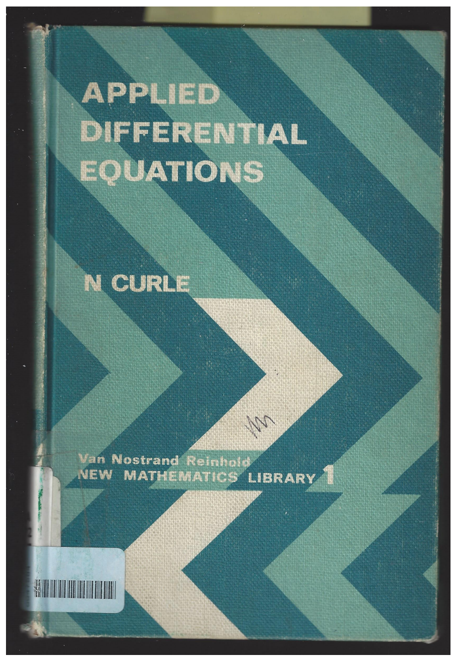
A Solution Manual For

Applied Differential equations, N Curle, 1971



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1 Examples, page 35

1.1 problem 1

Internal problem ID [2998]
Internal file name [OUTPUT/2490_Sunday_June_05_2022_03_16_17_AM_61902104/index.tex]

Book: Applied Differential equations, N Curle, 1971
Section: Examples, page 35
Problem number: 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y - y' - \frac{y'^2}{2} = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -1 + \sqrt{2y+1} \tag{1}$$

$$y' = -1 - \sqrt{2y+1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{-1+\sqrt{2y+1}} dy = \int dx$$

$$\frac{\ln(y)}{2} + \sqrt{2y+1} + \frac{\ln\left(-1+\sqrt{2y+1}\right)}{2} - \frac{\ln\left(1+\sqrt{2y+1}\right)}{2} = x + c_1$$

Summary
The solution(s) found are the following

$$\frac{\ln(y)}{2} + \sqrt{2y+1} + \frac{\ln\left(-1+\sqrt{2y+1}\right)}{2} - \frac{\ln\left(1+\sqrt{2y+1}\right)}{2} = x + c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln (y)}{2}+\sqrt{2y+1}+\frac{\ln \left(-1+\sqrt{2y+1}\right)}{2}-\frac{\ln \left(1+\sqrt{2y+1}\right)}{2}=x+c_{1}$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int \frac{1}{-1-\sqrt{2y+1}}dy=\int dx$$

$$\frac{\ln (y)}{2}-\sqrt{2y+1}-\frac{\ln \left(-1+\sqrt{2y+1}\right)}{2}+\frac{\ln \left(1+\sqrt{2y+1}\right)}{2}=x+c_{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln (y)}{2}-\sqrt{2y+1}-\frac{\ln \left(-1+\sqrt{2y+1}\right)}{2}+\frac{\ln \left(1+\sqrt{2y+1}\right)}{2}=x+c_{2} \tag{1}$$

Verification of solutions

$$\frac{\ln (y)}{2}-\sqrt{2y+1}-\frac{\ln \left(-1+\sqrt{2y+1}\right)}{2}+\frac{\ln \left(1+\sqrt{2y+1}\right)}{2}=x+c_{2}$$

Verified OK.

1.1.1 Maple step by step solution

Let's solve

$y-y'-\frac{y'^{2}}{2}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-1+\sqrt{2y+1}}=1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-1+\sqrt{2y+1}}dx=\int 1dx+c_{1}$

- Evaluate integral

  $\frac{\ln (y)}{2}+\sqrt{2y+1}+\frac{\ln \left(-1+\sqrt{2y+1}\right)}{2}-\frac{\ln \left(1+\sqrt{2y+1}\right)}{2}=x+c_{1}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 102

```
dsolve(y(x)=diff(y(x),x)+1/2*(diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = \frac{e^{2\,\mathrm{RootOf}\left(-\_Z-2x-2\,e^{\_Z}-2+2c_1-\ln(2)+\ln\left(e^{\_Z}\left(e^{\_Z}+2\right)^2\right)\right)}}{2} + e^{\mathrm{RootOf}\left(-\_Z-2x-2\,e^{\_Z}-2+2c_1-\ln(2)+\ln\left(e^{\_Z}\left(e^{\_Z}+2\right)^2\right)\right)}$$

$$y(x) = \frac{\mathrm{LambertW}\left(\sqrt{2}\,e^{-c_1+x-1}\right)\left(\mathrm{LambertW}\left(\sqrt{2}\,e^{-c_1+x-1}\right)+2\right)}{2}$$

✓ Solution by Mathematica

Time used: 18.04 (sec). Leaf size: 66

```
DSolve[y[x]==y'[x]+1/2*(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}W\left(-e^{x-1-c_1}\right)\left(2+W\left(-e^{x-1-c_1}\right)\right)$$
$$y(x) \to \frac{1}{2}W\left(e^{x-1+c_1}\right)\left(2+W\left(e^{x-1+c_1}\right)\right)$$
$$y(x) \to 0$$

1.2 problem 2

Internal problem ID [2999]
Internal file name [`OUTPUT/2491_Sunday_June_05_2022_03_16_22_AM_50717789/index.tex`]

Book: Applied Differential equations, N Curle, 1971
Section: Examples, page 35
Problem number: 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]`

$$\boxed{(-xy' + y)^2 - y'^2 = 1}$$

1.2.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$(-xp + y)^2 - p^2 = 1$$

Solving for y from the above results in

$$y = xp + \sqrt{p^2 + 1} \tag{1A}$$
$$y = xp - \sqrt{p^2 + 1} \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved. Solving ode 1A We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \sqrt{p^2 + 1} \\ &= xp + \sqrt{p^2 + 1} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \sqrt{p^2 + 1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \sqrt{c_1^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \sqrt{p^2 + 1}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x + \frac{p}{\sqrt{p^2 + 1}} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = -x\sqrt{-\frac{1}{x^2-1}}$$

Substituting the above back in (1) results in

$$y_1 = \sqrt{-\frac{1}{x^2-1}}\left(-x^2+1\right)$$

Solving ode 2A We start by replacing y' by p which gives

$$\begin{aligned} y &= xp - \sqrt{p^2+1} \\ &= xp - \sqrt{p^2+1} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -\sqrt{p^2+1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp+g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x+g')\frac{dp}{dx} \\ 0 &= (x+g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_2 x - \sqrt{c_2^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\sqrt{p^2 + 1}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - \frac{p}{\sqrt{p^2 + 1}} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = x\sqrt{-\frac{1}{x^2 - 1}}$$

Substituting the above back in (1) results in

$$y_1 = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1)$$

Summary

The solution(s) found are the following

$$y = c_1 x + \sqrt{c_1^2 + 1} \tag{1}$$

$$y = \sqrt{-\frac{1}{x^2 - 1}}\,(-x^2 + 1) \tag{2}$$

$$y = c_2 x - \sqrt{c_2^2 + 1} \tag{3}$$

$$y = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1) \tag{4}$$

Verification of solutions

$$y = c_1 x + \sqrt{c_1^2 + 1}$$

Verified OK.

$$y = \sqrt{-\frac{1}{x^2 - 1}}\,(-x^2 + 1)$$

Verified OK.

$$y = c_2 x - \sqrt{c_2^2 + 1}$$

Verified OK.

$$y = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 57

```
dsolve((y(x)-x*diff(y(x),x))^2=1+(diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = \sqrt{-x^2 + 1}$$
$$y(x) = -\sqrt{-x^2 + 1}$$
$$y(x) = c_1 x - \sqrt{c_1^2 + 1}$$
$$y(x) = c_1 x + \sqrt{c_1^2 + 1}$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 73

```
DSolve[(y[x]-x*y'[x])^2==1+(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \sqrt{1 + c_1{}^2}$$
$$y(x) \to c_1 x + \sqrt{1 + c_1{}^2}$$
$$y(x) \to -\sqrt{1 - x^2}$$
$$y(x) \to \sqrt{1 - x^2}$$

1.3 problem 3

Internal problem ID [3000]
Internal file name [`OUTPUT/2492_Sunday_June_05_2022_03_16_26_AM_69802190/index.tex`]

Book: Applied Differential equations, N Curle, 1971
Section: Examples, page 35
Problem number: 3.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y - y'^2\left(1 - \frac{2y'}{3}\right) = x}$$

1.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - p^2\left(1 - \frac{2p}{3}\right) = x$$

Solving for y from the above results in

$$y = x + p^2 - \frac{2}{3}p^3 \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 1$$
$$g = p^2 - \frac{2}{3}p^3$$

Hence (2) becomes

$$p - 1 = \left(-2p^2 + 2p\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 1 = 0$$

Solving for p from the above gives

$$p = 1$$

Substituting these in (1A) gives

$$y = x + \frac{1}{3}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - 1}{-2p(x)^2 + 2p(x)} \tag{3}$$

This ODE is now solved for $p(x)$. Integrating both sides gives

$$\int -2p dp = x + c_1$$
$$-p^2 = x + c_1$$

Solving for p gives these solutions

$$p_1 = \sqrt{-x - c_1}$$
$$p_2 = -\sqrt{-x - c_1}$$

Substituing the above solution for p in (2A) gives

$$y = -c_1 - \frac{2(-x - c_1)^{\frac{3}{2}}}{3}$$
$$y = -c_1 + \frac{2(-x - c_1)^{\frac{3}{2}}}{3}$$

Summary

The solution(s) found are the following

$$y = x + \frac{1}{3} \tag{1}$$

$$y = -c_1 - \frac{2(-x - c_1)^{\frac{3}{2}}}{3} \tag{2}$$

$$y = -c_1 + \frac{2(-x - c_1)^{\frac{3}{2}}}{3} \tag{3}$$

Verification of solutions

$$y = x + \frac{1}{3}$$

Verified OK.

$$y = -c_1 - \frac{2(-x - c_1)^{\frac{3}{2}}}{3}$$

Verified OK.

$$y = -c_1 + \frac{2(-x - c_1)^{\frac{3}{2}}}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 49

```
dsolve(y(x)-x=(diff(y(x),x))^2*(1-2/3* diff(y(x),x)),y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= x+\frac{1}{3} \\
y(x) &= \frac{(2x-2c_1)\sqrt{c_1-x}}{3}+c_1 \\
y(x) &= \frac{(-2x+2c_1)\sqrt{c_1-x}}{3}+c_1
\end{aligned}
$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y[x]-x==y'[x]^2*(1-2/3* y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

Timed out

1.4 problem 4

Internal problem ID [3001]
Internal file name [`OUTPUT/2493_Sunday_June_05_2022_03_16_41_AM_3764532/index.tex`]

Book: Applied Differential equations, N Curle, 1971
Section: Examples, page 35
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, _Riccati]
```

$$\boxed{y'x^2 - x(y-1) - (y-1)^2 = 0}$$

1.4.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{(Y(X) + y_0 - 1)(Y(X) + y_0 + X + x_0 - 1)}{(X + x_0)^2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X)X + Y(X)^2}{X^2}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= \frac{Y(X+Y)}{X^2}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = Y(X+Y)$ and $N = X^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= u^2 + u \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{u(X)^2}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{u(X)^2}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right) X - u(X)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= \frac{u^2}{X}
\end{aligned}
$$

Where $f(X) = \frac{1}{X}$ and $g(u) = u^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u^2}\, du &= \frac{1}{X}\, dX \\
\int \frac{1}{u^2}\, du &= \int \frac{1}{X}\, dX \\
-\frac{1}{u} &= \ln(X) + c_2
\end{aligned}
$$

The solution is

$$-\frac{1}{u(X)} - \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$-\frac{X}{Y(X)} - \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$-\frac{X}{Y(X)} - \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 1$$
$$X = x$$

Then the solution in y becomes

$$-\frac{x}{y-1} - \ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$-\frac{x}{y-1} - \ln(x) - c_2 = 0 \tag{1}$$

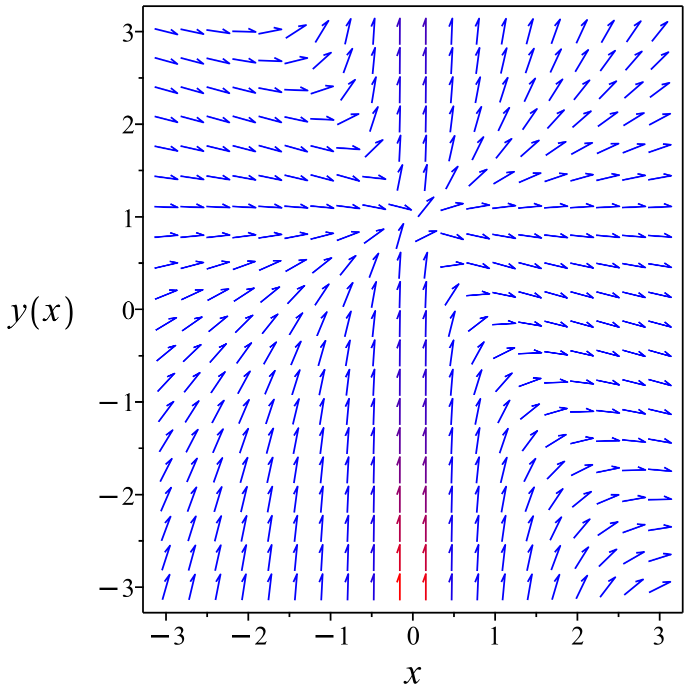


Figure 1: Slope field plot

Verification of solutions

$$-\frac{x}{y-1} - \ln(x) - c_2 = 0$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{xy + y^2 - x - 2y + 1}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(xy + y^2 - x - 2y + 1)(b_3 - a_2)}{x^2} - \frac{(xy + y^2 - x - 2y + 1)^2 a_3}{x^4} \\ & - \left(\frac{y-1}{x^2} - \frac{2(xy + y^2 - x - 2y + 1)}{x^3}\right)(xa_2 + ya_3 + a_1) \\ & - \frac{(2y + x - 2)(xb_2 + yb_3 + b_1)}{x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^3yb_2 - x^2y^2a_2 + x^2y^2b_3 + y^4a_3 + x^3b_1 - 2x^3b_2 + x^3b_3 - x^2ya_1 + 2x^2ya_2 - x^2ya_3 + 2x^2yb_1 - 2xy^2a_1 \ldots}{[illegible]}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^3yb_2 + x^2y^2a_2 - x^2y^2b_3 - y^4a_3 - x^3b_1 + 2x^3b_2 - x^3b_3 + x^2ya_1 - 2x^2ya_2 \\ & + x^2ya_3 - 2x^2yb_1 + 2xy^2a_1 + 2xy^2a_3 + 4y^3a_3 - x^2a_1 + x^2a_2 - x^2a_3 \\ & + 2x^2b_1 + x^2b_3 - 4xya_1 - 4xya_3 - 6y^2a_3 + 2xa_1 + 2xa_3 + 4ya_3 - a_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_2v_1^2v_2^2 - a_3v_2^4 - 2b_2v_1^3v_2 - b_3v_1^2v_2^2 + a_1v_1^2v_2 + 2a_1v_1v_2^2 - 2a_2v_1^2v_2 + a_3v_1^2v_2 \\ & + 2a_3v_1v_2^2 + 4a_3v_2^3 - b_1v_1^3 - 2b_1v_1^2v_2 + 2b_2v_1^3 - b_3v_1^3 - a_1v_1^2 - 4a_1v_1v_2 + a_2v_1^2 \\ & - a_3v_1^2 - 4a_3v_1v_2 - 6a_3v_2^2 + 2b_1v_1^2 + b_3v_1^2 + 2a_1v_1 + 2a_3v_1 + 4a_3v_2 - a_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-2b_2v_1^3v_2 + (-b_1 + 2b_2 - b_3)\, v_1^3 + (-b_3 + a_2)\, v_1^2v_2^2 + (a_1 - 2a_2 + a_3 - 2b_1)\, v_1^2v_2 + (-a_1 + a_2 - a_3 + 2b_1 + b_3)\, v_1^2 + (2a_1 + 2a_3)\, v_1v_2^2 + (-4a_1 - 4a_3)\, v_1v_2 + (2a_1 + 2a_3)\, v_1 - a_3v_2^4 + 4a_3v_2^3 - 6a_3v_2^2 + 4a_3v_2 - a_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-6a_3 &= 0 \\
-a_3 &= 0 \\
4a_3 &= 0 \\
-2b_2 &= 0 \\
-4a_1 - 4a_3 &= 0 \\
2a_1 + 2a_3 &= 0 \\
-b_3 + a_2 &= 0 \\
-b_1 + 2b_2 - b_3 &= 0 \\
a_1 - 2a_2 + a_3 - 2b_1 &= 0 \\
-a_1 + a_2 - a_3 + 2b_1 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= -b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y - 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - 1 - \left(\frac{xy + y^2 - x - 2y + 1}{x^2}\right)(x) \\
&= \frac{-y^2 + 2y - 1}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y^2+2y-1}{x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{x}{y - 1}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{xy + y^2 - x - 2y + 1}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{y-1} \\
S_y &= -\frac{x}{(y-1)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x}{y-1} = -\ln(x) + c_1$$

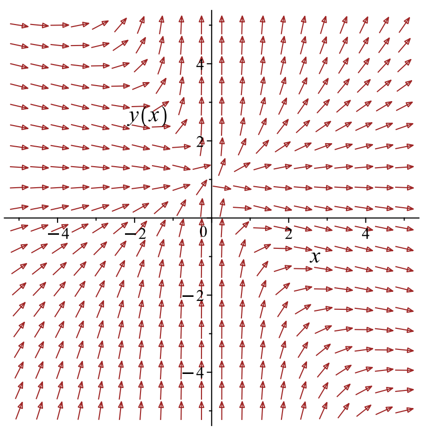
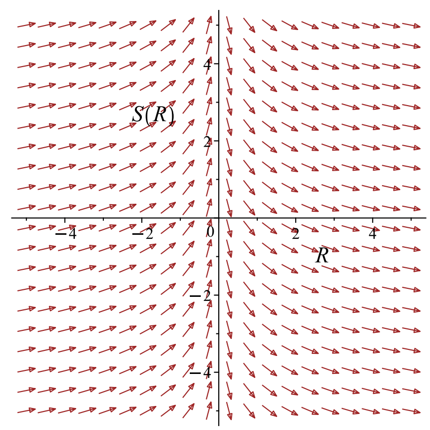
Which simplifies to

$$\frac{x}{y-1} = -\ln(x) + c_1$$

Which gives

$$y = \frac{\ln(x) - c_1 - x}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy+y^2-x-2y+1}{x^2}$ 	$R = x$ $S = \frac{x}{y-1}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{\ln(x) - c_1 - x}{\ln(x) - c_1} \tag{1}$$

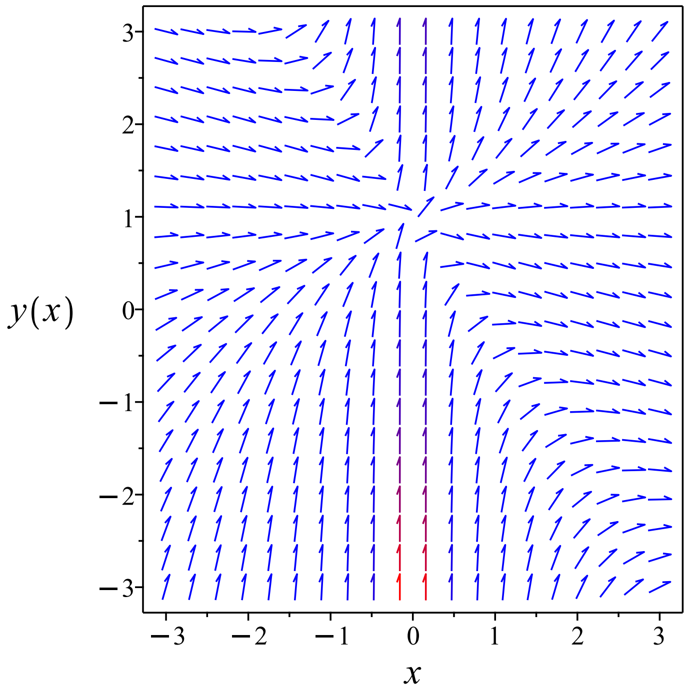


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{\ln(x) - c_1 - x}{\ln(x) - c_1}$$

Verified OK.

1.4.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{xy + y^2 - x - 2y + 1}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} + \frac{y^2}{x^2} - \frac{1}{x} - \frac{2y}{x^2} + \frac{1}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1-x}{x^2}$, $f_1(x) = \frac{x-2}{x^2}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{x^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{x^3} \\
f_1 f_2 &= \frac{x-2}{x^4} \\
f_2^2 f_0 &= \frac{1-x}{x^6}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{x^2} - \left(-\frac{2}{x^3} + \frac{x-2}{x^4}\right) u'(x) + \frac{(1-x)\, u(x)}{x^6} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \mathrm{e}^{\frac{1}{x}} (c_1 + c_2 \ln(x))
$$

The above shows that

$$
u'(x) = \frac{\mathrm{e}^{\frac{1}{x}} (-c_2 \ln(x) + c_2 x - c_1)}{x^2}
$$

Using the above in (1) gives the solution

$$
y = -\frac{-c_2 \ln(x) + c_2 x - c_1}{c_1 + c_2 \ln(x)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{\ln(x) - x + c_3}{c_3 + \ln(x)}
$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x) - x + c_3}{c_3 + \ln(x)} \tag{1}$$

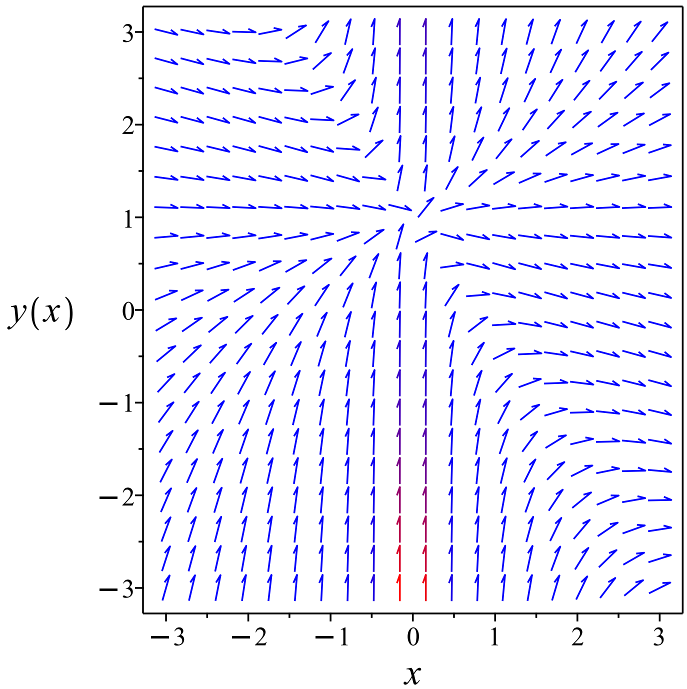


Figure 3: Slope field plot

Verification of solutions

$$y = \frac{\ln(x) - x + c_3}{c_3 + \ln(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x)=x*(y(x)-1)+(y(x)-1)^2,y(x), singsol=all)
```

$$y(x) = 1 - \frac{x}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.203 (sec). Leaf size: 23

```
DSolve[x^2*y'[x]==x*(y[x]-1)+(y[x]-1)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + \frac{x}{-\log(x) + c_1}$$
$$y(x) \to 1$$