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \frac{\mathrm{e}^{2x}}{3}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{3} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{3} \\ -\mathrm{e}^{-x} & \frac{2\,\mathrm{e}^{2x}}{3} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\frac{2\,\mathrm{e}^{2x}}{3}\right) - \left(\frac{\mathrm{e}^{2x}}{3}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-x}\mathrm{e}^{2x}$$

Which simplifies to

$$W = \mathrm{e}^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{4x}}{3}}{\mathrm{e}^{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{3x}}{3}dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{3x}}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-x}\mathrm{e}^{2x}}{\mathrm{e}^{x}}\,dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-x} e^{3x}}{9} + \frac{e^{2x} x}{3}$$

Which simplifies to

$$y_p(x) = \frac{(3x - 1) e^{2x}}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{2x}}{3}\right) + \left(\frac{(3x - 1) e^{2x}}{9}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} + \frac{(3x - 1) e^{2x}}{9} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + \frac{c_2}{3} - \frac{1}{9} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + \frac{2c_2 e^{2x}}{3} + \frac{e^{2x}}{3} + \frac{2(3x - 1) e^{2x}}{9}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -c_1 + \frac{2c_2}{3} + \frac{1}{9} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{7}{9}$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = \frac{2\,e^{2x}}{9} + \frac{7\,e^{-x}}{9} + \frac{e^{2x}x}{3}$$

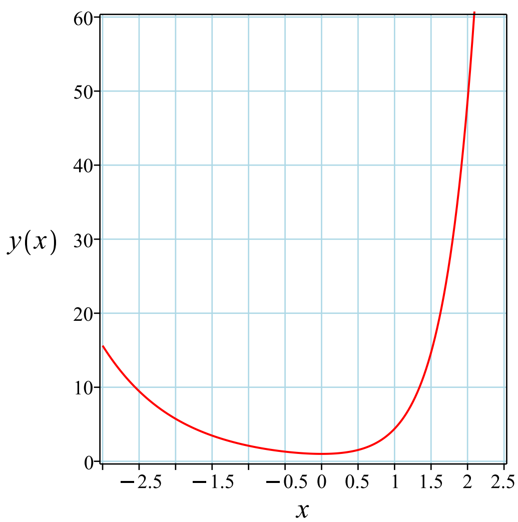
Which simplifies to

$$y = \frac{(2+3x)\,e^{2x}}{9} + \frac{7\,e^{-x}}{9}$$

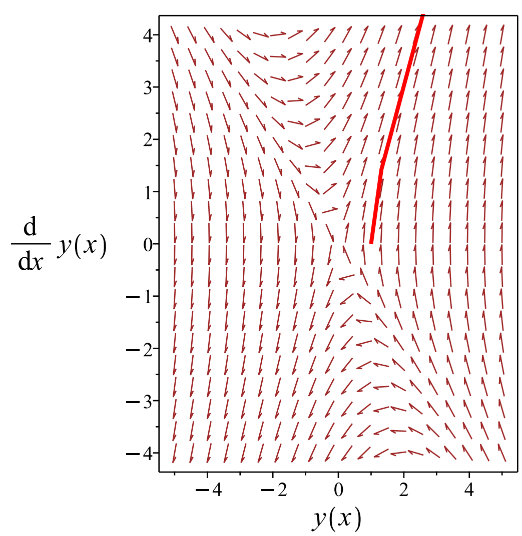
Summary

The solution(s) found are the following

$$y = \frac{(2+3x)\,e^{2x}}{9} + \frac{7\,e^{-x}}{9} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(2+3x)\,e^{2x}}{9} + \frac{7\,e^{-x}}{9}$$

Verified OK.

1.7.4 Maple step by step solution

Let's solve

$$\left[y'' - y' - 2y = e^{2x}, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{2x}\right]$$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2\,e^{2x} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,e^{x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\mathrm{e}^{-x} \left(\int \mathrm{e}^{3x}dx\right)}{3} + \frac{\mathrm{e}^{2x} \left(\int 1dx\right)}{3}$$

- Compute integrals

  $$y_p(x) = \frac{(3x-1)\,\mathrm{e}^{2x}}{9}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + \frac{(3x-1)\,\mathrm{e}^{2x}}{9}$$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + \frac{(3x-1)\,\mathrm{e}^{2x}}{9}$

- Use initial condition $y(0) = 1$

  $$1 = c_1 + c_2 - \frac{1}{9}$$

- Compute derivative of the solution

  $$y' = -c_1 \mathrm{e}^{-x} + 2c_2 \mathrm{e}^{2x} + \frac{\mathrm{e}^{2x}}{3} + \frac{2(3x-1)\,\mathrm{e}^{2x}}{9}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $$0 = -c_1 + 2c_2 + \frac{1}{9}$$

- Solve for c_1 and c_2

  $$\left\{c_1 = \frac{7}{9}, c_2 = \frac{1}{3}\right\}$$

- Substitute constant values into general solution and simplify

  $$y = \frac{(2+3x)\,\mathrm{e}^{2x}}{9} + \frac{7\,\mathrm{e}^{-x}}{9}$$

- Solution to the IVP

  $$y = \frac{(2+3x)\,\mathrm{e}^{2x}}{9} + \frac{7\,\mathrm{e}^{-x}}{9}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-2*y(x)=exp(2*x),y(0) = 1, D(y)(0) = 0],y(x), singsol=all
```

$$y(x) = \frac{(3x+2)\,\mathrm{e}^{2x}}{9} + \frac{7\,\mathrm{e}^{-x}}{9}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 27

```
DSolve[{y''[x]-y'[x]-2*y[x]==Exp[2*x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{9}e^{-x}\left(e^{3x}(3x+2)+7\right)$$

1.8 problem 10.2.11 (ii)

1.8.1 Existence and uniqueness analysis 66
1.8.2 Solving as second order linear constant coeff ode 66
1.8.3 Solving as linear second order ode solved by an integrating factor ode 70
1.8.4 Solving using Kovacic algorithm 72
1.8.5 Maple step by step solution 77

Internal problem ID [5052]
Internal file name [`OUTPUT/4545_Sunday_June_05_2022_03_00_38_PM_77665157/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.11 (ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 2\cos(x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

1.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 1 \\ F &= 2\cos(x) \end{aligned}$$

Hence the ode is

$$y'' - 2y' + y = 2\cos(x)$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2\cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.8.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 2\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2)^2 - (4)\,(1)\,(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 e^{x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 x\, e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x} x, e^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \sin(x) - 2A_2 \cos(x) = 2\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^x + c_2 x \,\mathrm{e}^x) + (-\sin(x)) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x (c_2 x + c_1) - \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^x (c_2 x + c_1) - \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \mathrm{e}^x (c_2 x + c_1) + c_2 \mathrm{e}^x - \cos(x)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -1 + c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

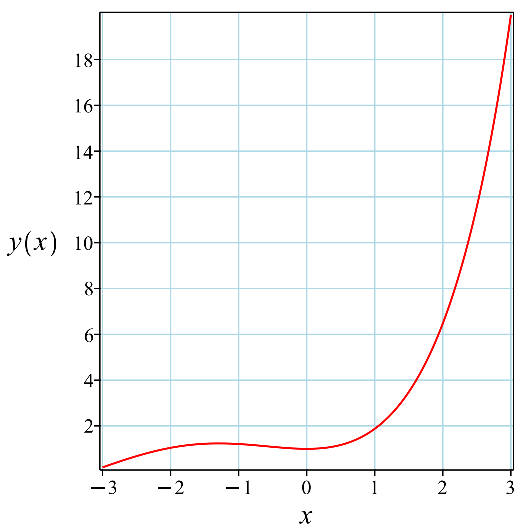
$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

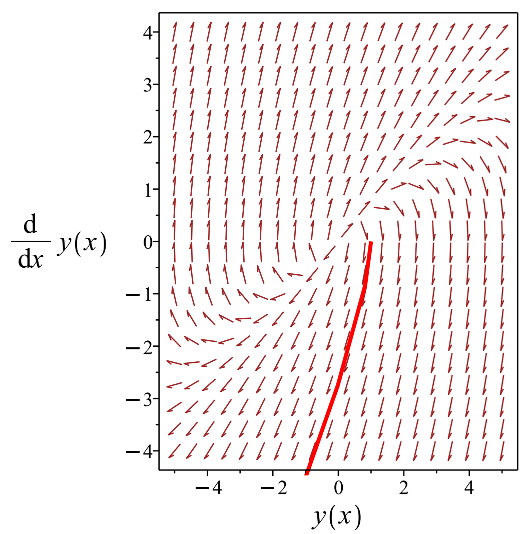
$$y = e^x - \sin(x)$$

Summary
The solution(s) found are the following

$$y = e^x - \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^x - \sin(x)$$

Verified OK.

1.8.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 2\,\mathrm{e}^{-x}\cos(x) \\ \left(\mathrm{e}^{-x}y\right)'' &= 2\,\mathrm{e}^{-x}\cos(x) \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x}y\right)' = -\mathrm{e}^{-x}(-\sin(x) + \cos(x)) + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x}y\right) = c_1 x - \mathrm{e}^{-x}\sin(x) + c_2$$

Hence the solution is

$$y = \frac{c_1 x - \mathrm{e}^{-x}\sin(x) + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} - \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} - \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x + c_1 x\, e^x + c_2 e^x - \cos(x)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -1 + c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

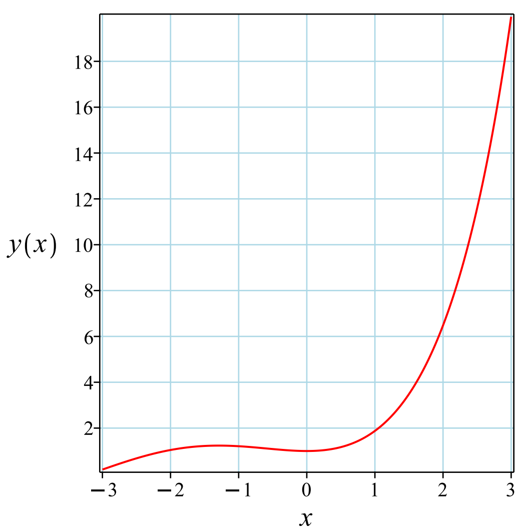
$$\begin{aligned} c_1 &= 0 \\ c_2 &= 1 \end{aligned}$$

Substituting these values back in above solution results in

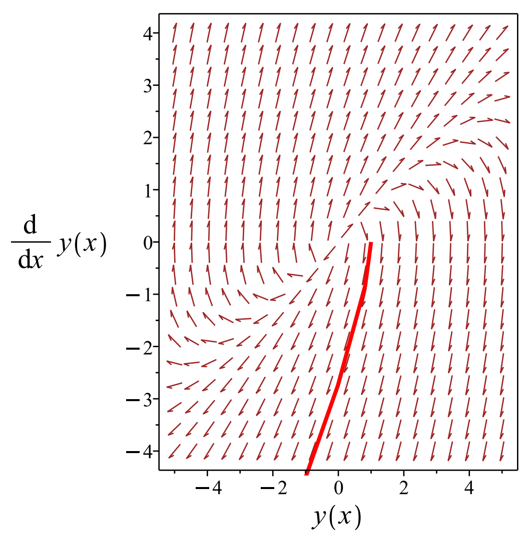
$$y = e^x - \sin(x)$$

Summary
The solution(s) found are the following

$$y = e^x - \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^x - \sin(x)$$

Verified OK.

1.8.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' - 2y' + y &= 0 & (1) \\
Ay'' + By' + Cy &= 0 & (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -2 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 12: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} \, dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + c_2 x\, \mathrm{e}^x
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x x, e^x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \sin(x) - 2A_2 \cos(x) = 2\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x\, e^x) + (-\sin(x)) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) - \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_2 x + c_1) - \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_2 x + c_1) + c_2 e^{x} - \cos(x)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -1 + c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

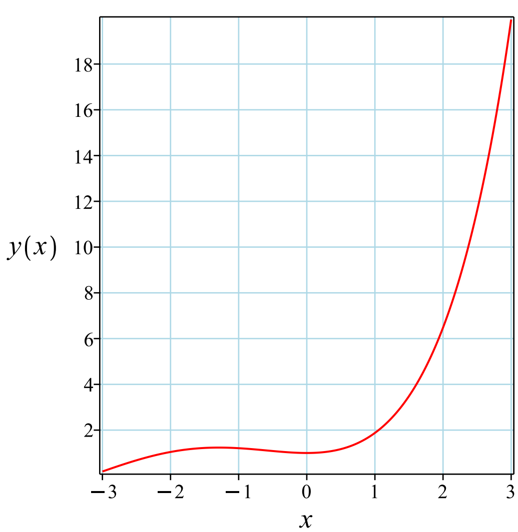
$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

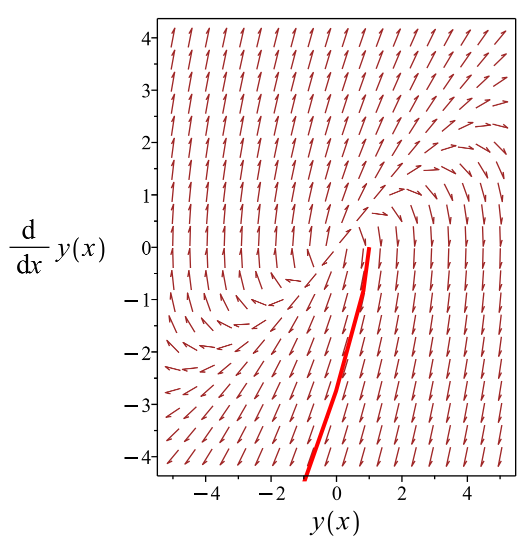
$$y = e^{x} - \sin(x)$$

Summary
The solution(s) found are the following

$$y = e^{x} - \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{x} - \sin(x)$$

Verified OK.

1.8.5 Maple step by step solution

Let's solve

$\left[y'' - 2y' + y = 2\cos(x), y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{x} x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{x} + c_2 x\, e^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\cos(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^x x \\ e^x & e^x x + e^x \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = 2\,e^x\left(-\left(\int \cos(x)\,x\,e^{-x}dx\right) + x\left(\int e^{-x}\cos(x)\,dx\right)\right)$

- Compute integrals

  $y_p(x) = -\sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,e^x + c_1 e^x - \sin(x)$

☐ Check validity of solution $y = c_2 x e^x + c_1 e^x - \sin(x)$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $y' = c_2 e^x + c_2 x\,e^x + c_1 e^x - \cos(x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = -1 + c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = e^x - \sin(x)$

- Solution to the IVP

  $y = e^x - \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=2*cos(x),y(0) = 1, D(y)(0) = 0],y(x), singsol=all
```

$$y(x) = \mathrm{e}^{x} - \sin(x)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 13

```
DSolve[{y''[x]-2*y'[x]+y[x]==2*Cos[x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{x} - \sin(x)$$

1.9 problem 10.2.11 (iii)

1.9.1 Existence and uniqueness analysis 80
1.9.2 Solving as second order linear constant coeff ode 81
1.9.3 Solving using Kovacic algorithm 85
1.9.4 Maple step by step solution 90

Internal problem ID [5053]
Internal file name [`OUTPUT/4546_Sunday_June_05_2022_03_00_39_PM_97533883/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.11 (iii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 16y = 16\cos(4x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

1.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 16 \\ F &= 16\cos(4x) \end{aligned}$$

Hence the ode is

$$y'' + 16y = 16\cos(4x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 16$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 16\cos(4x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.9.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 16, f(x) = 16\cos(4x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 16\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(16)} \\ &= \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +4i \\ \lambda_2 &= -4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4i \\ \lambda_2 &= -4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(4x) + c_2 \sin(4x))$$

Or

$$y = c_1 \cos(4x) + c_2 \sin(4x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(4x) + c_2 \sin(4x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16 \cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4x), \sin(4x)\}$$

Since $\cos(4x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(4x), x\sin(4x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(4x) + A_2 x\sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1\sin(4x) + 8A_2\cos(4x) = 16\cos(4x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\sin(4x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(4x) + c_2\sin(4x)) + (2x\sin(4x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\cos(4x) + c_2\sin(4x) + 2x\sin(4x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 \sin(4x) + 4c_2 \cos(4x) + 2\sin(4x) + 8x\cos(4x)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 0 \end{aligned}$$

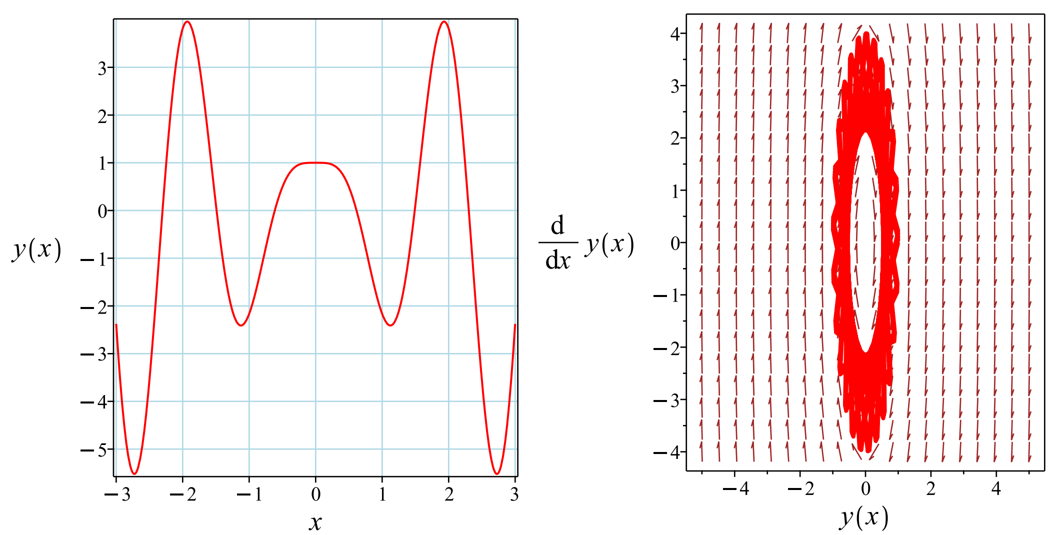
Substituting these values back in above solution results in

$$y = 2x\sin(4x) + \cos(4x)$$

Summary

The solution(s) found are the following

$$y = 2x\sin(4x) + \cos(4x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 2x\sin(4x) + \cos(4x)$$

Verified OK.

1.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 14: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(4x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(4x) \int \frac{1}{\cos(4x)^2}\,dx \\ &= \cos(4x) \left(\frac{\tan(4x)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(4x)) + c_2\left(\cos(4x)\left(\frac{\tan(4x)}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16 \cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(4x)}{4}, \cos(4x)\right\}$$

Since $\cos(4x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(4x), x \sin(4x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(4x) + A_2 x \sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \sin(4x) + 8A_2 \cos(4x) = 16 \cos(4x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x \sin(4x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}\right) + (2x \sin(4x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4} + 2x \sin(4x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 \sin(4x) + c_2 \cos(4x) + 2\sin(4x) + 8x \cos(4x)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

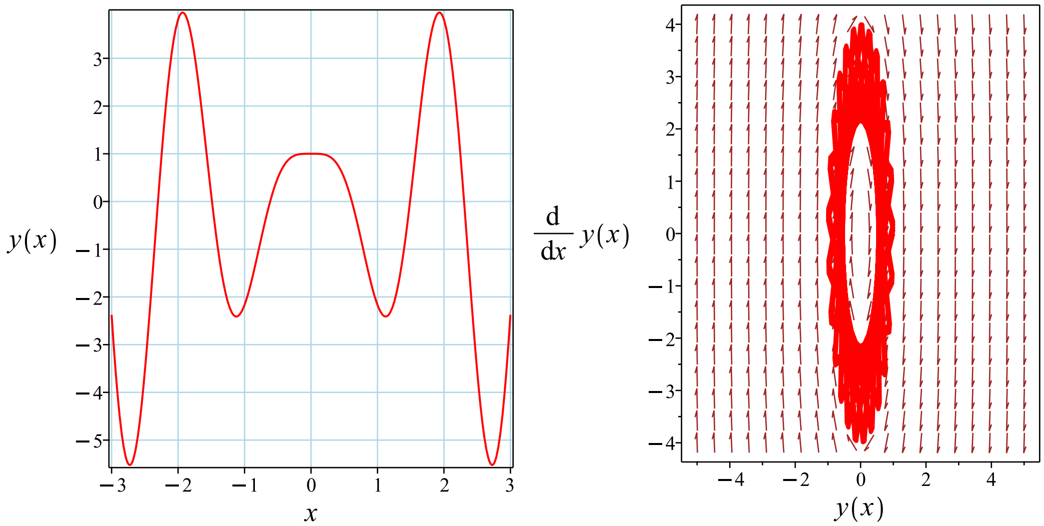
$$\begin{aligned} c_1 &= 1 \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = 2x \sin(4x) + \cos(4x)$$

Summary
The solution(s) found are the following

$$y = 2x \sin(4x) + \cos(4x) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 2x \sin(4x) + \cos(4x)$$

Verified OK.

1.9.4 Maple step by step solution

Let's solve

$$\left[y'' + 16y = 16\cos(4x), y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-4\,\mathrm{I}, 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(4x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4x) + c_2 \sin(4x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 16\cos(4x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(4x) & \sin(4x) \\ -4\sin(4x) & 4\cos(4x) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 4$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\cos(4x)\left(\int \sin(8x)\,dx\right) + 2\sin(4x)\left(\int (1+\cos(8x))\,dx\right)$

- Compute integrals

  $y_p(x) = \frac{\cos(4x)}{4} + 2x\sin(4x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(4x) + c_2 \sin(4x) + \frac{\cos(4x)}{4} + 2x\sin(4x)$

□ Check validity of solution $y = c_1 \cos(4x) + c_2 \sin(4x) + \frac{\cos(4x)}{4} + 2x\sin(4x)$

- Use initial condition $y(0) = 1$

  $1 = \frac{1}{4} + c_1$

- Compute derivative of the solution

  $y' = -4c_1 \sin(4x) + 4c_2 \cos(4x) + \sin(4x) + 8x\cos(4x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = 4c_2$

- Solve for c_1 and c_2

  $\{c_1 = \frac{3}{4}, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $y = 2x \sin(4x) + \cos(4x)$

- Solution to the IVP

  $y = 2x \sin(4x) + \cos(4x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)+16*y(x)=16*cos(4*x),y(0) = 1, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = \cos(4x) + 2\sin(4x)\, x$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 17

```
DSolve[{y''[x]+16*y[x]==16*Cos[4*x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to 2x \sin(4x) + \cos(4x)$$

1.10 problem 10.2.11 (iv)

1.10.1 Existence and uniqueness analysis 93
1.10.2 Solving as second order linear constant coeff ode 94
1.10.3 Solving using Kovacic algorithm 99
1.10.4 Maple step by step solution 106

Internal problem ID [5054]
Internal file name [OUTPUT/4547_Sunday_June_05_2022_03_00_40_PM_40071015/index.tex]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.2, ODEs with constant Coefficients. page 307
Problem number: 10.2.11 (iv).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \cosh(x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

1.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -1 \\ F &= \cosh(x) \end{aligned}$$

Hence the ode is

$$y'' - y = \cosh(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \cosh(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.10.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \cosh(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{-x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

Therefore

$$W = (e^x)(-e^{-x}) - (e^{-x})(e^x)$$

Which simplifies to

$$W = -2\,e^x e^{-x}$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-x}\cosh(x)}{-2}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-x}\cosh(x)}{2}dx$$

Hence

$$u_1 = \frac{x}{4} + \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^x \cosh(x)}{-2}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{e^x \cosh(x)}{2}dx$$

Hence

$$u_2 = -\frac{\cosh(x)^2}{4} - \frac{\cosh(x)\sinh(x)}{4} - \frac{x}{4}$$

Which simplifies to

$$u_1 = \frac{x}{4} + \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8}$$
$$u_2 = -\frac{x}{4} - \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8} - \frac{1}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{x}{4} + \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8}\right)e^x + \left(-\frac{x}{4} - \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8} - \frac{1}{8}\right)e^{-x}$$

Which simplifies to

$$y_p(x) = \frac{(-2x - 1 - \cosh(2x) - \sinh(2x))\, e^{-x}}{8} + \frac{\left(x + \frac{\sinh(2x)}{2} - \frac{\cosh(2x)}{2}\right)e^x}{4}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= (c_1 e^x + c_2 e^{-x}) + \left(\frac{(-2x - 1 - \cosh(2x) - \sinh(2x))\, e^{-x}}{8} + \frac{\left(x + \frac{\sinh(2x)}{2} - \frac{\cosh(2x)}{2}\right)e^x}{4}\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^x + c_2 e^{-x} + \frac{(-2x - 1 - \cosh(2x) - \sinh(2x))\,e^{-x}}{8} + \frac{\left(x + \frac{\sinh(2x)}{2} - \frac{\cosh(2x)}{2}\right) e^x}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 - \frac{3}{8} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x - c_2 e^{-x} + \frac{(-2 - 2\sinh(2x) - 2\cosh(2x))\,e^{-x}}{8} - \frac{(-2x - 1 - \cosh(2x) - \sinh(2x))\,e^{-x}}{8} + \frac{(1 +$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 - c_2 + \frac{1}{8} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{5}{8}$$
$$c_2 = \frac{3}{4}$$

Substituting these values back in above solution results in

$$y = \frac{5\,e^x}{8} + \frac{5\,e^{-x}}{8} - \frac{e^{-x}\sinh(2x)}{8} - \frac{e^{-x}\cosh(2x)}{8} - \frac{x\,e^{-x}}{4} + \frac{e^x x}{4} + \frac{e^x \sinh(2x)}{8} - \frac{e^x \cosh(2x)}{8}$$

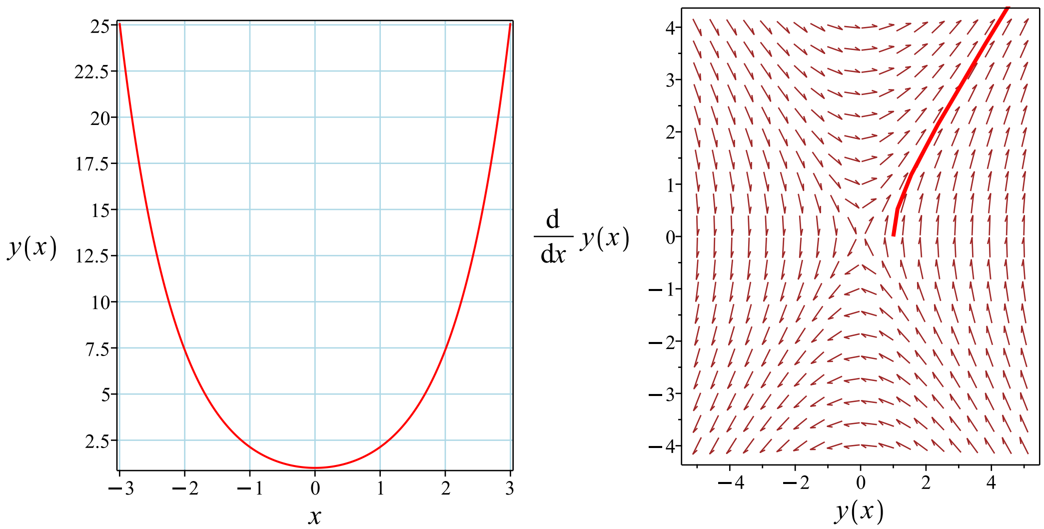
Which simplifies to

$$y = \frac{(-2x - \cosh(2x) - \sinh(2x) + 5)\,e^{-x}}{8} + \frac{e^x\left(x - \frac{\cosh(2x)}{2} + \frac{\sinh(2x)}{2} + \frac{5}{2}\right)}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(-2x - \cosh(2x) - \sinh(2x) + 5)\,e^{-x}}{8} + \frac{e^x\left(x - \frac{\cosh(2x)}{2} + \frac{\sinh(2x)}{2} + \frac{5}{2}\right)}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{(-2x-\cosh(2x)-\sinh(2x)+5)\,e^{-x}}{8} + \frac{e^{x}\left(x-\frac{\cosh(2x)}{2}+\frac{\sinh(2x)}{2}+\frac{5}{2}\right)}{4}$$

Verified OK.

1.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y''-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1\\ B &= 0\\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 16: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \frac{\mathrm{e}^{x}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ \frac{d}{dx}(e^{-x}) & \frac{d}{dx}\left(\frac{e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ -e^{-x} & \frac{e^x}{2} \end{vmatrix}$$

Therefore

$$W = (e^{-x})\left(\frac{e^x}{2}\right) - \left(\frac{e^x}{2}\right)(-e^{-x})$$

Which simplifies to

$$W = e^x e^{-x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^x \cosh(x)}{2}}{1} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{e^x \cosh(x)}{2} dx$$

Hence

$$u_1 = -\frac{\cosh(x)^2}{4} - \frac{\cosh(x)\sinh(x)}{4} - \frac{x}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x} \cosh(x)}{1} \, dx$$

Which simplifies to

$$u_2 = \int e^{-x} \cosh(x)\,dx$$

Hence

$$u_2 = \frac{x}{2} + \frac{\sinh(2x)}{4} - \frac{\cosh(2x)}{4}$$

Which simplifies to

$$u_1 = -\frac{x}{4} - \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8} - \frac{1}{8}$$
$$u_2 = \frac{x}{2} + \frac{\sinh(2x)}{4} - \frac{\cosh(2x)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x}{4} - \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8} - \frac{1}{8}\right) e^{-x} + \frac{\left(\frac{x}{2} + \frac{\sinh(2x)}{4} - \frac{\cosh(2x)}{4}\right) e^{x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{(-2x - 1 - \cosh(2x) - \sinh(2x))\,e^{-x}}{8} + \frac{\left(x + \frac{\sinh(2x)}{2} - \frac{\cosh(2x)}{2}\right) e^{x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{x}}{2}\right) \\ &\quad + \left(\frac{(-2x - 1 - \cosh(2x) - \sinh(2x))\,e^{-x}}{8} + \frac{\left(x + \frac{\sinh(2x)}{2} - \frac{\cosh(2x)}{2}\right) e^{x}}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \frac{(-2x - 1 - \cosh(2x) - \sinh(2x))\,e^{-x}}{8} + \frac{\left(x + \frac{\sinh(2x)}{2} - \frac{\cosh(2x)}{2}\right) e^{x}}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + \frac{c_2}{2} - \frac{3}{8} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + \frac{(-2 - 2\sinh(2x) - 2\cosh(2x))\, \mathrm{e}^{-x}}{8} - \frac{(-2x - 1 - \cosh(2x) - \sinh(2x))\, \mathrm{e}^{-x}}{8} + \frac{(1$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -c_1 + \frac{c_2}{2} + \frac{1}{8} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3}{4}$$
$$c_2 = \frac{5}{4}$$

Substituting these values back in above solution results in

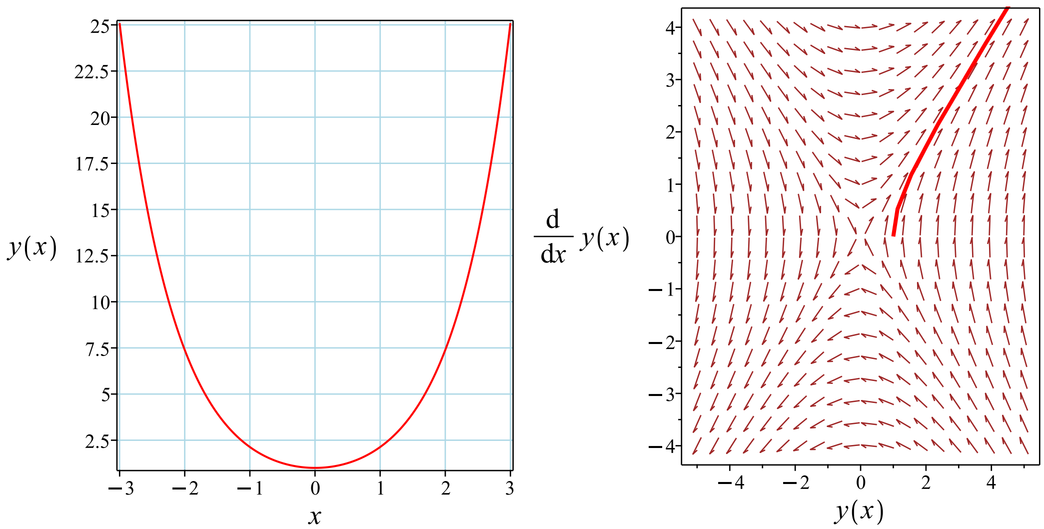
$$y = \frac{5\,\mathrm{e}^{x}}{8} + \frac{5\,\mathrm{e}^{-x}}{8} - \frac{\mathrm{e}^{-x}\sinh(2x)}{8} - \frac{\mathrm{e}^{-x}\cosh(2x)}{8} - \frac{x\,\mathrm{e}^{-x}}{4} + \frac{\mathrm{e}^{x} x}{4} + \frac{\mathrm{e}^{x}\sinh(2x)}{8} - \frac{\mathrm{e}^{x}\cosh(2x)}{8}$$

Which simplifies to

$$y = \frac{(-2x - \cosh(2x) - \sinh(2x) + 5)\, \mathrm{e}^{-x}}{8} + \frac{\mathrm{e}^{x}\left(x - \frac{\cosh(2x)}{2} + \frac{\sinh(2x)}{2} + \frac{5}{2}\right)}{4}$$

Summary
The solution(s) found are the following

$$y = \frac{(-2x - \cosh(2x) - \sinh(2x) + 5)\, \mathrm{e}^{-x}}{8} + \frac{\mathrm{e}^{x}\left(x - \frac{\cosh(2x)}{2} + \frac{\sinh(2x)}{2} + \frac{5}{2}\right)}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{(-2x - \cosh(2x) - \sinh(2x) + 5)\,\mathrm{e}^{-x}}{8} + \frac{\mathrm{e}^{x}\left(x - \frac{\cosh(2x)}{2} + \frac{\sinh(2x)}{2} + \frac{5}{2}\right)}{4}$$

Verified OK.

1.10.4 Maple step by step solution

Let's solve

$$\left[y'' - y = \cosh(x), y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cosh(x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{x} \cosh(x) dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int \mathrm{e}^{-x} \cosh(x) dx\right)}{2}$

- Compute integrals

  $y_p(x) = \frac{(-2x-1-\cosh(2x)-\sinh(2x))\mathrm{e}^{-x}}{8} + \frac{\left(x+\frac{\sinh(2x)}{2}-\frac{\cosh(2x)}{2}\right)\mathrm{e}^{x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{(-2x-1-\cosh(2x)-\sinh(2x))\mathrm{e}^{-x}}{8} + \frac{\left(x+\frac{\sinh(2x)}{2}-\frac{\cosh(2x)}{2}\right)\mathrm{e}^{x}}{4}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{(-2x-1-\cosh(2x)-\sinh(2x))\mathrm{e}^{-x}}{8} + \frac{\left(x+\frac{\sinh(2x)}{2}-\frac{\cosh(2x)}{2}\right)\mathrm{e}^{x}}{4}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 - \frac{3}{8}$

- Compute derivative of the solution

  $y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{(-2-2\sinh(2x)-2\cosh(2x))\mathrm{e}^{-x}}{8} - \frac{(-2x-1-\cosh(2x)-\sinh(2x))\mathrm{e}^{-x}}{8} + \frac{(1+\cosh(2x)-\sinh(2x))\mathrm{e}^{x}}{4}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

$0=-c_1+c_2+\frac{1}{8}$

- Solve for c_1 and c_2

  $\{c_1=\frac{3}{4}, c_2=\frac{5}{8}\}$

- Substitute constant values into general solution and simplify

  $$y=\frac{(-2x-\cosh(2x)-\sinh(2x)+5)\,e^{-x}}{8}+\frac{e^{x}\left(x-\frac{\cosh(2x)}{2}+\frac{\sinh(2x)}{2}+\frac{5}{2}\right)}{4}$$

- Solution to the IVP

  $$y=\frac{(-2x-\cosh(2x)-\sinh(2x)+5)\,e^{-x}}{8}+\frac{e^{x}\left(x-\frac{\cosh(2x)}{2}+\frac{\sinh(2x)}{2}+\frac{5}{2}\right)}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)-y(x)=cosh(x),y(0) = 1, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x)=\frac{(-x+2)\,e^{-x}}{4}+\frac{e^{x}(x+2)}{4}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 28

```
DSolve[{y''[x]-y[x]==Cosh[x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{4}e^{-x}\left(-x+e^{2x}(x+2)+2\right)$$

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2.1 problem 10.3.2

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Internal problem ID [5055]
Internal file name [`OUTPUT/4548_Sunday_June_05_2022_03_00_42_PM_28220764/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.3, ODEs with variable Coefficients. First order. page 315
Problem number: 10.3.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = \mathrm{e}^{2x}}$$

2.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = \mathrm{e}^{2x}$$

Hence the ode is

$$y' - y = \mathrm{e}^{2x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1)dx} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\mathrm{e}^{2x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x} y\right) &= \left(\mathrm{e}^{-x}\right)\left(\mathrm{e}^{2x}\right) \\
\mathrm{d}\left(\mathrm{e}^{-x} y\right) &= \mathrm{e}^{x}\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-x} y &= \int \mathrm{e}^{x}\,\mathrm{d}x \\
\mathrm{e}^{-x} y &= \mathrm{e}^{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$
y = \mathrm{e}^{2x} + c_1 \mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{2x} + c_1 \mathrm{e}^{x} \tag{1}
$$

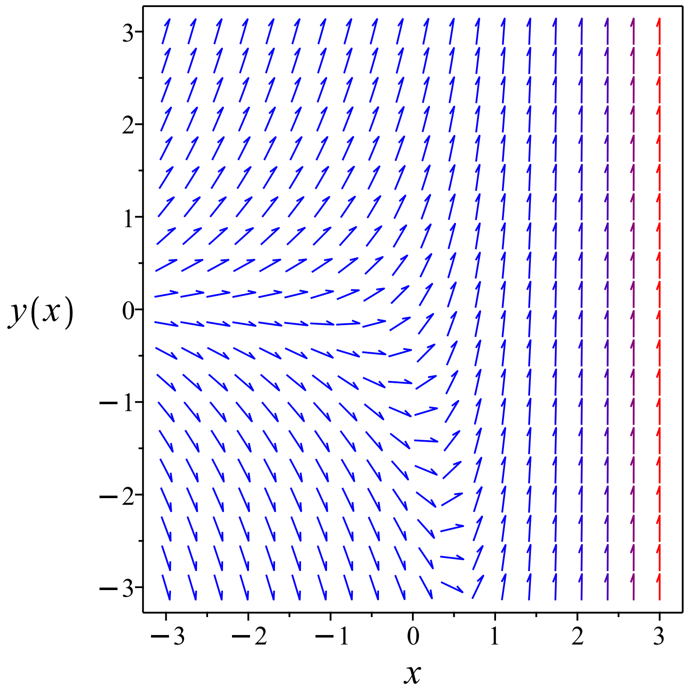


Figure 12: Slope field plot

Verification of solutions

$$y = e^{2x} + c_1 e^x$$

Verified OK.

2.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y + e^{2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 18: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^x} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y + e^{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\, e^{-x} = e^{x} + c_1$$

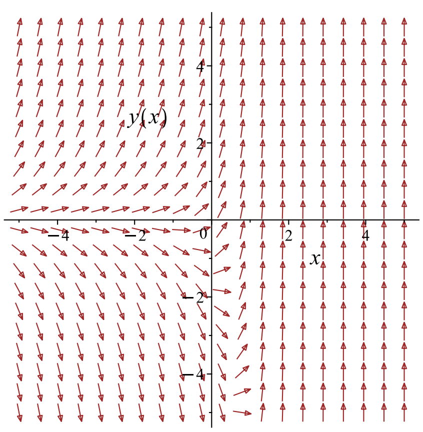
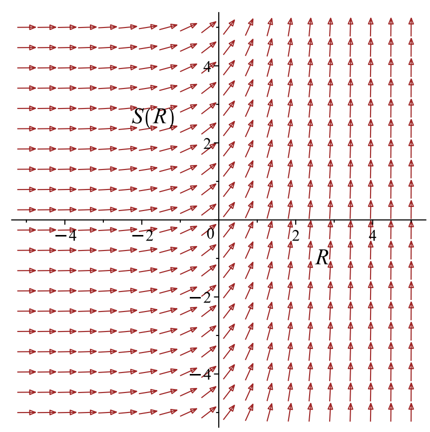
Which simplifies to

$$y\, e^{-x} = e^{x} + c_1$$

Which gives

$$y = e^{x}(e^{x} + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y + e^{2x}$ 	$R = x$ $S = e^{-x} y$	$\frac{dS}{dR} = e^{R}$ 

Summary

The solution(s) found are the following

$$y = e^{x}(e^{x} + c_1) \tag{1}$$

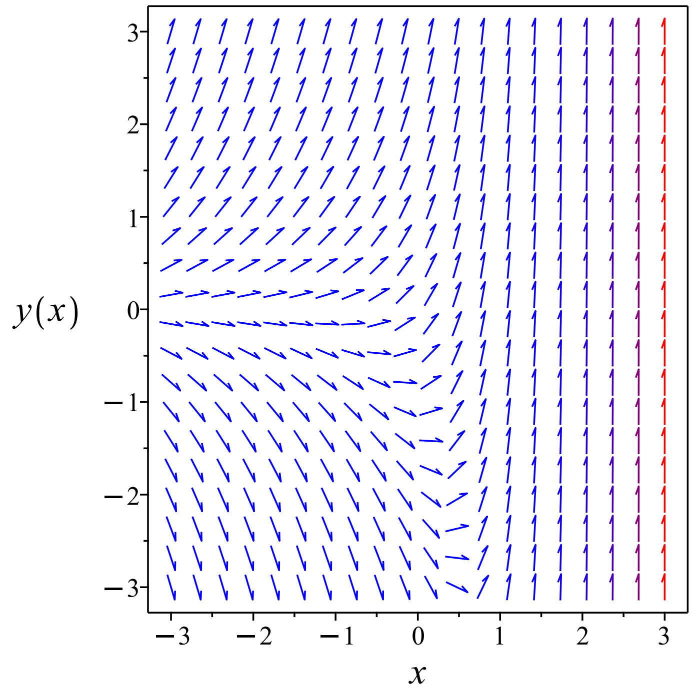


Figure 13: Slope field plot

Verification of solutions

$$y = e^x(e^x + c_1)$$

Verified OK.

2.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(y + \mathrm{e}^{2x}\right)\mathrm{d}x \\ \left(-y - \mathrm{e}^{2x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y - \mathrm{e}^{2x} \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - \mathrm{e}^{2x}\right) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}\left(-y - \mathrm{e}^{2x}\right) \\ &= -\mathrm{e}^{-x} y - \mathrm{e}^{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{-x} y - \mathrm{e}^{x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int -e^{-x} y - e^{x} \, dx$$
$$\phi = e^{-x} y - e^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{-x}$. Therefore equation (4) becomes

$$e^{-x} = e^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{-x} y - e^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{-x} y - e^{x}$$

The solution becomes

$$y = e^{x} (e^{x} + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{x} (e^{x} + c_1) \tag{1}$$

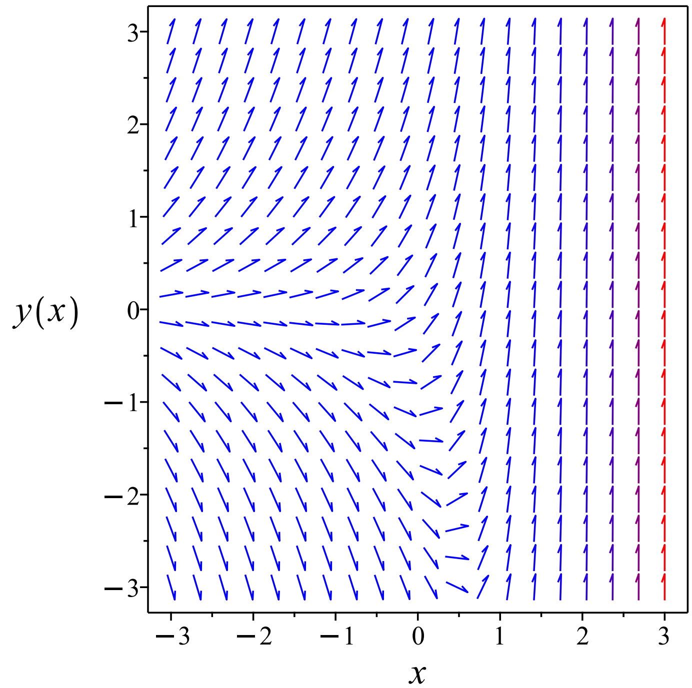


Figure 14: Slope field plot

Verification of solutions

$$y = e^{x}(e^{x} + c_1)$$

Verified OK.

2.1.4 Maple step by step solution

Let's solve

$y' - y = e^{2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + e^{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = e^{2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' - y) = \mu(x)e^{2x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'-y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^{2x}dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{2x}dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^{2x}dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int \mathrm{e}^{-x}\mathrm{e}^{2x}dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = \mathrm{e}^{x}(\mathrm{e}^{x} + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)-y(x)=exp(2*x),y(x), singsol=all)
```

$$y(x) = (e^x + c_1)\, e^x$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 15

```
DSolve[y'[x]-y[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(e^x + c_1)$$

2.2 problem 10.3.3

2.2.1 Existence and uniqueness analysis 123
2.2.2 Solving as linear ode 124
2.2.3 Solving as differentialType ode 126
2.2.4 Solving as first order ode lie symmetry lookup ode 127
2.2.5 Solving as exact ode 132
2.2.6 Maple step by step solution 135

Internal problem ID [5056]
Internal file name [`OUTPUT/4549_Sunday_June_05_2022_03_00_42_PM_87162301/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.3, ODEs with variable Coefficients. First order. page 315
Problem number: 10.3.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x^2y' + 2xy = x - 1}$$

With initial conditions

$$[y(1) = 0]$$

2.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{x-1}{x^2} \end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{x-1}{x^2}$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{x-1}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.2.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x-1}{x^2}\right) \\ \frac{d}{dx}(y\,x^2) &= (x^2)\left(\frac{x-1}{x^2}\right) \\ d(y\,x^2) &= (x-1)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^2 &= \int x - 1\,dx \\ y\,x^2 &= \frac{1}{2}x^2 - x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{\frac{1}{2}x^2 - x}{x^2} + \frac{c_1}{x^2}$$

which simplifies to

$$y = \frac{x^2 + 2c_1 - 2x}{2x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

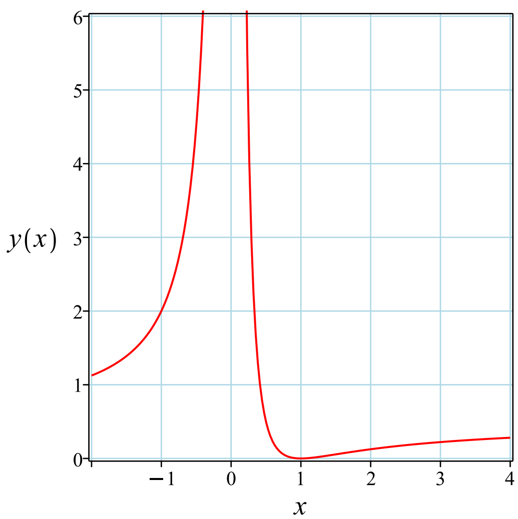
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 - 2x + 1}{2x^2}$$

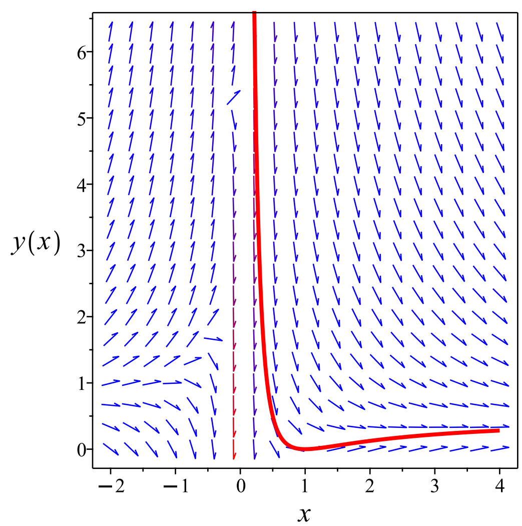
Summary

The solution(s) found are the following

$$y = \frac{x^2 - 2x + 1}{2x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 - 2x + 1}{2x^2}$$

Verified OK.

2.2.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2xy + x - 1}{x^2} \tag{1}$$

Which becomes

$$0 = (-x^2)\, dy + (-2xy + x - 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x^2)\, dy + (-2xy + x - 1)\, dx = d\left(-y\, x^2 + \frac{1}{2}x^2 - x\right)$$

Hence (2) becomes

$$0 = d\left(-y\, x^2 + \frac{1}{2}x^2 - x\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^2 + 2c_1 - 2x}{2x^2} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 2c_1 - \frac{1}{2}$$

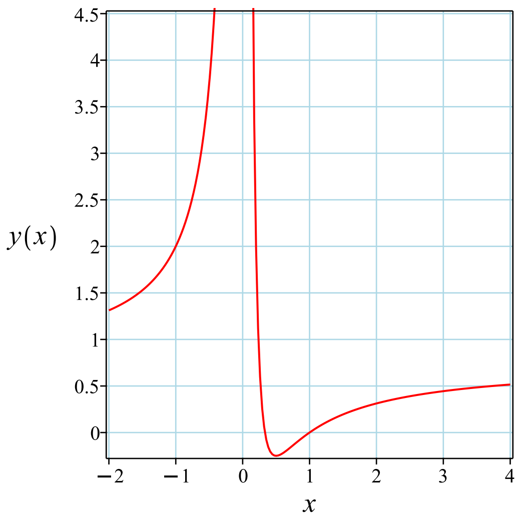
$$c_1 = \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

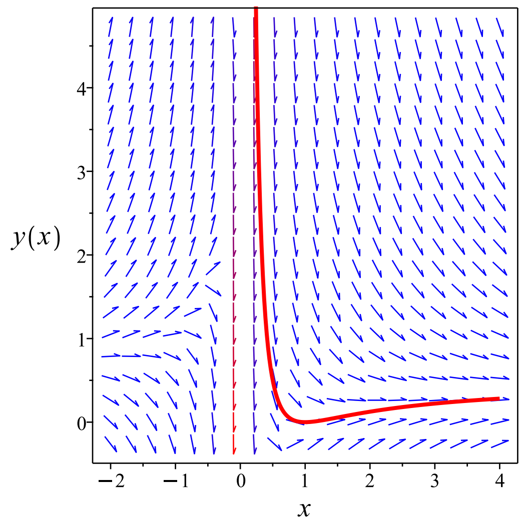
$$y = \frac{3x^2 - 4x + 1}{4x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{3x^2 - 4x + 1}{4x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{3x^2 - 4x + 1}{4x^2}$$

Verified OK.

2.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2xy - x + 1}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 21: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{x^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y\,x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2xy - x + 1}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x - 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R - 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 - R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^2 = \frac{1}{2}x^2 - x + c_1$$

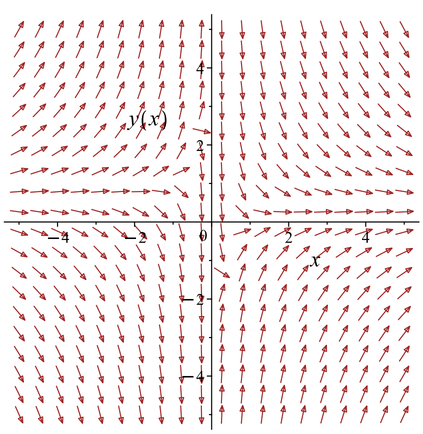
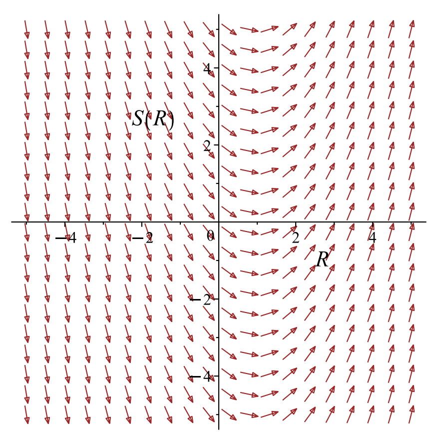
Which simplifies to

$$yx^2 = \frac{1}{2}x^2 - x + c_1$$

Which gives

$$y = \frac{x^2 + 2c_1 - 2x}{2x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2xy - x + 1}{x^2}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = R - 1$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

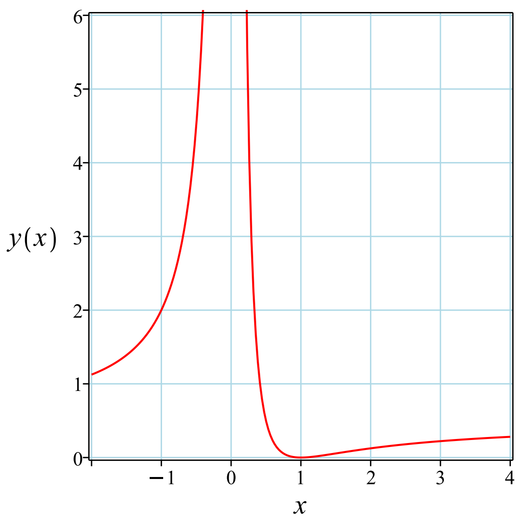
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 - 2x + 1}{2x^2}$$

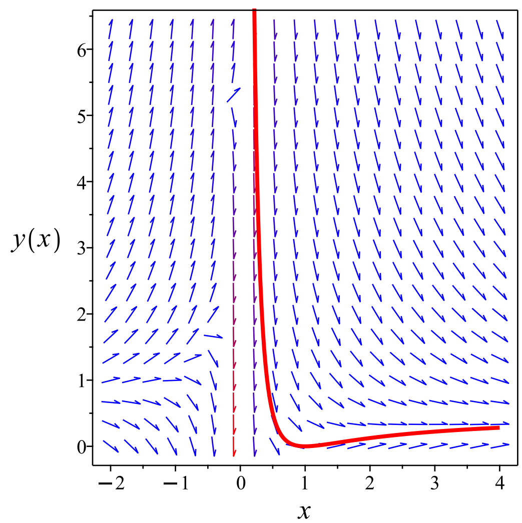
Summary

The solution(s) found are the following

$$y = \frac{x^2 - 2x + 1}{2x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 - 2x + 1}{2x^2}$$

Verified OK.

2.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y&=(-2xy+x-1)\,\mathrm{d}x\\ (2xy-x+1)\,\mathrm{d}x+(x^2)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2xy-x+1\\ N(x,y)&=x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy - x + 1)\\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 2xy - x + 1 \,\mathrm{d}x\\ \phi &= \frac{(2y-1)\,x^2}{2} + x + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(2y-1)\,x^2}{2} + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2y-1)\,x^2}{2} + x$$

The solution becomes

$$y = \frac{x^2 + 2c_1 - 2x}{2x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

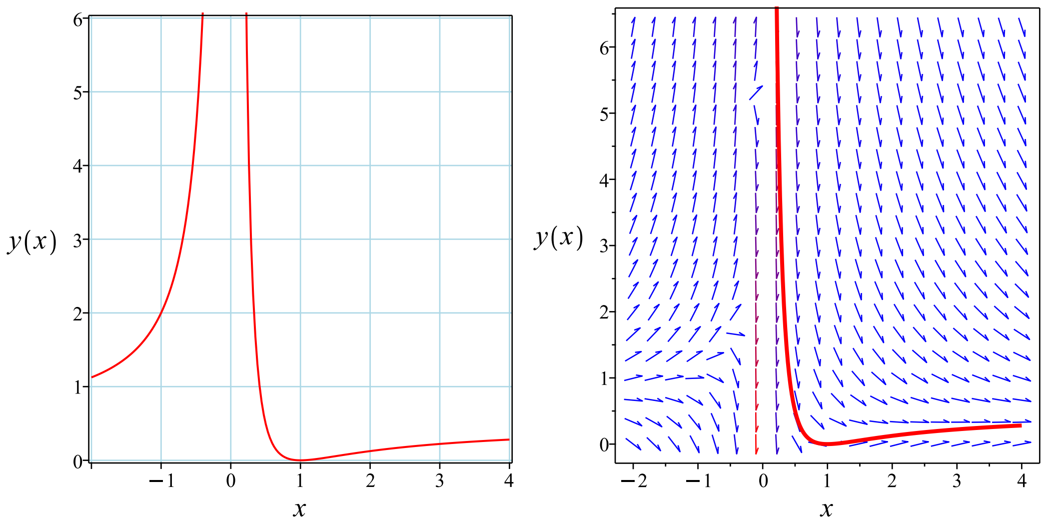
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 - 2x + 1}{2x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 - 2x + 1}{2x^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{x^2 - 2x + 1}{2x^2}$$

Verified OK.

2.2.6 Maple step by step solution

Let's solve

$[x^2y' + 2xy = x - 1, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + \frac{x-1}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = \frac{x-1}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \frac{\mu(x)(x-1)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = x^2$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(x-1)}{x^2} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)(x-1)}{x^2} dx + c_1$
- Solve for y

  $y = \frac{\int \frac{\mu(x)(x-1)}{x^2} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = x^2$

  $y = \frac{\int (x-1) dx + c_1}{x^2}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{1}{2}x^2 - x + c_1}{x^2}$
- Simplify

  $y = \frac{x^2 + 2c_1 - 2x}{2x^2}$
- Use initial condition $y(1) = 0$

  $0 = -\frac{1}{2} + c_1$
- Solve for c_1

  $c_1 = \frac{1}{2}$
- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = \frac{(x-1)^2}{2x^2}$
- Solution to the IVP

  $y = \frac{(x-1)^2}{2x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([x^2*diff(y(x),x)+2*x*y(x)-x+1=0,y(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{(x-1)^2}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 17

```
DSolve[{x^2*y'[x]+2*x*y[x]-x+1==0,{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(x-1)^2}{2x^2}$$

2.3 problem 10.3.4

2.3.1 Existence and uniqueness analysis 138
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Internal problem ID [5057]
Internal file name [`OUTPUT/4550_Sunday_June_05_2022_03_00_43_PM_3682306/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.3, ODEs with variable Coefficients. First order. page 315
Problem number: 10.3.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y + y' = (x+1)^2}$$

With initial conditions

$$[y(0) = 0]$$

2.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= (x+1)^2 \end{aligned}$$

Hence the ode is

$$y + y' = (x + 1)^2$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = (x + 1)^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.3.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dx} \\ &= e^x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left((x + 1)^2\right) \\ \frac{d}{dx}(y\, e^x) &= (e^x)\left((x + 1)^2\right) \\ d(y\, e^x) &= \left((x + 1)^2 e^x\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\, e^x &= \int (x + 1)^2 e^x \, dx \\ y\, e^x &= \left(x^2 + 1\right) e^x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(x^2 + 1\right) e^x + c_1 e^{-x}$$

which simplifies to

$$y = x^2 + 1 + c_1 e^{-x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 1$$

$$c_1 = -1$$

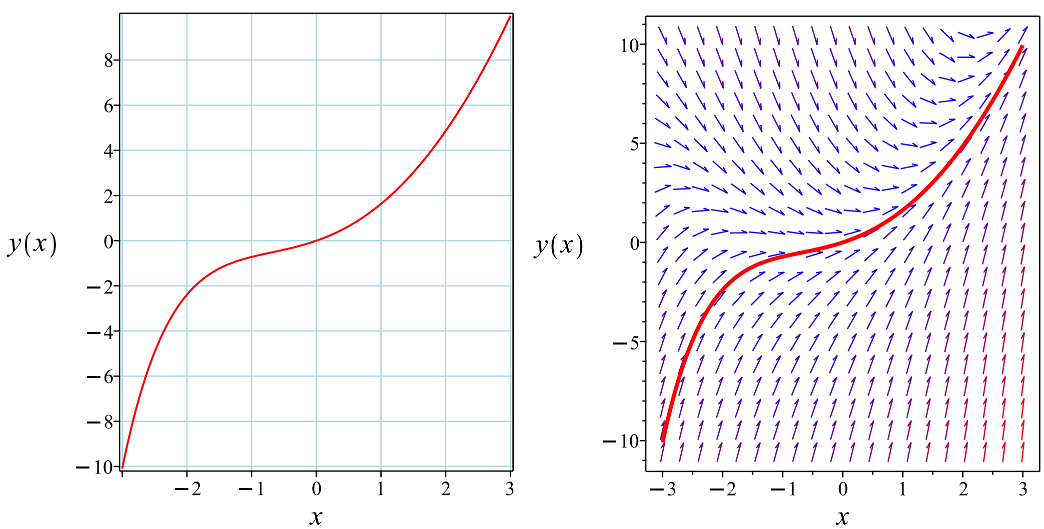
Substituting c_1 found above in the general solution gives

$$y = 1 + x^2 - \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = 1 + x^2 - \mathrm{e}^{-x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 + x^2 - \mathrm{e}^{-x}$$

Verified OK.

2.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= x^2 + 2x - y + 1 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-x}} dy
\end{aligned}
$$

Which results in

$$
S = y\,\mathrm{e}^{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = x^2 + 2x - y + 1
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y\,\mathrm{e}^{x} \\
S_y &= \mathrm{e}^{x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (x+1)^2 \, \mathrm{e}^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R+1)^2 \, \mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \left(R^2+1\right) \mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x} y = \left(x^2+1\right) \mathrm{e}^{x} + c_1$$

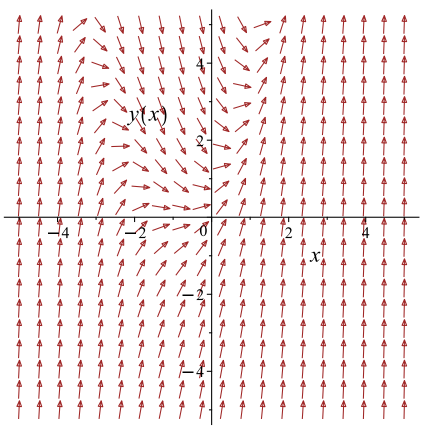
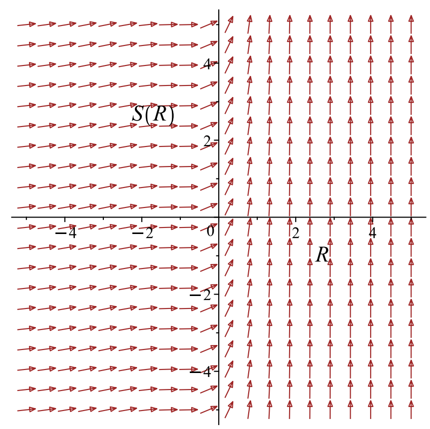
Which simplifies to

$$\mathrm{e}^{x} y = \left(x^2+1\right) \mathrm{e}^{x} + c_1$$

Which gives

$$y = \left(x^2 \mathrm{e}^{x} + \mathrm{e}^{x} + c_1\right) \mathrm{e}^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 + 2x - y + 1$ 	$R = x$ $S = y\,\mathrm{e}^{x}$	$\frac{dS}{dR} = (R+1)^2\,\mathrm{e}^{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 1$$

$$c_1 = -1$$

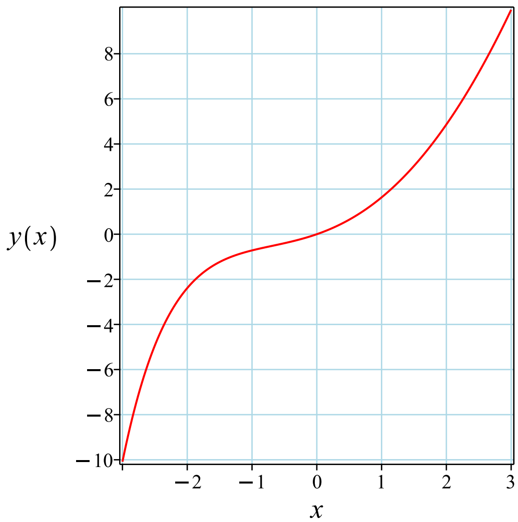
Substituting c_1 found above in the general solution gives

$$y = 1 + x^2 - \mathrm{e}^{-x}$$

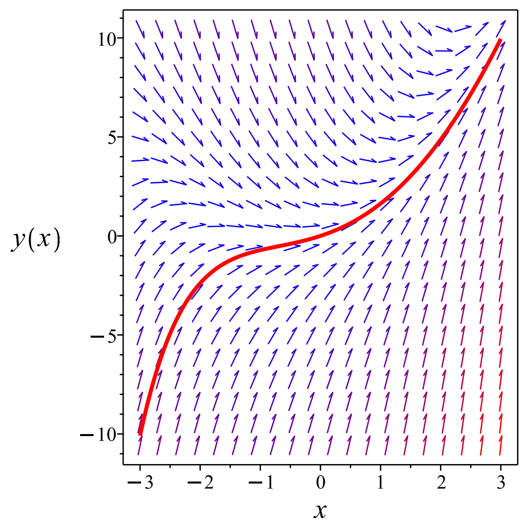
Summary

The solution(s) found are the following

$$y = 1 + x^2 - \mathrm{e}^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + x^2 - e^{-x}$$

Verified OK.

2.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}y &= \left(-y + (x+1)^2\right)\mathrm{d}x \\
\left(y - (x+1)^2\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= y - (x+1)^2 \\
N(x,y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - (x+1)^2\right) \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}\left(y-(x+1)^2\right) \\ &= -\mathrm{e}^{x}\left(x^2+2x-y+1\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{x}\left(x^2+2x-y+1\right)\right) + \left(\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\mathrm{e}^{x}\left(x^2+2x-y+1\right) \mathrm{d}x\end{aligned}$$

$$\phi = -\left(x^2-y+1\right)\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\left(x^2 - y + 1\right) \mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^2 - y + 1\right) \mathrm{e}^{x}$$

The solution becomes

$$y = \left(x^2 \mathrm{e}^{x} + \mathrm{e}^{x} + c_1\right) \mathrm{e}^{-x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 1$$

$$c_1 = -1$$

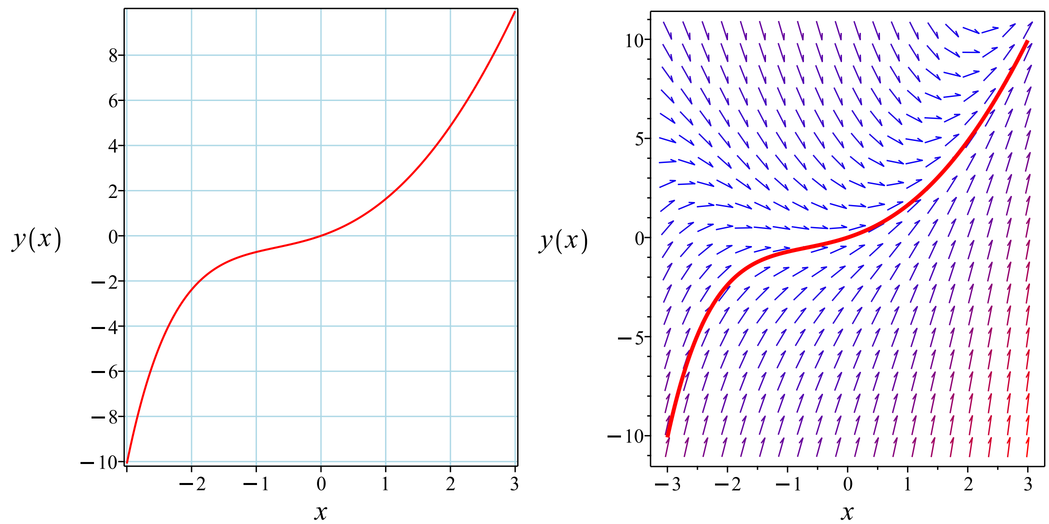
Substituting c_1 found above in the general solution gives

$$y = 1 + x^2 - \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = 1 + x^2 - e^{-x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 1 + x^2 - e^{-x}$$

Verified OK.

2.3.5 Maple step by step solution

Let's solve

$\left[y + y' = (x+1)^2, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + (x+1)^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y + y' = (x+1)^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y+y'\right)=\mu(x)\left(x+1\right)^{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}\left(\mu(x)\,y\right)$

  $\mu(x)\left(y+y'\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\mu(x)$

- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}\left(\mu(x)\,y\right)\right)dx=\int\mu(x)\left(x+1\right)^{2}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\left(x+1\right)^{2}dx+c_1$

- Solve for y

  $y=\frac{\int\mu(x)(x+1)^{2}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\mathrm{e}^{x}$

  $y=\frac{\int(x+1)^{2}\mathrm{e}^{x}dx+c_1}{\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y=\frac{(x^{2}+1)\mathrm{e}^{x}+c_1}{\mathrm{e}^{x}}$

- Simplify

  $y=x^{2}+1+c_1\mathrm{e}^{-x}$

- Use initial condition $y(0)=0$

  $0=c_1+1$

- Solve for c_1

  $c_1=-1$

- Substitute $c_1=-1$ into general solution and simplify

  $y=1+x^{2}-\mathrm{e}^{-x}$

- Solution to the IVP

  $y=1+x^{2}-\mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([diff(y(x),x)+y(x)=(x+1)^2,y(0) = 0],y(x), singsol=all)
```

$$y(x) = x^2 + 1 - \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 17

```
DSolve[{y'[x]+y[x]==(x+1)^2,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 - e^{-x} + 1$$

2.4 problem 10.3.5

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Internal problem ID [5058]
Internal file name [`OUTPUT/4551_Sunday_June_05_2022_03_00_45_PM_20469628/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.3, ODEs with variable Coefficients. First order. page 315
Problem number: 10.3.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x^2y' + 2xy = \sinh(x)}$$

With initial conditions

$$[y(1) = 2]$$

2.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{\sinh(x)}{x^2}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{\sinh(x)}{x^2}$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{\sinh(x)}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.4.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\sinh(x)}{x^2}\right) \\ \frac{d}{dx}(y\,x^2) &= (x^2)\left(\frac{\sinh(x)}{x^2}\right) \\ d(y\,x^2) &= \sinh(x)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^2 &= \int \sinh(x)\,dx \\ y\,x^2 &= \cosh(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{\cosh(x)}{x^2} + \frac{c_1}{x^2}$$

which simplifies to

$$y = \frac{\cosh(x) + c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \cosh(1) + c_1$$

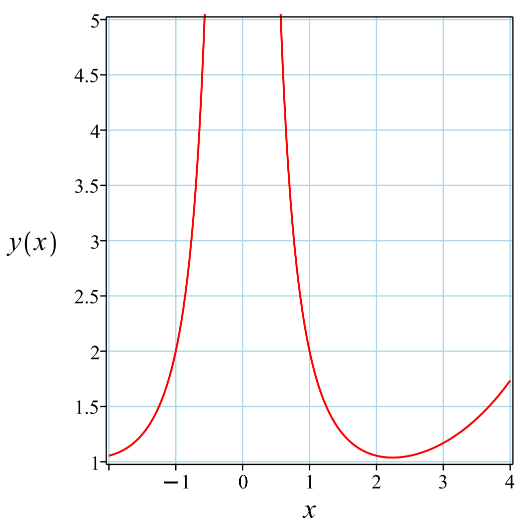
$$c_1 = -\cosh(1) + 2$$

Substituting c_1 found above in the general solution gives

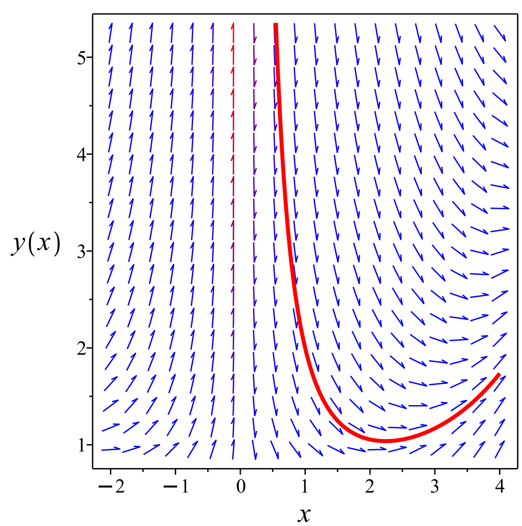
$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

Verified OK.

2.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-2xy + \sinh(x)}{x^2} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 27: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x^2}} dy
\end{aligned}
$$

Which results in

$$
S = y\,x^2
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-2xy + \sinh(x)}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 2xy \\
S_y &= x^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \sinh\left(x\right) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \sinh\left(R\right)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \cosh\left(R\right) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
y x^2 = \cosh\left(x\right) + c_1
$$

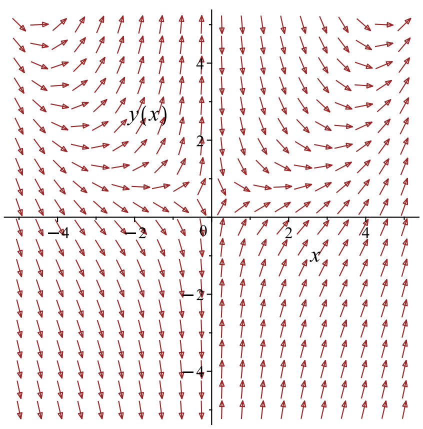
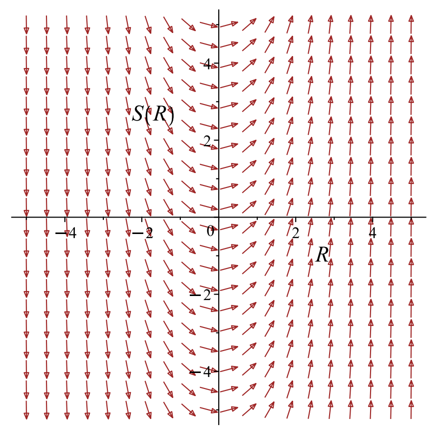
Which simplifies to

$$
y x^2 = \cosh\left(x\right) + c_1
$$

Which gives

$$
y = \frac{\cosh\left(x\right) + c_1}{x^2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-2xy+\sinh(x)}{x^2}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = \sinh(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \cosh(1) + c_1$$

$$c_1 = -\cosh(1) + 2$$

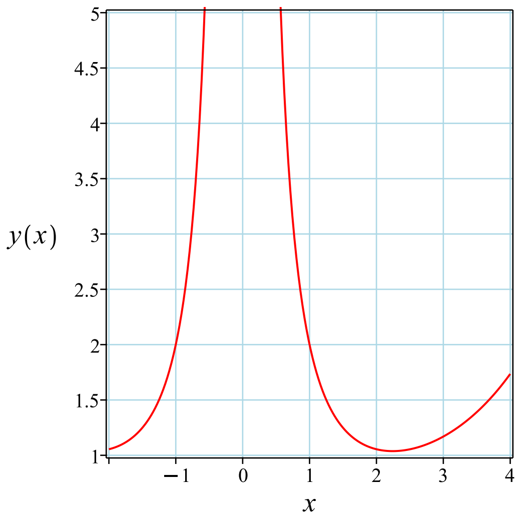
Substituting c_1 found above in the general solution gives

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

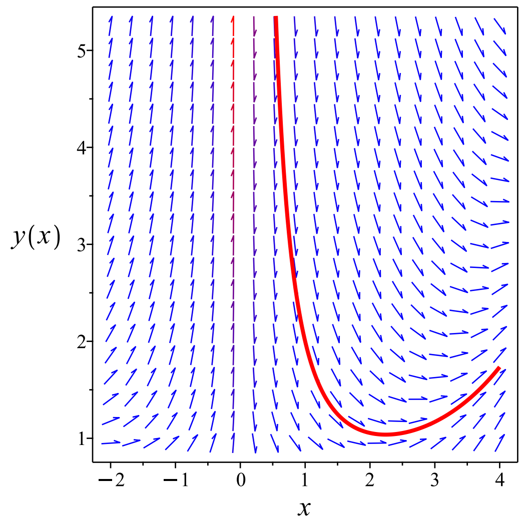
Summary

The solution(s) found are the following

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

Verified OK.

2.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y &= (-2xy + \sinh(x))\,\mathrm{d}x\\ (2xy - \sinh(x))\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2xy - \sinh(x)\\ N(x, y) &= x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy - \sinh(x))\\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2xy - \sinh(x) \, \mathrm{d}x$$
$$\phi = y\,x^2 - \cosh(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 - \cosh(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 - \cosh(x)$$

The solution becomes

$$y = \frac{\cosh(x) + c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \cosh(1) + c_1$$

$$c_1 = -\cosh(1) + 2$$

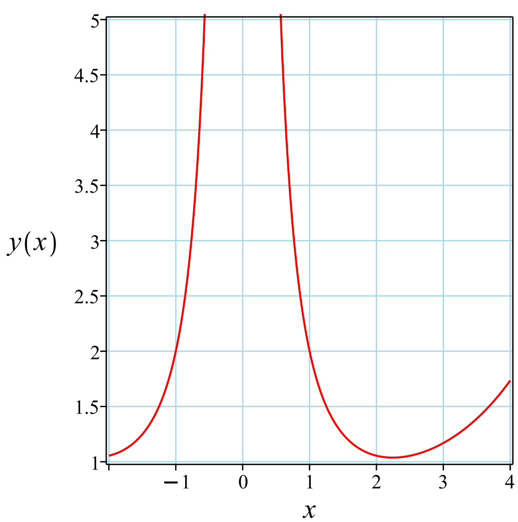
Substituting c_1 found above in the general solution gives

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

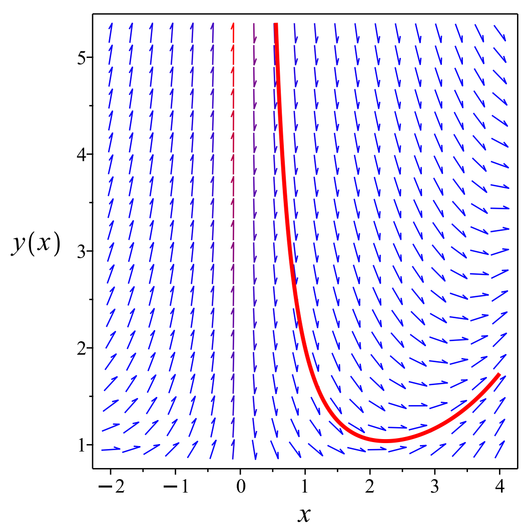
Summary

The solution(s) found are the following

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

Verified OK.

2.4.5 Maple step by step solution

Let's solve

$\left[x^2 y' + 2xy = \sinh\left(x\right), y(1) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + \frac{\sinh(x)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = \frac{\sinh(x)}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \frac{\mu(x)\sinh(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)\sinh(x)}{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)\sinh(x)}{x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\sinh(x)}{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int \sinh(x) dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{\cosh(x) + c_1}{x^2}$

- Use initial condition $y(1) = 2$

$$2 = \cosh(1) + c_1$$

- Solve for c_1

$$c_1 = -\cosh(1) + 2$$

- Substitute $c_1 = -\cosh(1) + 2$ into general solution and simplify

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

- Solution to the IVP

$$y = \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve([x^2*diff(y(x),x)+2*x*y(x)=sinh(x),y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 17

```
DSolve[{x^2*y'[x]+2*x*y[x]==Sinh[x],{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\cosh(x) + 2 - \cosh(1)}{x^2}$$

2.5 problem 10.3.6

2.5.1 Solving as linear ode 165
2.5.2 Solving as first order ode lie symmetry lookup ode 167
2.5.3 Solving as exact ode 171
2.5.4 Maple step by step solution 176

Internal problem ID [5059]
Internal file name [`OUTPUT/4552_Sunday_June_05_2022_03_00_46_PM_11495574/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.3, ODEs with variable Coefficients. First order. page 315
Problem number: 10.3.6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + \frac{y}{1-x} = x^2 - 2x}$$

2.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x-1}$$
$$q(x) = x(-2+x)$$

Hence the ode is

$$y' - \frac{y}{x-1} = x(-2+x)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x-1}dx} \\
&= \frac{1}{x-1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(x(-2+x)) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x-1}\right) &= \left(\frac{1}{x-1}\right)(x(-2+x)) \\
\mathrm{d}\left(\frac{y}{x-1}\right) &= \left(\frac{x(-2+x)}{x-1}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x-1} &= \int \frac{x(-2+x)}{x-1}\,\mathrm{d}x \\
\frac{y}{x-1} &= \frac{x^2}{2} - x - \ln(x-1) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x-1}$ results in

$$
y = (x-1)\left(\frac{x^2}{2} - x - \ln(x-1)\right) + c_1(x-1)
$$

which simplifies to

$$
y = \frac{(x-1)\,(x^2 - 2x - 2\ln(x-1) + 2c_1)}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(x-1)\,(x^2 - 2x - 2\ln(x-1) + 2c_1)}{2} \tag{1}
$$

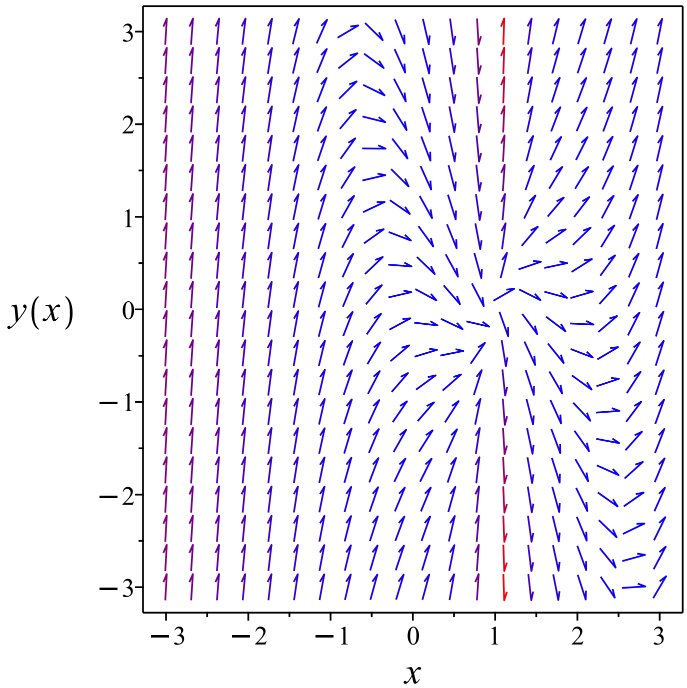


Figure 25: Slope field plot

Verification of solutions

$$y = \frac{(x-1)\left(x^2 - 2x - 2\ln(x-1) + 2c_1\right)}{2}$$

Verified OK.

2.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 - 3x^2 + 2x + y}{x-1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x - 1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x-1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x-1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 - 3x^2 + 2x + y}{x-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x-1)^2} \\ S_y &= \frac{1}{x-1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x(-2+x)}{x-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R(-2+R)}{R-1}$$

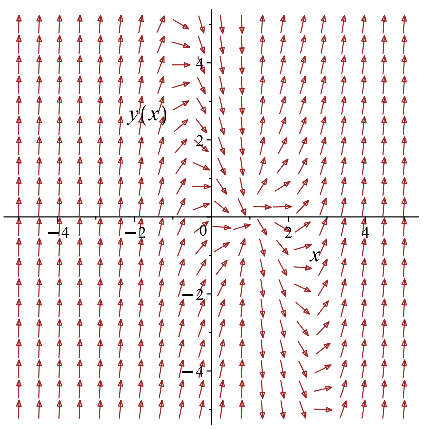
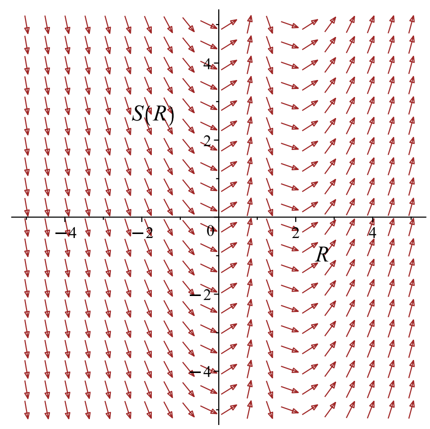
The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} - R - \ln(R-1) + c_1 \tag{4}$$

Which gives

$$y = -\frac{(x-1)\left(-x^2 + 2\ln(x-1) - 2c_1 + 2x\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3-3x^2+2x+y}{x-1}$ 	$R = x$ $S = \frac{y}{x-1}$	$\frac{dS}{dR} = \frac{R(-2+R)}{R-1}$ 

Summary

The solution(s) found are the following

$$y = -\frac{(x-1)\left(-x^2 + 2\ln(x-1) - 2c_1 + 2x\right)}{2} \tag{1}$$

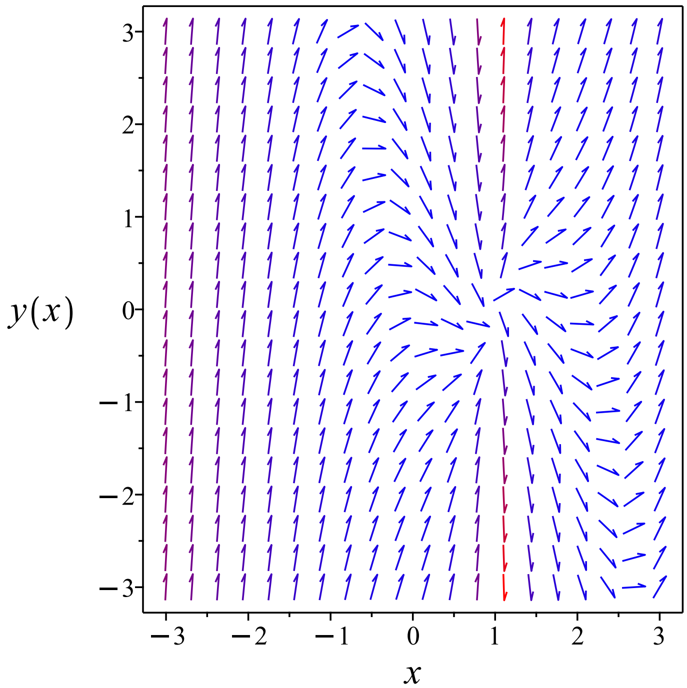


Figure 26: Slope field plot

Verification of solutions

$$y = -\frac{(x-1)\left(-x^2+2\ln(x-1)-2c_1+2x\right)}{2}$$

Verified OK.

2.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{y}{1-x} - 2x + x^2\right)\mathrm{d}x$$
$$\left(\frac{y}{1-x} + 2x - x^2\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{y}{1-x} + 2x - x^2$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{1-x} + 2x - x^2\right) \\ &= \frac{1}{1-x}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{1}{1-x}\right) - (0)\right) \\
&= \frac{1}{1-x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{1-x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(1-x)} \\
&= \frac{1}{1-x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{1-x}\left(\frac{y}{1-x} + 2x - x^2\right) \\
&= \frac{x^3 - 3x^2 + 2x + y}{(x-1)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{1-x}(1) \\
&= \frac{1}{1-x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^3 - 3x^2 + 2x + y}{(x-1)^2}\right) + \left(\frac{1}{1-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \frac{x^3 - 3x^2 + 2x + y}{(x-1)^2} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2}{2} - x - \ln(x-1) - \frac{y}{x-1} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{1}{x-1} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1-x}$. Therefore equation (4) becomes

$$
\frac{1}{1-x} = -\frac{1}{x-1} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{2} - x - \ln(x-1) - \frac{y}{x-1} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{2} - x - \ln(x-1) - \frac{y}{x-1}$$

The solution becomes

$$y = -\frac{(-x^2 + 2\ln(x-1) + 2c_1 + 2x)(x-1)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{(-x^2 + 2\ln(x-1) + 2c_1 + 2x)(x-1)}{2} \tag{1}$$

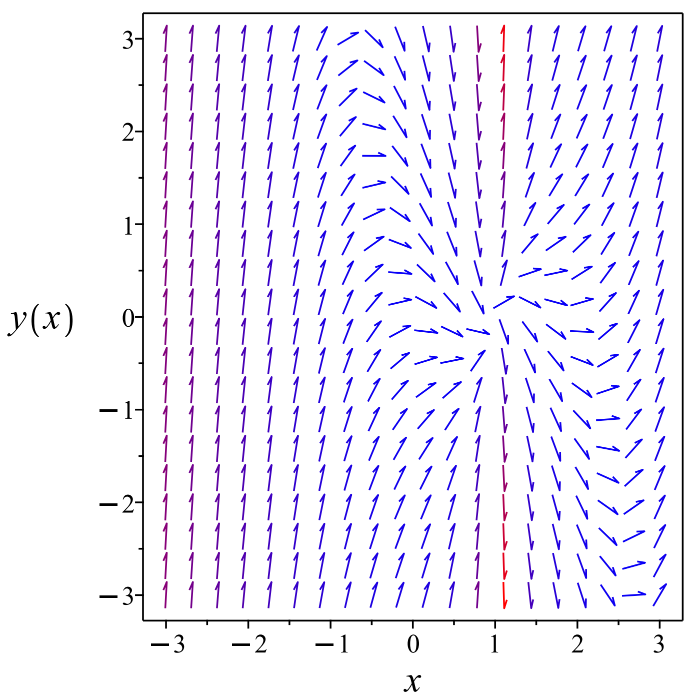


Figure 27: Slope field plot

Verification of solutions

$$y = -\frac{(-x^2 + 2\ln(x-1) + 2c_1 + 2x)(x-1)}{2}$$

Verified OK.

2.5.4 Maple step by step solution

Let's solve

$y' + \frac{y}{1-x} = x^2 - 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x-1} + x^2 - 2x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x-1} = x^2 - 2x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x-1}\right) = \mu(x)(x^2 - 2x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x-1}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x-1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x-1}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \mu(x)(x^2 - 2x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)(x^2 - 2x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)(x^2-2x)\,dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x-1}$

$$y = (x-1)\left(\int \frac{x^2-2x}{x-1}dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = (x-1)\left(\frac{x^2}{2} - x - \ln(x-1) + c_1\right)$$

- Simplify

$$y = \frac{(x-1)(x^2-2x-2\ln(x-1)+2c_1)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)+y(x)/(1-x)+2*x-x^2=0,y(x), singsol=all)
```

$$y(x) = \frac{(x^2-2x-2\ln(x-1)+2c_1)(x-1)}{2}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 27

```
DSolve[y'[x]+y[x]/(1-x)+2*x-x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x-1)\left(\frac{1}{2}(x-1)^2 - \log(x-1) + c_1\right)$$

2.6 problem 10.3.7

2.6.1 Solving as linear ode 178
2.6.2 Solving as first order ode lie symmetry lookup ode 180
2.6.3 Solving as exact ode 184
2.6.4 Maple step by step solution 189

Internal problem ID [5060]
Internal file name [`OUTPUT/4553_Sunday_June_05_2022_03_00_47_PM_35483196/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.3, ODEs with variable Coefficients. First order. page 315
Problem number: 10.3.7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{1-x} = x^2 - x}$$

2.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x-1}$$
$$q(x) = x(x-1)$$

Hence the ode is

$$y' - \frac{y}{x-1} = x(x-1)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x-1}dx} \\
&= \frac{1}{x-1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(x(x-1)) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x-1}\right) &= \left(\frac{1}{x-1}\right)(x(x-1)) \\
\mathrm{d}\left(\frac{y}{x-1}\right) &= x\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x-1} &= \int x\,\mathrm{d}x \\
\frac{y}{x-1} &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x-1}$ results in

$$
y = \frac{x^2(x-1)}{2} + c_1(x-1)
$$

which simplifies to

$$
y = \frac{(x-1)\,(x^2+2c_1)}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(x-1)\,(x^2+2c_1)}{2} \tag{1}
$$

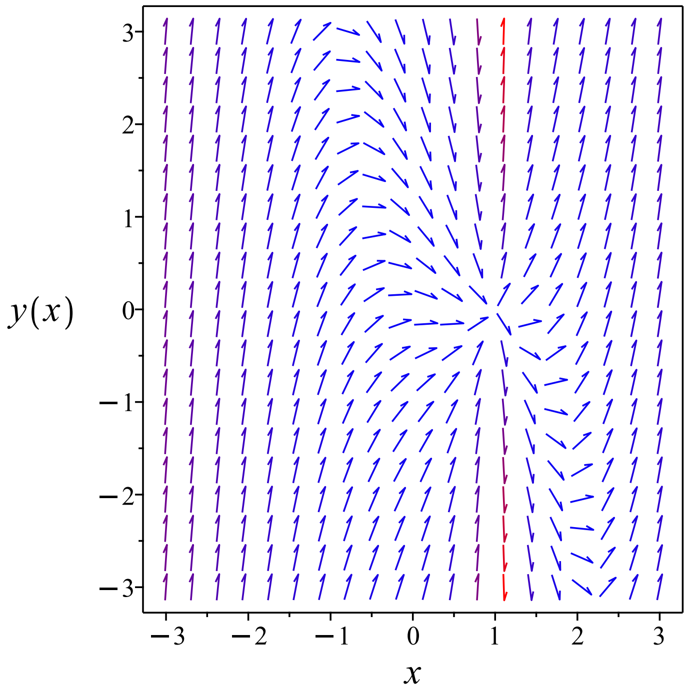


Figure 28: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{(x-1)(x^2+2c_1)}{2}$$

Verified OK.

2.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3-2x^2+x+y}{x-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x - 1 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x-1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x-1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 - 2x^2 + x + y}{x-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x-1)^2} \\ S_y &= \frac{1}{x-1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x-1} = \frac{x^2}{2} + c_1$$

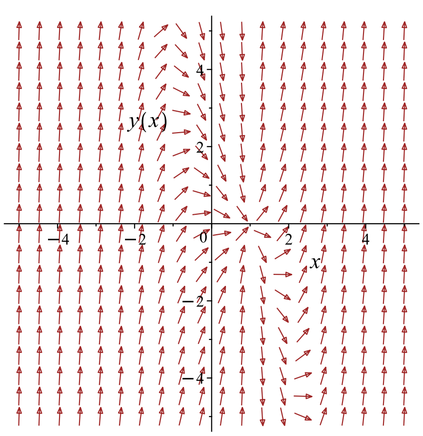
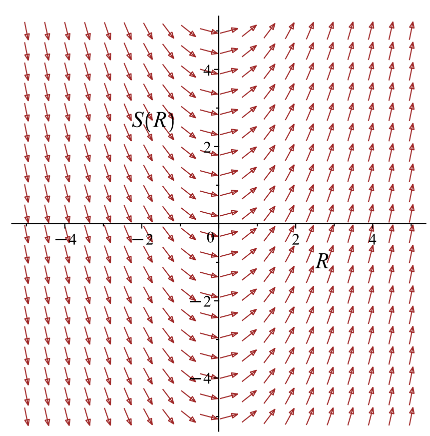
Which simplifies to

$$\frac{y}{x-1} = \frac{x^2}{2} + c_1$$

Which gives

$$y = \frac{(x-1)(x^2 + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3-2x^2+x+y}{x-1}$ 	$R = x$ $S = \frac{y}{x-1}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$y = \frac{(x-1)(x^2+2c_1)}{2} \tag{1}$$

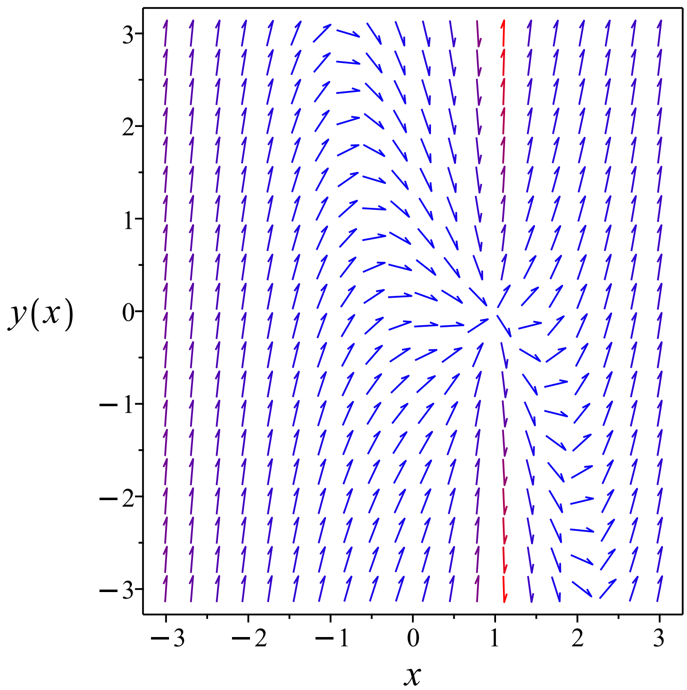


Figure 29: Slope field plot

Verification of solutions

$$y = \frac{(x-1)(x^2+2c_1)}{2}$$

Verified OK.

2.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(-\frac{y}{1-x}-x+x^2\right)\mathrm{d}x$$

$$\left(\frac{y}{1-x}+x-x^2\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \frac{y}{1-x}+x-x^2\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{1-x}+x-x^2\right)\\ &= \frac{1}{1-x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{1}{1-x}\right) - (0)\right)\\ &= \frac{1}{1-x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int \frac{1}{1-x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(1-x)}\\ &= \frac{1}{1-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{1-x}\left(\frac{y}{1-x} + x - x^2\right)\\ &= \frac{x^3 - 2x^2 + x + y}{(x-1)^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{1-x}(1)\\ &= \frac{1}{1-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{x^3 - 2x^2 + x + y}{(x-1)^2}\right) + \left(\frac{1}{1-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x^3 - 2x^2 + x + y}{(x-1)^2}\,\mathrm{d}x$$
$$\phi = \frac{x^2}{2} - \frac{y}{x-1} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x-1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1-x}$. Therefore equation (4) becomes

$$\frac{1}{1-x} = -\frac{1}{x-1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{2} - \frac{y}{x-1} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{2} - \frac{y}{x-1}$$

The solution becomes

$$y = -\frac{(-x^2 + 2c_1)(x-1)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{(-x^2 + 2c_1)(x-1)}{2} \tag{1}$$

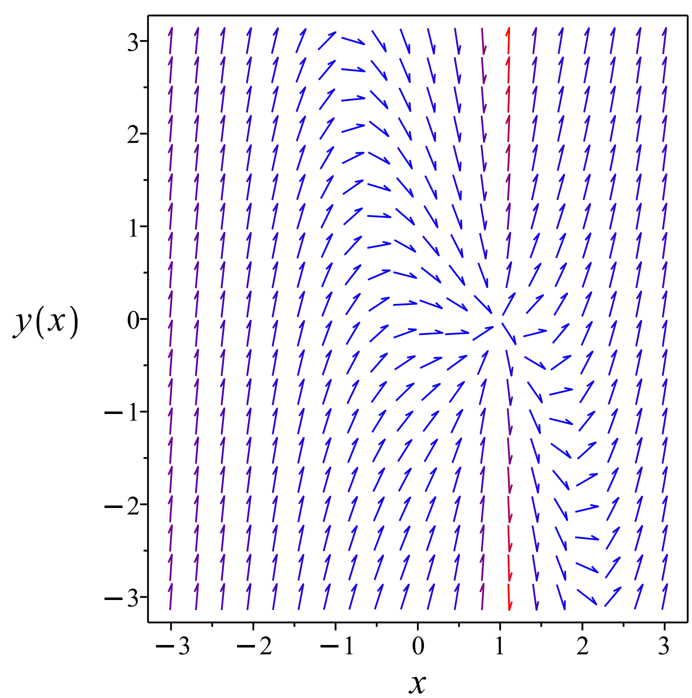


Figure 30: Slope field plot

Verification of solutions

$$y = -\frac{(-x^2 + 2c_1)(x - 1)}{2}$$

Verified OK.

2.6.4 Maple step by step solution

Let's solve

$y' + \frac{y}{1-x} = x^2 - x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x-1} + x^2 - x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x-1} = x^2 - x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x-1}\right) = \mu(x)(x^2 - x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x-1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x-1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x-1}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)(x^2 - x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)(x^2 - x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)(x^2-x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x-1}$

$$y = (x-1)\left(\int \frac{x^2-x}{x-1}dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \left(\frac{x^2}{2} + c_1\right)(x-1)$$

- Simplify

$$y = \frac{(x-1)(x^2+2c_1)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+y(x)/(1-x)+x-x^2=0,y(x), singsol=all)
```

$$y(x) = \frac{(x^2+2c_1)(x-1)}{2}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 20

```
DSolve[y'[x]+y[x]/(1-x)+x-x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(x-1)(x^2+2c_1)$$

2.7 problem 10.3.8

2.7.1 Solving as linear ode 191
2.7.2 Solving as homogeneousTypeD2 ode 193
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Internal problem ID [5061]
Internal file name [`OUTPUT/4554_Sunday_June_05_2022_03_00_48_PM_6281316/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.3, ODEs with variable Coefficients. First order. page 315
Problem number: 10.3.8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x^2+1)\,y' - xy = 1}$$

2.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{x}{x^2+1}$$
$$q(x) = \frac{1}{x^2+1}$$

Hence the ode is

$$y' - \frac{xy}{x^2+1} = \frac{1}{x^2+1}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{x}{x^2+1}dx}$$
$$= \frac{1}{\sqrt{x^2+1}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{x^2+1}\right)$$
$$\frac{d}{dx}\left(\frac{y}{\sqrt{x^2+1}}\right) = \left(\frac{1}{\sqrt{x^2+1}}\right)\left(\frac{1}{x^2+1}\right)$$
$$d\left(\frac{y}{\sqrt{x^2+1}}\right) = \frac{1}{(x^2+1)^{\frac{3}{2}}}\, dx$$

Integrating gives

$$\frac{y}{\sqrt{x^2+1}} = \int \frac{1}{(x^2+1)^{\frac{3}{2}}}\, dx$$
$$\frac{y}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x^2+1}}$ results in

$$y = c_1\sqrt{x^2+1} + x$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x^2+1} + x \tag{1}$$

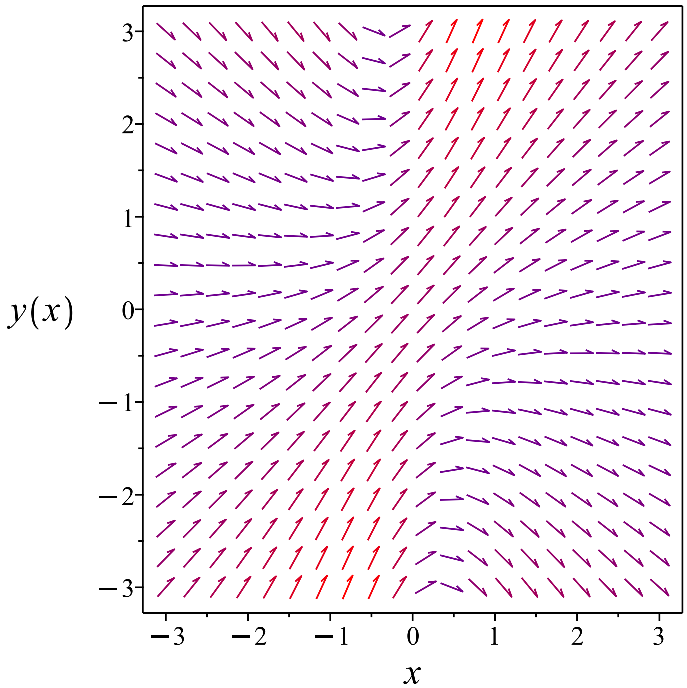


Figure 31: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x^2+1} + x$$

Verified OK.

2.7.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\left(x^2+1\right)\left(u'(x)\,x + u(x)\right) - x^2 u(x) = 1$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-u+1}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = \frac{1}{x(x^2+1)}$ and $g(u) = -u+1$. Integrating both sides gives

$$\frac{1}{-u+1}\,du = \frac{1}{x\,(x^2+1)}\,dx$$

$$
\int \frac{1}{-u+1}\, du = \int \frac{1}{x\left(x^2+1\right)}\, dx
$$

$$
-\ln\left(u-1\right) = -\frac{\ln\left(x^2+1\right)}{2} + \ln\left(x\right) + c_2
$$

Raising both side to exponential gives

$$
\frac{1}{u-1} = \mathrm{e}^{-\frac{\ln\left(x^2+1\right)}{2}+\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{1}{u-1} = c_3 \mathrm{e}^{-\frac{\ln\left(x^2+1\right)}{2}+\ln(x)}
$$

Which simplifies to

$$
u(x) = \frac{\left(\frac{c_3 \mathrm{e}^{c_2} x}{\sqrt{x^2+1}} + 1\right) \mathrm{e}^{-c_2} \sqrt{x^2+1}}{c_3 x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= \frac{\left(\frac{c_3 \mathrm{e}^{c_2} x}{\sqrt{x^2+1}} + 1\right) \mathrm{e}^{-c_2} \sqrt{x^2+1}}{c_3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(\frac{c_3 \mathrm{e}^{c_2} x}{\sqrt{x^2+1}} + 1\right) \mathrm{e}^{-c_2} \sqrt{x^2+1}}{c_3} \tag{1}
$$

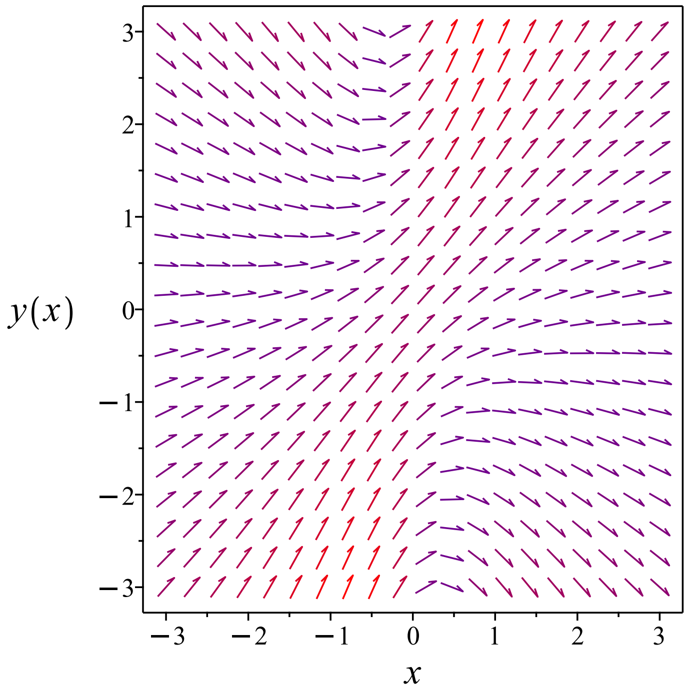


Figure 32: Slope field plot

Verification of solutions

$$y = \frac{\left(\frac{c_3 e^{c_2 x}}{\sqrt{x^2+1}} + 1\right) e^{-c_2} \sqrt{x^2+1}}{c_3}$$

Verified OK.

2.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{xy+1}{x^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 36: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sqrt{x^2 + 1}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x^2+1}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sqrt{x^2+1}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{xy+1}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{yx}{(x^2+1)^{\frac{3}{2}}} \\ S_y &= \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(x^2+1)^{\frac{3}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R^2+1)^{\frac{3}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{\sqrt{R^2+1}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} + c_1$$

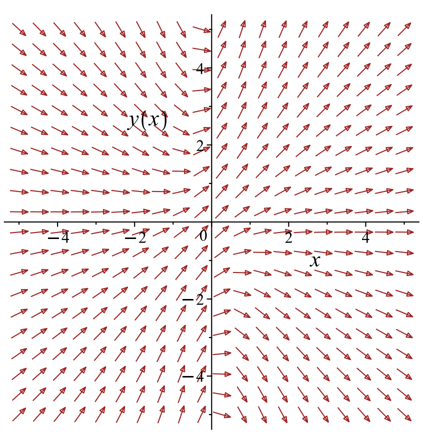
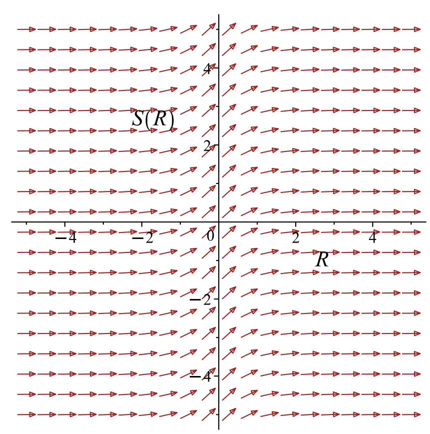
Which simplifies to

$$\frac{y}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} + c_1$$

Which gives

$$y = c_1\sqrt{x^2+1} + x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy+1}{x^2+1}$ 	$R = x$ $S = \frac{y}{\sqrt{x^2+1}}$	$\frac{dS}{dR} = \frac{1}{(R^2+1)^{\frac{3}{2}}}$ 

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x^2+1} + x \tag{1}$$

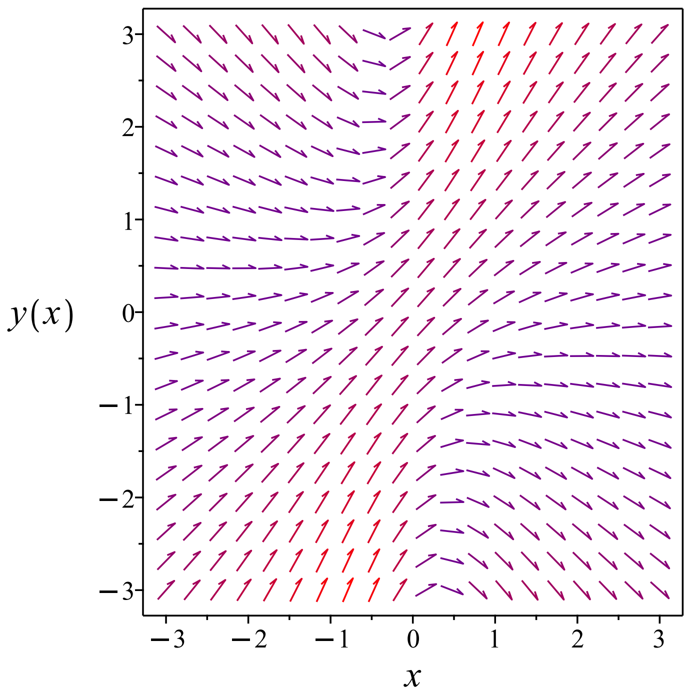


Figure 33: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x^2+1} + x$$

Verified OK.

2.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+1\right)\mathrm{d}y&=(xy+1)\,\mathrm{d}x\\ (-xy-1)\,\mathrm{d}x+\left(x^2+1\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-xy-1\\ N(x,y)&=x^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-xy-1)\\ &=-x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2+1)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2+1}((-x)-(2x))\\ &= -\frac{3x}{x^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{3x}{x^2+1}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{3\ln(x^2+1)}{2}}\\ &= \frac{1}{(x^2+1)^{\frac{3}{2}}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{(x^2+1)^{\frac{3}{2}}}(-xy-1)\\ &= -\frac{xy+1}{(x^2+1)^{\frac{3}{2}}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{(x^2+1)^{\frac{3}{2}}}(x^2+1)\\ &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-\frac{xy+1}{(x^2+1)^{\frac{3}{2}}}\right) + \left(\frac{1}{\sqrt{x^2+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{xy+1}{(x^2+1)^{\frac{3}{2}}}\,\mathrm{d}x$$
$$\phi = \frac{-x+y}{\sqrt{x^2+1}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{x^2+1}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{x^2+1}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x + y}{\sqrt{x^2 + 1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x + y}{\sqrt{x^2 + 1}}$$

The solution becomes

$$y = c_1\sqrt{x^2 + 1} + x$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x^2 + 1} + x \tag{1}$$

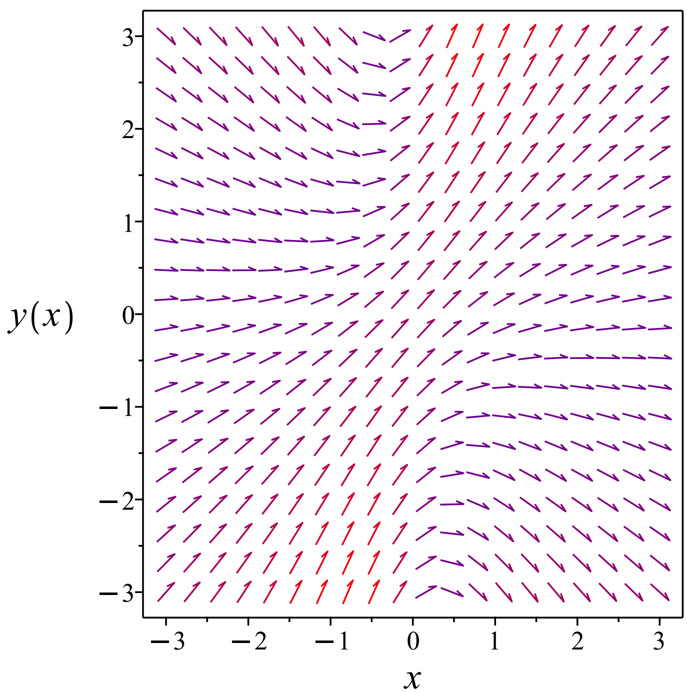


Figure 34: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x^2+1}+x$$

Verified OK.

2.7.5 Maple step by step solution

Let's solve

$(x^2+1)\, y' - xy = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{xy}{x^2+1} + \frac{1}{x^2+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{xy}{x^2+1} = \frac{1}{x^2+1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{xy}{x^2+1}\right) = \frac{\mu(x)}{x^2+1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{xy}{x^2+1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)x}{x^2+1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\sqrt{x^2+1}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x^2+1} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x^2+1} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^2+1} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\sqrt{x^2+1}}$

$$y = \sqrt{x^2+1}\left(\int \frac{1}{(x^2+1)^{\frac{3}{2}}}dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \sqrt{x^2+1}\left(\frac{x}{\sqrt{x^2+1}} + c_1\right)$$

- Simplify

$$y = c_1\sqrt{x^2+1} + x$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x)=1+x*y(x),y(x), singsol=all)
```

$$y(x) = \sqrt{x^2+1}\, c_1 + x$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 19

```
DSolve[(1+x^2)*y'[x]==1+x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x + c_1\sqrt{x^2+1}$$

2.8 problem 10.3.9 (a)

2.8.1 Solving as separable ode 206
2.8.2 Solving as first order ode lie symmetry lookup ode 208
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2.8.4 Solving as exact ode 215
2.8.5 Solving as riccati ode 219
2.8.6 Maple step by step solution 221

Internal problem ID [5062]
Internal file name [`OUTPUT/4555_Sunday_June_05_2022_03_00_49_PM_941248/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.3, ODEs with variable Coefficients. First order. page 315
Problem number: 10.3.9 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + xy - xy^2 = 0}$$

2.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= xy(y-1) \end{aligned}$$

Where $f(x) = x$ and $g(y) = y(y-1)$. Integrating both sides gives

$$\frac{1}{y\,(y-1)}\,dy = x\,dx$$

$$\int \frac{1}{y(y-1)}\,dy = \int x\,dx$$

$$\ln(y-1) - \ln(y) = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(y)} = e^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\frac{y-1}{y} = c_2 e^{\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{-1 + c_2 e^{\frac{x^2}{2}}} \tag{1}$$

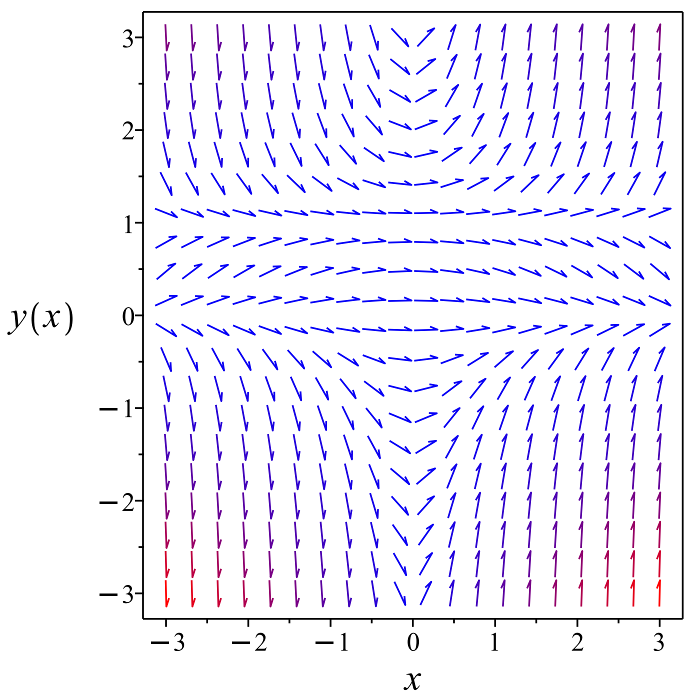


Figure 35: Slope field plot

Verification of solutions

$$y = -\frac{1}{-1 + c_2 \mathrm{e}^{\frac{x^2}{2}}}$$

Verified OK.

2.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y^2 x - xy \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \frac{1}{x} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y^2 x - xy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(y-1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(R-1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R-1) - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \ln(y-1) - \ln(y) + c_1$$

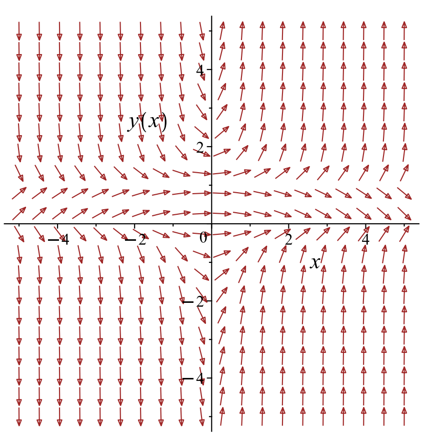
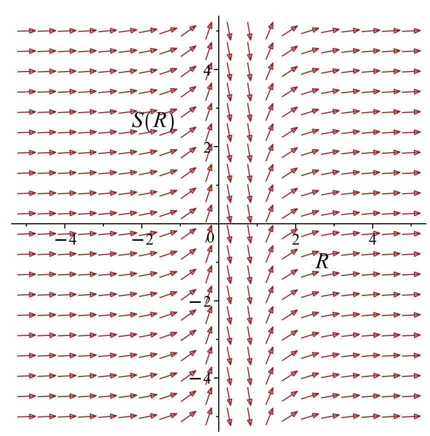
Which simplifies to

$$\frac{x^2}{2} = \ln(y-1) - \ln(y) + c_1$$

Which gives

$$y = \frac{e^{-\frac{x^2}{2}+c_1}}{-1+e^{-\frac{x^2}{2}+c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y^2x - xy$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R(R-1)}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-\frac{x^2}{2}+c_1}}{-1+e^{-\frac{x^2}{2}+c_1}} \tag{1}$$

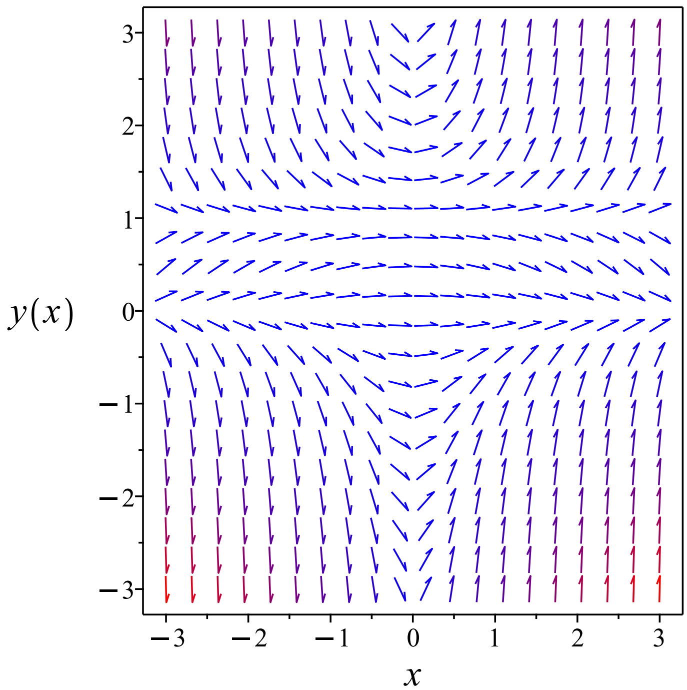


Figure 36: Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{x^2}{2}+c_1}}{-1+e^{-\frac{x^2}{2}+c_1}}$$

Verified OK.

2.8.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2x - xy \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -xy + xy^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -x \\ f_1(x) &= x \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{x}{y} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -w(x)\,x + x \\ w' &= xw - x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$p(x) = -x$$
$$q(x) = -x$$

Hence the ode is

$$w'(x) - w(x)\,x = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -x dx} \\ &= e^{-\frac{x^2}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\,(-x) \\ \frac{d}{dx}\left(e^{-\frac{x^2}{2}} w\right) &= \left(e^{-\frac{x^2}{2}}\right)(-x) \\ d\left(e^{-\frac{x^2}{2}} w\right) &= \left(-x\,e^{-\frac{x^2}{2}}\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-\frac{x^2}{2}} w &= \int -x\,e^{-\frac{x^2}{2}}\,dx \\ e^{-\frac{x^2}{2}} w &= e^{-\frac{x^2}{2}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x^2}{2}}$ results in

$$w(x) = e^{\frac{x^2}{2}} e^{-\frac{x^2}{2}} + c_1 e^{\frac{x^2}{2}}$$

which simplifies to

$$w(x) = 1 + c_1 e^{\frac{x^2}{2}}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = 1 + c_1 e^{\frac{x^2}{2}}$$

Or

$$y = \frac{1}{1 + c_1 e^{\frac{x^2}{2}}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{1 + c_1 e^{\frac{x^2}{2}}} \tag{1}$$

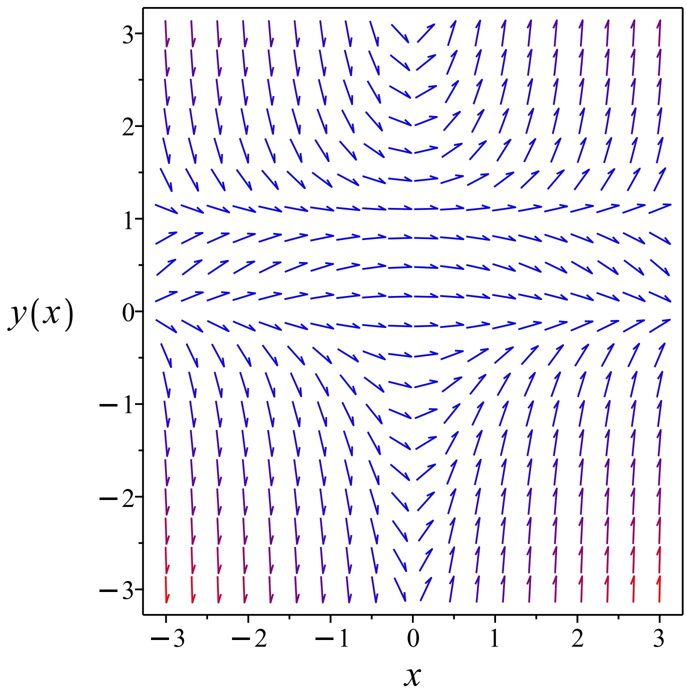


Figure 37: Slope field plot

Verification of solutions

$$y = \frac{1}{1 + c_1 e^{\frac{x^2}{2}}}$$

Verified OK.

2.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y\,(y-1)}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{y\,(y-1)}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{1}{y\,(y-1)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y\left(y-1\right)}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x
\end{aligned}$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y(y-1)}$. Therefore equation (4) becomes

$$\frac{1}{y\left(y-1\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y\left(y-1\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y\,(y-1)}\right) \mathrm{d}y$$
$$f(y) = \ln(y-1) - \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \ln(y-1) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \ln(y-1) - \ln(y)$$

The solution becomes

$$y = -\frac{1}{\mathrm{e}^{\frac{x^2}{2}+c_1} - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\mathrm{e}^{\frac{x^2}{2}+c_1} - 1} \tag{1}$$

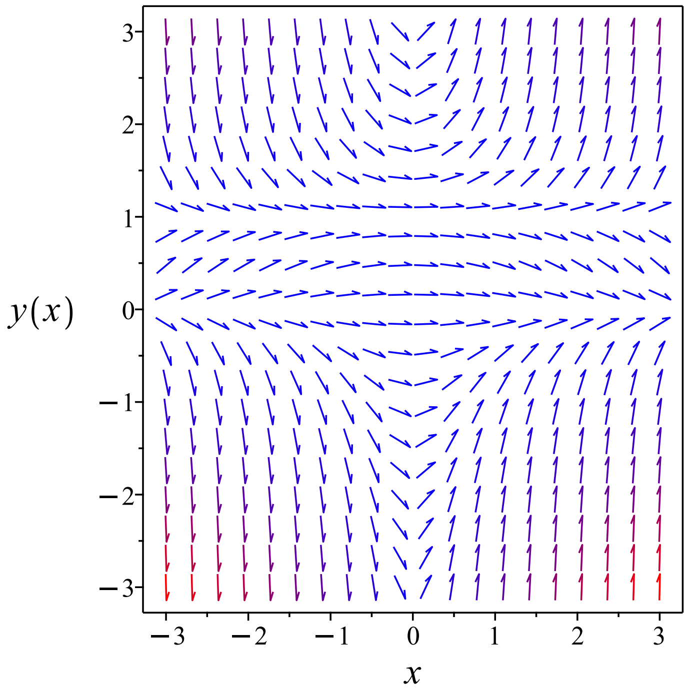


Figure 38: Slope field plot

Verification of solutions

$$y = -\frac{1}{e^{\frac{x^2}{2}+c_1} - 1}$$

Verified OK.

2.8.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 x - xy \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 x - xy$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -x$ and $f_2(x) = x$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{x u} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 1 \\
f_1 f_2 &= -x^2 \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
x u''(x) - \left(-x^2 + 1\right) u'(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + \mathrm{e}^{-\frac{x^2}{2}} c_2
$$

The above shows that

$$
u'(x) = -x\,\mathrm{e}^{-\frac{x^2}{2}} c_2
$$

Using the above in (1) gives the solution

$$
y = \frac{\mathrm{e}^{-\frac{x^2}{2}} c_2}{c_1 + \mathrm{e}^{-\frac{x^2}{2}} c_2}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{\mathrm{e}^{-\frac{x^2}{2}}}{c_3 + \mathrm{e}^{-\frac{x^2}{2}}}
$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-\frac{x^2}{2}}}{c_3 + e^{-\frac{x^2}{2}}} \tag{1}$$

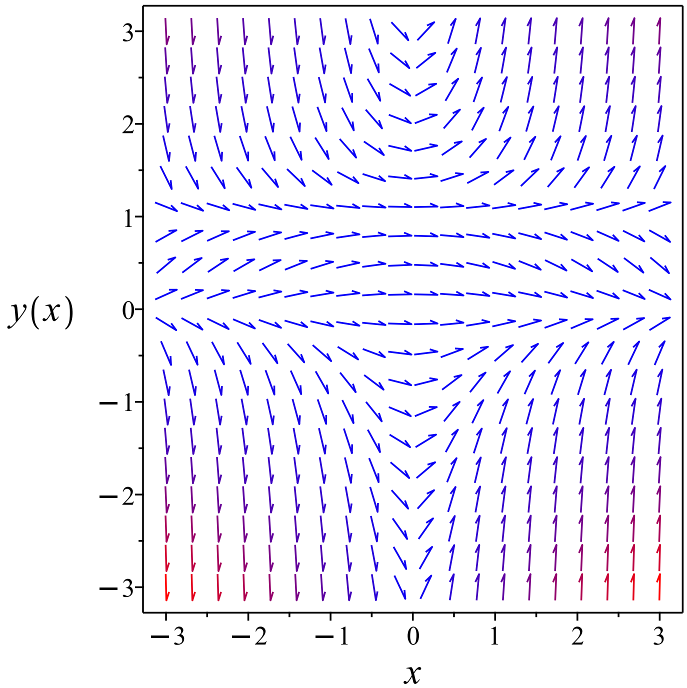


Figure 39: Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{x^2}{2}}}{c_3 + e^{-\frac{x^2}{2}}}$$

Verified OK.

2.8.6 Maple step by step solution

Let's solve
$y' + xy - xy^2 = 0$

- Highest derivative means the order of the ODE is 1
  y'

- Separate variables

  $\frac{y'}{y(y-1)} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y-1)}dx = \int x dx + c_1$

- Evaluate integral

  $\ln(y-1) - \ln(y) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = -\frac{1}{e^{\frac{x^2}{2}+c_1} - 1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+x*y(x)=x*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{1 + e^{\frac{x^2}{2}} c_1}$$

✓ Solution by Mathematica

Time used: 0.25 (sec). Leaf size: 31

```
DSolve[y'[x]+x*y[x]==x*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{1 + e^{\frac{x^2}{2}+c_1}}$$
$$y(x) \to 0$$
$$y(x) \to 1$$

2.9 problem 10.3.9 (b)

2.9.1 Solving as first order ode lie symmetry lookup ode 223
2.9.2 Solving as bernoulli ode 227

Internal problem ID [5063]
Internal file name [`OUTPUT/4556_Sunday_June_05_2022_03_00_50_PM_48381868/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.3, ODEs with variable Coefficients. First order. page 315
Problem number: 10.3.9 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{3xy' + y + y^4x^2 = 0}$$

2.9.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(y^3x^2 + 1)}{3x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^4 x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^4 x} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{3x\, y^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(y^3 x^2 + 1)}{3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{3x^2 y^3} \\ S_y &= \frac{1}{y^4 x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{3y^3x} = -\frac{x}{3} + c_1$$

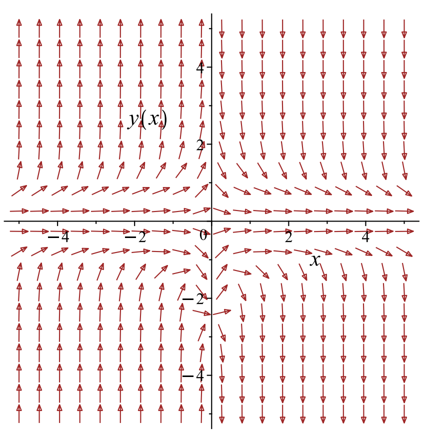
Which simplifies to

$$-\frac{1}{3y^3x} = -\frac{x}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{y(y^3x^2+1)}{3x}$



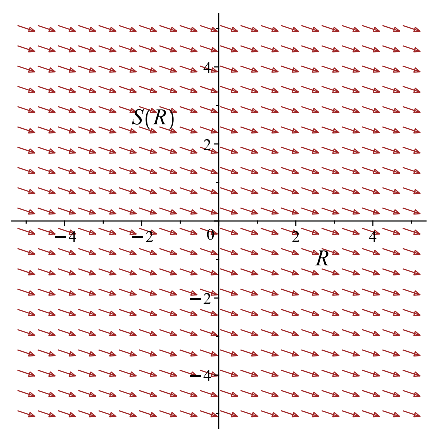
</td><td>

$R = x$

$S = -\frac{1}{3x\,y^3}$

</td><td>

$\frac{dS}{dR} = -\frac{1}{3}$



</td></tr>
</table>

Summary
The solution(s) found are the following

$$-\frac{1}{3y^3x} = -\frac{x}{3} + c_1 \tag{1}$$

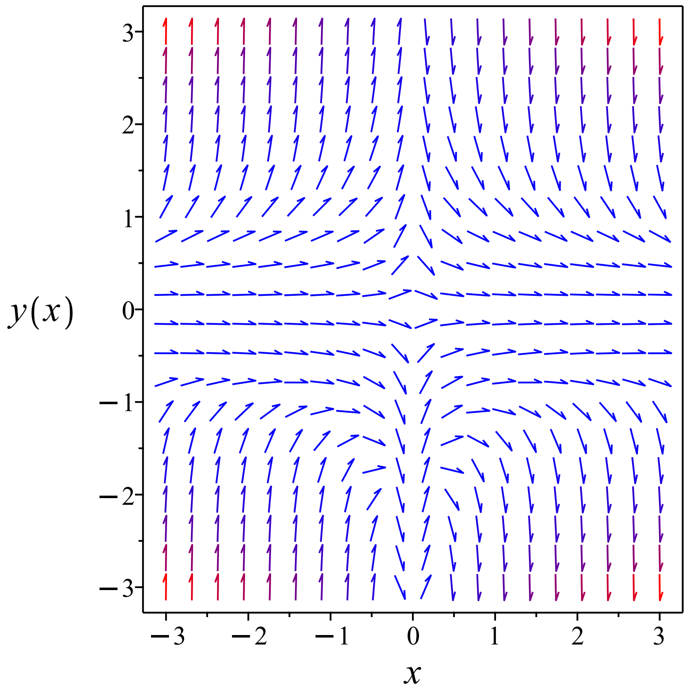


Figure 40: Slope field plot

Verification of solutions

$$-\frac{1}{3y^3x} = -\frac{x}{3} + c_1$$

Verified OK.

2.9.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(y^3x^2+1)}{3x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{3x}y - \frac{x}{3}y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{3x} \\ f_1(x) &= -\frac{x}{3} \\ n &= 4 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$y' \frac{1}{y^4} = -\frac{1}{3x\, y^3} - \frac{x}{3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^3} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{3}{y^4} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{3} &= -\frac{w(x)}{3x} - \frac{x}{3} \\ w' &= \frac{w}{x} + x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(x) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(x) \\ d\left(\frac{w}{x}\right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int dx \\ \frac{w}{x} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x + x^2$$

which simplifies to

$$w(x) = x(x + c_1)$$

Replacing w in the above by $\frac{1}{y^3}$ using equation (5) gives the final solution.

$$\frac{1}{y^3} = x(x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\left(x^2(x+c_1)^2\right)^{\frac{1}{3}}}{x\,(x+c_1)} \\ y(x) &= \frac{\left(x^2(x+c_1)^2\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2x\,(x+c_1)} \\ y(x) &= -\frac{\left(x^2(x+c_1)^2\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2x\,(x+c_1)}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(x^2 (x + c_1)^2\right)^{\frac{1}{3}}}{x (x + c_1)} \tag{1}$$

$$y = \frac{\left(x^2 (x + c_1)^2\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2x (x + c_1)} \tag{2}$$

$$y = -\frac{\left(x^2 (x + c_1)^2\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2x (x + c_1)} \tag{3}$$

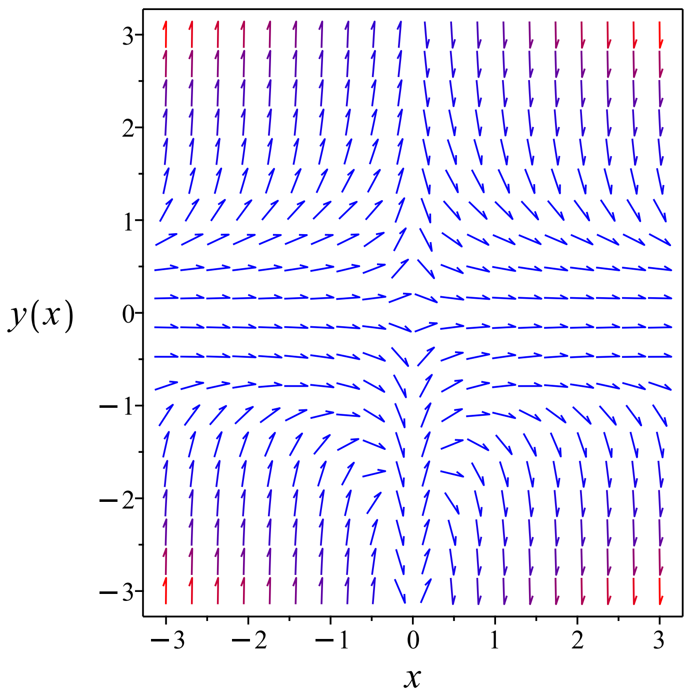


Figure 41: Slope field plot

Verification of solutions

$$y = \frac{\left(x^2(x+c_1)^2\right)^{\frac{1}{3}}}{x\,(x+c_1)}$$

Verified OK.

$$y = \frac{\left(x^2(x+c_1)^2\right)^{\frac{1}{3}}\,\left(i\sqrt{3}-1\right)}{2x\,(x+c_1)}$$

Verified OK.

$$y = -\frac{\left(x^2(x+c_1)^2\right)^{\frac{1}{3}}\,\left(1+i\sqrt{3}\right)}{2x\,(x+c_1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 88

```
dsolve(3*x*diff(y(x),x)+y(x)+x^2*y(x)^4=0,y(x), singsol=all)
```

$$y(x) = \frac{\left((x+c_1)^2\,x^2\right)^{\frac{1}{3}}}{(x+c_1)\,x}$$

$$y(x) = -\frac{\left((x+c_1)^2\,x^2\right)^{\frac{1}{3}}\,\left(1+i\sqrt{3}\right)}{2\,(x+c_1)\,x}$$

$$y(x) = \frac{\left((x+c_1)^2\,x^2\right)^{\frac{1}{3}}\,\left(i\sqrt{3}-1\right)}{2\,(x+c_1)\,x}$$

✓ Solution by Mathematica

Time used: 0.3 (sec). Leaf size: 61

```
DSolve[3*x*y'[x]+y[x]+x^2*y[x]^4==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\sqrt[3]{x(x+c_1)}}$$
$$y(x) \to -\frac{\sqrt[3]{-1}}{\sqrt[3]{x(x+c_1)}}$$
$$y(x) \to \frac{(-1)^{2/3}}{\sqrt[3]{x(x+c_1)}}$$
$$y(x) \to 0$$

3 Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318

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3.1 problem 10.4.8 (a)

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Internal problem ID [5064]
Internal file name [`OUTPUT/4557_Sunday_June_05_2022_03_00_51_PM_97582610/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.8 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x(x+1)^2 y'' + \left(-x^2+1\right) y' + (x-1) y = 0}$$

3.1.1 Solving using Kovacic algorithm

Writing the ode as

$$x(x+1)^2 y'' + \left(-x^2+1\right) y' + (x-1) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(x+1)^2 \\ B &= -x^2+1 \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= e^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2+1}{x(x+1)^2} \, dx}\\ &= z_1 e^{\ln(x+1) - \frac{\ln(x)}{2}}\\ &= z_1 \left(\frac{x+1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+1}{x(x+1)^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x+1)-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x+1) + c_2(x+1(\ln(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = (x+1)\, c_1 + c_2(x+1)\ln(x) \tag{1}
$$

Verification of solutions

$$
y = (x+1)\, c_1 + c_2(x+1)\ln(x)
$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$x(x+1)^2\, y'' + (-x^2+1)\, y' + (x-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x(x+1)^2} + \frac{(x-1)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x(x+1)} + \frac{(x-1)y}{x(x+1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=-\frac{x-1}{(x+1)x}, P_3(x)=\frac{x-1}{x(x+1)^2}\right]$

- $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-2$

- $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=2$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x(x+1)^2 y''-(x-1)(x+1)y'+(x-1)y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-u^2)\left(\frac{d^2}{du^2}y(u)\right)+(-u^2+2u)\left(\frac{d}{du}y(u)\right)+(u-2)y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+r)(-2+r)\, u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+r)(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2 = 0$$

- Shift index using $k->k+1$

$$-a_{k+1}(k+r)(k+r-1) + a_k(k+r-1)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-1)}{k+r}$$

- Recursion relation for $r=1$

$$a_{k+1} = \frac{a_k k}{k+1}$$

- Solution for $r=1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$$

- Recursion relation for $r=2$

$$a_{k+1} = \frac{a_k(k+1)}{k+2}$$

- Solution for $r=2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k(k+1)}{k+2}\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+1} = \frac{a_k (k+1)}{k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), a_{k+1} = \frac{a_k k}{k+1}, b_{k+1} = \frac{b_k (k+1)}{k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(x*(x+1)^2*diff(y(x),x$2)+(1-x^2)*diff(y(x),x)+(x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = (x+1)(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 17

```
DSolve[x*(x+1)^2*y''[x]+(1-x^2)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x+1)(c_2 \log(x) + c_1)$$

3.2 problem 10.4.8 (b)

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Internal problem ID [5065]
Internal file name [`OUTPUT/4558_Sunday_June_05_2022_03_00_52_PM_84210290/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.8 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(1-x)\,y''+2(1-2x)\,y'-2y=0}$$

3.2.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-4x+2}{-x^2+x} \\ q(x) &= -\frac{2}{-x^2+x} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= -\frac{2}{-x^2+x} - \frac{\left(\frac{-4x+2}{-x^2+x}\right)'}{2} - \frac{\left(\frac{-4x+2}{-x^2+x}\right)^2}{4} \\
&= -\frac{2}{-x^2+x} - \frac{\left(-\frac{4}{-x^2+x} - \frac{(-4x+2)(1-2x)}{(-x^2+x)^2}\right)}{2} - \frac{\left(\frac{(-4x+2)^2}{(-x^2+x)^2}\right)}{4} \\
&= -\frac{2}{-x^2+x} - \left(-\frac{2}{-x^2+x} - \frac{(-4x+2)(1-2x)}{2(-x^2+x)^2}\right) - \frac{(-4x+2)^2}{4(-x^2+x)^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{-4x+2}{-x^2+x}}{2}} \\
&= \frac{1}{x(x-1)}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{x(x-1)} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
-v''(x) = 0
$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$
v(x) = c_1 x + c_2
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\, z(x) \\
&= (c_1 x + c_2)\,(z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$z(x) = \frac{1}{x\,(x-1)}$$

Hence (7) becomes

$$y = \frac{c_1 x + c_2}{x\,(x-1)}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x + c_2}{x\,(x-1)} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x + c_2}{x\,(x-1)}$$

Verified OK.

3.2.2 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$\left(-x^2 + x\right) y'' + (-4x + 2)\, y' - 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{4x-2}{x\,(x-1)}$$
$$q(x) = \frac{2}{x\,(x-1)}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(4x-2)}{x^2(x-1)} + \frac{2}{x(x-1)} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(-\frac{2}{x} + \frac{4x-2}{x(x-1)}\right) v'(x) = 0$$
$$v''(x) + \frac{2v'(x)}{x-1} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x-1} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x-1} \end{aligned}$$

Where $f(x) = -\frac{2}{x-1}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x-1}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x-1}\,dx \\ \ln(u) &= -2\ln(x-1) + c_1 \\ u &= e^{-2\ln(x-1)+c_1} \\ &= \frac{c_1}{(x-1)^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{x-1} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \frac{-\frac{c_1}{x-1} + c_2}{x} \\ &= \frac{-\frac{c_1}{x-1} + c_2}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{-\frac{c_1}{x-1} + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{-\frac{c_1}{x-1} + c_2}{x}$$

Verified OK.

3.2.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int \left(\left(-x^2 + x\right) y'' + (-4x + 2)\, y' - 2y \right) dx &= 0 \\ -(2x - 1)\, y - \left(x^2 - x\right) y' &= c_1 \end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1 - 2x}{x\,(x - 1)} \\ q(x) &= -\frac{c_1}{x\,(x - 1)} \end{aligned}$$

Hence the ode is

$$y' - \frac{(1-2x)\,y}{x\,(x-1)} = -\frac{c_1}{x\,(x-1)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1-2x}{x(x-1)}dx}\\ &= x(x-1)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{c_1}{x\,(x-1)}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(yx(x-1)) &= (x(x-1))\left(-\frac{c_1}{x\,(x-1)}\right)\\ \mathrm{d}(yx(x-1)) &= (-c_1)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}yx(x-1) &= \int -c_1\,\mathrm{d}x\\ yx(x-1) &= -c_1x + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x(x-1)$ results in

$$y = -\frac{c_1}{x-1} + \frac{c_2}{x\,(x-1)}$$

which simplifies to

$$y = \frac{-c_1x + c_2}{x\,(x-1)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_1x + c_2}{x\,(x-1)} \tag{1}$$

Verification of solutions

$$y = \frac{-c_1x + c_2}{x\,(x-1)}$$

Verified OK.

3.2.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$\left(-x^2+x\right) y''+(-4x+2)\, y'-2y=0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(-x^2+x\right) y''+(-4x+2)\, y'-2y\right) dx=0$$
$$-(2x-1)\, y-\left(x^2-x\right) y'=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=-\frac{1-2x}{x\,(x-1)}$$
$$q(x)=-\frac{c_1}{x\,(x-1)}$$

Hence the ode is

$$y'-\frac{(1-2x)\, y}{x\,(x-1)}=-\frac{c_1}{x\,(x-1)}$$

The integrating factor μ is

$$\mu=\mathrm{e}^{\int -\frac{1-2x}{x(x-1)}dx}$$
$$=x(x-1)$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)=(\mu)\left(-\frac{c_1}{x\,(x-1)}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(yx(x-1))=(x(x-1))\left(-\frac{c_1}{x\,(x-1)}\right)$$
$$\mathrm{d}(yx(x-1))=(-c_1)\ \mathrm{d}x$$

Integrating gives

$$
\begin{aligned}
yx(x-1) &= \int -c_1 \,\mathrm{d}x \\
yx(x-1) &= -c_1 x + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x(x-1)$ results in

$$
y = -\frac{c_1}{x-1} + \frac{c_2}{x(x-1)}
$$

which simplifies to

$$
y = \frac{-c_1 x + c_2}{x(x-1)}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{-c_1 x + c_2}{x(x-1)} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{-c_1 x + c_2}{x(x-1)}
$$

Verified OK.

3.2.5 Solving using Kovacic algorithm

Writing the ode as

$$
(-x^2 + x)\, y'' + (-4x + 2)\, y' - 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= -x^2 + x \\
B &= -4x + 2 \\
C &= -2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x+2}{-x^2+x}\,dx} \\ &= z_1 e^{-\ln(x(x-1))} \\ &= z_1 \left(\frac{1}{x\,(x-1)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x\,(x-1)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x+2}{-x^2+x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x(x-1))}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x\left(x-1\right)}\right) + c_2\left(\frac{1}{x\left(x-1\right)}(x)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1}{x\left(x-1\right)} + \frac{c_2}{x-1} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1}{x\left(x-1\right)} + \frac{c_2}{x-1}
$$

Verified OK.

3.2.6 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= -x^2 + x \\
q(x) &= -4x + 2 \\
r(x) &= -2 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= -2 \\
q'(x) &= -4
\end{aligned}
$$

Therefore (1) becomes

$$
-2 - (-4) + (-2) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$\left(-x^2 + x\right) y' + (1 - 2x)\, y = c_1$$

We now have a first order ode to solve which is

$$\left(-x^2 + x\right) y' + (1 - 2x)\, y = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1 - 2x}{x\,(x - 1)} \\ q(x) &= -\frac{c_1}{x\,(x - 1)} \end{aligned}$$

Hence the ode is

$$y' - \frac{(1 - 2x)\, y}{x\,(x - 1)} = -\frac{c_1}{x\,(x - 1)}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1-2x}{x(x-1)} dx} \\ &= x(x - 1) \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(-\frac{c_1}{x\,(x - 1)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(yx(x - 1)) &= (x(x - 1)) \left(-\frac{c_1}{x\,(x - 1)}\right) \\ \mathrm{d}(yx(x - 1)) &= (-c_1)\ \mathrm{d}x \end{aligned}$$

Integrating gives

$$yx(x-1) = \int -c_1 \, \mathrm{d}x$$
$$yx(x-1) = -c_1 x + c_2$$

Dividing both sides by the integrating factor $\mu = x(x-1)$ results in

$$y = -\frac{c_1}{x-1} + \frac{c_2}{x(x-1)}$$

which simplifies to

$$y = \frac{-c_1 x + c_2}{x(x-1)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_1 x + c_2}{x(x-1)} \tag{1}$$

Verification of solutions

$$y = \frac{-c_1 x + c_2}{x(x-1)}$$

Verified OK.

3.2.7 Maple step by step solution

Let's solve

$(-x^2 + x) y'' + (-4x + 2) y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x(x-1)} - \frac{2(2x-1)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(2x-1)y'}{x(x-1)} + \frac{2y}{x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{2(2x-1)}{x(x-1)}, P_3(x)=\frac{2}{x(x-1)}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=2$$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(x-1)+(4x-2)\,y'+2y=0$

- Assume series solution for y

  $$y=\sum_{k=0}^{\infty}a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m\cdot y''$ to series expansion for $m=1..2$

  $$x^m\cdot y''=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0r(1+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+r+1)\,(k+r+2)+a_k(k+r+2)\,(k+r+1)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)(k+r+1)(-a_{k+1}+a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=a_k$

- Recursion relation for $r=-1$

  $a_{k+1}=a_k$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=a_k\right]$

- Recursion relation for $r=0$

  $a_{k+1}=a_k$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+1}=a_k\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+1}=a_k, b_{k+1}=b_k\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x*(1-x)*diff(y(x),x$2)+2*(1-2*x)*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x + c_2}{x (x-1)}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 22

```
DSolve[x*(1-x)*y''[x]+2*(1-2*x)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x + c_1}{x - x^2}$$

3.3 problem 10.4.8 (c)

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Internal problem ID [5066]
Internal file name [`OUTPUT/4559_Sunday_June_05_2022_03_00_53_PM_83484169/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.8 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + xy' - 9y = 0}$$

3.3.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - 9x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - 9x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)+r-9=0$$

Or

$$r^2-9=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1=-3$$
$$r_2=3$$

Since the roots are real and distinct, then the general solution is

$$y=c_1y_1+c_2y_2$$

Where $y_1=x^{r_1}$ and $y_2=x^{r_2}$. Hence

$$y=\frac{c_1}{x^3}+c_2x^3$$

Summary
The solution(s) found are the following

$$y=\frac{c_1}{x^3}+c_2x^3 \tag{1}$$

Verification of solutions

$$y=\frac{c_1}{x^3}+c_2x^3$$

Verified OK.

3.3.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y''+xy'-9y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{1}{x}$$
$$q(x)=-\frac{9}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{9}{x^2}}{\frac{1}{x^2}} \\
&= -9
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 9y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - 9\,e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-9)} \\ &= \pm 3\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3 \\ \lambda_2 &= -3\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{(3)\tau} + c_2 e^{(-3)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{3\tau} + c_2 e^{-3\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^6 + c_2}{x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^6 + c_2}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^6 + c_2}{x^3}$$

Verified OK.

3.3.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + x y' - 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{9}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{3\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{3}{c\sqrt{-\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{3}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{3\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{3\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 3\sqrt{-\frac{1}{x^2}}dx}{c} \\ &= \frac{3\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cosh(3\ln(x)) + ic_2 \sinh(3\ln(x))$$

Summary

The solution(s) found are the following

$$y = c_1 \cosh(3\ln(x)) + i c_2 \sinh(3\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cosh(3\ln(x)) + i c_2 \sinh(3\ln(x))$$

Verified OK.

3.3.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' + x y' - 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{9}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{7v'(x)}{x} &= 0 \\ v''(x) + \frac{7v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{7u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{7u}{x} \end{aligned}$$

Where $f(x) = -\frac{7}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{7}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{7}{x}\,dx \\ \ln(u) &= -7\ln(x) + c_1 \\ u &= \mathrm{e}^{-7\ln(x)+c_1} \\ &= \frac{c_1}{x^7} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{6x^6} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{6x^6} + c_2\right) x^3 \\
&= \frac{6c_2x^6 - c_1}{6x^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{6x^6} + c_2\right) x^3 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{6x^6} + c_2\right) x^3
$$

Verified OK.

3.3.5 Solving using Kovacic algorithm

Writing the ode as

$$
x^2y'' + xy' - 9y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -9
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 48: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{5}{2} - \left(-\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{5}{2x} + (-)\,(0) \\ &= -\frac{5}{2x} \\ &= -\frac{5}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x}\right)(0) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x}\right)^2 - \left(\frac{35}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{5}{2x} dx} \\
&= \frac{1}{x^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^6}{6}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^3}\right) + c_2 \left(\frac{1}{x^3} \left(\frac{x^6}{6}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1}{x^3} + \frac{c_2 x^3}{6} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1}{x^3} + \frac{c_2 x^3}{6}
$$

Verified OK.

3.3.6 Maple step by step solution

Let's solve

$x^2 y'' + x y' - 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{9y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{9y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + x y' - 9y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - 9y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 9y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 3)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-3t}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{3t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^3} + c_2 x^3$

- Simplify

  $y = \frac{c_1}{x^3} + c_2 x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-9*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^6 + c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+x*y'[x]-9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^6 + c_1}{x^3}$$

3.4 problem 10.4.8 (d)

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Internal problem ID [5067]
Internal file name [`OUTPUT/4560_Sunday_June_05_2022_03_00_54_PM_41197975/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.8 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{xy'' + \frac{y'}{2} + 2y = 0}$$

3.4.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$xy'' + \frac{y'}{2} + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{2x} \\
q(x) &= \frac{2}{x}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\tau''(x) + p(x)\,\tau'(x) = 0
$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int \frac{1}{2x} dx\right)} dx \\
&= \int e^{-\frac{\ln(x)}{2}} dx \\
&= \int \frac{1}{\sqrt{x}} dx \\
&= 2\sqrt{x} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{x}}{\frac{1}{x}} \\
&= 2 \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + 2y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda\tau} + 2\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 + 2 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(2)} \\
&= \pm i\sqrt{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +i\sqrt{2} \\
\lambda_2 &= -i\sqrt{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= i\sqrt{2} \\
\lambda_2 &= -i\sqrt{2}
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0\left(c_1 \cos\left(\sqrt{2}\,\tau\right) + c_2 \sin\left(\sqrt{2}\,\tau\right)\right)$$

Or

$$y(\tau) = c_1 \cos\left(\sqrt{2}\,\tau\right) + c_2 \sin\left(\sqrt{2}\,\tau\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(2\sqrt{2}\,\sqrt{x}\right) + c_2 \sin\left(2\sqrt{2}\,\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(2\sqrt{2}\,\sqrt{x}\right) + c_2 \sin\left(2\sqrt{2}\,\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(2\sqrt{2}\,\sqrt{x}\right) + c_2 \sin\left(2\sqrt{2}\,\sqrt{x}\right)$$

Verified OK.

3.4.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$xy'' + \frac{y'}{2} + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{2}{x}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{x}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{2c\sqrt{\frac{1}{x}}\,x^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{2}}{2c\sqrt{\frac{1}{x}}\,x^2} + \frac{1}{2x}\frac{\sqrt{2}\,\sqrt{\frac{1}{x}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{x}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1\cos(c\tau) + c_2\sin(c\tau)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{2}\,\sqrt{\frac{1}{x}}dx}{c} \\
&= \frac{2x\sqrt{2}\,\sqrt{\frac{1}{x}}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(2\sqrt{2}\,\sqrt{x}\right) + c_2 \sin\left(2\sqrt{2}\,\sqrt{x}\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos\left(2\sqrt{2}\,\sqrt{x}\right) + c_2 \sin\left(2\sqrt{2}\,\sqrt{x}\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos\left(2\sqrt{2}\,\sqrt{x}\right) + c_2 \sin\left(2\sqrt{2}\,\sqrt{x}\right)
$$

Verified OK.

3.4.3 Solving as second order bessel ode ode

Writing the ode as

$$
x^2y'' + \frac{xy'}{2} + 2xy = 0 \tag{1}
$$

Bessel ode has the form

$$
x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}
$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$
x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}
$$

With the standard solution

$$
y = x^\alpha(c_1 \,\mathrm{BesselJ}\left(n, \beta\, x^\gamma\right) + c_2 \,\mathrm{BesselY}\left(n, \beta\, x^\gamma\right)) \tag{4}
$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{4} \\
\beta &= 2\sqrt{2} \\
n &= \frac{1}{2} \\
\gamma &= \frac{1}{2}
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = \frac{c_1 x^{\frac{1}{4}} \sin\left(2\sqrt{2}\,\sqrt{x}\right)}{\sqrt{\pi}\,\sqrt{\sqrt{2}\,\sqrt{x}}} - \frac{c_2 x^{\frac{1}{4}} \cos\left(2\sqrt{2}\,\sqrt{x}\right)}{\sqrt{\pi}\,\sqrt{\sqrt{2}\,\sqrt{x}}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{4}} \sin\left(2\sqrt{2}\,\sqrt{x}\right)}{\sqrt{\pi}\,\sqrt{\sqrt{2}\,\sqrt{x}}} - \frac{c_2 x^{\frac{1}{4}} \cos\left(2\sqrt{2}\,\sqrt{x}\right)}{\sqrt{\pi}\,\sqrt{\sqrt{2}\,\sqrt{x}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{4}} \sin\left(2\sqrt{2}\,\sqrt{x}\right)}{\sqrt{\pi}\,\sqrt{\sqrt{2}\,\sqrt{x}}} - \frac{c_2 x^{\frac{1}{4}} \cos\left(2\sqrt{2}\,\sqrt{x}\right)}{\sqrt{\pi}\,\sqrt{\sqrt{2}\,\sqrt{x}}}
$$

Verified OK.

3.4.4 Solving using Kovacic algorithm

Writing the ode as

$$
xy'' + \frac{y'}{2} + 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x \\
B &= \frac{1}{2} \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{1}{(x - (0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1 + 32x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 4\sqrt{2}\,\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\, dx}\\ &= \mathrm{e}^{\int \frac{1+4\sqrt{2}\,\sqrt{-x}}{4x} dx}\\ &= x^{\frac{1}{4}}\mathrm{e}^{2\sqrt{2}\,\sqrt{-x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{2}}{x}\, dx}\\ &= z_1 e^{-\frac{\ln(x)}{4}}\\ &= z_1\left(\frac{1}{x^{\frac{1}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2\sqrt{2}\,\sqrt{-x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{1}{2}}{x} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{2}}}{(y_1)^2} dx \\ &= y_1 \left(\frac{\sqrt{2}\,\sqrt{-x}\left(-1+\mathrm{e}^{-4\sqrt{2}\,\sqrt{-x}}\right)}{4\sqrt{x}} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{2\sqrt{2}\,\sqrt{-x}}\right) + c_2 \left(\mathrm{e}^{2\sqrt{2}\,\sqrt{-x}} \left(\frac{\sqrt{2}\,\sqrt{-x}\left(-1+\mathrm{e}^{-4\sqrt{2}\,\sqrt{-x}}\right)}{4\sqrt{x}} \right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2\sqrt{2}\,\sqrt{-x}} - \frac{c_2 \sqrt{2}\,\sqrt{-x}\left(\mathrm{e}^{2\sqrt{2}\,\sqrt{-x}} - \mathrm{e}^{-2\sqrt{2}\,\sqrt{-x}}\right)}{4\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2\sqrt{2}\,\sqrt{-x}} - \frac{c_2 \sqrt{2}\,\sqrt{-x}\left(\mathrm{e}^{2\sqrt{2}\,\sqrt{-x}} - \mathrm{e}^{-2\sqrt{2}\,\sqrt{-x}}\right)}{4\sqrt{x}}$$

Verified OK.

3.4.5 Maple step by step solution

Let's solve

$y''x + \frac{y'}{2} + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y'}{2x} - \frac{2y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{y'}{2x} + \frac{2y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{2}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + 4y + y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k+1+2r) + 4a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k}{(2k+1+2r)(k+1+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{4a_k}{(2k+1)(k+1)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{4a_k}{(2k+1)(k+1)}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{4a_k}{(2k+2)\left(k+\frac{3}{2}\right)}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{4a_k}{(2k+1)(k+1)}, b_{k+1} = -\frac{4b_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x$2)+1/2*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(2\sqrt{x}\,\sqrt{2}\right) + c_2 \cos\left(2\sqrt{x}\,\sqrt{2}\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 38

```
DSolve[x*y''[x]+1/2*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(2\sqrt{2}\sqrt{x}\right) + c_2 \sin\left(2\sqrt{2}\sqrt{x}\right)$$

3.5 problem 10.4.8 (e)

3.5.1 Solving as second order euler ode ode 291
3.5.2 Solving as second order change of variable on x method 2 ode . 292
3.5.3 Solving as second order change of variable on x method 1 ode . 294
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Internal problem ID [5068]
Internal file name [`OUTPUT/4561_Sunday_June_05_2022_03_00_55_PM_87042529/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.8 (e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - xy' + y = 0}$$

3.5.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 1 = 0$$

Or

$$r^2 - 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 1$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x + c_2 x \ln(x)$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 x \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2 x \ln(x)$$

Verified OK.

3.5.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x)dx\right)}dx \\ &= \int e^{-\left(\int -\frac{1}{x}dx\right)}dx \\ &= \int e^{\ln(x)}\, dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{x^2} \\
&= \frac{1}{x^4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{x^4} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{1}{x^4} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since the roots are equal, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$
y(\tau) = c_1 \sqrt{\tau} + c_2 \sqrt{\tau}\, \ln(\tau)
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{x\sqrt{2}\,(c_1 + 2c_2 \ln(x) - c_2 \ln(2))}{2}
$$

Summary
The solution(s) found are the following

$$
y = \frac{x\sqrt{2}\,(c_1 + 2c_2 \ln(x) - c_2 \ln(2))}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{x\sqrt{2}\,(c_1 + 2c_2 \ln(x) - c_2 \ln(2))}{2}
$$

Verified OK.

3.5.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$
x^2 y'' - x y' + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x
$$

Summary

The solution(s) found are the following

$$
y = c_1 x \tag{1}
$$

Verification of solutions

$$
y = c_1 x
$$

Verified OK.

3.5.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$
x^2y'' - xy' + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\, dx + c_2\\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n\\
&= (c_1 \ln(x) + c_2)\, x\\
&= (c_1 \ln(x) + c_2)\, x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = (c_1 \ln(x) + c_2)\, x \tag{1}
$$

Verification of solutions

$$
y = (c_1 \ln(x) + c_2)\, x
$$

Verified OK.

3.5.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B\\
y'' &= B''v + B'v' + v''B + v'B'\\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0\\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (-x)(-1) + (1)(-x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x^3 v'' + \left(-x^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x^2 (u'(x)\, x + u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x) + c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-x)(c_1 \ln(x) + c_2) \\
&= -(c_1 \ln(x) + c_2)\, x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -(c_1 \ln(x) + c_2)\, x \tag{1}
$$

Verification of solutions

$$
y = -(c_1 \ln(x) + c_2)\, x
$$

Verified OK.

3.5.6 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' - x y' + y = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -x \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(\ln(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 x \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2 x \ln(x)$$

Verified OK.

3.5.7 Maple step by step solution

Let's solve

$x^2 y'' - x y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - x y' + y = 0$

- Make a change of variables

$t = \ln(x)$

- ☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - \frac{d}{dt}y(t) + y(t) = 0$
- Simplify

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$
- Factor the characteristic polynomial

  $(r-1)^2 = 0$
- Root of the characteristic polynomial

  $r = 1$
- 1st solution of the ODE

  $y_1(t) = e^t$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^t$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^t + c_2 t\,e^t$

- Change variables back using $t = \ln(x)$

  $y = c_1 x + c_2 x \ln(x)$

- Simplify

  $y = x(c_2 \ln(x) + c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = x(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 15

```
DSolve[x^2*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2 \log(x) + c_1)$$

3.6 problem 10.4.8 (f)

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3.6.2 Solving using Kovacic algorithm 309
3.6.3 Maple step by step solution 314

Internal problem ID [5069]
Internal file name [`OUTPUT/4562_Sunday_June_05_2022_03_00_56_PM_86182655/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.8 (f).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2xy'' - y' + 2y = 0}$$

3.6.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - \frac{xy'}{2} + xy = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{3}{4}$$
$$\beta = 2$$
$$n = \frac{3}{2}$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = -\frac{c_1 \left(2 \cos \left(2\sqrt{x}\right) \sqrt{x} - \sin \left(2\sqrt{x}\right)\right)}{2\sqrt{\pi}} - \frac{c_2 \left(2 \sin \left(2\sqrt{x}\right) \sqrt{x} + \cos \left(2\sqrt{x}\right)\right)}{2\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1 \left(2 \cos \left(2\sqrt{x}\right) \sqrt{x} - \sin \left(2\sqrt{x}\right)\right)}{2\sqrt{\pi}} - \frac{c_2 \left(2 \sin \left(2\sqrt{x}\right) \sqrt{x} + \cos \left(2\sqrt{x}\right)\right)}{2\sqrt{\pi}} \tag{1}$$

Verification of solutions

$$y = -\frac{c_1 \left(2 \cos \left(2\sqrt{x}\right) \sqrt{x} - \sin \left(2\sqrt{x}\right)\right)}{2\sqrt{\pi}} - \frac{c_2 \left(2 \sin \left(2\sqrt{x}\right) \sqrt{x} + \cos \left(2\sqrt{x}\right)\right)}{2\sqrt{\pi}}$$

Verified OK.

3.6.2 Solving using Kovacic algorithm

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right) w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x+1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x} dx} \\
&= \frac{\sqrt{2\sqrt{-x}-1}\,\sqrt{4x+1}\,\mathrm{e}^{2\sqrt{-x}}}{\sqrt{2\sqrt{-x}+1}\,x^{\frac{1}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{1}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)(4x+1)}}{\sqrt{2\sqrt{-x}+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)(4x+1)} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(4x+1\right)}}{\sqrt{2\sqrt{-x}+1}} \right) \\
&\quad + c_2 \left(\frac{\mathrm{e}^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(4x+1\right)}}{\sqrt{2\sqrt{-x}+1}} \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(4x+1\right)} dx \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 \mathrm{e}^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(4x+1\right)}}{\sqrt{2\sqrt{-x}+1}} \\
&\quad + \frac{c_2 \mathrm{e}^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(4x+1\right)} \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(4x+1\right)} dx \right)}{\sqrt{2\sqrt{-x}+1}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1 \mathrm{e}^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(4x+1\right)}}{\sqrt{2\sqrt{-x}+1}} \\
&\quad + \frac{c_2 \mathrm{e}^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(4x+1\right)} \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(4x+1\right)} dx \right)}{\sqrt{2\sqrt{-x}+1}}
\end{aligned}
$$

Verified OK.

3.6.3 Maple step by step solution

Let's solve

$2y''x - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2x} - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{y'}{2x} + \frac{y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{1}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $2y''x - y' + 2y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k - \frac{1}{2} + r\right)(k+1+r)\, a_{k+1} + 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k}{(2k-1+2r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}, b_{k+1} = -\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 36

```
dsolve(2*x*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(2\sqrt{x}\, c_1 + c_2\right) \cos\left(2\sqrt{x}\right) - \sin\left(2\sqrt{x}\right) \left(-2\sqrt{x}\, c_2 + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 59

```
DSolve[2*x*y''[x]-y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2i\sqrt{x}}\left(2\sqrt{x} + i\right) + \frac{1}{8} c_2 e^{-2i\sqrt{x}}\left(1 + 2i\sqrt{x}\right)$$

3.7 problem 10.4.8 (g)

3.7.1 Solving using Kovacic algorithm 318

Internal problem ID [5070]
Internal file name [`OUTPUT/4563_Sunday_June_05_2022_03_00_57_PM_84765408/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.8 (g).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + xy' - 2y = 0}$$

3.7.1 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{8+x}{4x} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 8+x \\
t &= 4x
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{8+x}{4x}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{4}{x^2} + \frac{16}{x^3} - \frac{80}{x^4} + \frac{448}{x^5} - \frac{2688}{x^6} + \frac{16896}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{8 + x}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x}\right) \\ &= \frac{1}{4} + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &= 2\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &= -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{8+x}{4x}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right)\\
&= 2 - (1)\\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{x} + \frac{1}{2} \\
&= \frac{1}{x} + \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{8 + x}{4x}\right)\right) &= 0 \\
\frac{2 - a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x + 2)\, e^{\int \left(\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= (x + 2)\, e^{\frac{x}{2} + \ln(x)} \\
&= (x + 2)\, x\, e^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x(x+2)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(-1-x)\,\mathrm{e}^{-x} + (x+2)\,x\ \mathrm{expIntegral}_1\,(x)}{2\,(x+2)\,x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x(x+2)) + c_2\left(x(x+2)\left(\frac{(-1-x)\,\mathrm{e}^{-x} + (x+2)\,x\ \mathrm{expIntegral}_1\,(x)}{2\,(x+2)\,x}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 x(x+2) + c_2\left(\frac{(-1-x)\,\mathrm{e}^{-x}}{2} + \frac{(x+2)\,x\ \mathrm{expIntegral}_1\,(x)}{2}\right) \tag{1}
$$

Verification of solutions

$$y = c_1 x(x+2) + c_2\left(\frac{(-1-x)\,\mathrm{e}^{-x}}{2} + \frac{(x+2)\,x\ \mathrm{expIntegral}_1\,(x)}{2}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(x*diff(y(x),x$2)+x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{(x+1)\,c_2\mathrm{e}^{-x}}{2} + (x+2)\,x\left(c_1 + \frac{\mathrm{expIntegral}_1\,(x)\,c_2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 39

```
DSolve[x*y''[x]+x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x(x+2) - \frac{1}{2}c_2 e^{-x}(e^x (x+2)x\,\mathrm{ExpIntegralEi}(-x) + x + 1)$$

3.8 problem 10.4.8 (h)

3.8.1 Solving as second order bessel ode ode 326
3.8.2 Solving using Kovacic algorithm 327
3.8.3 Maple step by step solution 332

Internal problem ID [5071]
Internal file name [`OUTPUT/4564_Sunday_June_05_2022_03_00_58_PM_93789482/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.8 (h).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x-1)^2 y''-2y=0}$$

3.8.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y''-\frac{2y}{x}=0 \tag{1}$$

Bessel ode has the form

$$x^2y''+xy'+\left(-n^2+x^2\right)y=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y''+(1-2\alpha)\,xy'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)y=0 \tag{3}$$

With the standard solution

$$y=x^{\alpha}(c_1\,\text{BesselJ}\left(n,\beta\,x^{\gamma}\right)+c_2\,\text{BesselY}\left(n,\beta\,x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2i\sqrt{2}$$
$$n = 1$$
$$\gamma = -\frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = ic_1\sqrt{x}\ \mathrm{BesselI}\left(1, \frac{2\sqrt{2}}{\sqrt{x}}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(1, \frac{2i\sqrt{2}}{\sqrt{x}}\right)$$

Summary

The solution(s) found are the following

$$y = ic_1\sqrt{x}\ \mathrm{BesselI}\left(1, \frac{2\sqrt{2}}{\sqrt{x}}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(1, \frac{2i\sqrt{2}}{\sqrt{x}}\right) \tag{1}$$

Verification of solutions

$$y = ic_1\sqrt{x}\ \mathrm{BesselI}\left(1, \frac{2\sqrt{2}}{\sqrt{x}}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(1, \frac{2i\sqrt{2}}{\sqrt{x}}\right)$$

Verified OK.

3.8.2 Solving using Kovacic algorithm

Writing the ode as

$$x(x-1)^2 y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(x-1)^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 57: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-1)^2} + \frac{2}{x} - \frac{2}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x\,(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 0 - (0) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} - \frac{1}{x-1} + (0) \\ &= \frac{1}{x} - \frac{1}{x-1} \\ &= -\frac{1}{x\,(x-1)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{x-1}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{x-1}\right)^2 - \left(\frac{2}{x\,(x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx} \\ &= \frac{x}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{x}{x-1} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \frac{x}{x-1} \int \frac{1}{\frac{x^2}{(x-1)^2}}\, dx \\
&= \frac{x}{x-1} \left(x - 2\ln(x) - \frac{1}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{x-1}\right) + c_2 \left(\frac{x}{x-1} \left(x - 2\ln(x) - \frac{1}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x}{x-1} + \frac{c_2(-2\ln(x)\, x + x^2 - 1)}{x-1} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x}{x-1} + \frac{c_2(-2\ln(x)\, x + x^2 - 1)}{x-1}
$$

Verified OK.

3.8.3 Maple step by step solution

Let's solve

$x(x-1)^2 y'' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2y}{x(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{2}{x(x-1)^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x-1)^2 y'' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r)\, x^{-1+r} + (a_1(1+r)\, r - 2a_0(r^2-r+1))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r) - 2a_k(k^2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 1\}$

- Each term must be 0

  $a_1(1+r) r - 2a_0(r^2 - r + 1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1}) k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1}) r + 2a_k - 3a_{k-1} + a_{k+1}) k + (-2a_k + a_{k-1}$ -

- Shift index using $k- >k+1$

  $(-2a_{k+1} + a_k + a_{k+2}) (k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2}) r + 2a_{k+1} - 3a_k + a_{k+2}) (k+1) + (-$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} - ka_k - 2ka_{k+1} - ra_k - 2ra_{k+1} - 2a_{k+1}}{k^2 + 2kr + r^2 + 3k + 3r + 2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - ka_k - 2ka_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - ka_k - 2ka_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0\right]$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + ka_k - 6ka_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + ka_k - 6ka_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}, 2a_1 - 2a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - ka_k - 2ka_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0, b_{k+2} = -\frac{k^2 b}{}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*(x-1)^2*diff(y(x),x$2)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2c_2x\ln(x) - c_2x^2 + c_1x + c_2}{x-1}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 33

```
DSolve[x*(x-1)^2*y''[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-c_2x^2 - c_1x + 2c_2x\log(x) + c_2}{x-1}$$

3.9 problem 10.4.9 (i)

3.9.1 Solving as linear ode 336
3.9.2 Solving as first order ode lie symmetry lookup ode 338
3.9.3 Solving as exact ode 342
3.9.4 Maple step by step solution 347

Internal problem ID [5072]
Internal file name [`OUTPUT/4565_Sunday_June_05_2022_03_00_59_PM_16506391/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.9 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{2y}{x} = x^2}$$

3.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = x^2$$

Hence the ode is

$$y' - \frac{2y}{x} = x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^2\right) \\
\frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(x^2\right) \\
d\left(\frac{y}{x^2}\right) &= dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^2} &= \int dx \\
\frac{y}{x^2} &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$
y = c_1 x^2 + x^3
$$

which simplifies to

$$
y = x^2(x + c_1)
$$

Summary

The solution(s) found are the following

$$
y = x^2(x + c_1) \tag{1}
$$

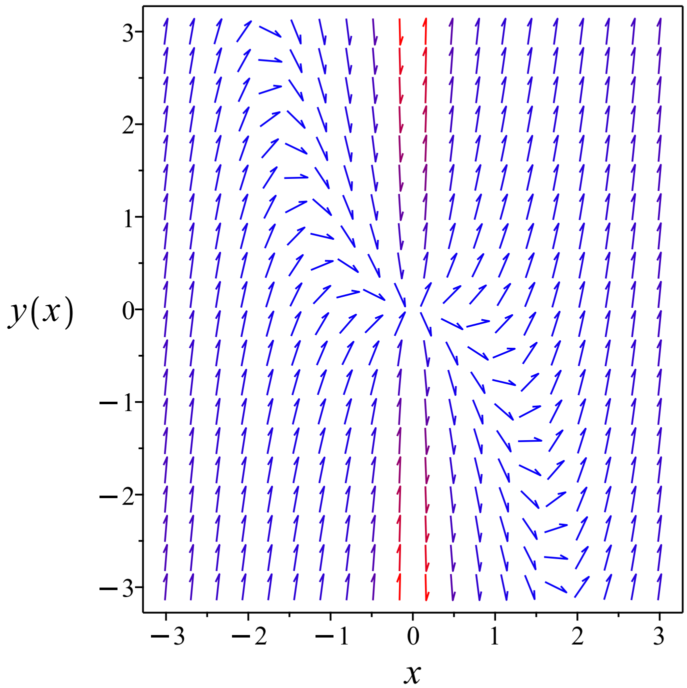


Figure 42: Slope field plot

Verification of solutions

$$y = x^2(x + c_1)$$

Verified OK.

3.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 + 2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 59: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^3 + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = x + c_1$$

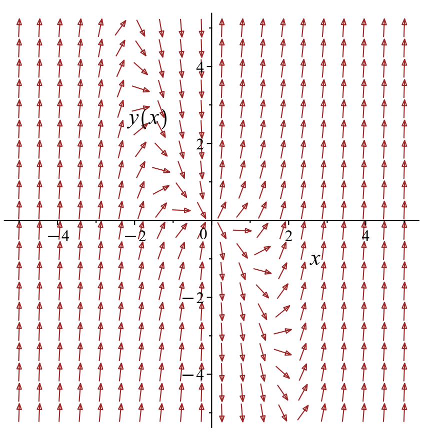
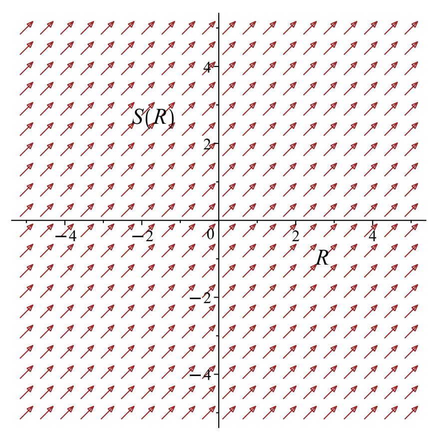
Which simplifies to

$$\frac{y}{x^2} = x + c_1$$

Which gives

$$y = x^2(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = 1$ 

Summary
The solution(s) found are the following

$$y = x^2(x + c_1) \tag{1}$$

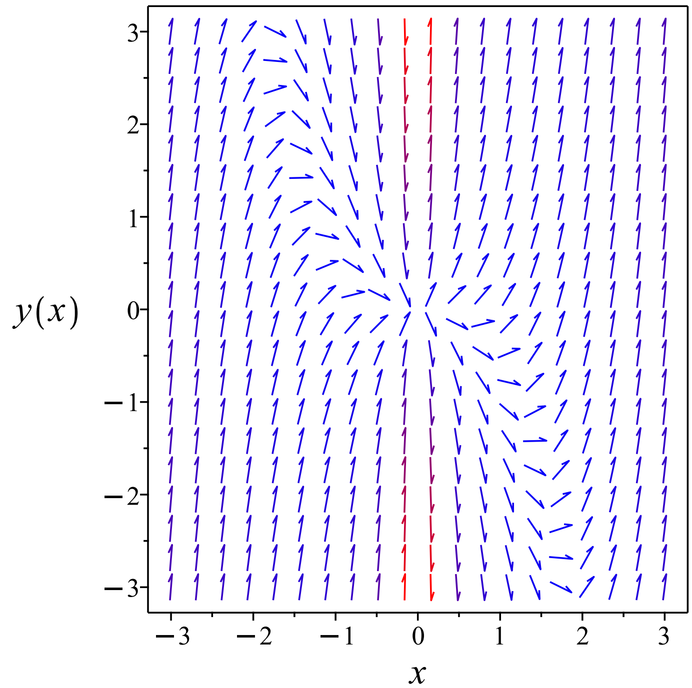


Figure 43: Slope field plot

Verification of solutions

$$y = x^2(x + c_1)$$

Verified OK.

3.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(\frac{2y}{x} + x^2\right) \mathrm{d}x \\
\left(-\frac{2y}{x} - x^2\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{2y}{x} - x^2 \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2y}{x} - x^2\right) \\
&= -\frac{2}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{2}{x}\right) - (0)\right) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(-\frac{2y}{x} - x^2\right) \\
&= \frac{-x^3 - 2y}{x^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(1) \\
&= \frac{1}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{gathered}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}=0\\ \left(\frac{-x^3-2y}{x^3}\right)+\left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0\end{gathered}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int\frac{-x^3-2y}{x^3}\,\mathrm{d}x\end{aligned}$$

$$\phi=-x+\frac{y}{x^2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=\frac{1}{x^2}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2}=\frac{1}{x^2}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \frac{y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \frac{y}{x^2}$$

The solution becomes

$$y = x^2(x + c_1)$$

Summary

The solution(s) found are the following

$$y = x^2(x + c_1) \tag{1}$$

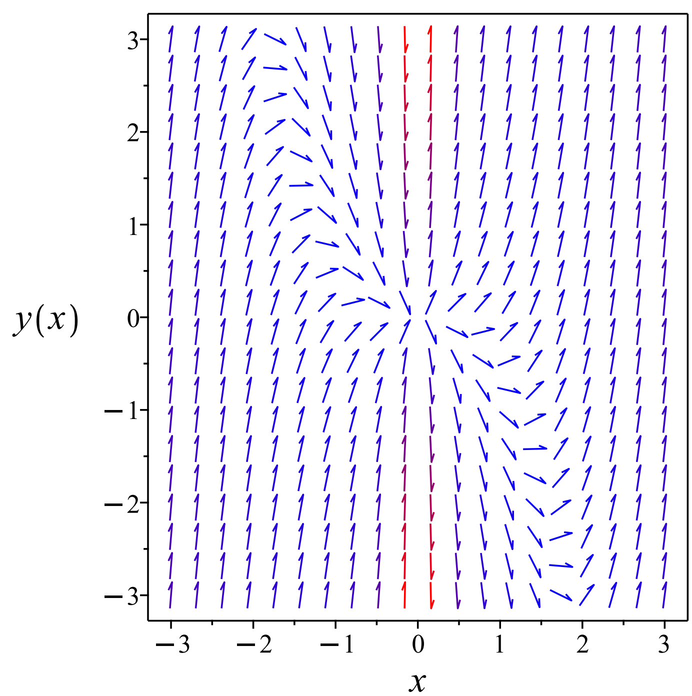


Figure 44: Slope field plot

Verification of solutions

$$y = x^2(x + c_1)$$

Verified OK.

3.9.4 Maple step by step solution

Let's solve

$y' - \frac{2y}{x} = x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2\left(\int 1dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2(x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)-2*y(x)/x-x^2=0,y(x), singsol=all)
```

$$y(x) = (x + c_1)\, x^2$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 13

```
DSolve[y'[x]-2*y[x]/x-x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(x + c_1)$$

3.10 problem 10.4.9 (ii)

3.10.1 Solving as linear ode 349
3.10.2 Solving as first order ode lie symmetry lookup ode 351
3.10.3 Solving as exact ode 355
3.10.4 Maple step by step solution 360

Internal problem ID [5073]
Internal file name [`OUTPUT/4566_Sunday_June_05_2022_03_01_00_PM_73400741/index.tex`]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.9 (ii).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + \frac{2y}{x} = x^3}$$

3.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = x^3$$

Hence the ode is

$$y' + \frac{2y}{x} = x^3$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x^3\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(y\, x^2\right) &= \left(x^2\right)\left(x^3\right) \\
\mathrm{d}\left(y\, x^2\right) &= x^5\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\, x^2 &= \int x^5\, \mathrm{d}x \\
y\, x^2 &= \frac{x^6}{6} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = \frac{x^4}{6} + \frac{c_1}{x^2}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{x^4}{6} + \frac{c_1}{x^2} \tag{1}
$$

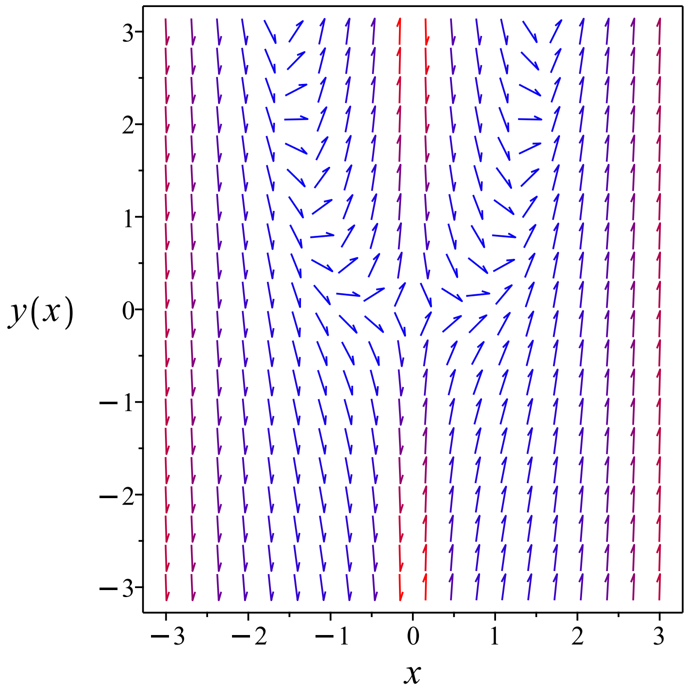


Figure 45: Slope field plot

Verification of solutions

$$y = \frac{x^4}{6} + \frac{c_1}{x^2}$$

Verified OK.

3.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^4 + 2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 62: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y\,x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^4 + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^5 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^5$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^6}{6} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^2 = \frac{x^6}{6} + c_1$$

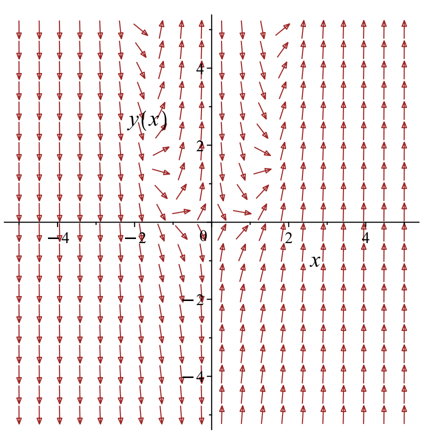
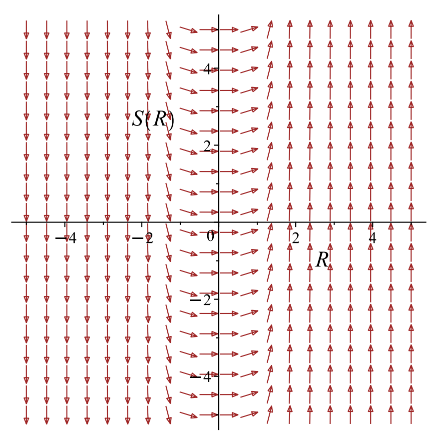
Which simplifies to

$$yx^2 = \frac{x^6}{6} + c_1$$

Which gives

$$y = \frac{x^6 + 6c_1}{6x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{-x^4+2y}{x}$
</td><td>$R = x$
$S = y\,x^2$</td><td>$\frac{dS}{dR} = R^5$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{x^6 + 6c_1}{6x^2} \tag{1}$$

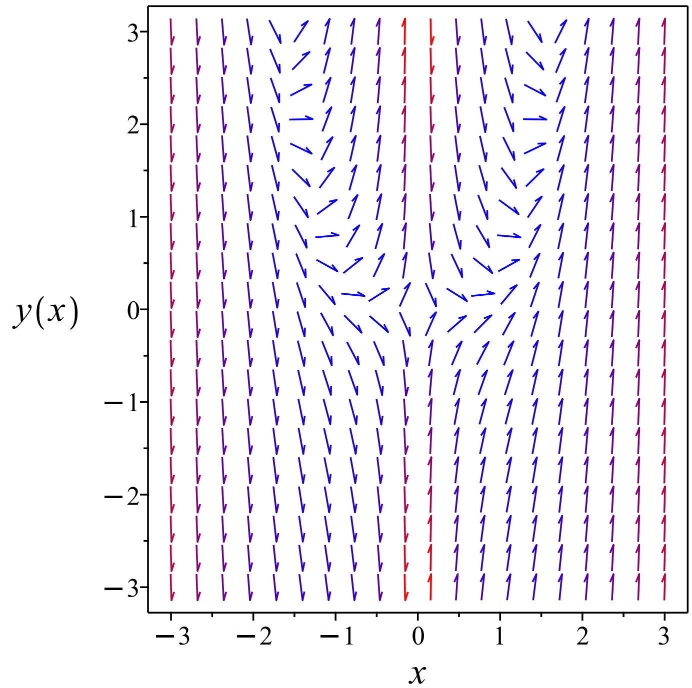


Figure 46: Slope field plot

Verification of solutions

$$y = \frac{x^6 + 6c_1}{6x^2}$$

Verified OK.

3.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-\frac{2y}{x} + x^3\right)\mathrm{d}x \\
\left(-x^3 + \frac{2y}{x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^3 + \frac{2y}{x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 + \frac{2y}{x}\right) \\
&= \frac{2}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{2}{x}\right) - (0)\right) \\
&= \frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(x)} \\
&= x^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x^2\left(-x^3 + \frac{2y}{x}\right) \\
&= -x\left(x^4 - 2y\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^2(1) \\
&= x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-x\left(x^4 - 2y\right)\right) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x\left(x^4 - 2y\right)\mathrm{d}x$$
$$\phi = -\frac{1}{6}x^6 + y\,x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{6}x^6 + y\,x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{6}x^6 + y\,x^2$$

The solution becomes

$$y = \frac{x^6 + 6c_1}{6x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^6 + 6c_1}{6x^2} \tag{1}$$

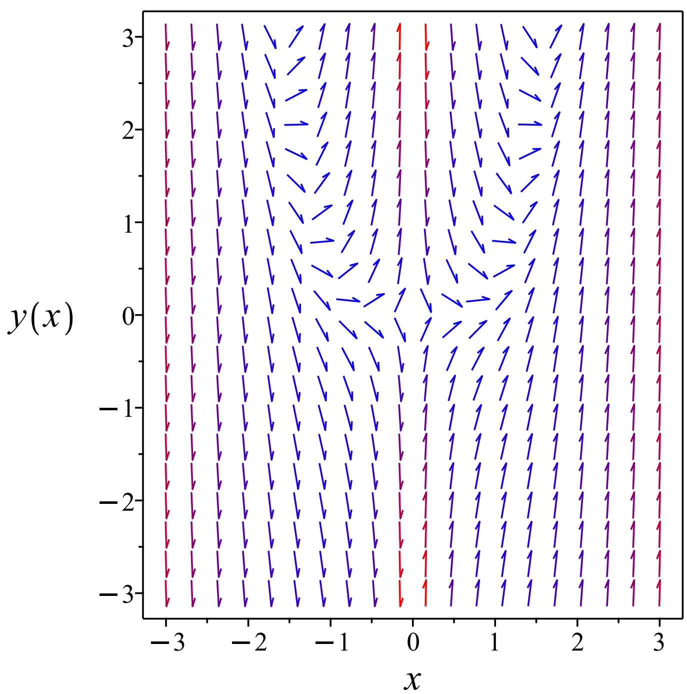


Figure 47: Slope field plot

Verification of solutions

$$y = \frac{x^6 + 6c_1}{6x^2}$$

Verified OK.

3.10.4 Maple step by step solution

Let's solve

$y' + \frac{2y}{x} = x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + x^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = x^3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{2y}{x}\right) = \mu(x)\, x^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^3 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^3 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^3 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

$y = \frac{\int x^5 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^6}{6} + c_1}{x^2}$

- Simplify

  $y = \frac{x^6 + 6c_1}{6x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+2*y(x)/x-x^3=0,y(x), singsol=all)
```

$$y(x) = \frac{x^6 + 6c_1}{6x^2}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 13

```
DSolve[y'[x]-2*y[x]/x-x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(x + c_1)$$

3.11 problem 10.4.10

3.11.1 Maple step by step solution 362

Internal problem ID [5074]
Internal file name [OUTPUT/4567_Sunday_June_05_2022_03_01_01_PM_94712491/index.tex]

Book: Basic Training in Mathematics. By R. Shankar. Plenum Press. NY. 1995
Section: Chapter 10, Differential equations. Section 10.4, ODEs with variable Coefficients. Second order and Homogeneous. page 318
Problem number: 10.4.10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Laguerre]`

Unable to solve or complete the solution.

$$xy'' + (1 - x)\, y' + my = 0$$

3.11.1 Maple step by step solution

Let's solve

$y''x + (1 - x)\, y' + my = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y'}{x} - \frac{my}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} + \frac{my}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = \frac{m}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (1 - x)\, y' + my = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k + 1 + r)^2 - a_k(-m + k + r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_k(-m+k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(-m+k)}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(-m+k)}{(k+1)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(-m+k)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x$2)+(1-x)*diff(y(x),x)+m*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{KummerM}(-m, 1, x) + c_2 \operatorname{KummerU}(-m, 1, x)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 21

```
DSolve[x*y''[x]+(1-x)*y'[x]+m*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{HypergeometricU}(-m, 1, x) + c_2 \operatorname{LaguerreL}(m, x)$$