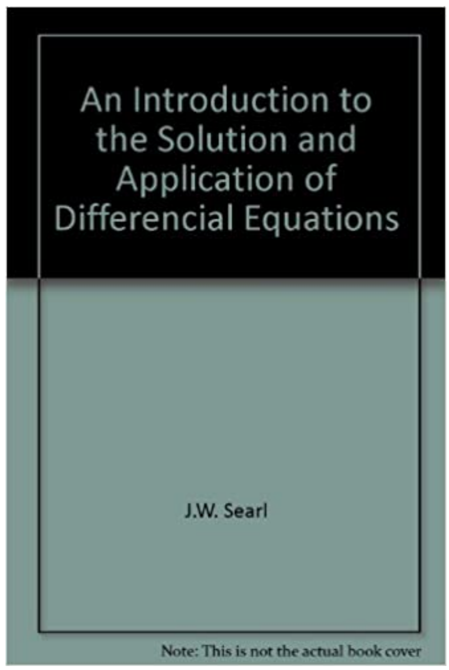
A Solution Manual For

# An introduction to the solution and applications of differential equations, J.W. Searl, 1966

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May 15, 2024

# Contents

# 1 Chapter 4, Ex. 4.1

## 1.1 problem 1



Internal problem ID [3134]
Internal file name [`OUTPUT/2626_Sunday_June_05_2022_03_23_18_AM_18850583/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.1
**Problem number**: 1.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x^2 + 2yx = 0}$$

### 1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{2y}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\frac{2}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{2}{x}\,dx \\
\ln(y) &= -2\ln(x) + c_1 \\
y &= e^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} \tag{1}$$

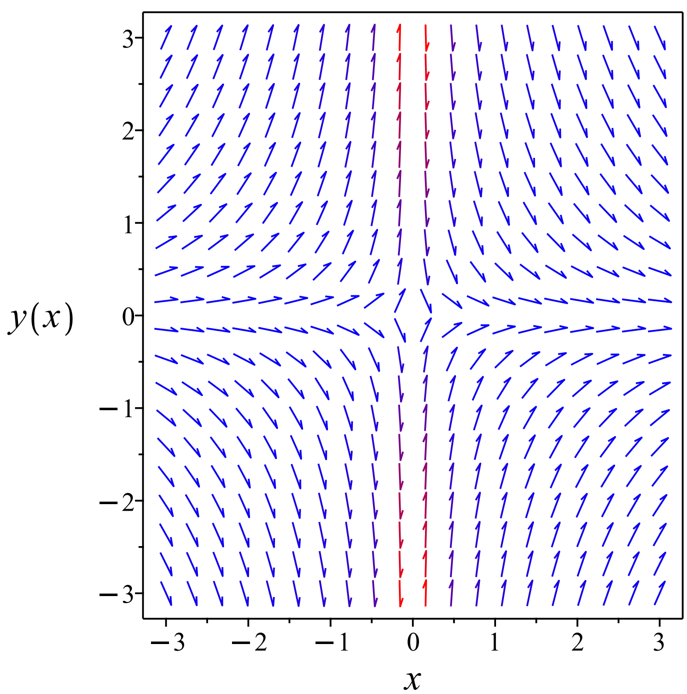



Figure 1: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^2}$$

Verified OK.

### 1.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x} = 0$$

The integrating factor $\mu$ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2}{x}dx} \\ &= x^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 y\right) &= 0 \end{aligned}$$

Integrating gives

$$x^2 y = c_1$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^2} \tag{1}$$

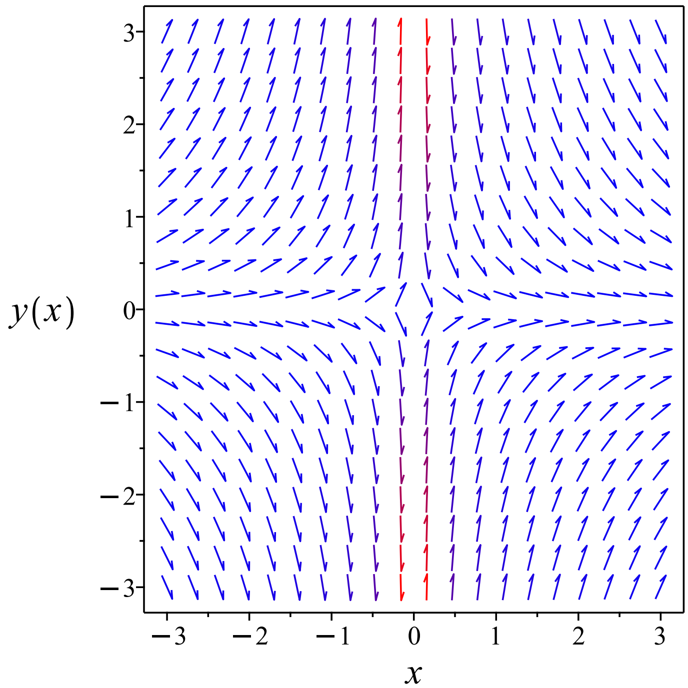



Figure 2: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^2}$$

Verified OK.

### 1.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 + 2u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{3}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\
\ln(u) &= -3\ln(x) + c_2 \\
u &= e^{-3\ln(x)+c_2} \\
&= \frac{c_2}{x^3}
\end{aligned}$$

Therefore the solution $y$ is

$$\begin{aligned}
y &= xu \\
&= \frac{c_2}{x^2}
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_2}{x^2} \tag{1}$$

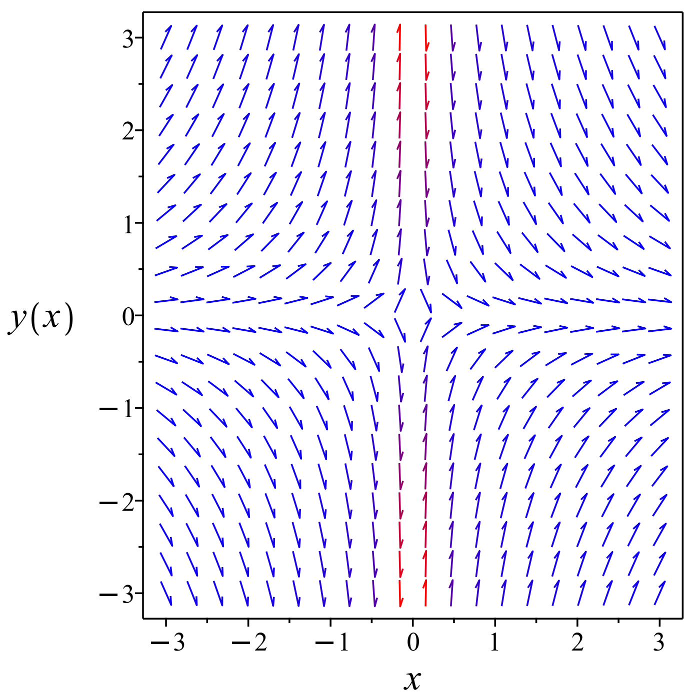



Figure 3: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x^2}$$

Verified OK.

### 1.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | 0 | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\, g(y)$ | $\frac{1}{f}$ | 0 |
| quadrature ode | $y' = f(x)$ | 0 | 1 |
| quadrature ode | $y' = g(y)$ | 1 | 0 |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a + bx + cy)^{\frac{n}{m}}$ | 1 | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\, e^{h(x)+by} + f(x)$ | $\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$ | $\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$ | $\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$ | $\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$ |
| Bernoulli ode | $y' = f(x)\, y + g(x)\, y^n$ | 0 | $e^{-\int (n-1) f(x) dx} y^n$ |
| Reduced Riccati | $y' = f_1(x)\, y + f_2(x)\, y^2$ | 0 | $e^{-\int f_1 dx}$ |

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x, y) \to (R, S)$ where $(R, S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

$S$ is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = x^2 y$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates $R, S$. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$x^2 y = c_1$$

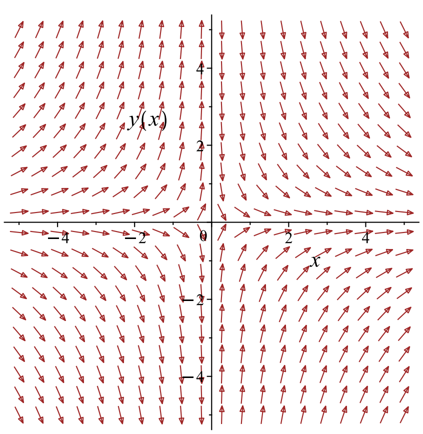
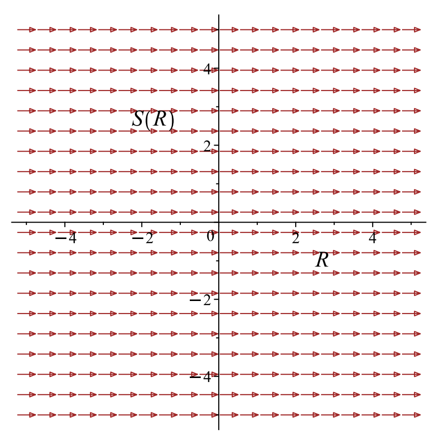
Which simplifies to

$$x^2 y = c_1$$

Which gives

$$y = \frac{c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = -\frac{2y}{x}$   | $R = x$<br>$S = x^2 y$ | $\frac{dS}{dR} = 0$   |

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^2} \tag{1}$$

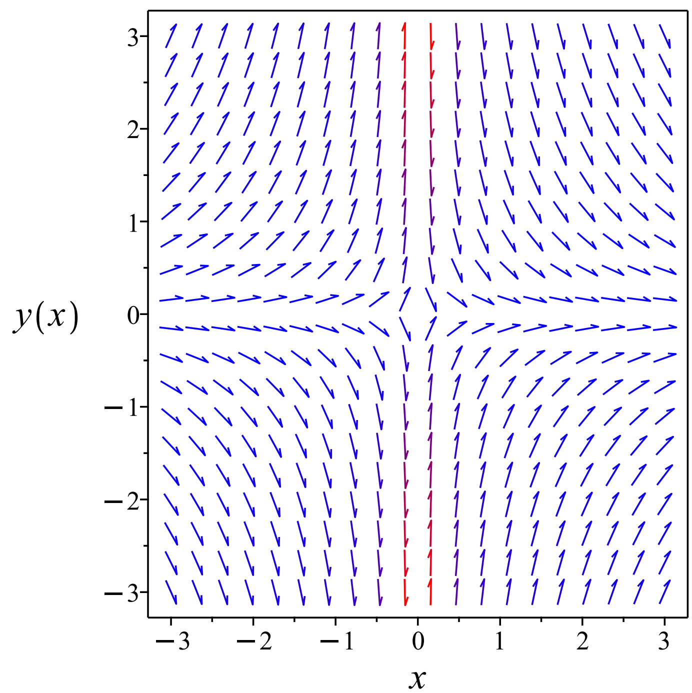



Figure 4: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^2}$$

Verified OK.

### 1.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{2y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(-\frac{1}{2y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= -\frac{1}{2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{2y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{x}\,\mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y}$. Therefore equation (4) becomes

$$-\frac{1}{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{2y}$$

Integrating the above w.r.t $y$ gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{2y}\right)\mathrm{d}y$$
$$f(y) = -\frac{\ln(y)}{2} + c_1$$

Where $c_1$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$\phi = -\ln(x) - \frac{\ln(y)}{2} + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y)}{2}$$

The solution becomes

$$y = \frac{e^{-2c_1}}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2c_1}}{x^2} \tag{1}$$

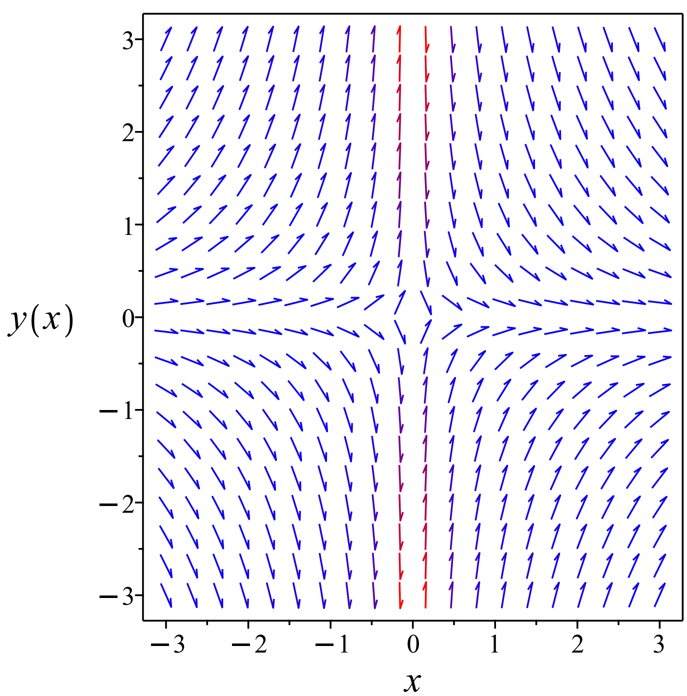



Figure 5: Slope field plot

Verification of solutions

$$y = \frac{e^{-2c_1}}{x^2}$$

Verified OK.

### 1.1.6 Maple step by step solution

Let's solve

$y'x^2 + 2yx = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Integrate both sides with respect to $x$

  $\int (y'x^2 + 2yx)\, dx = \int 0 dx + c_1$

- Evaluate integral

  $x^2y = c_1$

- Solve for $y$

  $y = \frac{c_1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(2*x*y(x)+x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 16

```
DSolve[2*x*y[x]+x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{c_1}{x^2} \\
y(x) &\to 0
\end{aligned}
$$

## 1.2 problem 2



Internal problem ID [3135]
Internal file name [`OUTPUT/2627_Sunday_June_05_2022_03_23_20_AM_64957984/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.1
**Problem number**: 2.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (-y + x)\, y' = -x}$$

With initial conditions

$$[y(0) = 0]$$

### 1.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y + x}{y - x} \end{aligned}$$

The $x$ domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The $y$ domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y+x}{y-x}\right) \\ &= \frac{1}{y-x} - \frac{y+x}{(y-x)^2}\end{aligned}$$

The $x$ domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

### 1.2.2 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y + (-y + x)\, p = -x$$

Solving for $y$ from the above results in

$$y = \frac{x(p+1)}{-1+p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where $f, g$ are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. $x$ gives

$$\begin{aligned}p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned}f &= \frac{p+1}{-1+p} \\ g &= 0\end{aligned}$$

Hence (2) becomes

$$p-\frac{p+1}{-1+p}=x\left(\frac{1}{-1+p}-\frac{p+1}{\left(-1+p\right)^{2}}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx}=0$ in the above which gives

$$p-\frac{p+1}{-1+p}=0$$

Solving for $p$ from the above gives

$$\begin{aligned}p&=1+\sqrt{2}\\ p&=-\sqrt{2}+1\end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned}y&=x+\sqrt{2}\,x\\ y&=x-\sqrt{2}\,x\end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x}\neq 0$. From eq. (2A). This results in

$$p'(x)=\frac{p(x)-\frac{p(x)+1}{-1+p(x)}}{x\left(\frac{1}{-1+p(x)}-\frac{p(x)+1}{\left(-1+p(x)\right)^{2}}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p)=\frac{x(p)\left(\frac{1}{-1+p}-\frac{p+1}{\left(-1+p\right)^{2}}\right)}{p-\frac{p+1}{-1+p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p)+p(p)x(p)=q(p)$$

Where here

$$\begin{aligned}p(p)&=\frac{2}{\left(p^{2}-2p-1\right)\left(-1+p\right)}\\ q(p)&=0\end{aligned}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{(p^2 - 2p - 1)(-1 + p)} = 0$$

The integrating factor $\mu$ is

$$\begin{aligned}
\mu &= e^{\int \frac{2}{(p^2-2p-1)(-1+p)}dp} \\
&= e^{\frac{\ln(p^2-2p-1)}{2} - \ln(-1+p)}
\end{aligned}$$

Which simplifies to

$$\mu = \frac{\sqrt{p^2 - 2p - 1}}{-1 + p}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p^2 - 2p - 1}\, x}{-1 + p}\right) &= 0
\end{aligned}$$

Integrating gives

$$\frac{\sqrt{p^2 - 2p - 1}\, x}{-1 + p} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p^2-2p-1}}{-1+p}$ results in

$$x(p) = \frac{c_3(-1 + p)}{\sqrt{p^2 - 2p - 1}}$$

Now we need to eliminate $p$ between the above and (1A). One way to do this is by solving (1) for $p$. This results in

$$p = -\frac{y + x}{-y + x}$$

Substituting the above in the solution for $x$ found above gives

$$x = \frac{c_3\sqrt{2}\, x}{(y - x)\sqrt{\frac{-y^2+2yx+x^2}{(-y+x)^2}}}$$

Initial conditions are used to solve for $c_3$. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -c_3\sqrt{2}$$

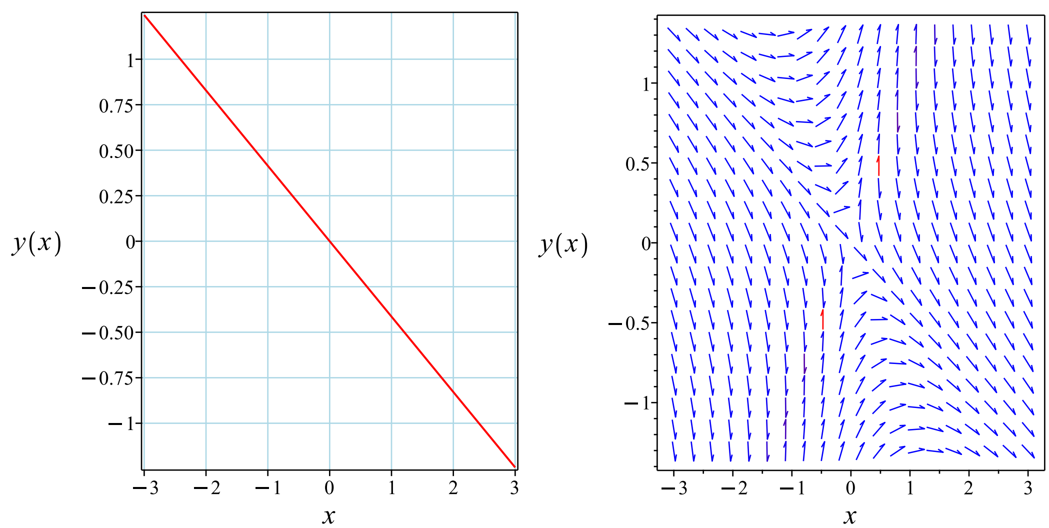
$$c_3 = 0$$

Substituting $c_3$ found above in the general solution gives

$$x = 0$$

<u>Summary</u>
The solution(s) found are the following

$$y = x + \sqrt{2}\,x \tag{1}$$
$$y = x - \sqrt{2}\,x \tag{2}$$
$$x = 0 \tag{3}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x + \sqrt{2}\, x$$

Verified OK.

$$y = x - \sqrt{2}\, x$$

Verified OK.

$$x = 0$$

Verified OK.

### 1.2.3 Maple step by step solution

Let's solve

$[y + (-y + x)\, y' = -x, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  $y'$

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a $C^2$ function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $1 = 1$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\, , N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to $x$

  $F(x, y) = \int (y + x)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = xy + \frac{x^2}{2} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to $y$

$N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $-y + x = x + \frac{d}{dy} f_1(y)$
- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -y$
- Solve for $f_1(y)$

  $f_1(y) = -\frac{y^2}{2}$
- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = xy + \frac{1}{2}x^2 - \frac{1}{2}y^2$
- Substitute $F(x, y)$ into the solution of the ODE

  $xy + \frac{1}{2}x^2 - \frac{1}{2}y^2 = c_1$
- Solve for $y$

  $\left\{y = x - \sqrt{2x^2 - 2c_1}, y = x + \sqrt{2x^2 - 2c_1}\right\}$
- Use initial condition $y(0) = 0$

  $0 = -\sqrt{-2c_1}$
- Solve for $c_1$

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = x\left(1 - \sqrt{2}\,\mathrm{csgn}(x)\right)$
- Use initial condition $y(0) = 0$

  $0 = \sqrt{-2c_1}$
- Solve for $c_1$

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = x\left(1 + \sqrt{2}\,\mathrm{csgn}(x)\right)$
- Solutions to the IVP

  $\left\{y = x\left(1 - \sqrt{2}\,\mathrm{csgn}(x)\right), y = x\left(1 + \sqrt{2}\,\mathrm{csgn}(x)\right)\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 22

```
dsolve([(x+y(x))+(x-y(x))*diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

$$y(x) = x\left(1+\sqrt{2}\right)$$
$$y(x) = -x\left(\sqrt{2}-1\right)$$

✓ Solution by Mathematica

Time used: 0.482 (sec). Leaf size: 40

```
DSolve[{(x+y[x])+(x-y[x])*y'[x]==0,y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x-\sqrt{2}\sqrt{x^2}$$
$$y(x) \to \sqrt{2}\sqrt{x^2}+x$$

## 1.3 problem 3



Internal problem ID [3136]
Internal file name [`OUTPUT/2628_Sunday_June_05_2022_03_23_26_AM_80798607/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.1
**Problem number**: 3.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' \ln(x) + \frac{y + x}{x} = 0}$$

### 1.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x \ln(x)}$$
$$q(x) = -\frac{1}{\ln(x)}$$

Hence the ode is

$$y' + \frac{y}{\ln(x)\, x} = -\frac{1}{\ln(x)}$$

The integrating factor $\mu$ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x \ln(x)}dx} \\
&= \ln(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{1}{\ln(x)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)\, y) &= (\ln(x))\left(-\frac{1}{\ln(x)}\right) \\
\mathrm{d}(\ln(x)\, y) &= -1\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\ln(x)\, y &= \int -1\,\mathrm{d}x \\
\ln(x)\, y &= -x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \ln(x)$ results in

$$
y = -\frac{x}{\ln(x)} + \frac{c_1}{\ln(x)}
$$

which simplifies to

$$
y = \frac{-x + c_1}{\ln(x)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{-x + c_1}{\ln(x)} \tag{1}
$$

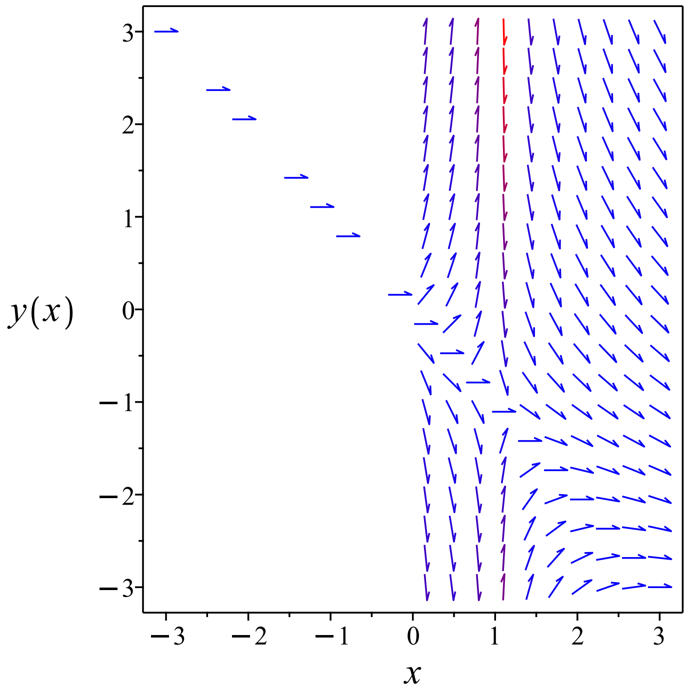


Figure 7: Slope field plot

Verification of solutions

$$y = \frac{-x + c_1}{\ln(x)}$$

Verified OK.

### 1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y + x}{x \ln(x)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 5: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | $0$ | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\,g(y)$ | $\frac{1}{f}$ | $0$ |
| quadrature ode | $y' = f(x)$ | $0$ | $1$ |
| quadrature ode | $y' = g(y)$ | $1$ | $0$ |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a + bx + cy)^{\frac{n}{m}}$ | $1$ | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\,e^{h(x)+by} + f(x)$ | $\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$ | $\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ | $\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$ | $\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$ |
| Bernoulli ode | $y' = f(x)\,y + g(x)\,y^n$ | $0$ | $e^{-\int (n-1)f(x)dx}y^n$ |
| Reduced Riccati | $y' = f_1(x)\,y + f_2(x)\,y^2$ | $0$ | $e^{-\int f_1 dx}$ |

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{\ln(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x,y) \to (R,S)$ where $(R,S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

$S$ is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \ln(x) y$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y + x}{x \ln(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x} \\ S_y &= \ln(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates $R, S$. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$y \ln(x) = -x + c_1$$

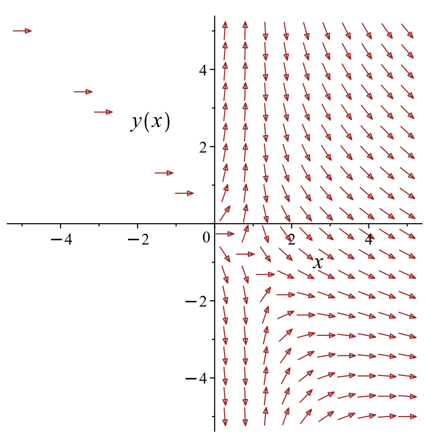
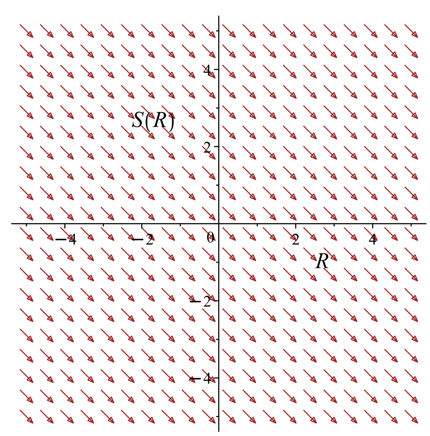
Which simplifies to

$$y \ln(x) = -x + c_1$$

Which gives

$$y = \frac{-x + c_1}{\ln(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = -\frac{y+x}{x \ln(x)}$   | $R = x$<br>$S = \ln(x)\, y$ | $\frac{dS}{dR} = -1$   |

Summary

The solution(s) found are the following

$$y = \frac{-x + c_1}{\ln(x)} \tag{1}$$

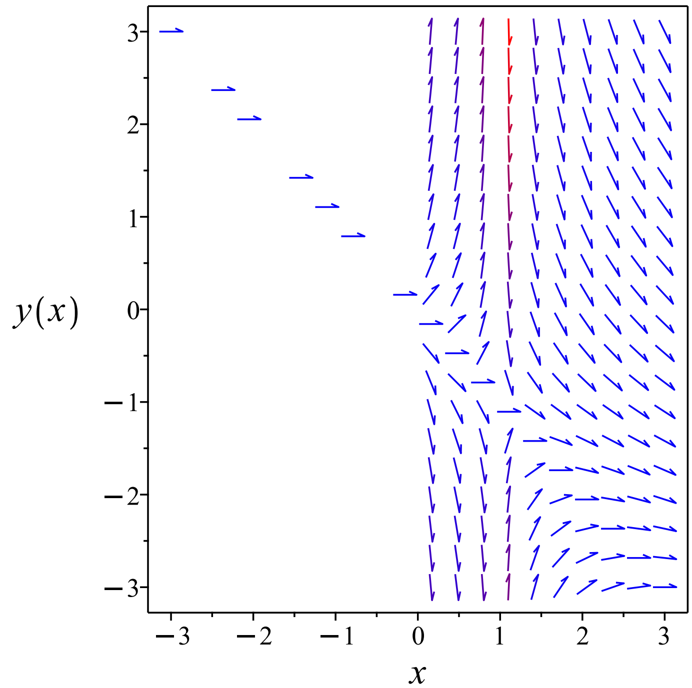


Figure 8: Slope field plot

Verification of solutions

$$y = \frac{-x + c_1}{\ln(x)}$$

Verified OK.

### 1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(\ln (x)) \,\mathrm{d}y &= \left(-\frac{y + x}{x}\right) \mathrm{d}x \\
\left(\frac{y + x}{x}\right) \mathrm{d}x + (\ln (x)) \,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \frac{y + x}{x} \\
N(x, y) &= \ln (x)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y + x}{x}\right) \\
&= \frac{1}{x}
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\ln(x)) \\ &= \frac{1}{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \frac{y + x}{x} \, dx$$

$$\phi = x + \ln(x)\, y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = \ln(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \ln(x)$. Therefore equation (4) becomes

$$\ln(x) = \ln(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where $c_1$ is constant of integration. Substituting this result for $f(y)$ into equation (3) gives $\phi$

$$\phi = x + \ln(x)\, y + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = x + \ln(x)\, y$$

The solution becomes

$$y = \frac{-x + c_1}{\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{-x + c_1}{\ln(x)} \tag{1}$$

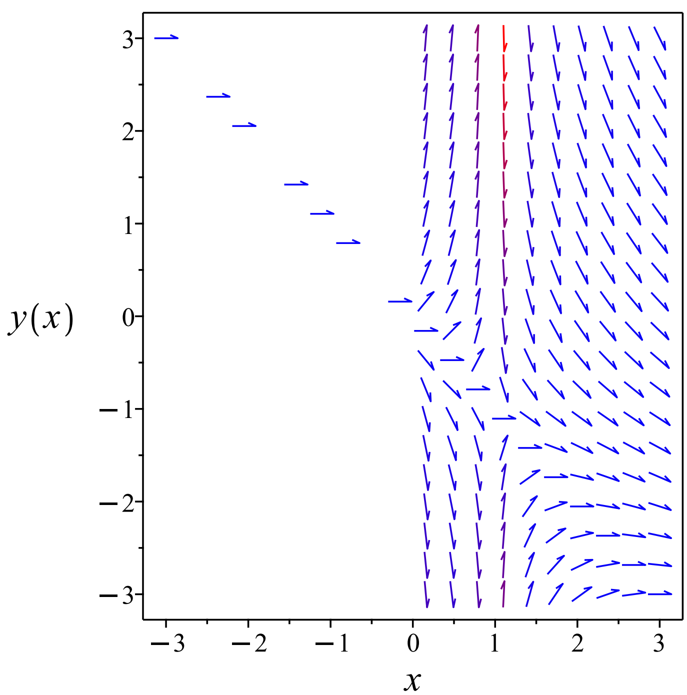



Figure 9: Slope field plot

Verification of solutions

$$y = \frac{-x + c_1}{\ln(x)}$$

Verified OK.

### 1.3.4 Maple step by step solution

Let's solve

$y' \ln(x) + \frac{y+x}{x} = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Isolate the derivative

  $y' = -\frac{y}{\ln(x) x} - \frac{1}{\ln(x)}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\ln(x) x} = -\frac{1}{\ln(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{\ln(x) x}\right) = -\frac{\mu(x)}{\ln(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{\ln(x) x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x\ln(x)}$

- Solve to find the integrating factor

  $\mu(x) = \ln(x)$

- Integrate both sides with respect to $x$

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\frac{\mu(x)}{\ln(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\frac{\mu(x)}{\ln(x)} dx + c_1$

- Solve for $y$

  $y = \frac{\int -\frac{\mu(x)}{\ln(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \ln(x)$

  $y = \frac{\int (-1) dx + c_1}{\ln(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{-x + c_1}{\ln(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(ln(x)*diff(y(x),x)+(x+y(x))/x=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 - x}{\ln(x)}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 16

```
DSolve[Log[x]*y'[x]+(x+y[x])/x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-x + c_1}{\log(x)}$$

## 1.4 problem 4



Internal problem ID [3137]
Internal file name [OUTPUT/2629_Sunday_June_05_2022_03_23_31_AM_42488425/index.tex]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.1
**Problem number**: 4.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact]
```

$$\boxed{\cos(y) - x\sin(y)\,y' = \sec(x)^2}$$

With initial conditions

$$[y(0) = 0]$$

### 1.4.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{-\sec(x)^2 + \cos(y)}{x\sin(y)} \end{aligned}$$

$f(x,y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

### 1.4.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x\sin(y))\,\mathrm{d}y&=\left(-\cos(y)+\sec(x)^2\right)\mathrm{d}x\\ \left(-\sec(x)^2+\cos(y)\right)\mathrm{d}x+(-x\sin(y))\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\sec(x)^2+\cos(y)\\ N(x,y)&=-x\sin(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\sec\left(x\right)^{2}+\cos\left(y\right)\right)\\ &= -\sin\left(y\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x\sin\left(y\right))\\ &= -\sin\left(y\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\sec\left(x\right)^{2}+\cos\left(y\right)\mathrm{d}x\end{aligned}$$

$$\phi = -\tan\left(x\right)+x\cos\left(y\right)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = -x\sin\left(y\right)+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x\sin\left(y\right)$. Therefore equation (4) becomes

$$-x\sin\left(y\right) = -x\sin\left(y\right)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where $c_1$ is constant of integration. Substituting this result for $f(y)$ into equation (3) gives $\phi$

$$\phi = -\tan(x) + x\cos(y) + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = -\tan(x) + x\cos(y)$$

Initial conditions are used to solve for $c_1$. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting $c_1$ found above in the general solution gives

$$-\tan(x) + x\cos(y) = 0$$

Solving for $y$ from the above gives

$$y = \arccos\left(\frac{\tan(x)}{x}\right)$$

Summary

The solution(s) found are the following

$$y = \arccos\left(\frac{\tan(x)}{x}\right) \tag{1}$$

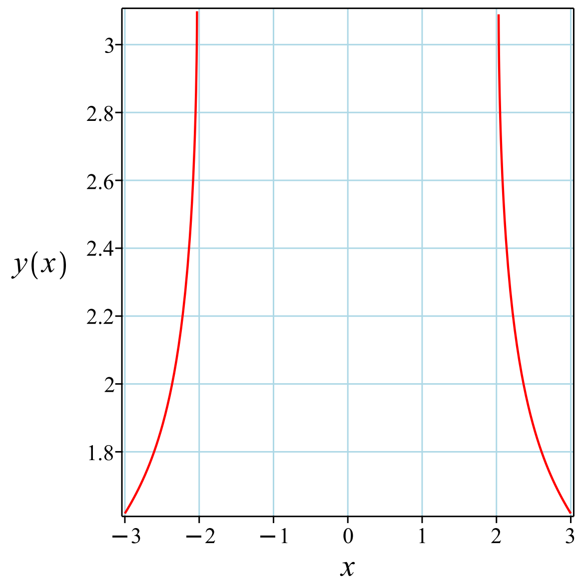



Figure 10: Solution plot

Verification of solutions

$$y = \arccos\left(\frac{\tan(x)}{x}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.985 (sec). Leaf size: 23

```
dsolve([cos(y(x))-x*sin(y(x))*diff(y(x),x)=sec(x)^2,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{\tan(x)}{x}\right)$$
$$y(x) = -\arccos\left(\frac{\tan(x)}{x}\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{Cos[y[x]]-x*Sin[y[x]]*y'[x]==Sec[x]^2,y[0]==0},y[x],x,IncludeSingularSolutions -> Tr
```

{}

## 1.5 problem 5



Internal problem ID [3138]
Internal file name [`OUTPUT/2630_Sunday_June_05_2022_03_23_38_AM_66712369/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.1
**Problem number**: 5.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{y\sin\left(\frac{x}{y}\right)+x\cos\left(\frac{x}{y}\right)+\left(x\sin\left(\frac{x}{y}\right)-\frac{x^2\cos\left(\frac{x}{y}\right)}{y}\right)y'=1}$$

### 1.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x \sin\left(\frac{x}{y}\right) - \frac{x^2 \cos\left(\frac{x}{y}\right)}{y}\right)\mathrm{d}y = \left(-y \sin\left(\frac{x}{y}\right) - x \cos\left(\frac{x}{y}\right) + 1\right)$$

$$\left(y \sin\left(\frac{x}{y}\right) + x \cos\left(\frac{x}{y}\right) - 1\right)\mathrm{d}x + \left(x \sin\left(\frac{x}{y}\right) - \frac{x^2 \cos\left(\frac{x}{y}\right)}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y \sin\left(\frac{x}{y}\right) + x \cos\left(\frac{x}{y}\right) - 1\\ N(x,y) &= x \sin\left(\frac{x}{y}\right) - \frac{x^2 \cos\left(\frac{x}{y}\right)}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y \sin\left(\frac{x}{y}\right) + x \cos\left(\frac{x}{y}\right) - 1\right)\\ &= \frac{(x^2+y^2)\sin\left(\frac{x}{y}\right) - \cos\left(\frac{x}{y}\right) xy}{y^2}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\sin\left(\frac{x}{y}\right) - \frac{x^2\cos\left(\frac{x}{y}\right)}{y}\right) \\
&= \frac{(x^2+y^2)\sin\left(\frac{x}{y}\right) - \cos\left(\frac{x}{y}\right)xy}{y^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. $x$ gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y\sin\left(\frac{x}{y}\right) + x\cos\left(\frac{x}{y}\right) - 1\,\mathrm{d}x \\
\phi &= x\left(y\sin\left(\frac{x}{y}\right) - 1\right) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$
\frac{\partial \phi}{\partial y} = x\left(\sin\left(\frac{x}{y}\right) - \frac{x\cos\left(\frac{x}{y}\right)}{y}\right) + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x\sin\left(\frac{x}{y}\right) - \frac{x^2\cos\left(\frac{x}{y}\right)}{y}$. Therefore equation (4) becomes

$$
x\sin\left(\frac{x}{y}\right) - \frac{x^2\cos\left(\frac{x}{y}\right)}{y} = x\left(\sin\left(\frac{x}{y}\right) - \frac{x\cos\left(\frac{x}{y}\right)}{y}\right) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where $c_1$ is constant of integration. Substituting this result for $f(y)$ into equation (3) gives $\phi$

$$\phi = x\left(y\sin\left(\frac{x}{y}\right) - 1\right) + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = x\left(y\sin\left(\frac{x}{y}\right) - 1\right)$$

Summary

The solution(s) found are the following

$$x\left(y\sin\left(\frac{x}{y}\right) - 1\right) = c_1 \tag{1}$$

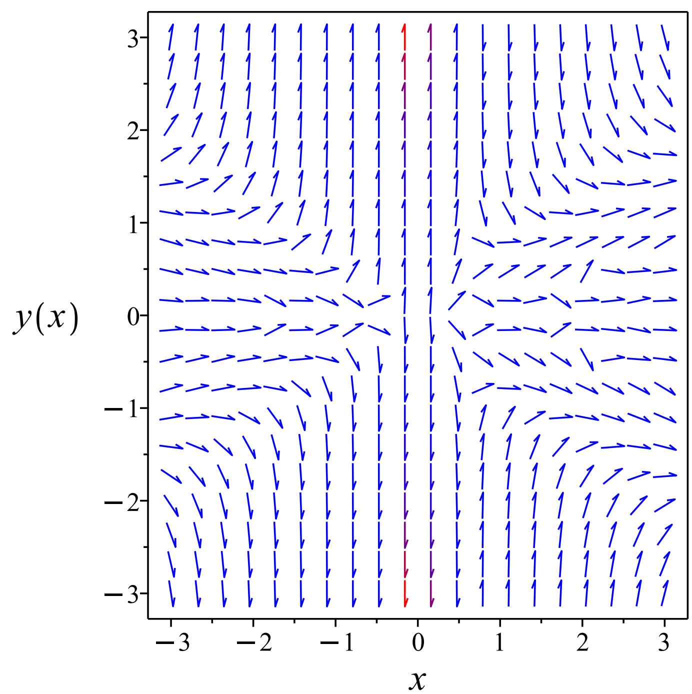



Figure 11: Slope field plot

Verification of solutions

$$x\left(y\sin\left(\frac{x}{y}\right)-1\right)=c_1$$

Verified OK.

### 1.5.2 Maple step by step solution

Let's solve

$y\sin\left(\frac{x}{y}\right)+x\cos\left(\frac{x}{y}\right)+\left(x\sin\left(\frac{x}{y}\right)-\frac{x^2\cos\left(\frac{x}{y}\right)}{y}\right)y'=1$

- Highest derivative means the order of the ODE is 1

  $y'$

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a $C^2$ function

    $F'(x,y)=0$
  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - Evaluate derivatives

    $\sin\left(\frac{x}{y}\right)-\frac{x\cos\left(\frac{x}{y}\right)}{y}+\frac{x^2\sin\left(\frac{x}{y}\right)}{y^2}=\sin\left(\frac{x}{y}\right)-\frac{x\cos\left(\frac{x}{y}\right)}{y}+\frac{x^2\sin\left(\frac{x}{y}\right)}{y^2}$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to $x$

  $F(x,y)=\int\left(y\sin\left(\frac{x}{y}\right)+x\cos\left(\frac{x}{y}\right)-1\right)dx+f_1(y)$
- Evaluate integral

  $F(x,y)=-x+y^2\left(\cos\left(\frac{x}{y}\right)+\frac{x\sin\left(\frac{x}{y}\right)}{y}\right)-\cos\left(\frac{x}{y}\right)y^2+f_1(y)$
- Take derivative of $F(x,y)$ with respect to $y$

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative

$$x\sin\left(\frac{x}{y}\right)-\frac{x^2\cos\left(\frac{x}{y}\right)}{y}=2y\left(\cos\left(\frac{x}{y}\right)+\frac{x\sin\left(\frac{x}{y}\right)}{y}\right)-\frac{x^2\cos\left(\frac{x}{y}\right)}{y}-x\sin\left(\frac{x}{y}\right)-2\cos\left(\frac{x}{y}\right)y+\frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y)=2x\sin\left(\frac{x}{y}\right)-2y\left(\cos\left(\frac{x}{y}\right)+\frac{x\sin\left(\frac{x}{y}\right)}{y}\right)+2\cos\left(\frac{x}{y}\right)y$$

- Solve for $f_1(y)$

$$f_1(y)=0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y)=-x+y^2\left(\cos\left(\frac{x}{y}\right)+\frac{x\sin\left(\frac{x}{y}\right)}{y}\right)-\cos\left(\frac{x}{y}\right)y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$-x+y^2\left(\cos\left(\frac{x}{y}\right)+\frac{x\sin\left(\frac{x}{y}\right)}{y}\right)-\cos\left(\frac{x}{y}\right)y^2=c_1$$

- Solve for $y$

$$y=\frac{x}{\mathit{RootOf}\left(-x^2\sin\left(\_Z\right)+c_1\_Z+\_Zx\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve((y(x)*sin(x/y(x))+x*cos(x/y(x))-1)+(x*sin(x/y(x))-x^2/y(x)*cos(x/y(x)))*diff(y(x),x)=
```

$$y(x) = \frac{x}{\text{RootOf}\left(x^2 \sin\left(\_Z\right) + \_Z c_1 - x\_Z\right)}$$

✓ Solution by Mathematica

Time used: 0.444 (sec). Leaf size: 20

```
DSolve[(y[x]*Sin[x/y[x]]+x*Cos[x/y[x]]-1)+(x*Sin[x/y[x]]-x^2/y[x]*Cos[x/y[x]])*y'[x]==0,y[x]
```

$$\text{Solve}\left[x - xy(x)\sin\left(\frac{x}{y(x)}\right) = c_1, y(x)\right]$$

## 1.6 problem 6



Internal problem ID [3139]
Internal file name [`OUTPUT/2631_Sunday_June_05_2022_03_23_42_AM_16016819/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.1
**Problem number**: 6.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{\frac{x}{y^2+x^2}+\frac{y}{x^2}+\left(\frac{y}{y^2+x^2}-\frac{1}{x}\right)y'=0}$$

With initial conditions

$$[y(1)=0]$$

### 1.6.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{x^3+x^2y+y^3}{x\,(x^2-xy+y^2)} \end{aligned}$$

The $x$ domain of $f(x,y)$ when $y=0$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0 = 1$ is inside this domain. The $y$ domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^3 + x^2 y + y^3}{x(x^2 - xy + y^2)}\right) \\ &= \frac{x^2 + 3y^2}{x(x^2 - xy + y^2)} - \frac{(x^3 + x^2 y + y^3)(-x + 2y)}{x(x^2 - xy + y^2)^2}\end{aligned}$$

The $x$ domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The $y$ domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

### 1.6.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{x}{u(x)^2 x^2 + x^2} + \frac{u(x)}{x} + \left(\frac{u(x)\,x}{u(x)^2 x^2 + x^2} - \frac{1}{x}\right)(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u^2 + 1}{(u^2 - u + 1)\,x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u^2+1}{u^2-u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u^2-u+1}}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+1}{u^2-u+1}}\,du = \int \frac{1}{x}\,dx$$

$$u - \frac{\ln(u^2 + 1)}{2} = \ln(x) + c_2$$

The solution is

$$u(x) - \frac{\ln\left(u(x)^2+1\right)}{2} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for $y$ in implicit form

$$\frac{y}{x} - \frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \ln(x) - c_2 = 0$$

$$\frac{y}{x} - \frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \ln(x) - c_2 = 0$$

Substituting initial conditions and solving for $c_2$ gives $c_2 = 0$. Hence the solution be-

Summary

The solution(s) found are the following

comes

$$\frac{y}{x} - \frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \ln(x) = 0 \tag{1}$$

Verification of solutions

$$\frac{y}{x} - \frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \ln(x) = 0$$

Verified OK.

### 1.6.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^3+x^2y+y^3}{x\left(x^2-xy+y^2\right)}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine $\xi, \eta$ then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and $\omega$ into (A) gives

$$\begin{aligned} &b_2 + \frac{(x^3+x^2y+y^3)(b_3-a_2)}{x(x^2-xy+y^2)} - \frac{(x^3+x^2y+y^3)^2 a_3}{x^2(x^2-xy+y^2)^2} - \left(\frac{3x^2+2xy}{x(x^2-xy+y^2)}\right. \\ &\left. - \frac{x^3+x^2y+y^3}{x^2(x^2-xy+y^2)} - \frac{(x^3+x^2y+y^3)(2x-y)}{x(x^2-xy+y^2)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{x^2+3y^2}{x(x^2-xy+y^2)} - \frac{(x^3+x^2y+y^3)(-x+2y)}{x(x^2-xy+y^2)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^6a_2 + x^6a_3 + x^6b_2 - x^6b_3 - 2x^5ya_2 + 2x^5ya_3 + 2x^5yb_3 + 2x^4y^2a_2 - x^4y^2a_3 - x^4y^2b_2 - 2x^4y^2b_3 + 4x^3y^3 \cdots}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^6a_2 - x^6a_3 - x^6b_2 + x^6b_3 + 2x^5ya_2 - 2x^5ya_3 - 2x^5yb_3 - 2x^4y^2a_2 + x^4y^2a_3 \\ &+ x^4y^2b_2 + 2x^4y^2b_3 - 4x^3y^3a_3 - x^2y^4a_2 + x^2y^4b_3 - 2x\,y^5a_3 - 2x^5b_1 + 2x^4ya_1 \\ &+2x^4yb_1 - 2x^3y^2a_1 - 2x^3y^2b_1 + 2x^2y^3a_1 + 2x^2y^3b_1 - 2x\,y^4a_1 - x\,y^4b_1 + y^5a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a_2v_1^6 + 2a_2v_1^5v_2 - 2a_2v_1^4v_2^2 - a_2v_1^2v_2^4 - a_3v_1^6 - 2a_3v_1^5v_2 + a_3v_1^4v_2^2 \\ &- 4a_3v_1^3v_2^3 - 2a_3v_1v_2^5 - b_2v_1^6 + b_2v_1^4v_2^2 + b_3v_1^6 - 2b_3v_1^5v_2 \\ &+ 2b_3v_1^4v_2^2 + b_3v_1^2v_2^4 + 2a_1v_1^4v_2 - 2a_1v_1^3v_2^2 + 2a_1v_1^2v_2^3 - 2a_1v_1v_2^4 \\ &+ a_1v_2^5 - 2b_1v_1^5 + 2b_1v_1^4v_2 - 2b_1v_1^3v_2^2 + 2b_1v_1^2v_2^3 - b_1v_1v_2^4 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms $v_i$ introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-a_2 - a_3 - b_2 + b_3) v_1^6 + (2a_2 - 2a_3 - 2b_3) v_1^5 v_2 - 2b_1 v_1^5 \\ &+(-2a_2 + a_3 + b_2 + 2b_3) v_1^4 v_2^2 + (2a_1 + 2b_1) v_1^4 v_2 - 4a_3 v_1^3 v_2^3 + (-2a_1 - 2b_1) v_1^3 v_2^2 \\ &+ (b_3 - a_2) v_1^2 v_2^4 + (2a_1 + 2b_1) v_1^2 v_2^3 - 2a_3 v_1 v_2^5 + (-2a_1 - b_1) v_1 v_2^4 + a_1 v_2^5 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -4a_3 &= 0 \\ -2a_3 &= 0 \\ -2b_1 &= 0 \\ -2a_1 - 2b_1 &= 0 \\ -2a_1 - b_1 &= 0 \\ 2a_1 + 2b_1 &= 0 \\ b_3 - a_2 &= 0 \\ 2a_2 - 2a_3 - 2b_3 &= 0 \\ -2a_2 + a_3 + b_2 + 2b_3 &= 0 \\ -a_2 - a_3 - b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x^3 + x^2 y + y^3}{x\,(x^2 - xy + y^2)}\right)(x) \\
&= \frac{-x^3 - x\,y^2}{x^2 - xy + y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x, y) \to (R, S)$ where $(R, S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

$S$ is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^3 - x\,y^2}{x^2 - xy + y^2}} dy
\end{aligned}$$

Which results in

$$S = -\frac{y}{x} + \frac{\ln(x^2 + y^2)}{2}$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^3 + x^2 y + y^3}{x\,(x^2 - xy + y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x^3 + x^2 y + y^3}{(x^2 + y^2)\, x^2} \\
S_y &= \frac{\frac{2yx}{x^2+y^2} - 2}{2x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates $R, S$. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

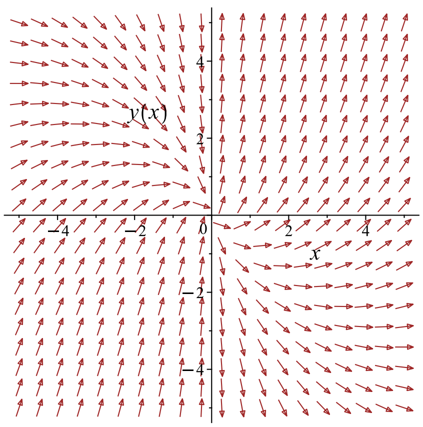
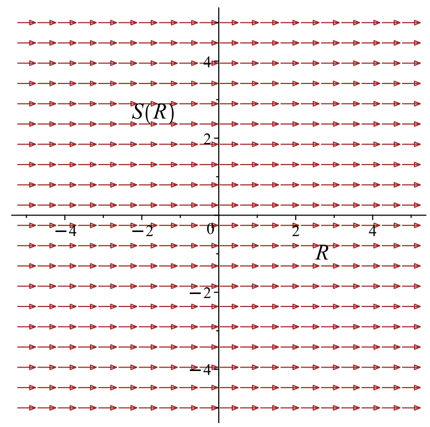
To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$
\frac{\ln(y^2 + x^2)\, x - 2y}{2x} = c_1
$$

Which simplifies to

$$
\frac{\ln(y^2 + x^2)\, x - 2y}{2x} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = \frac{x^3+x^2y+y^3}{x(x^2-xy+y^2)}$   | $R = x$ <br> $S = \frac{\ln(x^2+y^2)\,x - 2y}{2x}$ | $\frac{dS}{dR} = 0$   |

Initial conditions are used to solve for $c_1$. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting $c_1$ found above in the general solution gives

$$\frac{\ln(x^2+y^2)\,x - 2y}{2x} = 0$$

The above simplifies to

$$\ln\left(x^2+y^2\right)x - 2y = 0$$

Summary

The solution(s) found are the following

$$\ln\left(y^2+x^2\right)x - 2y = 0 \tag{1}$$

Verification of solutions

$$\ln\left(y^2+x^2\right)x - 2y = 0$$

Verified OK.

### 1.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{x^2+y^2}-\frac{1}{x}\right)\mathrm{d}y&=\left(-\frac{x}{x^2+y^2}-\frac{y}{x^2}\right)\mathrm{d}x\\ \left(\frac{x}{x^2+y^2}+\frac{y}{x^2}\right)\mathrm{d}x+\left(\frac{y}{x^2+y^2}-\frac{1}{x}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \frac{x}{x^2+y^2} + \frac{y}{x^2} \\
N(x,y) &= \frac{y}{x^2+y^2} - \frac{1}{x}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2} + \frac{y}{x^2}\right) \\
&= -\frac{2xy}{(x^2+y^2)^2} + \frac{1}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2} - \frac{1}{x}\right) \\
&= -\frac{2xy}{(x^2+y^2)^2} + \frac{1}{x^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x}{x^2+y^2} + \frac{y}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$\phi = -\frac{y}{x} + \frac{\ln(x^2+y^2)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2+y^2} - \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{x^2+y^2} - \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{y}{x^2+y^2} - \frac{1}{x} = \frac{y}{x^2+y^2} - \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where $c_1$ is constant of integration. Substituting this result for $f(y)$ into equation (3) gives $\phi$

$$\phi = -\frac{y}{x} + \frac{\ln(x^2+y^2)}{2} + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = -\frac{y}{x} + \frac{\ln(x^2+y^2)}{2}$$

Initial conditions are used to solve for $c_1$. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting $c_1$ found above in the general solution gives

$$-\frac{y}{x} + \frac{\ln(x^2+y^2)}{2} = 0$$

The above simplifies to

$$\ln\left(x^2+y^2\right)x-2y=0$$

Summary
The solution(s) found are the following

$$\ln\left(y^2+x^2\right)x-2y=0 \tag{1}$$

Verification of solutions

$$\ln\left(y^2+x^2\right)x-2y=0$$

Verified OK.

### 1.6.5 Maple step by step solution

Let's solve

$\left[\frac{x}{y^2+x^2}+\frac{y}{x^2}+\left(\frac{y}{y^2+x^2}-\frac{1}{x}\right)y'=0, y(1)=0\right]$

- Highest derivative means the order of the ODE is 1

  $y'$

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a $C^2$ function

    $F'(x,y)=0$
  - ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - ◦ Evaluate derivatives

    $-\frac{2xy}{(x^2+y^2)^2}+\frac{1}{x^2}=-\frac{2xy}{(x^2+y^2)^2}+\frac{1}{x^2}$
  - ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to $x$

  $F(x,y)=\int\left(\frac{x}{x^2+y^2}+\frac{y}{x^2}\right)dx+f_1(y)$

- Evaluate integral

$$F(x,y) = -\frac{y}{x} + \frac{\ln(x^2+y^2)}{2} + f_1(y)$$

- Take derivative of $F(x,y)$ with respect to $y$

$$N(x,y) = \frac{\partial}{\partial y}F(x,y)$$

- Compute derivative

$$\frac{y}{x^2+y^2} - \frac{1}{x} = -\frac{1}{x} + \frac{y}{x^2+y^2} + \frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = 0$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = -\frac{y}{x} + \frac{\ln(x^2+y^2)}{2}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$-\frac{y}{x} + \frac{\ln(x^2+y^2)}{2} = c_1$$

- Solve for $y$

$$y = \frac{x\left(-2c_1 + \mathit{RootOf}\left(4c_1^2x^2 - 4c_1x^2\_Z + x^2\_Z^2 + 4x^2 - 4\,\mathrm{e}^{\_Z}\right)\right)}{2}$$

- Use initial condition $y(1) = 0$

$$0 = -c_1 + \frac{\mathit{RootOf}\left(4c_1^2 - 4c_1\_Z + \_Z^2 + 4 - 4\,\mathrm{e}^{\_Z}\right)}{2}$$

- Solve for $c_1$

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \frac{\mathit{RootOf}\left(x^2\_Z^2 + 4x^2 - 4\,\mathrm{e}^{\_Z}\right)x}{2}$$

- Solution to the IVP

$$y = \frac{\mathit{RootOf}\left(x^2\_Z^2 + 4x^2 - 4\,\mathrm{e}^{\_Z}\right)x}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 33

```
dsolve([(x/(x^2+y(x)^2)+y(x)/x^2)+(y(x)/(x^2+y(x)^2)-1/x)*diff(y(x),x)=0,y(1) = 0],y(x), sin
```

$$y(x) = \frac{x\left(\operatorname{RootOf}\left(4+4\ln\left(x\right)^2+4\ln\left(x\right)\_Z+\_Z^2-4\,\mathrm{e}^{\_Z}\right)+2\ln\left(x\right)\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.175 (sec). Leaf size: 28

```
DSolve[{(x/(x^2+y[x]^2)+y[x]/x^2)+(y[x]/(x^2+y[x]^2)-1/x)*y'[x]==0,y[1]==0},y[x],x,IncludeSi
```

$$\text{Solve}\left[\frac{y(x)}{x}-\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right)=\log(x),y(x)\right]$$

# 2 Chapter 4, Ex. 4.2

## 2.1 problem 1



Internal problem ID [3140]
Internal file name [`OUTPUT/2632_Sunday_June_05_2022_08_37_44_AM_69209561/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.2
**Problem number**: 1.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2 (y^2+1) y'+y^2 (x^2+1)=0}$$

### 2.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{y^2(x^2+1)}{x^2(y^2+1)}
\end{aligned}$$

Where $f(x)=-\frac{x^2+1}{x^2}$ and $g(y)=\frac{y^2}{y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{y^2}{y^2+1}}\,dy = -\frac{x^2+1}{x^2}\,dx$$

$$\int \frac{1}{\frac{y^2}{y^2+1}}\,dy = \int -\frac{x^2+1}{x^2}\,dx$$

$$y-\frac{1}{y} = -x+\frac{1}{x}+c_1$$

Which results in

$$y = \frac{c_1 x - x^2 + 1 + \sqrt{c_1^2 x^2 - 2c_1 x^3 + x^4 + 2c_1 x + 2x^2 + 1}}{2x}$$
$$y = -\frac{-c_1 x + x^2 + \sqrt{c_1^2 x^2 - 2c_1 x^3 + x^4 + 2c_1 x + 2x^2 + 1} - 1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - x^2 + 1 + \sqrt{c_1^2 x^2 - 2c_1 x^3 + x^4 + 2c_1 x + 2x^2 + 1}}{2x} \tag{1}$$
$$y = -\frac{-c_1 x + x^2 + \sqrt{c_1^2 x^2 - 2c_1 x^3 + x^4 + 2c_1 x + 2x^2 + 1} - 1}{2x} \tag{2}$$

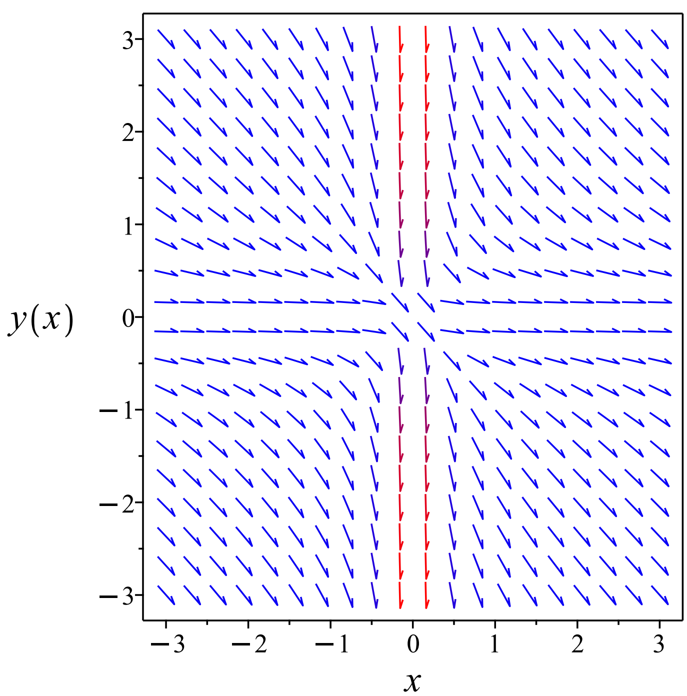



Figure 12: Slope field plot

Verification of solutions

$$y = \frac{c_1 x - x^2 + 1 + \sqrt{c_1^2 x^2 - 2c_1 x^3 + x^4 + 2c_1 x + 2x^2 + 1}}{2x}$$

Verified OK.

$$y = -\frac{-c_1 x + x^2 + \sqrt{c_1^2 x^2 - 2c_1 x^3 + x^4 + 2c_1 x + 2x^2 + 1} - 1}{2x}$$

Verified OK.

### 2.1.2 Maple step by step solution

Let's solve

$x^2(y^2+1)\, y' + y^2(x^2+1) = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'(y^2+1)}{y^2} = -\frac{x^2+1}{x^2}$

- Integrate both sides with respect to $x$

  $\int \frac{y'(y^2+1)}{y^2} dx = \int -\frac{x^2+1}{x^2} dx + c_1$

- Evaluate integral

  $y - \frac{1}{y} = -x + \frac{1}{x} + c_1$

- Solve for $y$

  $\left\{y = \frac{c_1 x - x^2 + 1 + \sqrt{c_1^2 x^2 - 2c_1 x^3 + x^4 + 2c_1 x + 2x^2 + 1}}{2x}, y = -\frac{-c_1 x + x^2 + \sqrt{c_1^2 x^2 - 2c_1 x^3 + x^4 + 2c_1 x + 2x^2 + 1} - 1}{2x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 93

```
dsolve(x^2*(1+y(x)^2)*diff(y(x),x)+y(x)^2*(x^2+1)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x^2 - c_1 x + \sqrt{1 + x^4 + 2c_1 x^3 + (c_1^2 + 2)\, x^2 - 2c_1 x + 1}}{2x}$$
$$y(x) = \frac{-x^2 - c_1 x - \sqrt{1 + x^4 + 2c_1 x^3 + (c_1^2 + 2)\, x^2 - 2c_1 x + 1}}{2x}$$

✓ Solution by Mathematica

Time used: 1.162 (sec). Leaf size: 95

```
DSolve[x^2*(1+y[x]^2)*y'[x]+y[x]^2*(x^2+1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2 + \sqrt{4x^2 + (-x^2 + c_1 x + 1)^2} - c_1 x - 1}{2x}$$
$$y(x) \to \frac{-x^2 + \sqrt{4x^2 + (-x^2 + c_1 x + 1)^2} + c_1 x + 1}{2x}$$
$$y(x) \to 0$$

## 2.2 problem 2



Internal problem ID [3141]
Internal file name [`OUTPUT/2633_Sunday_June_05_2022_08_37_45_AM_26296455/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.2
**Problem number**: 2.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x(x-1)\,y'-\cot(y)=0}$$

### 2.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{\cot(y)}{x(x-1)} \end{aligned}$$

Where $f(x)=\frac{1}{x(x-1)}$ and $g(y)=\cot(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cot(y)}\,dy &= \frac{1}{x(x-1)}\,dx\\ \int\frac{1}{\cot(y)}\,dy &= \int\frac{1}{x(x-1)}\,dx\\ -\ln(\cos(y)) &= \ln(x-1)-\ln(x)+c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = e^{\ln(x-1)-\ln(x)+c_1}$$

Which simplifies to

$$\sec(y) = c_2 e^{\ln(x-1)-\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \operatorname{arcsec}\left(\frac{c_2 e^{c_1}(x-1)}{x}\right) \tag{1}$$

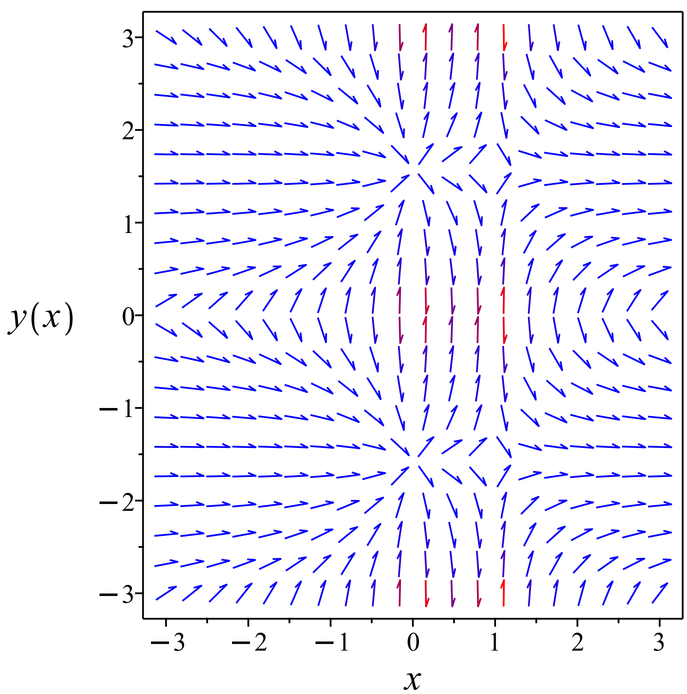



Figure 13: Slope field plot

Verification of solutions

$$y = \operatorname{arcsec}\left(\frac{c_2 e^{c_1}(x-1)}{x}\right)$$

Verified OK.

### 2.2.2 Maple step by step solution

Let's solve

$x(x-1)\,y' - \cot(y) = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'}{\cot(y)} = \frac{1}{x(x-1)}$

- Integrate both sides with respect to $x$

  $\int \frac{y'}{\cot(y)}dx = \int \frac{1}{x(x-1)}dx + c_1$

- Evaluate integral

  $-\ln(\cos(y)) = \ln(x-1) - \ln(x) + c_1$

- Solve for $y$

  $y = \arccos\left(\frac{x}{e^{c_1}(x-1)}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(x*(x-1)*diff(y(x),x)=cot(y(x)),y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{x}{c_1(x-1)}\right)$$

✓ Solution by Mathematica

Time used: 52.823 (sec). Leaf size: 59

```
DSolve[x*(x-1)*y'[x]==Cot[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arccos\left(-\frac{e^{-c_1}x}{x-1}\right)$$
$$y(x) \to \arccos\left(-\frac{e^{-c_1}x}{x-1}\right)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

## 2.3 problem 3



Internal problem ID [3142]
Internal file name [`OUTPUT/2634_Sunday_June_05_2022_08_37_47_AM_71070688/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.2
**Problem number**: 3.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{ry' - \frac{(a^2 - r^2)\tan(y)}{a^2 + r^2} = 0}$$

### 2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(r, y) \\ &= f(r)g(y) \\ &= \frac{(a^2 - r^2)\tan(y)}{(a^2 + r^2)\, r} \end{aligned}$$

Where $f(r) = \frac{a^2 - r^2}{(a^2 + r^2)r}$ and $g(y) = \tan(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\tan(y)}\, dy &= \frac{a^2 - r^2}{(a^2 + r^2)\, r}\, dr \\ \int \frac{1}{\tan(y)}\, dy &= \int \frac{a^2 - r^2}{(a^2 + r^2)\, r}\, dr \\ \ln(\sin(y)) &= -\ln\left(a^2 + r^2\right) + \ln(r) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sin(y) = e^{-\ln(a^2+r^2)+\ln(r)+c_1}$$

Which simplifies to

$$\sin(y) = c_2 e^{-\ln(a^2+r^2)+\ln(r)}$$

Summary

The solution(s) found are the following

$$y = \arcsin\left(\frac{c_2 e^{c_1} r}{a^2+r^2}\right) \tag{1}$$

Verification of solutions

$$y = \arcsin\left(\frac{c_2 e^{c_1} r}{a^2+r^2}\right)$$

Verified OK.

### 2.3.2 Maple step by step solution

Let's solve

$r y' - \frac{(a^2-r^2)\tan(y)}{a^2+r^2} = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'}{\tan(y)} = \frac{a^2-r^2}{(a^2+r^2)r}$

- Integrate both sides with respect to $r$

  $\int \frac{y'}{\tan(y)} dr = \int \frac{a^2-r^2}{(a^2+r^2)r} dr + c_1$

- Evaluate integral

  $\ln(\sin(y)) = -\ln(a^2+r^2) + \ln(r) + c_1$

- Solve for $y$

  $y = \arcsin\left(\frac{e^{c_1} r}{a^2+r^2}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(r*diff(y(r),r)= (a^2-r^2)/(a^2+r^2)*tan(y(r)),y(r), singsol=all)
```

$$y(r) = \arcsin\left(\frac{rc_1}{a^2 + r^2}\right)$$

✓ Solution by Mathematica

Time used: 23.337 (sec). Leaf size: 26

```
DSolve[r*y'[r]== (a^2-r^2)/(a^2+r^2)*Tan[y[r]],y[r],r,IncludeSingularSolutions -> True]
```

$$y(r) \to \arcsin\left(\frac{e^{c_1}r}{a^2 + r^2}\right)$$
$$y(r) \to 0$$

## 2.4 problem 4



Internal problem ID [3143]
Internal file name [OUTPUT/2635_Sunday_June_05_2022_08_37_48_AM_26764279/index.tex]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.2
**Problem number**: 4.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sqrt{x^2+1}\,y'+\sqrt{y^2+1}=0}$$

### 2.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= -\frac{\sqrt{y^2+1}}{\sqrt{x^2+1}}\end{aligned}$$

Where $f(x)=-\frac{1}{\sqrt{x^2+1}}$ and $g(y)=\sqrt{y^2+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{y^2+1}}\,dy &= -\frac{1}{\sqrt{x^2+1}}\,dx\\ \int\frac{1}{\sqrt{y^2+1}}\,dy &= \int-\frac{1}{\sqrt{x^2+1}}\,dx\\ \operatorname{arcsinh}(y) &= -\operatorname{arcsinh}(x)+c_1\end{aligned}$$

Which results in

$$y = \sinh(-\operatorname{arcsinh}(x) + c_1)$$

Summary
The solution(s) found are the following

$$y = \sinh(-\operatorname{arcsinh}(x) + c_1) \tag{1}$$

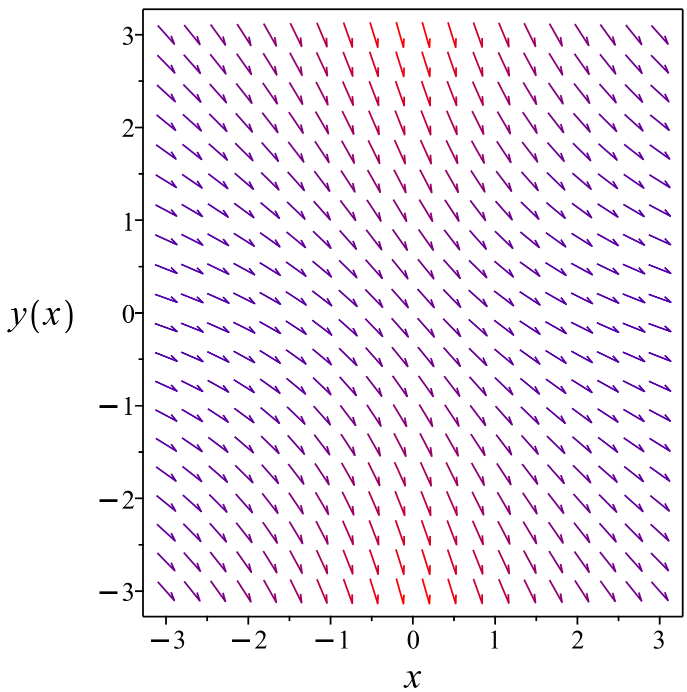



Figure 14: Slope field plot

Verification of solutions

$$y = \sinh(-\operatorname{arcsinh}(x) + c_1)$$

Verified OK.

### 2.4.2 Maple step by step solution

Let's solve

$\sqrt{x^2+1}\, y' + \sqrt{y^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'}{\sqrt{y^2+1}} = -\frac{1}{\sqrt{x^2+1}}$

- Integrate both sides with respect to $x$

  $\int \frac{y'}{\sqrt{y^2+1}} dx = \int -\frac{1}{\sqrt{x^2+1}} dx + c_1$

- Evaluate integral

  $\operatorname{arcsinh}(y) = -\operatorname{arcsinh}(x) + c_1$

- Solve for $y$

  $y = \sinh\left(-\operatorname{arcsinh}(x) + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(sqrt(1+x^2)*diff(y(x),x)+sqrt(1+y(x)^2)=0,y(x), singsol=all)
```

$$y(x) = -\sinh\left(\operatorname{arcsinh}(x) + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.349 (sec). Leaf size: 59

```
DSolve[Sqrt[1+x^2]*y'[x]+Sqrt[1+y[x]^2]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{1}{2}e^{-c_1}\left(\left(-1+e^{2c_1}\right)\sqrt{x^2+1}-\left(1+e^{2c_1}\right)x\right)$$
$$y(x)\to -i$$
$$y(x)\to i$$

## 2.5 problem 5



Internal problem ID [3144]
Internal file name [`OUTPUT/2636_Sunday_June_05_2022_08_37_49_AM_16493042/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.2
**Problem number**: 5.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x(y^2+1)}{y\,(x^2+1)} = 0}$$

With initial conditions

$$[y(0) = 1]$$

### 2.5.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{x(y^2+1)}{y\,(x^2+1)} \end{aligned}$$

The $x$ domain of $f(x,y)$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The $y$ domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x(y^2+1)}{y\,(x^2+1)}\right) \\
&= \frac{2x}{x^2+1} - \frac{x(y^2+1)}{y^2\,(x^2+1)}
\end{aligned}$$

The $x$ domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The $y$ domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

### 2.5.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{x(y^2+1)}{y\,(x^2+1)}
\end{aligned}$$

Where $f(x) = \frac{x}{x^2+1}$ and $g(y) = \frac{y^2+1}{y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{y^2+1}{y}}\,dy &= \frac{x}{x^2+1}\,dx \\
\int \frac{1}{\frac{y^2+1}{y}}\,dy &= \int \frac{x}{x^2+1}\,dx \\
\frac{\ln\left(y^2+1\right)}{2} &= \frac{\ln\left(x^2+1\right)}{2} + c_1
\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = e^{\frac{\ln\left(x^2+1\right)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = c_2\sqrt{x^2+1}$$

Which can be simplified to become

$$\sqrt{y^2+1} = c_2 e^{c_1}\sqrt{x^2+1}$$

The solution is

$$\sqrt{y^2+1} = c_2 e^{c_1}\sqrt{x^2+1}$$

Initial conditions are used to solve for $c_1$. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\sqrt{2} = e^{c_1} c_2$$

$$c_1 = \frac{\ln\left(\frac{2}{c_2^2}\right)}{2}$$

Substituting $c_1$ found above in the general solution gives

$$\sqrt{y^2+1} = \sqrt{2}\, c_2 \sqrt{\frac{1}{c_2^2}}\, \sqrt{x^2+1}$$

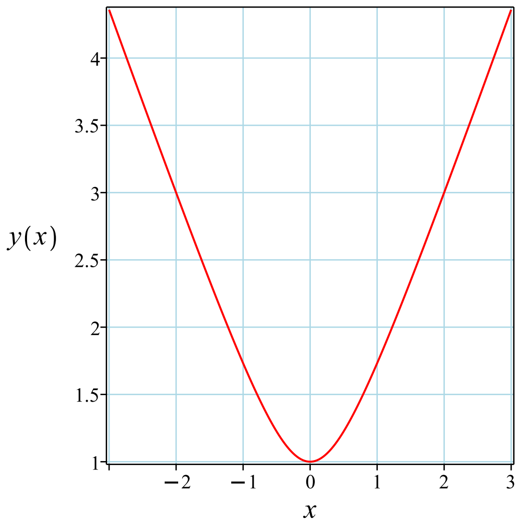
Solving for $y$ from the above gives

$$y = \sqrt{2x^2+1}$$

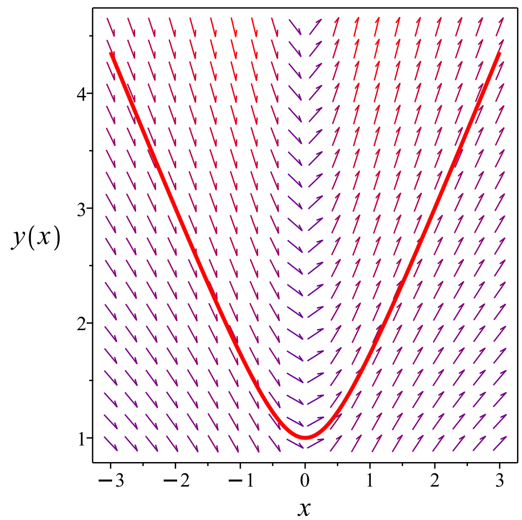
<ins>Summary</ins>

The solution(s) found are the following

$$y = \sqrt{2x^2+1} \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = \sqrt{2x^2 + 1}$$

Verified OK. `{positive}`

### 2.5.3 Maple step by step solution

Let's solve

$\left[y' - \frac{x(y^2+1)}{y(x^2+1)} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'y}{y^2+1} = \frac{x}{x^2+1}$

- Integrate both sides with respect to $x$

  $\int \frac{y'y}{y^2+1}dx = \int \frac{x}{x^2+1}dx + c_1$

- Evaluate integral

  $\frac{\ln(y^2+1)}{2} = \frac{\ln(x^2+1)}{2} + c_1$

- Solve for $y$

$$\left\{y = \frac{\sqrt{-\mathrm{e}^{-2c_1}\left(-x^2+\mathrm{e}^{-2c_1}-1\right)}}{\mathrm{e}^{-2c_1}}, y = -\frac{\sqrt{-\mathrm{e}^{-2c_1}\left(-x^2+\mathrm{e}^{-2c_1}-1\right)}}{\mathrm{e}^{-2c_1}}\right\}$$

- Use initial condition $y(0) = 1$

  $1 = \frac{\sqrt{-\mathrm{e}^{-2c_1}\left(\mathrm{e}^{-2c_1}-1\right)}}{\mathrm{e}^{-2c_1}}$

- Solve for $c_1$

  $c_1 = \frac{\ln(2)}{2}$

- Substitute $c_1 = \frac{\ln(2)}{2}$ into general solution and simplify

  $y = \sqrt{2x^2+1}$

- Use initial condition $y(0) = 1$

  $1 = -\frac{\sqrt{-\mathrm{e}^{-2c_1}\left(\mathrm{e}^{-2c_1}-1\right)}}{\mathrm{e}^{-2c_1}}$

- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \sqrt{2x^2+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=(x*(1+y(x)^2))/(y(x)*(1+x^2)),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \sqrt{2x^2+1}$$

✓ Solution by Mathematica

Time used: 0.549 (sec). Leaf size: 16

```
DSolve[{y'[x]==(x*(1+y[x]^2))/(y[x]*(1+x^2)),y[0]==1},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \sqrt{2x^2 + 1}$$

## 2.6 problem 6



Internal problem ID [3145]
Internal file name [`OUTPUT/2637_Sunday_June_05_2022_08_37_50_AM_9569087/index.tex`]

**Book**: An introduction to the solution and applications of differential equations, J.W. Searl, 1966
**Section**: Chapter 4, Ex. 4.2
**Problem number**: 6.
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y^2y' - 3y^6 = 2}$$

With initial conditions

$$[y(0) = 0]$$

### 2.6.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{3y^6 + 2}{y^2} \end{aligned}$$

The $y$ domain of $f(x, y)$ when $x = 0$ is

$$\{y < 0 \lor 0 < y\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

### 2.6.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{3y^6+2}{y^2}
\end{aligned}
$$

Where $f(x) = 1$ and $g(y) = \frac{3y^6+2}{y^2}$. Integrating both sides gives

$$
\frac{1}{\frac{3y^6+2}{y^2}}\,dy = 1\,dx
$$

$$
\int \frac{1}{\frac{3y^6+2}{y^2}}\,dy = \int 1\,dx
$$

$$
\frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,y^3}{2}\right)}{18} = x + c_1
$$

Which results in

$$
\begin{aligned}
y &= \frac{9^{\frac{1}{3}}\left(\sqrt{6}\,\tan\left(3(x+c_1)\sqrt{6}\right)\right)^{\frac{1}{3}}}{3} \\
y &= -\frac{9^{\frac{1}{3}}\left(\sqrt{6}\,\tan\left(3(x+c_1)\sqrt{6}\right)\right)^{\frac{1}{3}}}{6} + \frac{i\sqrt{3}\,9^{\frac{1}{3}}\left(\sqrt{6}\,\tan\left(3(x+c_1)\sqrt{6}\right)\right)^{\frac{1}{3}}}{6} \\
y &= -\frac{9^{\frac{1}{3}}\left(\sqrt{6}\,\tan\left(3(x+c_1)\sqrt{6}\right)\right)^{\frac{1}{3}}}{6} - \frac{i\sqrt{3}\,9^{\frac{1}{3}}\left(\sqrt{6}\,\tan\left(3(x+c_1)\sqrt{6}\right)\right)^{\frac{1}{3}}}{6}
\end{aligned}
$$

Initial conditions are used to solve for $c_1$. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = -\frac{i3^{\frac{1}{6}}\tan\left(3c_1\sqrt{6}\right)^{\frac{1}{3}}6^{\frac{1}{6}}}{2} - \frac{\tan\left(3c_1\sqrt{6}\right)^{\frac{1}{3}}3^{\frac{2}{3}}6^{\frac{1}{6}}}{6}
$$

$$
c_1 = 0
$$

Substituting $c_1$ found above in the general solution gives

$$
y = -\frac{i3^{\frac{1}{6}}6^{\frac{1}{6}}\tan\left(3\sqrt{6}\,x\right)^{\frac{1}{3}}}{2} - \frac{6^{\frac{1}{6}}3^{\frac{2}{3}}\tan\left(3\sqrt{6}\,x\right)^{\frac{1}{3}}}{6}
$$

Initial conditions are used to solve for $c_1$. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{i 3^{\frac{1}{6}} \tan\left(3 c_1 \sqrt{6}\right)^{\frac{1}{3}} 6^{\frac{1}{6}}}{2} - \frac{\tan\left(3 c_1 \sqrt{6}\right)^{\frac{1}{3}} 3^{\frac{2}{3}} 6^{\frac{1}{6}}}{6}$$

$$c_1 = 0$$

Substituting $c_1$ found above in the general solution gives

$$y = \frac{i 3^{\frac{1}{6}} 6^{\frac{1}{6}} \tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}}}{2} - \frac{6^{\frac{1}{6}} 3^{\frac{2}{3}} \tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}}}{6}$$

Initial conditions are used to solve for $c_1$. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\tan\left(3 c_1 \sqrt{6}\right)^{\frac{1}{3}} 6^{\frac{1}{6}} 9^{\frac{1}{3}}}{3}$$

$$c_1 = 0$$

Substituting $c_1$ found above in the general solution gives

$$y = \frac{\tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}} 6^{\frac{1}{6}} 9^{\frac{1}{3}}}{3}$$

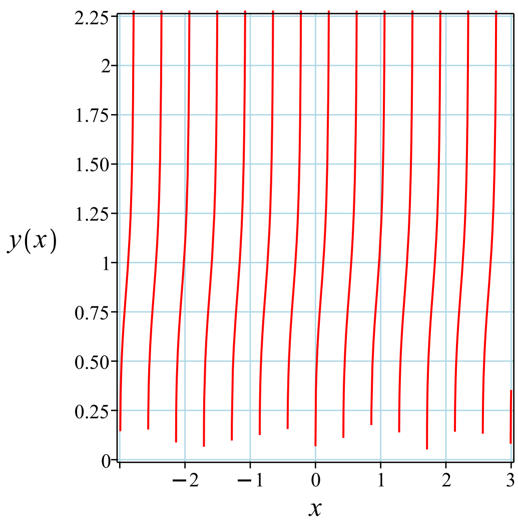
<u>Summary</u>

The solution(s) found are the following

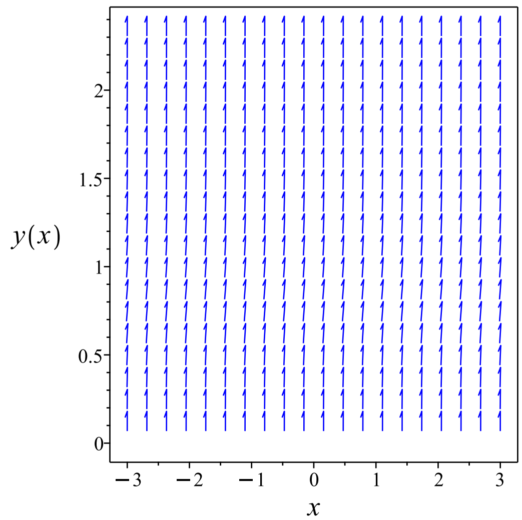
$$y = \frac{\tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}} 6^{\frac{1}{6}} 9^{\frac{1}{3}}}{3} \tag{1}$$

$$y = \frac{i 3^{\frac{1}{6}} 6^{\frac{1}{6}} \tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}}}{2} - \frac{6^{\frac{1}{6}} 3^{\frac{2}{3}} \tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}}}{6} \tag{2}$$

$$y = -\frac{i 3^{\frac{1}{6}} 6^{\frac{1}{6}} \tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}}}{2} - \frac{6^{\frac{1}{6}} 3^{\frac{2}{3}} \tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}}}{6} \tag{3}$$




(a) Solution plot




(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\tan\left(3\sqrt{6}\,x\right)^{\frac{1}{3}} 6^{\frac{1}{6}} 9^{\frac{1}{3}}}{3}$$

Verified OK.

$$y = \frac{i 3^{\frac{1}{6}} 6^{\frac{1}{6}} \tan\left(3\sqrt{6}\,x\right)^{\frac{1}{3}}}{2} - \frac{6^{\frac{1}{6}} 3^{\frac{2}{3}} \tan\left(3\sqrt{6}\,x\right)^{\frac{1}{3}}}{6}$$

Verified OK.

$$y = -\frac{i 3^{\frac{1}{6}} 6^{\frac{1}{6}} \tan\left(3\sqrt{6}\,x\right)^{\frac{1}{3}}}{2} - \frac{6^{\frac{1}{6}} 3^{\frac{2}{3}} \tan\left(3\sqrt{6}\,x\right)^{\frac{1}{3}}}{6}$$

Verified OK.

### 2.6.3 Maple step by step solution

Let's solve

$[y^2 y' - 3y^6 = 2, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

$$\frac{y'y^2}{2+3y^6} = 1$$

- Integrate both sides with respect to $x$

$$\int \frac{y'y^2}{2+3y^6}dx = \int 1dx + c_1$$

- Evaluate integral

$$\frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,y^3}{2}\right)}{18} = x + c_1$$

- Solve for $y$

$$y = \frac{9^{\frac{1}{3}}\left(\sqrt{6}\,\tan\left(3(x+c_1)\sqrt{6}\right)\right)^{\frac{1}{3}}}{3}$$

- Use initial condition $y(0) = 0$

$$0 = \frac{9^{\frac{1}{3}}\left(\tan\left(3c_1\sqrt{6}\right)\sqrt{6}\right)^{\frac{1}{3}}}{3}$$

- Solve for $c_1$

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \frac{3^{\frac{5}{6}}2^{\frac{1}{6}}\tan\left(3\sqrt{6}\,x\right)^{\frac{1}{3}}}{3}$$

- Solution to the IVP

$$y = \frac{3^{\frac{5}{6}}2^{\frac{1}{6}}\tan\left(3\sqrt{6}\,x\right)^{\frac{1}{3}}}{3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.235 (sec). Leaf size: 77

```
dsolve([y(x)^2*diff(y(x),x)=2+3*y(x)^6,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{3^{\frac{5}{6}} 2^{\frac{1}{6}} \tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}}}{3}$$
$$y(x) = \frac{\tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}} \left(3i3^{\frac{1}{6}} - 3^{\frac{2}{3}}\right) 6^{\frac{1}{6}}}{6}$$
$$y(x) = -\frac{\tan\left(3\sqrt{6}\, x\right)^{\frac{1}{3}} \left(3i3^{\frac{1}{6}} + 3^{\frac{2}{3}}\right) 6^{\frac{1}{6}}}{6}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 87

```
DSolve[{y[x]^2*y'[x]==2+3*y[x]^6,y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[6]{\frac{2}{3}}\sqrt[3]{\tan\left(3\sqrt{6}x\right)}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[6]{\frac{2}{3}}\sqrt[3]{\tan\left(3\sqrt{6}x\right)}$$
$$y(x) \to (-1)^{2/3}\sqrt[6]{\frac{2}{3}}\sqrt[3]{\tan\left(3\sqrt{6}x\right)}$$