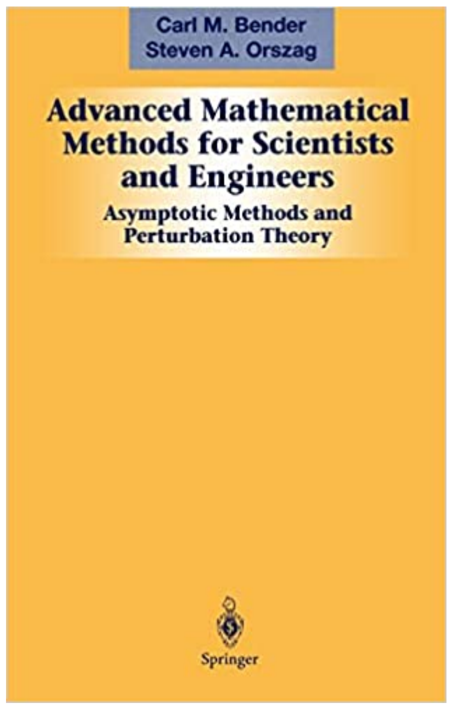
A Solution Manual For

Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999



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May 15, 2024

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1.1 problem 3.5

Internal problem ID [5480]
Internal file name [`OUTPUT/4728_Sunday_June_05_2022_03_04_13_PM_17619304/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x-1)(-2+x)\,y''+(4x-6)\,y'+2y=0}$$

With initial conditions

$$[y(0)=2, y'(0)=1]$$

With the expansion point for the power series method at $x=0$.

1.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{4x-6}{x^2-3x+2} \\ q(x) &= \frac{2}{x^2-3x+2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(4x-6)\,y'}{x^2-3x+2} + \frac{2y}{x^2-3x+2} = 0$$

The domain of $p(x) = \frac{4x-6}{x^2-3x+2}$ is

$$\{-\infty \le x < 1, 1 < x < 2, 2 < x \le \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{x^2-3x+2}$ is

$$\{-\infty \le x < 1, 1 < x < 2, 2 < x \le \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2}f|_{x_0,y_0,y'_0} + \frac{x^3}{3!}f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{2}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(2xy'-3y'+y)}{x^2-3x+2}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
&= \frac{\left(18x^2-54x+42\right)y' + (12x-18)\,y}{\left(x^2-3x+2\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1\\
&= \frac{\left(-96x^3+432x^2-672x+360\right)y' - 72y\left(x^2-3x+\frac{7}{3}\right)}{\left(x^2-3x+2\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2\\
&= \frac{\left(600x^4-3600x^3+8400x^2-9000x+3720\right)y' + 480y\left(x-\frac{3}{2}\right)\left(x^2-3x+\frac{5}{2}\right)}{\left(x^2-3x+2\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3\\
&= \frac{-4320\left(x-\frac{3}{2}\right)\left(x^2-3x+3\right)\left(x^2-3x+\frac{7}{3}\right)y' - 3600y\left(x^4-6x^3+14x^2-15x+\frac{31}{5}\right)}{\left(x^2-3x+2\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=2$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= 1\\
F_1 &= \frac{3}{2}\\
F_2 &= 3\\
F_3 &= \frac{15}{2}\\
F_4 &= \frac{45}{2}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = x + 2 + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} + \frac{x^5}{16} + \frac{x^6}{32} + O(x^6)$$

$$y = x + 2 + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} + \frac{x^5}{16} + \frac{x^6}{32} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$y''(x^2 - 3x + 2) + (4x - 6)\, y' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)(x^2 - 3x + 2) + (4x - 6)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-3n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2}\right) \tag{2}$$
$$+ \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \sum_{n=1}^{\infty} \left(-6n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-3n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-3(n+1)\, a_{n+1} n\, x^n\right)$$

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-6n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-6(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-3(n+1)\, a_{n+1} n\, x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) \\ &+ \sum_{n=0}^{\infty} \left(-6(n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \end{aligned} \tag{3}$$

$n = 0$ gives

$$4a_2 - 6a_1 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{2} + \frac{3a_1}{2}$$

$n = 1$ gives

$$-18a_2 + 12a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{3a_0}{4} + \frac{7a_1}{4}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) - 3(n+1) a_{n+1} n + 2(n+2) a_{n+2} (n+1) + 4 n a_n - 6(n+1) a_{n+1} + 2 a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{a_n}{2} + \frac{3 a_{n+1}}{2} \\ &= -\frac{a_n}{2} + \frac{3 a_{n+1}}{2} \end{aligned} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$12 a_2 - 36 a_3 + 24 a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{7 a_0}{8} + \frac{15 a_1}{8}$$

For $n = 3$ the recurrence equation gives

$$20 a_3 - 60 a_4 + 40 a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{15 a_0}{16} + \frac{31 a_1}{16}$$

For $n = 4$ the recurrence equation gives

$$30 a_4 - 90 a_5 + 60 a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{31 a_0}{32} + \frac{63 a_1}{32}$$

For $n = 5$ the recurrence equation gives

$$42 a_5 - 126 a_6 + 84 a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{63a_0}{64} + \frac{127a_1}{64}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x &+ \left(-\frac{a_0}{2} + \frac{3a_1}{2}\right) x^2 + \left(-\frac{3a_0}{4} + \frac{7a_1}{4}\right) x^3 \\ &+ \left(-\frac{7a_0}{8} + \frac{15a_1}{8}\right) x^4 + \left(-\frac{15a_0}{16} + \frac{31a_1}{16}\right) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{3}{4}x^3 - \frac{7}{8}x^4 - \frac{15}{16}x^5\right) a_0 + \left(x + \frac{3}{2}x^2 + \frac{7}{4}x^3 + \frac{15}{8}x^4 + \frac{31}{16}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{3}{4}x^3 - \frac{7}{8}x^4 - \frac{15}{16}x^5\right) c_1 + \left(x + \frac{3}{2}x^2 + \frac{7}{4}x^3 + \frac{15}{8}x^4 + \frac{31}{16}x^5\right) c_2 + O(x^6)$$

$$y = 2 + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} + \frac{x^5}{16} + x + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = x + 2 + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} + \frac{x^5}{16} + \frac{x^6}{32} + O(x^6) \tag{1}$$

$$y = 2 + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} + \frac{x^5}{16} + x + O(x^6) \tag{2}$$

Verification of solutions

$$y = x + 2 + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} + \frac{x^5}{16} + \frac{x^6}{32} + O(x^6)$$

Verified OK.

$$y = 2 + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} + \frac{x^5}{16} + x + O(x^6)$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$\left[y''(x^2 - 3x + 2) + (4x - 6)\, y' + 2y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x^2-3x+2} - \frac{2(2x-3)y'}{x^2-3x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(2x-3)y'}{x^2-3x+2} + \frac{2y}{x^2-3x+2} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2(2x-3)}{x^2-3x+2}, P_3(x) = \frac{2}{x^2-3x+2}\right]$

- ○ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = 2$

- ○ $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

- ○ $x = 1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 1$

- Multiply by denominators

$y''(x^2 - 3x + 2) + (4x - 6) y' + 2y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

$(u^2 - u) \left(\frac{d^2}{du^2} y(u)\right) + (4u - 2) \left(\frac{d}{du} y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r) u^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r) u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) u^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

  $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k + 2 - m + r) (k + 1 - m + r) u^{k+r}$

Rewrite ODE with series expansions

$-a_0 r(1 + r) u^{-1+r} + \left(\sum_{k=0}^{\infty} (-a_{k+1}(k + r + 1) (k + r + 2) + a_k (k + r + 2) (k + r + 1)) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(1 + r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term in the series must be 0, giving the recursion relation

$(k + r + 2) (k + r + 1) (-a_{k+1} + a_k) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = a_k$

- Recursion relation for $r = -1$

$a_{k+1} = a_k$

- Solution for $r = -1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1} = a_k\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k-1}, a_{k+1} = a_k\right]$$

- Recursion relation for $r = 0$

$a_{k+1} = a_k$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k}, a_{k+1} = a_k\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k}, a_{k+1} = a_k\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k}\right), a_{k+1} = a_k, b_{k+1} = b_k\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(x-1)*(x-2)*diff(y(x),x$2)+(4*x-6)*diff(y(x),x)+2*y(x)=0,y(0) = 2, D(y)(0) = 1],y(x)
```

$$y(x) = 2 + x + \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{8}x^4 + \frac{1}{16}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{(x-1)*(x-2)*y''[x]+(4*x-6)*y'[x]+2*y[x]==0,{y[0]==2,y'[0]==1}},y[x],{
```

$$y(x) \to \frac{x^5}{16} + \frac{x^4}{8} + \frac{x^3}{4} + \frac{x^2}{2} + x + 2$$

1.2 problem 3.6 (a)

Internal problem ID [5481]
Internal file name [OUTPUT/4729_Sunday_June_05_2022_03_04_14_PM_5913645/index.tex]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.6 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2xy' + 8y = 0}$$

With initial conditions

$$[y(0) = 4, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

1.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 8 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2xy' + 8y = 0$$

The domain of $p(x) = -2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 8$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (4)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2xy' - 8y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2 y' - 16xy - 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 8y'x^3 - 32yx^2 - 20xy' + 32y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 - 48x^2 + 12\right) y' + \left(-64x^3 + 96x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 32x\left(\left(x^4 - 3x^2 + \frac{3}{4}\right) y' + \left(-4x^3 + 6x\right) y\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 4$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= -32 \\
F_1 &= 0 \\
F_2 &= 128 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -16x^2 + 4 + \frac{16x^4}{3} + O\left(x^6\right)
$$

$$
y = -16x^2 + 4 + \frac{16x^4}{3} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 8\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 8a_0 = 0$$

$$a_2 = -4a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2n a_n + 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n(n-4)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -a_1$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{210}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 4a_0 x^2 - a_1 x^3 + \frac{4}{3} a_0 x^4 + \frac{1}{10} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 4x^2 + \frac{4}{3}x^4\right) a_0 + \left(x - x^3 + \frac{1}{10}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 4x^2 + \frac{4}{3}x^4\right) c_1 + \left(x - x^3 + \frac{1}{10}x^5\right) c_2 + O(x^6)$$

$$y = -16x^2 + 4 + \frac{16x^4}{3} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -16x^2 + 4 + \frac{16x^4}{3} + O(x^6) \tag{1}$$

$$y = -16x^2 + 4 + \frac{16x^4}{3} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -16x^2 + 4 + \frac{16x^4}{3} + O(x^6)$$

Verified OK.

$$y = -16x^2 + 4 + \frac{16x^4}{3} + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(0) = 4, D(y)(0) = 0],y(x),type='series',x
```

$$y(x) = 4 - 16x^2 + \frac{16}{3}x^4 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 17

```
AsymptoticDSolveValue[{y''[x]-2*x*y'[x]+8*y[x]==0,{y[0]==4,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{16x^4}{3} - 16x^2 + 4$$

1.3 problem 3.6 (b)

Internal problem ID [5482]
Internal file name [OUTPUT/4730_Sunday_June_05_2022_03_04_16_PM_25200396/index.tex]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.6 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2xy' + 8y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 4]$$

With the expansion point for the power series method at $x = 0$.

1.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 8 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2xy' + 8y = 0$$

The domain of $p(x) = -2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 8$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{7}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{8}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2xy' - 8y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 4x^2 y' - 16xy - 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 8y'x^3 - 32yx^2 - 20xy' + 32y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(16x^4 - 48x^2 + 12\right) y' + \left(-64x^3 + 96x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 32x\left(\left(x^4 - 3x^2 + \frac{3}{4}\right) y' + \left(-4x^3 + 6x\right) y\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 4$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -24 \\
F_2 &= 0 \\
F_3 &= 48 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -4x^3 + 4x + \frac{2x^5}{5} + O\left(x^6\right)
$$

$$
y = -4x^3 + 4x + \frac{2x^5}{5} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 8\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 8a_0 = 0$$

$$a_2 = -4a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2na_n + 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2a_n(n-4)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -a_1$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{210}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 4a_0 x^2 - a_1 x^3 + \frac{4}{3} a_0 x^4 + \frac{1}{10} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 4x^2 + \frac{4}{3}x^4\right) a_0 + \left(x - x^3 + \frac{1}{10}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 4x^2 + \frac{4}{3}x^4\right) c_1 + \left(x - x^3 + \frac{1}{10}x^5\right) c_2 + O(x^6)$$

$$y = -4x^3 + 4x + \frac{2x^5}{5} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -4x^3 + 4x + \frac{2x^5}{5} + O(x^6) \tag{1}$$

$$y = -4x^3 + 4x + \frac{2x^5}{5} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -4x^3 + 4x + \frac{2x^5}{5} + O(x^6)$$

Verified OK.

$$y = -4x^3 + 4x + \frac{2x^5}{5} + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(0) = 0, D(y)(0) = 4],y(x),type='series',x
```

$$y(x) = 4x - 4x^3 + \frac{2}{5}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]-2*x*y'[x]+8*y[x]==0,{y[0]==0,y'[0]==4}},y[x],{x,0,5}]
```

$$y(x) \rightarrow \frac{2x^5}{5} - 4x^3 + 4x$$

1.4 problem 3.6 (c)

Internal problem ID [5483]
Internal file name [`OUTPUT/4731_Sunday_June_05_2022_03_04_17_PM_33109312/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.6 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(-x^2+1)\,y''-2xy'+12y=0}$$

With initial conditions

$$[y(0)=0, y'(0)=3]$$

With the expansion point for the power series method at $x=0$.

1.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$p(x)=-\frac{2x}{-x^2+1}$$
$$q(x)=\frac{12}{-x^2+1}$$
$$F=0$$

Hence the ode is

$$y'' - \frac{2xy'}{-x^2+1} + \frac{12y}{-x^2+1} = 0$$

The domain of $p(x) = -\frac{2x}{-x^2+1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{12}{-x^2+1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (10)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (11)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(xy'-6y)}{x^2-1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{18x^2y'-48xy-10y'}{\left(x^2-1\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{-120y'x^3+360yx^2+72xy'-72y}{\left(x^2-1\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{192(5y'x^3-15yx^2-3xy'+3y)\,x}{\left(x^2-1\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= -\frac{8640\left(\left(x^3-\frac{3}{5}x\right)y'+y\left(-3x^2+\frac{3}{5}\right)\right)\left(x^2+\frac{1}{9}\right)}{\left(x^2-1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=0$ and $y'(0)=3$ gives

$$
\begin{aligned}
F_0 &= 0\\
F_1 &= -30\\
F_2 &= 0\\
F_3 &= 0\\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y=-5x^3+3x+O\left(x^6\right)
$$

$$y = -5x^3 + 3x + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-x^2 + 1\right) y'' - 2xy' + 12y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-x^2 + 1\right) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - 2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 12 \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n\right) + \sum_{n=1}^{\infty} \left(-2n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 + 12a_0 = 0$$

$$a_2 = -6a_0$$

$n = 1$ gives

$$6a_3 + 10a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{5a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$-na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 2na_n + 12a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 + n - 12)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$6a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 3a_0$$

For $n = 3$ the recurrence equation gives

$$20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$-8a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{4a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$-18a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 6a_0 x^2 - \frac{5}{3} a_1 x^3 + 3a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = (3x^4 - 6x^2 + 1) a_0 + \left(x - \frac{5}{3}x^3\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (3x^4 - 6x^2 + 1) c_1 + \left(x - \frac{5}{3}x^3\right) c_2 + O(x^6)$$

$$y = -5x^3 + 3x + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = -5x^3 + 3x + O(x^6) \tag{1}$$

$$y = -5x^3 + 3x + O(x^6) \tag{2}$$

Verification of solutions

$$y = -5x^3 + 3x + O(x^6)$$

Verified OK.

$$y = -5x^3 + 3x + O(x^6)$$

Verified OK.

1.4.2 Maple step by step solution

Let's solve

$\left[(-x^2+1)\, y'' - 2xy' + 12y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2xy'}{x^2-1} + \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{12}{x^2-1}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $((1+x) \cdot P_2(x))\Big|_{x=-1} = 1$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $((1+x)^2 \cdot P_3(x))\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' + 2xy' - 12y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u - 2)\left(\frac{d}{du}y(u)\right) - 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k- > k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+4)(k+r-3)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+4)(k-3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -6a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{5a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{15a_0}{2}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0

  $a_3 = -\frac{5a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 6u + \frac{15}{2}u^2 - \frac{5}{2}u^3\right)$
- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(\frac{3}{2}x - \frac{5}{2}x^3\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
Order:=6;
dsolve([(1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+12*y(x)=0,y(0) = 0, D(y)(0) = 3],y(x),type='
```

$$y(x) = -5x^3 + 3x$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 12

```
AsymptoticDSolveValue[{(1-x^2)*y''[x]-2*x*y'[x]+12*y[x]==0,{y[0]==0,y'[0]==3}},y[x],{x,0,5}]
```

$$y(x) \rightarrow 3x - 5x^3$$

1.5 problem 3.6 (d)

Internal problem ID [5484]
Internal file name [`OUTPUT/4732_Sunday_June_05_2022_03_04_19_PM_13770176/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.6 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__bessel__ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - (x-1)\, y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

1.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 - x \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + (1 - x)\, y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1 - x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} &\quad (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' &\quad (13)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &\quad (14)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f &\quad (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f &\quad (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= (x-1)\,y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= y + (x-1)\,y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= 2y' + (x-1)^2\,y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= (x-1)\left((x-1)\,y' + 4y\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= (x-1)^3\,y + (6x-6)\,y' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= -1 \\
F_1 &= 1 \\
F_2 &= 1 \\
F_3 &= -4 \\
F_4 &= 3
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + \frac{x^6}{240} + O(x^6)
$$

$$
y = 1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + \frac{x^6}{240} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = (x-1)\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_n - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{a_n - a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{a_n}{(n+2)\,(1+n)} + \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_1 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6} + \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_2 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24} + \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_3 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120} - \frac{a_0}{30}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_4 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{240} - \frac{a_1}{120}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_5 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{560} + \frac{a_0}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{a_0 x^2}{2} + \left(-\frac{a_1}{6} + \frac{a_0}{6}\right) x^3 + \left(\frac{a_0}{24} + \frac{a_1}{12}\right) x^4 + \left(\frac{a_1}{120} - \frac{a_0}{30}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

$$y = 1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + \frac{x^6}{240} + O(x^6) \tag{1}$$

$$y = 1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + \frac{x^6}{240} + O(x^6)$$

Verified OK.

$$y = 1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + O(x^6)$$

Verified OK.

1.5.2 Maple step by step solution

Let's solve

$$\left[y'' = (x-1)\,y, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (1-x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$2a_2 + a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 + a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} + a_k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} + a_{k+1} - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{-a_{k+1} + a_k}{k^2 + 5k + 6}, 2a_2 + a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
Order:=6;
dsolve([diff(y(x),x$2)=(x-1)*y(x),y(0) = 1, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 33

```
AsymptoticDSolveValue[{y''[x]==(x-1)*y[x],{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{30} + \frac{x^4}{24} + \frac{x^3}{6} - \frac{x^2}{2} + 1$$

1.6 problem 3.24 (a)

Internal problem ID [5485]
Internal file name [OUTPUT/4733_Sunday_June_05_2022_03_04_21_PM_58288000/index.tex]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.24 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x+2)\,y''+2(1+x)\,y'-2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2+2x\right)y''+(2+2x)\,y'-2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2+2x}{x\,(x+2)}$$
$$q(x)=-\frac{2}{x\,(x+2)}$$

Table 4: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2+2x}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{2}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x+2)\,y'' + (2+2x)\,y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x+2)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (2+2x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1} (n+r-1) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} (n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r}r(-1+r)+2r\,x^{-1+r}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^{-1+r}r^2=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1&=0\\ r_2&=0\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^{-1+r}r^2=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x)=y_1(x)\ln(x)+\left(\sum_{n=1}^{\infty}b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y=c_1y_1(x)+c_2y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0=1$, and $\{c_1,c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}&a_{n-1}(n+r-1)(n+r-2)+2a_n(n+r)(n+r-1)\\&+2a_{n-1}(n+r-1)+2a_n(n+r)-2a_{n-1}=0\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - n - r - 2)}{2\,(n^2 + 2nr + r^2)} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}(n^2 - n - 2)}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 - r + 2}{2\,(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r+3)\,r(-1+r)}{4\,(r+2)\,(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(-1+r)\, r(r+4)}{8\,(r+3)\,(r+1)\,(r+2)}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(-1+r)\,(r+5)}{16\,(r+4)\,(r+1)\,(r+3)}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(-1+r)\, r(r+6)}{32\,(r+5)\,(r+1)\,(r+4)}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1
a_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0
a_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0
a_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0
a_5	$-\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + x + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{-r^2-r+2}{2(r+1)^2}$	1	$\frac{-r-5}{2(r+1)^3}$	$-\frac{5}{2}$
b_2	$\frac{(r+3)r(-1+r)}{4(r+2)(r+1)^2}$	0	$\frac{r^3+7r^2+7r-3}{2(r+2)^2(r+1)^3}$	$-\frac{3}{8}$
b_3	$-\frac{(-1+r)r(r+4)}{8(r+3)(r+1)(r+2)}$	0	$\frac{3-\frac{9}{2}r-\frac{75}{8}r^2-\frac{3}{8}r^4-\frac{15}{4}r^3}{(r+3)^2(r+1)^2(r+2)^2}$	$\frac{1}{12}$
b_4	$\frac{r(-1+r)(r+5)}{16(r+4)(r+1)(r+3)}$	0	$\frac{r^4+12r^3+38r^2+24r-15}{4(r+4)^2(r+1)^2(r+3)^2}$	$-\frac{5}{192}$
b_5	$-\frac{(-1+r)r(r+6)}{32(r+5)(r+1)(r+4)}$	0	$\frac{\frac{15}{4}-\frac{25}{4}r-\frac{265}{32}r^2-\frac{5}{32}r^4-\frac{35}{16}r^3}{(r+5)^2(r+1)^2(r+4)^2}$	$\frac{3}{320}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\ &= \left(1 + x + O(x^6)\right)\ln(x) - \frac{5x}{2} - \frac{3x^2}{8} + \frac{x^3}{12} - \frac{5x^4}{192} + \frac{3x^5}{320} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\left(1 + x + O(x^6)\right) + c_2\left(\left(1 + x + O(x^6)\right)\ln(x) - \frac{5x}{2} - \frac{3x^2}{8} + \frac{x^3}{12} - \frac{5x^4}{192} + \frac{3x^5}{320} \right. \\ &\quad \left. + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\left(1+x+O(x^6)\right) + c_2\left(\left(1+x+O(x^6)\right)\ln(x) - \frac{5x}{2} - \frac{3x^2}{8} + \frac{x^3}{12} - \frac{5x^4}{192} + \frac{3x^5}{320} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\left(1 + x + O(x^6)\right) \\ &\quad + c_2\left(\left(1 + x + O(x^6)\right)\ln(x) - \frac{5x}{2} - \frac{3x^2}{8} + \frac{x^3}{12} - \frac{5x^4}{192} + \frac{3x^5}{320} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1\left(1+x+O\left(x^6\right)\right)+c_2\left(\left(1+x+O\left(x^6\right)\right)\ln\left(x\right)-\frac{5x}{2}-\frac{3x^2}{8}+\frac{x^3}{12}-\frac{5x^4}{192}+\frac{3x^5}{320}+O\left(x^6\right)\right)$$

Verified OK.

1.6.1 Maple step by step solution

Let's solve

$x(x+2)\,y''+(2+2x)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y}{x(x+2)}-\frac{2(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(1+x)y'}{x(x+2)}-\frac{2y}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2(1+x)}{x(x+2)}, P_3(x)=-\frac{2}{x(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=1$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $x(x+2)\,y''+(2+2x)\,y'-2y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-2+2u)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r^2 u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2+a_k(k+2)\,(k-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u + 1)$
- Revert the change of variables $u = x + 2$

  $[y = a_0(-1 - x)]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
Order:=6;
dsolve(x*(x+2)*diff(y(x),x$2)+2*(x+1)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)(1 + x + \mathrm{O}(x^6)) + \left(-\frac{5}{2}x - \frac{3}{8}x^2 + \frac{1}{12}x^3 - \frac{5}{192}x^4 + \frac{3}{320}x^5 + \mathrm{O}(x^6)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 53

```
AsymptoticDSolveValue[x*(x+2)*y''[x]+2*(x+1)*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{3x^5}{320} - \frac{5x^4}{192} + \frac{x^3}{12} - \frac{3x^2}{8} - \frac{5x}{2} + (x+1)\log(x)\right) + c_1(x+1)$$

1.7 problem 3.24 (b)

Internal problem ID [5486]
Internal file name [`OUTPUT/4734_Sunday_June_05_2022_03_04_22_PM_53719317/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.24 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1}{x}$$

Table 6: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}}{(n+1)\,n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{(1+r)\,r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)^2\,r\,(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2r(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(1+r)^2\,r\,(2+r)^2\,(3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$
a_5	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$	$-\frac{1}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -\frac{1}{(1+r)\,r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{1}{(1+r)\,r} &= \lim_{r\to 0} -\frac{1}{(1+r)\,r} \\ &= \mathit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$

$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$

$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$

$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $xy'' + y = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left(\left(y_1''(x)\,x + y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)xC + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^n a_n (n+1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1)\right) + \sum_{n=0}^{\infty} \left(-C\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^n a_n\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right)+\sum_{n\,=1}^{\infty}\left(-C a_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right)+\left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C+1=0$$

Which is solved for C. Solving for C gives

$$C=-1$$

For $n=2$, Eq (2B) gives

$$3 C a_1+b_1+2 b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 b_2+\frac{3}{2}=0$$

Solving the above for b_2 gives

$$b_2=-\frac{3}{4}$$

For $n=3$, Eq (2B) gives

$$5 C a_2+b_2+6 b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6 b_3-\frac{7}{6}=0$$

Solving the above for b_3 gives

$$b_3=\frac{7}{36}$$

For $n=4$, Eq (2B) gives

$$7 C a_3+b_3+12 b_4=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12 b_4+\frac{35}{144}=0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{1728}$$

For $n = 5$, Eq (2B) gives

$$9Ca_4 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 - \frac{101}{4320} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{101}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = (-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) \\ + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left((-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) + 1 \right. \\ &\quad \left. - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left(-x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\ln(x) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} \right. \\ &\quad \left. - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right) + c_2 \left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}+\frac{7x^3}{36}-\frac{35x^4}{1728}+\frac{101x^5}{86400}+O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right) + c_2 \left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}+\frac{7x^3}{36}-\frac{35x^4}{1728}+\frac{101x^5}{86400}+O(x^6)\right)$$

Verified OK.

1.7.1 Maple step by step solution

Let's solve

$y''x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r-1}$

  - Shift index using $k- >k + 1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r) (k + r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1 + r) x^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k + 1 + r) (k + r) + a_k) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k + 1 + r) (k + r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{(k+1)k}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)k}\right]$$

- Recursion relation for $r = 1$

  $$a_{k+1} = -\frac{a_k}{(k+2)(k+1)}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{(k+2)(k+1)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{a_k}{(k+1)k}, b_{k+1} = -\frac{b_k}{(k+2)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x\left(1-\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{144}x^3+\frac{1}{2880}x^4-\frac{1}{86400}x^5+\mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(-x+\frac{1}{2}x^2-\frac{1}{12}x^3+\frac{1}{144}x^4-\frac{1}{2880}x^5+\mathrm{O}\left(x^6\right)\right) + \left(1-\frac{3}{4}x^2+\frac{7}{36}x^3-\frac{35}{1728}x^4+\frac{101}{86400}x^5+\mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{144}x\left(x^3-12x^2+72x-144\right)\log(x) + \frac{-47x^4+480x^3-2160x^2+1728x+1728}{1728}\right) + c_2\left(\frac{x^5}{2880}-\frac{x^4}{144}+\frac{x^3}{12}-\frac{x^2}{2}+x\right)$$

1.8 problem 3.24 (c)

Internal problem ID [5487]
Internal file name [`OUTPUT/4735_Sunday_June_05_2022_03_04_24_PM_70902532/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.24 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode_form_A", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (e^{x} - 1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (18)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (19)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -(e^x - 1)\, y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y' e^x - e^x y + y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= e^{2x} y - 2y' e^x - 3\, e^x y + y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= (4y + y')\, e^{2x} + (-5\, e^x + 1)\, y' - 5\, e^x y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (14y + 6y')\, e^{2x} - e^{3x} y - 10 y' e^x + (-11\, e^x + 1)\, y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -y(0) - 2y'(0) \\
F_3 &= -y(0) - 3y'(0) \\
F_4 &= 3y(0) - 4y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{240}x^6\right) y(0) + \left(x - \frac{1}{12}x^4 - \frac{1}{40}x^5 - \frac{1}{180}x^6\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -(\mathrm{e}^x - 1)\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Expanding $\mathrm{e}^x - 1$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\mathrm{e}^x - 1 &= x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \dots \\ &= x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the second term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + x \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^4}{24} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^6}{720} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+2} a_n}{2}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+3} a_n}{6}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+4} a_n}{24}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+6} a_n}{720}\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} \frac{x^{n+2} a_n}{2} = \sum_{n=2}^{\infty} \frac{a_{n-2} x^n}{2}$$

$$\sum_{n=0}^{\infty} \frac{x^{n+3} a_n}{6} = \sum_{n=3}^{\infty} \frac{a_{n-3} x^n}{6}$$

$$\sum_{n=0}^{\infty} \frac{x^{n+4} a_n}{24} = \sum_{n=4}^{\infty} \frac{a_{n-4} x^n}{24}$$

$$\sum_{n=0}^{\infty} \frac{x^{n+5} a_n}{120} = \sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}$$

$$\sum_{n=0}^{\infty} \frac{x^{n+6} a_n}{720} = \sum_{n=6}^{\infty} \frac{a_{n-6} x^n}{720}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) + \left(\sum_{n=2}^{\infty} \frac{a_{n-2} x^n}{2}\right) + \left(\sum_{n=3}^{\infty} \frac{a_{n-3} x^n}{6}\right) + \left(\sum_{n=4}^{\infty} \frac{a_{n-4} x^n}{24}\right) + \left(\sum_{n=5}^{\infty} \frac{a_{n-5} x^n}{120}\right) + \left(\sum_{n=6}^{\infty} \frac{a_{n-6} x^n}{720}\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6}$$

$n = 2$ gives

$$12a_4 + a_1 + \frac{a_0}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{a_0}{24} - \frac{a_1}{12}$$

$n = 3$ gives

$$20a_5 + a_2 + \frac{a_1}{2} + \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = -\frac{a_0}{120} - \frac{a_1}{40}$$

$n = 4$ gives

$$30a_6 + a_3 + \frac{a_2}{2} + \frac{a_1}{6} + \frac{a_0}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{240} - \frac{a_1}{180}$$

$n = 5$ gives

$$42a_7 + a_4 + \frac{a_3}{2} + \frac{a_2}{6} + \frac{a_1}{24} + \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{a_0}{360} + \frac{a_1}{1008}$$

For $6 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} + \frac{a_{n-2}}{2} + \frac{a_{n-3}}{6} + \frac{a_{n-4}}{24} + \frac{a_{n-5}}{120} + \frac{a_{n-6}}{720} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{720a_{n-1} + 360a_{n-2} + 120a_{n-3} + 30a_{n-4} + 6a_{n-5} + a_{n-6}}{720\,(n+2)\,(1+n)} \\
&= -\frac{a_{n-6}}{720\,(n+2)\,(1+n)} - \frac{a_{n-5}}{120\,(n+2)\,(1+n)} - \frac{a_{n-4}}{24\,(n+2)\,(1+n)} \\
&\quad - \frac{a_{n-3}}{6\,(n+2)\,(1+n)} - \frac{a_{n-2}}{2\,(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)}
\end{aligned} \tag{5}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1x - \frac{a_0x^3}{6} + \left(-\frac{a_0}{24} - \frac{a_1}{12}\right)x^4 + \left(-\frac{a_0}{120} - \frac{a_1}{40}\right)x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5\right)a_0 + \left(x - \frac{1}{12}x^4 - \frac{1}{40}x^5\right)a_1 + O(x^6) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5\right)c_1 + \left(x - \frac{1}{12}x^4 - \frac{1}{40}x^5\right)c_2 + O(x^6)
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{240}x^6\right)y(0) \\
& + \left(x - \frac{1}{12}x^4 - \frac{1}{40}x^5 - \frac{1}{180}x^6\right)y'(0) + O(x^6)
\end{aligned} \tag{1}
$$

$$
y = \left(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5\right)c_1 + \left(x - \frac{1}{12}x^4 - \frac{1}{40}x^5\right)c_2 + O(x^6) \tag{2}
$$

Verification of solutions

$$y = \left(1-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5+\frac{1}{240}x^6\right)y(0)+\left(x-\frac{1}{12}x^4-\frac{1}{40}x^5-\frac{1}{180}x^6\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5\right)c_1+\left(x-\frac{1}{12}x^4-\frac{1}{40}x^5\right)c_2+O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)]
   Linear ODE actually solved:
      (t-1)*u(t)+t*diff(u(t),t)+t^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
Order:=6;
dsolve(diff(y(x),x$2)+(exp(x)-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5\right)y(0)+\left(x-\frac{1}{12}x^4-\frac{1}{40}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 49

```
AsymptoticDSolveValue[y''[x]+(Exp[x]-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{x^5}{40}-\frac{x^4}{12}+x\right)+c_1\left(-\frac{x^5}{120}-\frac{x^4}{24}-\frac{x^3}{6}+1\right)$$

1.9 problem 3.24 (d)

Internal problem ID [5488]
Internal file name [`OUTPUT/4736_Sunday_June_05_2022_03_04_26_PM_31832984/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.24 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(1-x)\,y''-3xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^2+x\right)y''-3xy'-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x-1}$$
$$q(x)=\frac{1}{x\,(x-1)}$$

Table 8: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x-1}$	
singularity	type
$x = 1$	"regular"

$q(x) = \frac{1}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[1, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) - 3xy' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) - 3x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)\,(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-3x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\,(n+r-1)\right)&=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right)\\ \sum_{n=0}^{\infty}\left(-3x^{n+r}a_n(n+r)\right)&=\sum_{n=1}^{\infty}\left(-3a_{n-1}(n+r-1)\,x^{n+r-1}\right)\\ \sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)&=\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)\,(n+r-1)\right) \\ +\sum_{n=1}^{\infty}\left(-3a_{n-1}(n+r-1)\,x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)\,(n+r-1)=0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0 r(-1+r)=0$$

Or

$$x^{-1+r}a_0 r(-1+r)=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) - 3a_{n-1}(n+r-1) - a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r)\,a_{n-1}}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(n+1)\,a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+r}{r}$$

Which for the root $r = 1$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2+r}{r}$$

Which for the root $r = 1$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{3 + r}{r}$$

Which for the root $r = 1$ becomes

$$a_3 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4 + r}{r}$$

Which for the root $r = 1$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4
a_4	$\frac{4+r}{r}$	5

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{5+r}{r}$$

Which for the root $r = 1$ becomes

$$a_5 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4
a_4	$\frac{4+r}{r}$	5
a_5	$\frac{5+r}{r}$	6

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{1+r}{r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1+r}{r} &= \lim_{r \to 0} \frac{1+r}{r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)\end{aligned}$$

Substituting these back into the given ode $-y''x(x-1) - 3xy' - y = 0$ gives

$$\begin{aligned}&-\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \right. \\ &\left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x(x-1) \\ &- 3x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\ &- Cy_1(x)\ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0\end{aligned}$$

Which can be written as

$$
\begin{aligned}
&\left(\left(-y_1''(x)\, x(x-1) - 3y_1'(x)\, x - y_1(x)\right) \ln(x) - \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x(x-1)\right. \\
&\left. - 3y_1(x)\right) C - \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x(x-1) \\
&- 3x \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
-y_1''(x)\, x(x-1) - 3y_1'(x)\, x - y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(-\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x(x-1) - 3y_1(x)\right) C \\
&- \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x(x-1) \\
&- 3x \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\frac{\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) + (-1-2x)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} \\
&+ \frac{(-x^3+x^2)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) - 3\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{x} \\
&= 0
\end{aligned}
\tag{9}
$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^n a_n(n+1)\right)+(-1-2x)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right)C}{x} + \frac{(-x^3+x^2)\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right)-3\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right)x^2-\left(\sum_{n=0}^{\infty} b_n x^n\right)x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{gathered}\sum_{n=0}^{\infty}\left(-2C\,x^{n+1}a_n(n+1)\right)+\left(\sum_{n=0}^{\infty}2C\,x^n a_n(n+1)\right)\\ +\sum_{n=0}^{\infty}\left(-Ca_n x^n\right)+\sum_{n=0}^{\infty}\left(-2C\,x^{n+1}a_n\right)+\sum_{n=0}^{\infty}\left(-x^n b_n n(n-1)\right)\\ +\left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right)+\sum_{n=0}^{\infty}\left(-3x^n b_n n\right)+\sum_{n=0}^{\infty}\left(-b_n x^n\right)=0\end{gathered} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2C\,x^{n+1}a_n(n+1)\right)=\sum_{n=2}^{\infty}\left(-2Ca_{-2+n}(n-1)\,x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}2C\,x^n a_n(n+1)=\sum_{n=1}^{\infty}2Ca_{n-1}n\,x^{n-1}$$

$$\sum_{n=0}^{\infty}\left(-Ca_n x^n\right)=\sum_{n=1}^{\infty}\left(-Ca_{n-1}x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-2C\,x^{n+1}a_n\right)=\sum_{n=2}^{\infty}\left(-2Ca_{-2+n}x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-x^n b_n n(n-1)\right)=\sum_{n=1}^{\infty}\left(-(n-1)\,b_{n-1}(-2+n)\,x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-3x^{n} b_{n} n\right)=\sum_{n=1}^{\infty}\left(-3(n-1)\, b_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-b_{n} x^{n}\right)=\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(-2C a_{-2+n}(n-1)\, x^{n-1}\right)+\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right)\\
&+\sum_{n=1}^{\infty}\left(-C a_{n-1} x^{n-1}\right)+\sum_{n=2}^{\infty}\left(-2C a_{-2+n} x^{n-1}\right)\\
&+\sum_{n=1}^{\infty}\left(-(n-1)\, b_{n-1}(-2+n)\, x^{n-1}\right)+\left(\sum_{n=0}^{\infty} n\, x^{n-1} b_{n}(n-1)\right)\\
&+\sum_{n=1}^{\infty}\left(-3(n-1)\, b_{n-1} x^{n-1}\right)+\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)=0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C-1=0$$

Which is solved for C. Solving for C gives

$$C=1$$

For $n=2$, Eq (2B) gives

$$\left(-4a_0+3a_1\right) C-4b_1+2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2+2b_2=0$$

Solving the above for b_2 gives

$$b_2=-1$$

For $n=3$, Eq (2B) gives

$$\left(-6a_1+5a_2\right) C-9b_2+6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12 + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -2$$

For $n = 4$, Eq (2B) gives

$$(-8a_2 + 7a_3) C - 16b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$36 + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -3$$

For $n = 5$, Eq (2B) gives

$$(-10a_3 + 9a_4) C - 25b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$80 + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -4$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = {} & 1\left(x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right)\right) \ln(x) \\ & + 1 - x^2 - 2x^3 - 3x^4 - 4x^5 + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right) \\ &\quad + c_2\left(1\left(x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right)\right) \ln(x) + 1 - x^2 - 2x^3 \right. \\ &\quad \left. - 3x^4 - 4x^5 + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right)\ln(x)+1-x^2-2x^3-3x^4-4x^5 \right.\\
&\quad \left. +O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right)\ln(x)+1-x^2-2x^3-3x^4 \right.\\
&\quad \left. -4x^5+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right)\ln(x)+1-x^2-2x^3-3x^4-4x^5 \right.\\
&\quad \left. +O(x^6)\right)
\end{aligned}
$$

Verified OK.

1.9.1 Maple step by step solution

Let's solve

$-y''x(x-1)-3xy'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(x-1)}-\frac{3y'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x-1}+\frac{y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3}{x-1}, P_3(x)=\frac{1}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x(x-1) + 3xy' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+r+1)\,(k+r) + a_k(k+r+1)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\,(-a_{k+1}(k+r) + a_k(k+r+1)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+1)}{k+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+1)}{k}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(k+1)}{k}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k(k+2)}{k+1}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k(k+2)}{k+1}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = \frac{a_k(k+1)}{k}, b_{k+1} = \frac{b_k(k+2)}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 60

```
Order:=6;
dsolve(x*(1-x)*diff(y(x),x$2)-3*x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & \ln(x)\left(x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \mathrm{O}\left(x^6\right)\right) c_2 \\
& + c_1 x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + \left(1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 63

```
AsymptoticDSolveValue[x*(1-x)*y''[x]-3*x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(x^4+x^3+x^2+\left(4x^3+3x^2+2x+1\right)x\log(x)+x+1\right) + c_2\left(5x^5+4x^4+3x^3+2x^2+x\right)$$

1.10 problem 3.24 (e)

Internal problem ID [5489]
Internal file name [`OUTPUT/4737_Sunday_June_05_2022_03_04_28_PM_3695263/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.24 (e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2xy'' - y' + yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' - y' + yx^2 = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \frac{x}{2}$$

Table 10: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x}{2}$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' - y' + yx^2 = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x^2 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{2+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{2+n+r} a_n = \sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) - (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) - r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-3+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

For $3 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-3} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-3}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{a_{n-3}}{n(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{2r^2 + 9r + 9}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -\frac{1}{27}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{1}{2r^2+9r+9}$	$-\frac{1}{27}$

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{1}{2r^2+9r+9}$	$-\frac{1}{27}$
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{1}{2r^2+9r+9}$	$-\frac{1}{27}$
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - \frac{x^3}{27} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$b_2 = 0$$

For $3 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - (n+r)b_n + b_{n-3} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-3}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{b_{n-3}}{n(2n-3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{2r^2 + 9r + 9}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{1}{2r^2+9r+9}$	$-\frac{1}{9}$

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{1}{2r^2+9r+9}$	$-\frac{1}{9}$
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{1}{2r^2+9r+9}$	$-\frac{1}{9}$
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 - \frac{x^3}{9} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}} \left(1 - \frac{x^3}{27} + O(x^6)\right) + c_2 \left(1 - \frac{x^3}{9} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}} \left(1 - \frac{x^3}{27} + O(x^6)\right) + c_2 \left(1 - \frac{x^3}{9} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{3}{2}} \left(1 - \frac{x^3}{27} + O(x^6)\right) + c_2 \left(1 - \frac{x^3}{9} + O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{3}{2}} \left(1 - \frac{x^3}{27} + O(x^6)\right) + c_2 \left(1 - \frac{x^3}{9} + O(x^6)\right)
$$

Verified OK.

1.10.1 Maple step by step solution

Let's solve

$2y''x - y' + yx^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2x} - \frac{xy}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} + \frac{xy}{2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{x}{2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x - y' + yx^2 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^2 \cdot y$ to series expansion

  $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+2}$

- ○ Shift index using $k- > k - 2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+2r)\, x^{-1+r} + a_1 (1+r)\,(-1+2r)\, x^r + a_2 (2+r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k+1}(k+1+r)\,($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{3}{2}\right\}$$

- The coefficients of each power of x must be 0

  $$[a_1(1+r)\,(-1+2r) = 0, a_2(2+r)\,(1+2r) = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_1 = 0, a_2 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k - \frac{1}{2} + r\right)(k+1+r)\, a_{k+1} + a_{k-2} = 0$$

- Shift index using $k->k+2$

  $$2\left(k + \frac{3}{2} + r\right)(k+3+r)\, a_{k+3} + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3} = -\frac{a_k}{(2k+3+2r)(k+3+r)}$$

- Recursion relation for $r = 0$

$a_{k+3} = -\frac{a_k}{(2k+3)(k+3)}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k}{(2k+3)(k+3)}, a_1 = 0, a_2 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

$a_{k+3} = -\frac{a_k}{(2k+6)\left(k+\frac{9}{2}\right)}$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+3} = -\frac{a_k}{(2k+6)\left(k+\frac{9}{2}\right)}, a_1 = 0, a_2 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+3} = -\frac{a_k}{(2k+3)(k+3)}, a_1 = 0, a_2 = 0, b_{k+3} = -\frac{b_k}{(2k+6)\left(k+\frac{9}{2}\right)}, b_1 =$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)-diff(y(x),x)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{3}{2}}\left(1 - \frac{1}{27}x^3 + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 - \frac{1}{9}x^3 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 33

```
AsymptoticDSolveValue[2*x*y''[x]-y'[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(1 - \frac{x^3}{9}\right) + c_1\left(1 - \frac{x^3}{27}\right)x^{3/2}$$

1.11 problem 3.24 (f)

Internal problem ID [5490]
Internal file name [`OUTPUT/4738_Sunday_June_05_2022_03_04_29_PM_85478439/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.24 (f).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\sin(x)\, y'' - 2\cos(x)\, y' - \sin(x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\sin(x)\, y'' - 2\cos(x)\, y' - \sin(x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2\cos(x)}{\sin(x)}$$
$$q(x) = -1$$

Table 12: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2\cos(x)}{\sin(x)}$	
singularity	type
$x = \pi Z$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\pi Z]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\sin(x) \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) - 2\cos(x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) - \sin(x) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \end{aligned}$$

Expanding $-2\cos(x)$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}-2\cos(x) &= -2+x^2-\frac{1}{12}x^4+\frac{1}{360}x^6+\dots\\ &= -2+x^2-\frac{1}{12}x^4+\frac{1}{360}x^6\end{aligned}$$

Expanding $-\sin(x)$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}-\sin(x) &= -x+\frac{1}{6}x^3-\frac{1}{120}x^5+\frac{1}{5040}x^7+\dots\\ &= -x+\frac{1}{6}x^3-\frac{1}{120}x^5+\frac{1}{5040}x^7\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+5}a_n(n+r)(n+r-1)}{5040}\right)\\ &+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+3}a_n(n+r)(n+r-1)}{120}\right)\\ &+\sum_{n=0}^{\infty}\left(-\frac{x^{1+n+r}a_n(n+r)(n+r-1)}{6}\right)\\ &+\left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+5}a_n(n+r)}{360}\right)\\ &+\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+3}a_n(n+r)}{12}\right)+\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n(n+r)\right)\\ &+\sum_{n=0}^{\infty}\left(-2(n+r)a_nx^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)\\ &+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+3}a_n}{6}\right)+\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+5}a_n}{120}\right)+\left(\sum_{n=0}^{\infty}\frac{x^{n+r+7}a_n}{5040}\right)=0\end{aligned}\tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+5}a_n(n+r)(n+r-1)}{5040}\right)=\sum_{n=6}^{\infty}\left(-\frac{a_{n-6}(n+r-6)(n-7+r)x^{n+r-1}}{5040}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+3}a_n(n+r)(n+r-1)}{120}=\sum_{n=4}^{\infty}\frac{a_{n-4}(-4+n+r)(n-5+r)x^{n+r-1}}{120}$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{1+n+r}a_n(n+r)(n+r-1)}{6}\right)=\sum_{n=2}^{\infty}\left(-\frac{a_{n-2}(n+r-2)(n-3+r)x^{n+r-1}}{6}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+5}a_n(n+r)}{360}=\sum_{n=6}^{\infty}\frac{a_{n-6}(n+r-6)x^{n+r-1}}{360}$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+3}a_n(n+r)}{12}\right)=\sum_{n=4}^{\infty}\left(-\frac{a_{n-4}(-4+n+r)x^{n+r-1}}{12}\right)$$

$$\sum_{n=0}^{\infty}x^{1+n+r}a_n(n+r)=\sum_{n=2}^{\infty}a_{n-2}(n+r-2)x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+3}a_n}{6}=\sum_{n=4}^{\infty}\frac{a_{n-4}x^{n+r-1}}{6}$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+5}a_n}{120}\right)=\sum_{n=6}^{\infty}\left(-\frac{a_{n-6}x^{n+r-1}}{120}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+7}a_n}{5040}=\sum_{n=8}^{\infty}\frac{a_{n-8}x^{n+r-1}}{5040}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=6}^{\infty}\left(-\frac{a_{n-6}(n+r-6)(n-7+r)x^{n+r-1}}{5040}\right) + \left(\sum_{n=4}^{\infty}\frac{a_{n-4}(-4+n+r)(n-5+r)x^{n+r-1}}{120}\right) + \sum_{n=2}^{\infty}\left(-\frac{a_{n-2}(n+r-2)(n-3+r)x^{n+r-1}}{6}\right) + \left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=6}^{\infty}\frac{a_{n-6}(n+r-6)x^{n+r-1}}{360}\right) + \sum_{n=4}^{\infty}\left(-\frac{a_{n-4}(-4+n+r)x^{n+r-1}}{12}\right) + \left(\sum_{n=2}^{\infty}a_{n-2}(n+r-2)x^{n+r-1}\right) + \sum_{n=0}^{\infty}\left(-2(n+r)a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r-1}\right) + \left(\sum_{n=4}^{\infty}\frac{a_{n-4}x^{n+r-1}}{6}\right) + \sum_{n=6}^{\infty}\left(-\frac{a_{n-6}x^{n+r-1}}{120}\right) + \left(\sum_{n=8}^{\infty}\frac{a_{n-8}x^{n+r-1}}{5040}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1) - 2(n+r)a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0 r(-1+r) - 2ra_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r}r(-1+r) - 2r\,x^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 3 \\
r_2 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
r\,x^{-1+r}(-3+r) = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^{3}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

Substituting $n = 2$ in Eq. (2B) gives

$$
a_2 = \frac{r-6}{12+6r}
$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{7r^3 - 63r^2 + 146r + 120}{360\left(2+r\right)\left(4+r\right)\left(1+r\right)}$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = 0$$

Substituting $n = 6$ in Eq. (2B) gives

$$a_6 = \frac{31r^5 - 248r^4 + 497r^3 + 1508r^2 - 6324r - 3024}{15120\left(2+r\right)\left(4+r\right)\left(1+r\right)\left(6+r\right)\left(3+r\right)}$$

Substituting $n = 7$ in Eq. (2B) gives

$$a_7 = 0$$

For $8 \leq n$ the recursive equation is

$$\begin{aligned} &-\frac{a_{n-6}(n+r-6)\left(n-7+r\right)}{5040} + \frac{a_{n-4}(-4+n+r)\left(n-5+r\right)}{120} \\ &- \frac{a_{n-2}(n+r-2)\left(n-3+r\right)}{6} + a_n(n+r)\left(n+r-1\right) \\ &+ \frac{a_{n-6}(n+r-6)}{360} - \frac{a_{n-4}(-4+n+r)}{12} + a_{n-2}(n+r-2) \\ &- 2a_n(n+r) - a_{n-2} + \frac{a_{n-4}}{6} - \frac{a_{n-6}}{120} + \frac{a_{n-8}}{5040} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n^2a_{n-6} - 42n^2a_{n-4} + 840n^2a_{n-2} + 2nra_{n-6} - 84nra_{n-4} + 1680nra_{n-2} + r^2a_{n-6} - 42r^2a_{n-4} + 840r^2}{5040n^2 + 10}$$ (4)

Which for the root $r = 3$ becomes

$$a_n = \frac{\left(a_{n-6} - 42a_{n-4} + 840a_{n-2}\right)n^2 + \left(-21a_{n-6} + 546a_{n-4} - 4200a_{n-2}\right)n - a_{n-8} + 96a_{n-6} - 1344a_{n-4}}{5040n\left(n+3\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r-6}{12+6r}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{7r^3-63r^2+146r+120}{360(2+r)(4+r)(1+r)}$	$\frac{1}{280}$
a_5	0	0
a_6	$\frac{31r^5-248r^4+497r^3+1508r^2-6324r-3024}{15120(2+r)(4+r)(1+r)(6+r)(3+r)}$	$-\frac{1}{15120}$
a_7	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^3\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^3\left(1 - \frac{x^2}{10} + \frac{x^4}{280} - \frac{x^6}{15120} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_3 \\ &= 0\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} 0 &= \lim_{r\to 0} 0 \\ &= 0\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = \frac{r-6}{12+6r}$$

Substituting $n = 3$ in Eq(3) gives

$$b_3 = 0$$

Substituting $n = 4$ in Eq(3) gives

$$b_4 = \frac{7r^3 - 63r^2 + 146r + 120}{360\,(2+r)\,(r^2+5r+4)}$$

Substituting $n = 5$ in Eq(3) gives

$$b_5 = 0$$

Substituting $n = 6$ in Eq(3) gives

$$b_6 = \frac{31r^5 - 248r^4 + 497r^3 + 1508r^2 - 6324r - 3024}{15120\,(2+r)\,(r^2+5r+4)\,(r^2+9r+18)}$$

Substituting $n = 7$ in Eq(3) gives

$$b_7 = 0$$

For $8 \leq n$ the recursive equation is

$$\begin{aligned}
&-\frac{b_{n-6}(n+r-6)\,(n-7+r)}{5040} + \frac{b_{n-4}(-4+n+r)\,(n-5+r)}{120} \\
&- \frac{b_{n-2}(n+r-2)\,(n-3+r)}{6} + b_n(n+r)\,(n+r-1) \\
&+ \frac{b_{n-6}(n+r-6)}{360} - \frac{b_{n-4}(-4+n+r)}{12} + b_{n-2}(n+r-2) \\
&- 2(n+r)\,b_n - b_{n-2} + \frac{b_{n-4}}{6} - \frac{b_{n-6}}{120} + \frac{b_{n-8}}{5040} = 0
\end{aligned} \tag{4}$$

Which for for the root $r = 0$ becomes

$$\begin{aligned}
&-\frac{b_{n-6}(n-6)\,(n-7)}{5040} + \frac{b_{n-4}(n-4)\,(n-5)}{120} \\
&- \frac{b_{n-2}(n-2)\,(n-3)}{6} + b_n n(n-1) + \frac{b_{n-6}(n-6)}{360} - \frac{b_{n-4}(n-4)}{12} \\
&+ b_{n-2}(n-2) - 2nb_n - b_{n-2} + \frac{b_{n-4}}{6} - \frac{b_{n-6}}{120} + \frac{b_{n-8}}{5040} = 0
\end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{n^2 b_{n-6} - 42n^2 b_{n-4} + 840n^2 b_{n-2} + 2nrb_{n-6} - 84nrb_{n-4} + 1680nrb_{n-2} + r^2 b_{n-6} - 42r^2 b_{n-4} + 840r^2 b_{n} \cdots}{5040n^2 + 1008 \cdots} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{n^2 b_{n-6} - 42n^2 b_{n-4} + 840n^2 b_{n-2} - 27nb_{n-6} + 798nb_{n-4} - 9240nb_{n-2} - b_{n-8} + 168b_{n-6} - 3360b_{n-4} + \cdots}{5040n^2 - 15120n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{r-6}{12+6r}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{7r^3-63r^2+146r+120}{360(2+r)(4+r)(1+r)}$	$\frac{1}{24}$
b_5	0	0
b_6	$\frac{31r^5-248r^4+497r^3+1508r^2-6324r-3024}{15120(2+r)(4+r)(1+r)(6+r)(3+r)}$	$-\frac{1}{720}$
b_7	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3 \left(1 - \frac{x^2}{10} + \frac{x^4}{280} - \frac{x^6}{15120} + O(x^6)\right) + c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^3 \left(1 - \frac{x^2}{10} + \frac{x^4}{280} - \frac{x^6}{15120} + O(x^6)\right) + c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^3 \left(1 - \frac{x^2}{10} + \frac{x^4}{280} - \frac{x^6}{15120} + O(x^6)\right) + c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^6)\right) \quad (1)$$

Verification of solutions

$$y = c_1 x^3 \left(1 - \frac{x^2}{10} + \frac{x^4}{280} - \frac{x^6}{15120} + O(x^6)\right) + c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^6)\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      -u(t)+t*diff(u(t),t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 32

```
Order:=6;
dsolve(sin(x)*diff(y(x),x$2)-2*cos(x)*diff(y(x),x)-sin(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 - \frac{1}{10}x^2 + \frac{1}{280}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(12 - 6x^2 + \frac{1}{2}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 44

```
AsymptoticDSolveValue[Sin[x]*y''[x]-2*Cos[x]*y'[x]-Sin[x]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{24} - \frac{x^2}{2} + 1\right) + c_2\left(\frac{x^7}{280} - \frac{x^5}{10} + x^3\right)$$

1.12 problem 3.24 (g)

Internal problem ID [5491]
Internal file name [`OUTPUT/4739_Sunday_June_05_2022_03_04_31_PM_2570596/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.24 (g).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (24) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (25) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= yx^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x(xy' + 2y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^4 + 4xy' + 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + 8yx^3 + 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12y'x^3 + x^2 y\left(x^4 + 30\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 2y(0) \\
F_3 &= 6y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) - a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{12} a_0 x^4 + \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^4}{12}\right) a_0 + \left(x + \frac{1}{20} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{20} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Verified OK.

1.12.1 Maple step by step solution

Let's solve

$y'' = yx^2$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - y x^2 = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

- ○ Shift index using $k- >k-2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- ○ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$[2a_2 = 0, 6a_3 = 0]$

- Solve for the dependent coefficient(s)

$\{a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} - a_{k-2} = 0$

- Shift index using $k- >k+2$

$\left((k+2)^2 + 3k + 8\right) a_{k+4} - a_k = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)-x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{20} x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]-x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^5}{20} + x\right) + c_1 \left(\frac{x^4}{12} + 1\right)$$

1.13 problem 3.24 (h)

Internal problem ID [5492]
Internal file name [`OUTPUT/4740_Sunday_June_05_2022_03_04_32_PM_12686391/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.24 (h).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x(x+2)\, y'' + (1+x)\, y' - 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^2 + 2x)\, y'' + (1+x)\, y' - 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1+x}{x\,(x+2)}$$
$$q(x) = -\frac{4}{x\,(x+2)}$$

Table 14: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+x}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{4}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x+2)\,y'' + (1+x)\,y' - 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x+2)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (1+x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-4a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-4a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r}r(-1+r)+r\,x^{-1+r}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(2r-1)=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2-r=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1&=\frac{1}{2}\\ r_2&=0\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2r-1)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned}y_1(x)&=x^{r_1}\left(\sum_{n=0}^{\infty}a_n x^n\right)\\ y_2(x)&=x^{r_2}\left(\sum_{n=0}^{\infty}b_n x^n\right)\end{aligned}$$

Or

$$\begin{aligned}y_1(x)&=\sum_{n=0}^{\infty}a_n x^{n+\frac{1}{2}}\\ y_2(x)&=\sum_{n=0}^{\infty}b_n x^n\end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}&a_{n-1}(n+r-1)(n+r-2)+2a_n(n+r)(n+r-1)\\ &+a_{n-1}(n+r-1)+a_n(n+r)-4a_{n-1}=0\end{aligned}\tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 3)}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}(4n^2 - 4n - 15)}{8n^2 + 4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 + 4}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{5}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{2r^2+3r+1}$	$\frac{5}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^3 - 7r + 6}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{7}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{2r^2+3r+1}$	$\frac{5}{4}$
a_2	$\frac{r^3-7r+6}{4r^3+12r^2+11r+3}$	$\frac{7}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r+4)\, r(-1+r)\,(r-2)}{8r^4+44r^3+82r^2+61r+15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{3}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{2r^2+3r+1}$	$\frac{5}{4}$
a_2	$\frac{r^3-7r+6}{4r^3+12r^2+11r+3}$	$\frac{7}{32}$
a_3	$-\frac{(r+4)r(-1+r)(r-2)}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{3}{128}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r-2)\,(-1+r)\, r(r+5)}{16r^4+128r^3+344r^2+352r+105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{11}{2048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{2r^2+3r+1}$	$\frac{5}{4}$
a_2	$\frac{r^3-7r+6}{4r^3+12r^2+11r+3}$	$\frac{7}{32}$
a_3	$-\frac{(r+4)r(-1+r)(r-2)}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{3}{128}$
a_4	$\frac{(r-2)(-1+r)r(r+5)}{16r^4+128r^3+344r^2+352r+105}$	$\frac{11}{2048}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r+6)(r+2)\, r(r-2)(-1+r)}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{13}{8192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+4}{2r^2+3r+1}$	$\frac{5}{4}$
a_2	$\frac{r^3-7r+6}{4r^3+12r^2+11r+3}$	$\frac{7}{32}$
a_3	$-\frac{(r+4)r(-1+r)(r-2)}{8r^4+44r^3+82r^2+61r+15}$	$-\frac{3}{128}$
a_4	$\frac{(r-2)(-1+r)r(r+5)}{16r^4+128r^3+344r^2+352r+105}$	$\frac{11}{2048}$
a_5	$-\frac{(r+6)(r+2)r(r-2)(-1+r)}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{13}{8192}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= \sqrt{x}\left(1 + \frac{5x}{4} + \frac{7x^2}{32} - \frac{3x^3}{128} + \frac{11x^4}{2048} - \frac{13x^5}{8192} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \le n$ the recursive equation is

$$\begin{aligned} &b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) \\ &+ b_{n-1}(n+r-1) + (n+r)\, b_n - 4b_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2 + 2nr + r^2 - 2n - 2r - 3)}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{b_{n-1}(n^2 - 2n - 3)}{n\,(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-r^2 + 4}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{2r^2+3r+1}$	4

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^3 - 7r + 6}{4r^3 + 12r^2 + 11r + 3}$$

Which for the root $r = 0$ becomes

$$b_2 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{2r^2+3r+1}$	4
b_2	$\frac{r^3-7r+6}{4r^3+12r^2+11r+3}$	2

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(r+4)\, r(-1+r)\,(r-2)}{8r^4 + 44r^3 + 82r^2 + 61r + 15}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{2r^2+3r+1}$	4
b_2	$\frac{r^3-7r+6}{4r^3+12r^2+11r+3}$	2
b_3	$-\frac{(r+4)r(-1+r)(r-2)}{8r^4+44r^3+82r^2+61r+15}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r-2)(-1+r)\,r(r+5)}{16r^4+128r^3+344r^2+352r+105}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{2r^2+3r+1}$	4
b_2	$\frac{r^3-7r+6}{4r^3+12r^2+11r+3}$	2
b_3	$-\frac{(r+4)r(-1+r)(r-2)}{8r^4+44r^3+82r^2+61r+15}$	0
b_4	$\frac{(r-2)(-1+r)r(r+5)}{16r^4+128r^3+344r^2+352r+105}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r+6)(r+2)\,r(r-2)(-1+r)}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+4}{2r^2+3r+1}$	4
b_2	$\frac{r^3-7r+6}{4r^3+12r^2+11r+3}$	2
b_3	$-\frac{(r+4)r(-1+r)(r-2)}{8r^4+44r^3+82r^2+61r+15}$	0
b_4	$\frac{(r-2)(-1+r)r(r+5)}{16r^4+128r^3+344r^2+352r+105}$	0
b_5	$-\frac{(r+6)(r+2)r(r-2)(-1+r)}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 + 4x + 2x^2 + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1+\frac{5x}{4}+\frac{7x^2}{32}-\frac{3x^3}{128}+\frac{11x^4}{2048}-\frac{13x^5}{8192}+O(x^6)\right)+c_2\left(1+4x+2x^2+O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1+\frac{5x}{4}+\frac{7x^2}{32}-\frac{3x^3}{128}+\frac{11x^4}{2048}-\frac{13x^5}{8192}+O(x^6)\right)+c_2\left(1+4x+2x^2+O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\left(1+\frac{5x}{4}+\frac{7x^2}{32}-\frac{3x^3}{128}+\frac{11x^4}{2048}-\frac{13x^5}{8192}+O(x^6)\right)+c_2\left(1+4x+2x^2+O(x^6)\right) \quad (1)$$

Verification of solutions

$$y = c_1 \sqrt{x}\left(1+\frac{5x}{4}+\frac{7x^2}{32}-\frac{3x^3}{128}+\frac{11x^4}{2048}-\frac{13x^5}{8192}+O(x^6)\right)+c_2\left(1+4x+2x^2+O(x^6)\right)$$

Verified OK.

1.13.1 Maple step by step solution

Let's solve

$x(x+2)\, y'' + (1+x)\, y' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y}{x(x+2)} - \frac{(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x(x+2)} - \frac{4y}{x(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+x}{x(x+2)}, P_3(x) = -\frac{4}{x(x+2)}\right]$

  ◦ $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

  $\left((x+2) \cdot P_2(x)\right)\bigg|_{x=-2} = \frac{1}{2}$

  ◦ $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

  $\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$

  ◦ $x = -2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $x(x+2)\, y'' + (1+x)\, y' - 4y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (-1+u)\left(\frac{d}{du}y(u)\right) - 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+1+2r) + a_k(k+r+2)(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + a_k(k+r+2)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+2)(k+r-2)}{(2k+1+2r)(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k+2)(k-2)}{(2k+1)(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -4a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{2}$
- Express in terms of a_0

  $a_2 = 2a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (2u^2 - 4u + 1)$

- Revert the change of variables $u = x + 2$

  $[y = a_0(1 + 4x + 2x^2)]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k (k+\frac{5}{2})(k-\frac{3}{2})}{(2k+2)(k+\frac{3}{2})}$

- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k (k+\frac{5}{2})(k-\frac{3}{2})}{(2k+2)(k+\frac{3}{2})}\right]$

- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k (k+\frac{5}{2})(k-\frac{3}{2})}{(2k+2)(k+\frac{3}{2})}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0(1 + 4x + 2x^2) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{1}{2}}\right), b_{k+1} = \frac{b_k (k+\frac{5}{2})(k-\frac{3}{2})}{(2k+2)(k+\frac{3}{2})}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   Solution is available but has compositions of trig with ln functions of radicals. Attempt
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 38

```
Order:=6;
dsolve(x*(x+2)*diff(y(x),x$2)+(x+1)*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{5}{4}x+\frac{7}{32}x^2-\frac{3}{128}x^3+\frac{11}{2048}x^4-\frac{13}{8192}x^5+\mathrm{O}\left(x^6\right)\right) + c_2\left(1+4x+2x^2+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*(x+2)*y''[x]+(x+1)*y'[x]-4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(2x^2+4x+1\right)+c_1\sqrt{x}\left(-\frac{13x^5}{8192}+\frac{11x^4}{2048}-\frac{3x^3}{128}+\frac{7x^2}{32}+\frac{5x}{4}+1\right)$$

1.14 problem 3.24 (i)

Internal problem ID [5493]
Internal file name [`OUTPUT/4741_Sunday_June_05_2022_03_04_34_PM_42135889/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.24 (i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy'' + \left(\frac{1}{2} - x\right) y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + \left(\frac{1}{2} - x\right) y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x - 1}{2x}$$
$$q(x) = -\frac{1}{x}$$

Table 16: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x-1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + \left(\frac{1}{2} - x\right) y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\frac{1}{2} - x\right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} \frac{(n+r) a_n x^{n+r-1}}{2}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} \frac{(n+r) a_n x^{n+r-1}}{2}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + \frac{(n+r) a_n x^{n+r-1}}{2} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + \frac{r a_0 x^{-1+r}}{2} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + \frac{r\, x^{-1+r}}{2}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}\left(-\frac{1}{2}+r\right) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{2}r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}\left(-\frac{1}{2}+r\right) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + \frac{a_n(n+r)}{2} - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}}{2n-1+2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2}{1 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{1+2r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{1+2r}$	1
a_2	$\frac{4}{4r^2+8r+3}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{1+2r}$	1
a_2	$\frac{4}{4r^2+8r+3}$	$\frac{1}{2}$
a_3	$\frac{8}{8r^3+36r^2+46r+15}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{1+2r}$	1
a_2	$\frac{4}{4r^2+8r+3}$	$\frac{1}{2}$
a_3	$\frac{8}{8r^3+36r^2+46r+15}$	$\frac{1}{6}$
a_4	$\frac{16}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{1+2r}$	1
a_2	$\frac{4}{4r^2+8r+3}$	$\frac{1}{2}$
a_3	$\frac{8}{8r^3+36r^2+46r+15}$	$\frac{1}{6}$
a_4	$\frac{16}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{24}$
a_5	$\frac{32}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{1}{120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= \sqrt{x}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + \frac{(n+r)\,b_n}{2} - b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{2b_{n-1}}{2n-1+2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{2b_{n-1}}{2n-1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2}{1 + 2r}$$

Which for the root $r = 0$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{1+2r}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^2 + 8r + 3}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{4}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{1+2r}$	2
b_2	$\frac{4}{4r^2+8r+3}$	$\frac{4}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{8}{15}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{1+2r}$	2
b_2	$\frac{4}{4r^2+8r+3}$	$\frac{4}{3}$
b_3	$\frac{8}{8r^3+36r^2+46r+15}$	$\frac{8}{15}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{16}{105}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{1+2r}$	2
b_2	$\frac{4}{4r^2+8r+3}$	$\frac{4}{3}$
b_3	$\frac{8}{8r^3+36r^2+46r+15}$	$\frac{8}{15}$
b_4	$\frac{16}{16r^4+128r^3+344r^2+352r+105}$	$\frac{16}{105}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{32}{945}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{1+2r}$	2
b_2	$\frac{4}{4r^2+8r+3}$	$\frac{4}{3}$
b_3	$\frac{8}{8r^3+36r^2+46r+15}$	$\frac{8}{15}$
b_4	$\frac{16}{16r^4+128r^3+344r^2+352r+105}$	$\frac{16}{105}$
b_5	$\frac{32}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{32}{945}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 + 2x + \frac{4x^2}{3} + \frac{8x^3}{15} + \frac{16x^4}{105} + \frac{32x^5}{945} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x} \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2 \left(1 + 2x + \frac{4x^2}{3} + \frac{8x^3}{15} + \frac{16x^4}{105} + \frac{32x^5}{945} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x} \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2 \left(1 + 2x + \frac{4x^2}{3} + \frac{8x^3}{15} + \frac{16x^4}{105} + \frac{32x^5}{945} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \sqrt{x} \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
+ c_2 \left(1 + 2x + \frac{4x^2}{3} + \frac{8x^3}{15} + \frac{16x^4}{105} + \frac{32x^5}{945} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)$$
$$+ c_2\left(1+2x+\frac{4x^2}{3}+\frac{8x^3}{15}+\frac{16x^4}{105}+\frac{32x^5}{945}+O(x^6)\right)$$

Verified OK.

1.14.1 Maple step by step solution

Let's solve

$y''x + \left(\frac{1}{2} - x\right) y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x} + \frac{(2x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{2x} - \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x-1}{2x}, P_3(x) = -\frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2y''x + (1-2x)y' - 2y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) - 2a_k(k+1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(\left(k+\frac{1}{2}+r\right) a_{k+1} - a_k\right)(k+1+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{2a_k}{2k+1+2r}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{2a_k}{2k+1}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{2a_k}{2k+1}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = \frac{2a_k}{2k+2}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{2a_k}{2k+1}, b_{k+1} = \frac{2b_k}{2k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 44

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(1/2-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 + 2x + \frac{4}{3}x^2 + \frac{8}{15}x^3 + \frac{16}{105}x^4 + \frac{32}{945}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 79

```
AsymptoticDSolveValue[x*y''[x]+(1/2-x)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + c_2\left(\frac{32x^5}{945} + \frac{16x^4}{105} + \frac{8x^3}{15} + \frac{4x^2}{3} + 2x + 1\right)$$

1.15 problem 3.25 v=1/2

Internal problem ID [5494]
Internal file name [`OUTPUT/4742_Sunday_June_05_2022_03_04_36_PM_88820856/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.25 v=1/2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 + \frac{1}{4}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 + \frac{1}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4x^2 + 1}{4x^2}$$

Table 18: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2+1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 + \frac{1}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 + \frac{1}{4}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} \frac{a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) + \frac{a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r + \frac{a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r + \frac{x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2+1)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + \frac{1}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{i}{2}$$
$$r_2 = -\frac{i}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 + 1)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{i}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{i}{2}}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_n(n + r) + a_{n-2} + \frac{a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 + 1} \tag{4}$$

Which for the root $r = \frac{i}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n\left(i+n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{i}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2+16r+17}$$

Which for the root $r = \frac{i}{2}$ becomes

$$a_2 = -\frac{1}{5}+\frac{i}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+17}$	$-\frac{1}{5}+\frac{i}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+17}$	$-\frac{1}{5}+\frac{i}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 17)(4r^2 + 32r + 65)}$$

Which for the root $r = \frac{i}{2}$ becomes

$$a_4 = \frac{7}{680} - \frac{3i}{340}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+17}$	$-\frac{1}{5} + \frac{i}{10}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+17)(4r^2+32r+65)}$	$\frac{7}{680} - \frac{3i}{340}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+17}$	$-\frac{1}{5} + \frac{i}{10}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+17)(4r^2+32r+65)}$	$\frac{7}{680} - \frac{3i}{340}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{i}{2}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{i}{2}} \left(1 + \left(-\frac{1}{5} + \frac{i}{10}\right) x^2 + \left(\frac{7}{680} - \frac{3i}{340}\right) x^4 + O(x^6)\right) \end{aligned}$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$y_2(x) = x^{-\frac{i}{2}}\left(1+\left(-\frac{1}{5}-\frac{i}{10}\right)x^2+\left(\frac{7}{680}+\frac{3i}{340}\right)x^4+O(x^6)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x)\\
&= c_1x^{\frac{i}{2}}\left(1+\left(-\frac{1}{5}+\frac{i}{10}\right)x^2+\left(\frac{7}{680}-\frac{3i}{340}\right)x^4+O(x^6)\right)\\
&\quad+c_2x^{-\frac{i}{2}}\left(1+\left(-\frac{1}{5}-\frac{i}{10}\right)x^2+\left(\frac{7}{680}+\frac{3i}{340}\right)x^4+O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h\\
&= c_1x^{\frac{i}{2}}\left(1+\left(-\frac{1}{5}+\frac{i}{10}\right)x^2+\left(\frac{7}{680}-\frac{3i}{340}\right)x^4+O(x^6)\right)\\
&\quad+c_2x^{-\frac{i}{2}}\left(1+\left(-\frac{1}{5}-\frac{i}{10}\right)x^2+\left(\frac{7}{680}+\frac{3i}{340}\right)x^4+O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1x^{\frac{i}{2}}\left(1+\left(-\frac{1}{5}+\frac{i}{10}\right)x^2+\left(\frac{7}{680}-\frac{3i}{340}\right)x^4+O(x^6)\right)\\
&\quad+c_2x^{-\frac{i}{2}}\left(1+\left(-\frac{1}{5}-\frac{i}{10}\right)x^2+\left(\frac{7}{680}+\frac{3i}{340}\right)x^4+O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1x^{\frac{i}{2}}\left(1+\left(-\frac{1}{5}+\frac{i}{10}\right)x^2+\left(\frac{7}{680}-\frac{3i}{340}\right)x^4+O(x^6)\right)\\
&\quad+c_2x^{-\frac{i}{2}}\left(1+\left(-\frac{1}{5}-\frac{i}{10}\right)x^2+\left(\frac{7}{680}+\frac{3i}{340}\right)x^4+O(x^6)\right)
\end{aligned}$$

Verified OK.

1.15.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 + \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2+1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2+1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(4r^2+1)\, x^r + a_1(4r^2+8r+5)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k^2+8kr+4r^2+1) + 4a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r^2+1=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{I}{2}, \frac{I}{2}\right\}$
- Each term must be 0

  $a_1(4r^2+8r+5) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2+1) + 4a_{k-2} = 0$
- Shift index using $k- > k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 + 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+17}$
- Recursion relation for $r = -\frac{I}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2-4\,\mathrm{I}k+16-8\,\mathrm{I}+16k}$
- Solution for $r = -\frac{I}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{I}{2}}, a_{k+2} = -\frac{4a_k}{4k^2-4\,\mathrm{I}k+16-8\,\mathrm{I}+16k}, a_1 = 0\right]$

- Recursion relation for $r = \frac{I}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+4\,\mathrm{I}k+16+8\,\mathrm{I}+16k}$

- Solution for $r = \frac{I}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{I}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+4\,\mathrm{I}k+16+8\,\mathrm{I}+16k}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{I}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{I}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2-4\,\mathrm{I}k+16-8\,\mathrm{I}+16k}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+4\,\mathrm{I}k+16+8\,\mathrm{I}+\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2+(1/2)^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{-\frac{i}{2}} \left(1 + \left(-\frac{1}{5} - \frac{i}{10}\right) x^2 + \left(\frac{7}{680} + \frac{3i}{340}\right) x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{i}{2}} \left(1 + \left(-\frac{1}{5} + \frac{i}{10}\right) x^2 + \left(\frac{7}{680} - \frac{3i}{340}\right) x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 66

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2+1/4)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \rightarrow & \left(\frac{7}{680}+\frac{3i}{340}\right) c_2 x^{-\frac{i}{2}}\left(x^4-(16-4i)x^2+(56-48i)\right) \\
& +\left(\frac{7}{680}-\frac{3i}{340}\right) c_1 x^{\frac{i}{2}}\left(x^4-(16+4i)x^2+(56+48i)\right)
\end{aligned}
$$

1.16 problem 3.25 v=3/2

Internal problem ID [5495]
Internal file name [`OUTPUT/4743_Sunday_June_05_2022_03_04_40_PM_72257281/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.25 v=3/2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 + \frac{9}{4}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 + \frac{9}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4x^2 + 9}{4x^2}$$

Table 20: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2+9}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 + \frac{9}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(x^2 + \frac{9}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{9 a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} \frac{9 a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) + \frac{9 a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r + \frac{9 a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r + \frac{9 x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2+9)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + \frac{9}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3i}{2}$$
$$r_2 = -\frac{3i}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 + 9)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3i}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{3i}{2}}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)\,(n + r - 1) + a_n(n + r) + a_{n-2} + \frac{9a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 + 9} \tag{4}$$

Which for the root $r = \frac{3i}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n(3i+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3i}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2+16r+25}$$

Which for the root $r = \frac{3i}{2}$ becomes

$$a_2 = -\frac{1}{13} + \frac{3i}{26}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+25}$	$-\frac{1}{13} + \frac{3i}{26}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+25}$	$-\frac{1}{13} + \frac{3i}{26}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 25)(4r^2 + 32r + 73)}$$

Which for the root $r = \frac{3i}{2}$ becomes

$$a_4 = -\frac{1}{2600} - \frac{9i}{1300}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+25}$	$-\frac{1}{13} + \frac{3i}{26}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+25)(4r^2+32r+73)}$	$-\frac{1}{2600} - \frac{9i}{1300}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+25}$	$-\frac{1}{13} + \frac{3i}{26}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+25)(4r^2+32r+73)}$	$-\frac{1}{2600} - \frac{9i}{1300}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3i}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3i}{2}}\left(1 + \left(-\frac{1}{13} + \frac{3i}{26}\right)x^2 + \left(-\frac{1}{2600} - \frac{9i}{1300}\right)x^4 + O(x^6)\right) \end{aligned}$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$y_2(x) = x^{-\frac{3i}{2}}\left(1+\left(-\frac{1}{13}-\frac{3i}{26}\right)x^2+\left(-\frac{1}{2600}+\frac{9i}{1300}\right)x^4+O(x^6)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1x^{\frac{3i}{2}}\left(1+\left(-\frac{1}{13}+\frac{3i}{26}\right)x^2+\left(-\frac{1}{2600}-\frac{9i}{1300}\right)x^4+O(x^6)\right) \\
&\quad + c_2x^{-\frac{3i}{2}}\left(1+\left(-\frac{1}{13}-\frac{3i}{26}\right)x^2+\left(-\frac{1}{2600}+\frac{9i}{1300}\right)x^4+O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1x^{\frac{3i}{2}}\left(1+\left(-\frac{1}{13}+\frac{3i}{26}\right)x^2+\left(-\frac{1}{2600}-\frac{9i}{1300}\right)x^4+O(x^6)\right) \\
&\quad + c_2x^{-\frac{3i}{2}}\left(1+\left(-\frac{1}{13}-\frac{3i}{26}\right)x^2+\left(-\frac{1}{2600}+\frac{9i}{1300}\right)x^4+O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1x^{\frac{3i}{2}}\left(1+\left(-\frac{1}{13}+\frac{3i}{26}\right)x^2+\left(-\frac{1}{2600}-\frac{9i}{1300}\right)x^4+O(x^6)\right) \\
+ c_2x^{-\frac{3i}{2}}\left(1+\left(-\frac{1}{13}-\frac{3i}{26}\right)x^2+\left(-\frac{1}{2600}+\frac{9i}{1300}\right)x^4+O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y = c_1x^{\frac{3i}{2}}\left(1+\left(-\frac{1}{13}+\frac{3i}{26}\right)x^2+\left(-\frac{1}{2600}-\frac{9i}{1300}\right)x^4+O(x^6)\right) \\
+ c_2x^{-\frac{3i}{2}}\left(1+\left(-\frac{1}{13}-\frac{3i}{26}\right)x^2+\left(-\frac{1}{2600}+\frac{9i}{1300}\right)x^4+O(x^6)\right)
\end{aligned}$$

Verified OK.

1.16.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 + \frac{9}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2+9)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2+9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2+9}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{9}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 + 9) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (4r^2+9)\, x^r + a_1 (4r^2+8r+13)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (4k^2+8kr+4r^2+9) + 4a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r^2+9=0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{3\,\mathrm{I}}{2}, \frac{3\,\mathrm{I}}{2}\right\}$$

- Each term must be 0

  $$a_1 (4r^2+8r+13) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k (4k^2+8kr+4r^2+9) + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$a_{k+2} \left(4(k+2)^2 + 8(k+2)\, r + 4r^2 + 9\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+25}$$

- Recursion relation for $r = -\frac{3\,\mathrm{I}}{2}$

  $$a_{k+2} = -\frac{4a_k}{4k^2-12\,\mathrm{I}k+16-24\,\mathrm{I}+16k}$$

- Solution for $r = -\frac{3\,\mathrm{I}}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3\,\mathrm{I}}{2}}, a_{k+2} = -\frac{4a_k}{4k^2-12\,\mathrm{I}k+16-24\,\mathrm{I}+16k}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3\,\mathrm{I}}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12\,\mathrm{I}k+16+24\,\mathrm{I}+16k}$

- Solution for $r = \frac{3\,\mathrm{I}}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3\,\mathrm{I}}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12\,\mathrm{I}k+16+24\,\mathrm{I}+16k}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3\,\mathrm{I}}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3\,\mathrm{I}}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2-12\,\mathrm{I}k+16-24\,\mathrm{I}+16k}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+12\,\mathrm{I}k+16}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2+(3/2)^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{-\frac{3i}{2}} \left(1 + \left(-\frac{1}{13} - \frac{3i}{26}\right) x^2 + \left(-\frac{1}{2600} + \frac{9i}{1300}\right) x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{3i}{2}} \left(1 + \left(-\frac{1}{13} + \frac{3i}{26}\right) x^2 + \left(-\frac{1}{2600} - \frac{9i}{1300}\right) x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 66

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2+9/4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \left(-\frac{1}{2600} - \frac{9i}{1300}\right) c_1 x^{\frac{3i}{2}} \left(x^4 - (16 + 12i)x^2 - (8 - 144i)\right) - \left(\frac{1}{2600} - \frac{9i}{1300}\right) c_2 x^{-\frac{3i}{2}} \left(x^4 - (16 - 12i)x^2 - (8 + 144i)\right)$$

1.17 problem 3.25 v=5/2

Internal problem ID [5496]
Internal file name [`OUTPUT/4744_Sunday_June_05_2022_03_04_46_PM_28423381/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.25 v=5/2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 + \frac{25}{4}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 + \frac{25}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4x^2 + 25}{4x^2}$$

Table 22: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2+25}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x y' + \left(x^2 + \frac{25}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(x^2 + \frac{25}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{25 a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} \frac{25 a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) + \frac{25 a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r + \frac{25 a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r + \frac{25 x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2+25)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + \frac{25}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{5i}{2}$$
$$r_2 = -\frac{5i}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 + 25)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{5i}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{5i}{2}}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)\,(n + r - 1) + a_n(n + r) + a_{n-2} + \frac{25a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 + 25} \tag{4}$$

Which for the root $r = \frac{5i}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n(5i+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5i}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2+16r+41}$$

Which for the root $r = \frac{5i}{2}$ becomes

$$a_2 = -\frac{1}{29} + \frac{5i}{58}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+41}$	$-\frac{1}{29}+\frac{5i}{58}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+41}$	$-\frac{1}{29}+\frac{5i}{58}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 41)(4r^2 + 32r + 89)}$$

Which for the root $r = \frac{5i}{2}$ becomes

$$a_4 = -\frac{17}{9512} - \frac{15i}{4756}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+41}$	$-\frac{1}{29} + \frac{5i}{58}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+41)(4r^2+32r+89)}$	$-\frac{17}{9512} - \frac{15i}{4756}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+41}$	$-\frac{1}{29} + \frac{5i}{58}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+41)(4r^2+32r+89)}$	$-\frac{17}{9512} - \frac{15i}{4756}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{5i}{2}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{5i}{2}} \left(1 + \left(-\frac{1}{29} + \frac{5i}{58}\right) x^2 + \left(-\frac{17}{9512} - \frac{15i}{4756}\right) x^4 + O(x^6)\right) \end{aligned}$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$y_2(x) = x^{-\frac{5i}{2}}\left(1+\left(-\frac{1}{29}-\frac{5i}{58}\right)x^2+\left(-\frac{17}{9512}+\frac{15i}{4756}\right)x^4+O(x^6)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x)+c_2y_2(x) \\ &= c_1x^{\frac{5i}{2}}\left(1+\left(-\frac{1}{29}+\frac{5i}{58}\right)x^2+\left(-\frac{17}{9512}-\frac{15i}{4756}\right)x^4+O(x^6)\right) \\ &\quad + c_2x^{-\frac{5i}{2}}\left(1+\left(-\frac{1}{29}-\frac{5i}{58}\right)x^2+\left(-\frac{17}{9512}+\frac{15i}{4756}\right)x^4+O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^{\frac{5i}{2}}\left(1+\left(-\frac{1}{29}+\frac{5i}{58}\right)x^2+\left(-\frac{17}{9512}-\frac{15i}{4756}\right)x^4+O(x^6)\right) \\ &\quad + c_2x^{-\frac{5i}{2}}\left(1+\left(-\frac{1}{29}-\frac{5i}{58}\right)x^2+\left(-\frac{17}{9512}+\frac{15i}{4756}\right)x^4+O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1x^{\frac{5i}{2}}\left(1+\left(-\frac{1}{29}+\frac{5i}{58}\right)x^2+\left(-\frac{17}{9512}-\frac{15i}{4756}\right)x^4+O(x^6)\right) \\ &\quad + c_2x^{-\frac{5i}{2}}\left(1+\left(-\frac{1}{29}-\frac{5i}{58}\right)x^2+\left(-\frac{17}{9512}+\frac{15i}{4756}\right)x^4+O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1x^{\frac{5i}{2}}\left(1+\left(-\frac{1}{29}+\frac{5i}{58}\right)x^2+\left(-\frac{17}{9512}-\frac{15i}{4756}\right)x^4+O(x^6)\right) \\ &\quad + c_2x^{-\frac{5i}{2}}\left(1+\left(-\frac{1}{29}-\frac{5i}{58}\right)x^2+\left(-\frac{17}{9512}+\frac{15i}{4756}\right)x^4+O(x^6)\right) \end{aligned}$$

Verified OK.

1.17.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 + \frac{25}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2+25)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2+25)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2+25}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{25}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 + 25) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(4r^{2}+25)\,x^{r}+a_{1}(4r^{2}+8r+29)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_{k}(4k^{2}+8kr+4r^{2}+25)+4a_{k-2})\,x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $4r^{2}+25=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{5\,\mathrm{I}}{2},\frac{5\,\mathrm{I}}{2}\right\}$

- Each term must be 0

  $a_{1}(4r^{2}+8r+29)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(4k^{2}+8kr+4r^{2}+25)+4a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^{2}+8(k+2)\,r+4r^{2}+25\right)+4a_{k}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_{k}}{4k^{2}+8kr+4r^{2}+16k+16r+41}$

- Recursion relation for $r=-\frac{5\,\mathrm{I}}{2}$

  $a_{k+2}=-\frac{4a_{k}}{4k^{2}-20\,\mathrm{I}k+16-40\,\mathrm{I}+16k}$

- Solution for $r=-\frac{5\,\mathrm{I}}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k-\frac{5\,\mathrm{I}}{2}},a_{k+2}=-\frac{4a_{k}}{4k^{2}-20\,\mathrm{I}k+16-40\,\mathrm{I}+16k},a_{1}=0\right]$

- Recursion relation for $r = \frac{5\,\mathrm{I}}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20\,\mathrm{I}k+16+40\,\mathrm{I}+16k}$

- Solution for $r = \frac{5\,\mathrm{I}}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5\,\mathrm{I}}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20\,\mathrm{I}k+16+40\,\mathrm{I}+16k}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5\,\mathrm{I}}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5\,\mathrm{I}}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2-20\,\mathrm{I}k+16-40\,\mathrm{I}+16k}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20\,\mathrm{I}k+16}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2+(5/2)^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{-\frac{5i}{2}} \left(1 + \left(-\frac{1}{29} - \frac{5i}{58}\right) x^2 + \left(-\frac{17}{9512} + \frac{15i}{4756}\right) x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{5i}{2}} \left(1 + \left(-\frac{1}{29} + \frac{5i}{58}\right) x^2 + \left(-\frac{17}{9512} - \frac{15i}{4756}\right) x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 66

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2+(5/2)^2)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to & \left(-\frac{17}{9512}-\frac{15i}{4756}\right)c_1x^{\frac{5i}{2}}\left(x^4-(16+20i)x^2-(136-240i)\right) \\
& -\left(\frac{17}{9512}-\frac{15i}{4756}\right)c_2x^{-\frac{5i}{2}}\left(x^4-(16-20i)x^2-(136+240i)\right)
\end{aligned}
$$

1.18 problem 3.26

Internal problem ID [5497]
Internal file name [`OUTPUT/4745_Sunday_June_05_2022_03_04_50_PM_29781487/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\, y''-xy'+y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (32) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (33) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{-y+xy'}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{-y+xy'}{x-1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{-y+xy'}{x-1} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{-y+xy'}{x-1} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{-y+xy'}{x-1}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y(0) \\
F_2 &= y(0) \\
F_3 &= y(0) \\
F_4 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6\right) y(0) + xy'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x-1)\, y'' - xy' + y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x-1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}\right)+\sum_{n\,=0}^{\infty} \left(-(n+2)\, a_{n+2}(n+1)\, x^{n}\right) \\
&+\sum_{n\,=1}^{\infty} \left(-n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)=0
\end{aligned}
\tag{3}
$$

$n=0$ gives

$$-2a_{2}+a_{0}=0$$

$$a_{2}=\frac{a_{0}}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n-(n+2)\, a_{n+2}(n+1)-n a_{n}+a_{n}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{n^{2} a_{n+1}-n a_{n}+n a_{n+1}+a_{n}}{(n+2)\,(n+1)} \\
&= \frac{(-n+1)\, a_{n}}{(n+2)\,(n+1)}+\frac{(n^{2}+n)\, a_{n+1}}{(n+2)\,(n+1)}
\end{aligned}
\tag{5}
$$

For $n=1$ the recurrence equation gives

$$2a_{2}-6a_{3}=0$$

Which after substituting the earlier terms found becomes

$$a_{3}=\frac{a_{0}}{6}$$

For $n=2$ the recurrence equation gives

$$6a_{3}-12a_{4}-a_{2}=0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$12a_4 - 20a_5 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

For $n = 4$ the recurrence equation gives

$$20a_5 - 30a_6 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$30a_6 - 42a_7 - 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + c_2 x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6\right) y(0) + xy'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + c_2 x + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6\right) y(0) + xy'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) c_1 + c_2 x + O(x^6)$$

Verified OK.

1.18.1 Maple step by step solution

Let's solve

$(x-1)\, y'' - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- ○ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

- ○ $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

- ○ $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $(x-1)\, y'' - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2} y(u)\right) + (-u-1)\left(\frac{d}{du} y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- ○ Shift index using $k- > k + 1 - m$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$

- ○ Convert $u \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$

- ○ Shift index using $k- > k + 1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) y(0) + D(y)(0)\, x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 41

```
AsymptoticDSolveValue[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + 1\right) + c_2 x$$

1.19 problem 3.48 (a)

Internal problem ID [5498]
Internal file name [`OUTPUT/4746_Sunday_June_05_2022_03_04_51_PM_15119847/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.48 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + xy = \cos(x)}$$

With the expansion point for the power series method at $x = 0$.

1.19.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n=2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}
$$

Hence

$$
\begin{aligned}
F_0 &= -xy + \cos(x) \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}F_0 \\
&= yx^2 - \cos(x)\,x - y - \sin(x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}F_1 \\
&= \left(x^2-3\right)\cos(x) - x\left(\left(x^2-3\right)y - \sin(x)\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}F_2 \\
&= \left(x^4-6x^2+3\right)y + \left(-x^3+6x\right)\cos(x) + \left(-x^2+4\right)\sin(x) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}F_3 \\
&= \left(x^4-10x^2+13\right)\cos(x) - \left(\left(x^4-10x^2+15\right)y + \left(-x^2+8\right)\sin(x)\right)x
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= -y(0) \\
F_2 &= -3 \\
F_3 &= 3y(0) \\
F_4 &= 13
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + x - \frac{x^3}{2} + \frac{13x^5}{120} + O(x^6)
$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' + xy &= \cos(x)
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= x \\
p(x) &= \cos(x)
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^n\right) = \cos(x) \tag{1}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\cos(x) &= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 + \dots \\ &= \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^n\right) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n \\ \sum_{n=0}^{\infty} x^{1+n} a_n &= \sum_{n=1}^{\infty} a_{n-1} x^n\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \tag{3}$$

$n = 0$ gives

$$
\begin{aligned}
(a_1)\, 1 &= 1 \\
a_1 &= 1
\end{aligned}
$$

Or

$$a_1 = 1$$

For $1 \leq n$, the recurrence equation is

$$((1+n)\, a_{1+n} + a_{n-1})\, x^n = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$
\begin{aligned}
(2a_2 + a_0)\, x &= 0 \\
2a_2 + a_0 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$
\begin{aligned}
(3a_3 + a_1)\, x^2 &= -\frac{x^2}{2} \\
3a_3 + a_1 &= -\frac{1}{2}
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{1}{2}$$

For $n = 3$ the recurrence equation gives

$$
\begin{aligned}
(4a_4 + a_2)\, x^3 &= 0 \\
4a_4 + a_2 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(5a_5 + a_3)\,x^4 &= \frac{x^4}{24}\\ 5a_5 + a_3 &= \frac{1}{24}\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{13}{120}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(6a_6 + a_4)\,x^5 &= 0\\ 6a_6 + a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{48}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + x - \frac{1}{2}a_0x^2 - \frac{1}{2}x^3 + \frac{1}{8}a_0x^4 + \frac{13}{120}x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right)a_0 + x - \frac{x^3}{2} + \frac{13x^5}{120} + O\left(x^6\right) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + x - \frac{x^3}{2} + \frac{13x^5}{120} + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + x - \frac{x^3}{2} + \frac{13x^5}{120} + O(x^6) \tag{2}$$

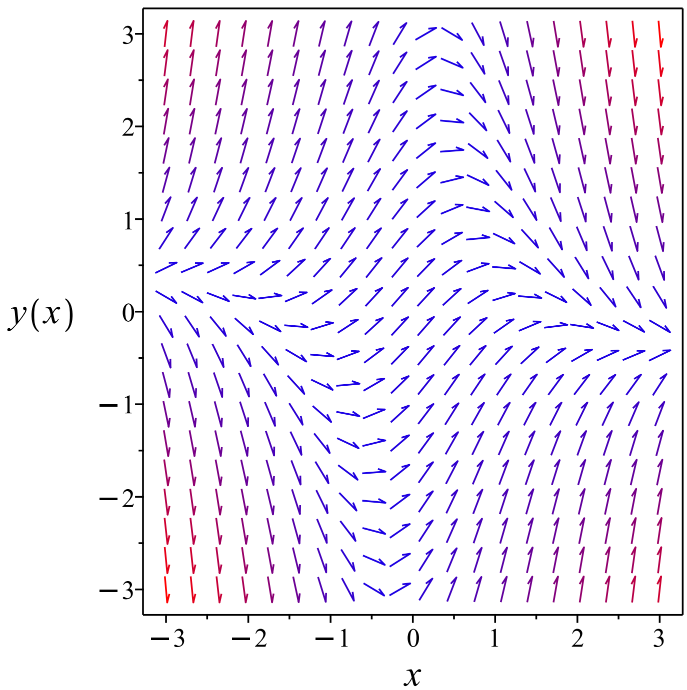


Figure 1: Slope field plot

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + x - \frac{x^3}{2} + \frac{13x^5}{120} + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + x - \frac{x^3}{2} + \frac{13x^5}{120} + O(x^6)$$

Verified OK.

1.19.2 Maple step by step solution

Let's solve

$y' + xy = \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -xy + \cos(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + xy = \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' + xy) = \mu(x)\cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)(y' + xy) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\,x$

- Solve to find the integrating factor

  $\mu(x) = e^{\frac{x^2}{2}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\cos(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\cos(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\cos(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{\frac{x^2}{2}}$

  $y = \frac{\int e^{\frac{x^2}{2}}\cos(x)dx + c_1}{e^{\frac{x^2}{2}}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{I\sqrt{\pi}\,e^{\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{I\sqrt{2}\,x}{2} - \frac{\sqrt{2}}{2}\right)}{4} - \frac{I\sqrt{\pi}\,e^{\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{I\sqrt{2}\,x}{2} + \frac{\sqrt{2}}{2}\right)}{4} + c_1}{e^{\frac{x^2}{2}}}$

- Simplify

$$y = -\frac{\left(\mathrm{I}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(\mathrm{I}x-1)}{2}\right)+\mathrm{I}\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(1+\mathrm{I}x)}{2}\right)-4c_1\right)\mathrm{e}^{-\frac{x^2}{2}}}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
Order:=6;
dsolve(diff(y(x),x)+x*y(x)=cos(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + x - \frac{x^3}{2} + \frac{13x^5}{120} + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 38

```
AsymptoticDSolveValue[y'[x]+x*y[x]==Cos[x],y[x],{x,0,5}]
```

$$y(x) \to \frac{13x^5}{120} - \frac{x^3}{2} + c_1\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right) + x$$

1.20 problem 3.48 (b)

Internal problem ID [5499]
Internal file name [`OUTPUT/4747_Sunday_June_05_2022_03_04_52_PM_79415664/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.48 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' + xy = \frac{1}{x^3}}$$

1.20.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = x$$
$$q(x) = \frac{1}{x^3}$$

Hence the ode is

$$y' + xy = \frac{1}{x^3}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int x dx} \\
&= \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x^3}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x^2}{2}} y\right) &= \left(\mathrm{e}^{\frac{x^2}{2}}\right)\left(\frac{1}{x^3}\right) \\
\mathrm{d}\left(\mathrm{e}^{\frac{x^2}{2}} y\right) &= \left(\frac{\mathrm{e}^{\frac{x^2}{2}}}{x^3}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{x^2}{2}} y &= \int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^3} \mathrm{d}x \\
\mathrm{e}^{\frac{x^2}{2}} y &= -\frac{\mathrm{e}^{\frac{x^2}{2}}}{2x^2} - \frac{\mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right)}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{x^2}{2}}$ results in

$$
y = \mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{2x^2} - \frac{\mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right)}{4}\right) + c_1 \mathrm{e}^{-\frac{x^2}{2}}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{2x^2} - \frac{\mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right)}{4}\right) + c_1 \mathrm{e}^{-\frac{x^2}{2}} \tag{1}
$$

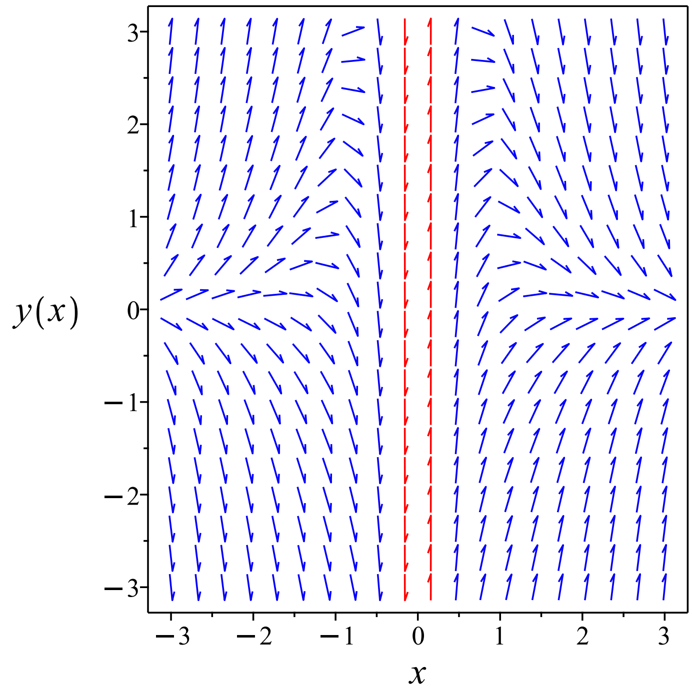


Figure 2: Slope field plot

Verification of solutions

$$y = e^{-\frac{x^2}{2}} \left(-\frac{e^{\frac{x^2}{2}}}{2x^2} - \frac{\mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right)}{4} \right) + c_1 e^{-\frac{x^2}{2}}$$

Verified OK.

1.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y\,x^4 - 1}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 26: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{x^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{x^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y\,x^4 - 1}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{\frac{x^2}{2}} xy \\ S_y &= e^{\frac{x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{\frac{x^2}{2}}}{x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{\frac{R^2}{2}}}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{e^{\frac{R^2}{2}}}{2R^2} - \frac{\text{expIntegral}_1\left(-\frac{R^2}{2}\right)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\frac{x^2}{2}} y = -\frac{e^{\frac{x^2}{2}}}{2x^2} - \frac{\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{4} + c_1$$

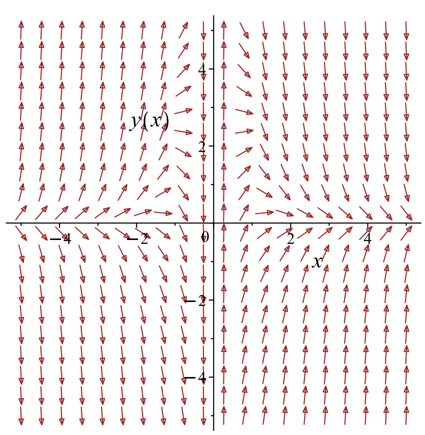
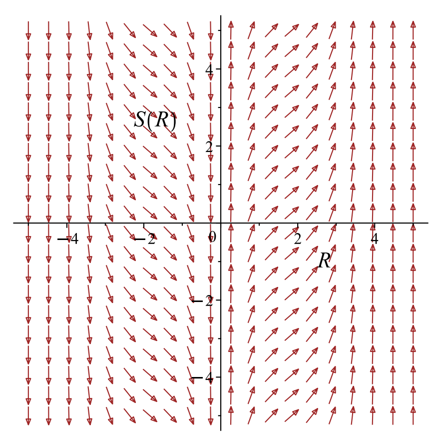
Which simplifies to

$$e^{\frac{x^2}{2}} y = -\frac{e^{\frac{x^2}{2}}}{2x^2} - \frac{\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{4} + c_1$$

Which gives

$$y = -\frac{e^{-\frac{x^2}{2}}\left(\text{expIntegral}_1\left(-\frac{x^2}{2}\right) x^2 - 4c_1 x^2 + 2\,e^{\frac{x^2}{2}}\right)}{4x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y\,x^4-1}{x^3}$ 	$R = x$ $S = e^{\frac{x^2}{2}} y$	$\frac{dS}{dR} = \frac{e^{\frac{R^2}{2}}}{R^3}$ 

Summary
The solution(s) found are the following

$$y = -\frac{e^{-\frac{x^2}{2}}\left(\text{expIntegral}_1\left(-\frac{x^2}{2}\right)x^2 - 4c_1x^2 + 2\,e^{\frac{x^2}{2}}\right)}{4x^2} \tag{1}$$

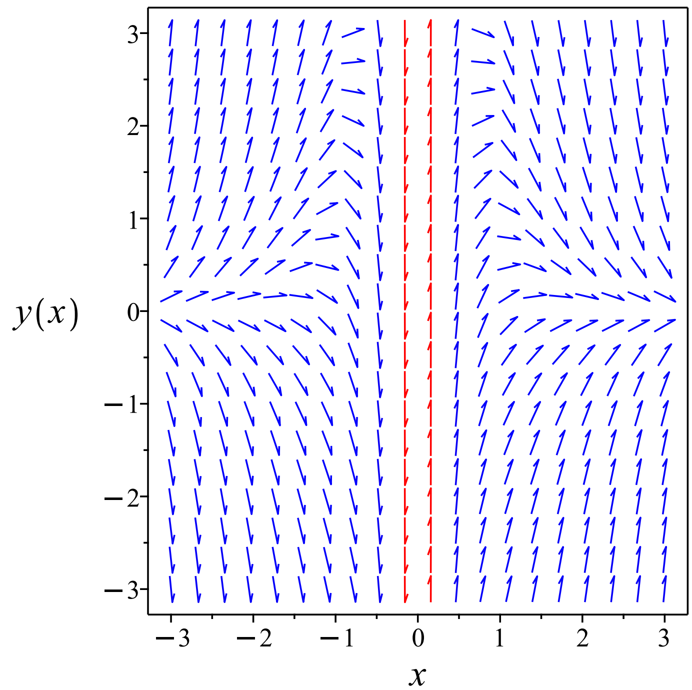


Figure 3: Slope field plot

Verification of solutions

$$y = -\frac{e^{-\frac{x^2}{2}}\left(\text{expIntegral}_1\left(-\frac{x^2}{2}\right)x^2 - 4c_1x^2 + 2\,e^{\frac{x^2}{2}}\right)}{4x^2}$$

Verified OK.

1.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(-xy+\frac{1}{x^3}\right)\mathrm{d}x$$

$$\left(xy-\frac{1}{x^3}\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=xy-\frac{1}{x^3}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(xy - \frac{1}{x^3}\right)\\ &= x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((x) - (0))\\ &= x\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int x\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{x^2}{2}}\\ &= \mathrm{e}^{\frac{x^2}{2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{\frac{x^2}{2}}\left(xy - \frac{1}{x^3}\right)\\ &= \frac{\mathrm{e}^{\frac{x^2}{2}}(y\,x^4 - 1)}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{\frac{x^2}{2}}(1)\\ &= \mathrm{e}^{\frac{x^2}{2}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{\mathrm{e}^{\frac{x^2}{2}} (y\,x^4 - 1)}{x^3}\right) + \left(\mathrm{e}^{\frac{x^2}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{\mathrm{e}^{\frac{x^2}{2}} (y\,x^4 - 1)}{x^3}\,\mathrm{d}x$$
$$\phi = \frac{4\,\mathrm{e}^{\frac{x^2}{2}} y\,x^2 + \mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right) x^2 + 2\,\mathrm{e}^{\frac{x^2}{2}}}{4x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{x^2}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{x^2}{2}}$. Therefore equation (4) becomes

$$\mathrm{e}^{\frac{x^2}{2}} = \mathrm{e}^{\frac{x^2}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{4\,\mathrm{e}^{\frac{x^2}{2}} y\,x^2 + \mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right) x^2 + 2\,\mathrm{e}^{\frac{x^2}{2}}}{4x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{4\,\mathrm{e}^{\frac{x^2}{2}} y\,x^2 + \mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right) x^2 + 2\,\mathrm{e}^{\frac{x^2}{2}}}{4x^2}$$

The solution becomes

$$y = -\frac{\mathrm{e}^{-\frac{x^2}{2}}\left(\mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right) x^2 - 4c_1x^2 + 2\,\mathrm{e}^{\frac{x^2}{2}}\right)}{4x^2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-\frac{x^2}{2}}\left(\mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right) x^2 - 4c_1x^2 + 2\,\mathrm{e}^{\frac{x^2}{2}}\right)}{4x^2} \tag{1}$$

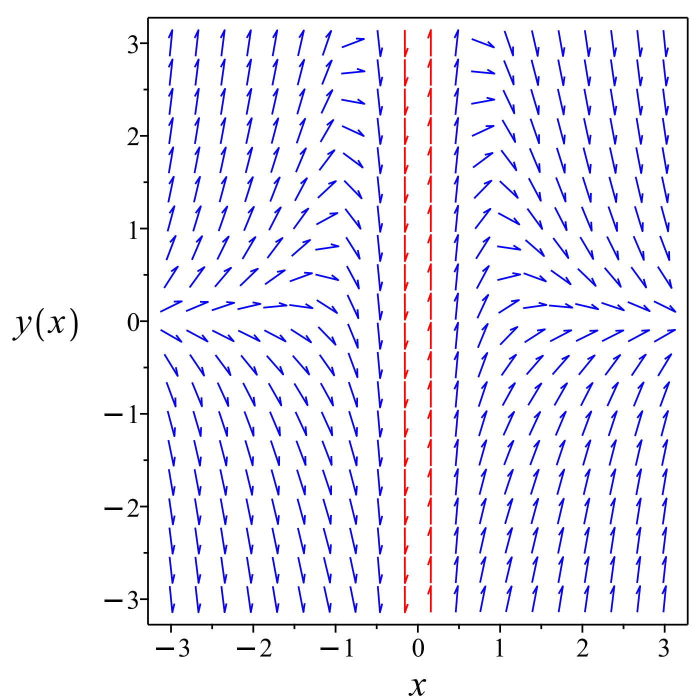


Figure 4: Slope field plot

Verification of solutions

$$y = -\frac{e^{-\frac{x^2}{2}} \left(\text{expIntegral}_1\left(-\frac{x^2}{2}\right) x^2 - 4c_1 x^2 + 2\, e^{\frac{x^2}{2}}\right)}{4x^2}$$

Verified OK.

1.20.4 Maple step by step solution

Let's solve

$y' + xy = \frac{1}{x^3}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -xy + \frac{1}{x^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + xy = \frac{1}{x^3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + xy) = \frac{\mu(x)}{x^3}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y' + xy) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\, x$

- Solve to find the integrating factor

  $\mu(x) = e^{\frac{x^2}{2}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x^3} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x^3} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^3} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{\frac{x^2}{2}}$

$$y = \frac{\int \frac{e^{\frac{x^2}{2}}}{x^3}dx + c_1}{e^{\frac{x^2}{2}}}$$

- Evaluate the integrals on the rhs

$$y = \frac{-\frac{e^{\frac{x^2}{2}}}{2x^2} - \frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{4} + c_1}{e^{\frac{x^2}{2}}}$$

- Simplify

$$y = \frac{4c_1 x^2 e^{-\frac{x^2}{2}} - \mathrm{Ei}_1\left(-\frac{x^2}{2}\right) x^2 e^{-\frac{x^2}{2}} - 2}{4x^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x)+x*y(x)=1/x^3,y(x), singsol=all)
```

$$y(x) = \frac{4c_1 x^2 e^{-\frac{x^2}{2}} - \mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right) x^2 e^{-\frac{x^2}{2}} - 2}{4x^2}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 46

```
DSolve[y'[x]+x*y[x]==1/x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-\frac{x^2}{2}} \text{ExpIntegralEi}\left(\frac{x^2}{2}\right) - \frac{1}{2x^2} + c_1 e^{-\frac{x^2}{2}}$$

1.21 problem 3.48 (c)

Internal problem ID [5500]
Internal file name [`OUTPUT/4748_Sunday_June_05_2022_03_04_54_PM_50846618/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.48 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + y = \frac{1}{x^4}}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^3y'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1}{x^3}$$

Table 29: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*diff(y(x),x$2)+y(x)=1/x^4,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.364 (sec). Leaf size: 800

```
AsymptoticDSolveValue[x^3*y''[x]+y[x]==1/x^4,y[x],{x,0,5}]
```

$$y(x) \to e^{-\frac{2i}{\sqrt{x}}} x^{3/4} \left(\frac{33424574007825x^5}{281474976710656} - \frac{468131288625ix^{9/2}}{8796093022208} - \frac{14783093325x^4}{549755813888} + \frac{66891825ix^{7/2}}{4294967296} + \frac{2837835x^3}{268435456} - \frac{72765ix^{5/2}}{8388608} - \frac{4725x^2}{524288} + \frac{105ix^{3/2}}{8192} + \frac{15x}{512} - \frac{3i\sqrt{x}}{16} + 1 \right) c_1 + e^{\frac{2i}{\sqrt{x}}} x^{3/4} \left(\frac{33424574007825x^5}{281474976710656} + \frac{468131288625ix^{9/2}}{8796093022208} - \frac{14783093325x^4}{549755813888} - \frac{66891825ix^{7/2}}{4294967296} + \frac{28378\ldots}{26843\ldots} \right.$$

1.22 problem 3.48 (d)

Internal problem ID [5501]
Internal file name [`OUTPUT/4749_Sunday_June_05_2022_03_04_54_PM_8074703/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.48 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{xy'' - 2y' + y = \cos(x)}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - 2y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{1}{x}$$

Table 30: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - 2y' + y = \cos(x)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $xy'' - 2y' + y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - 2\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 2(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(x^{-1+m} m(-1+m) - 2m\, x^{-1+m}\right) c_0 = \cos\left(x\right)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln\left(x\right) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}$$

Or

$$\begin{aligned}
y_1(x) &= x^3\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln\left(x\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}$$

Or

$$\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\
y_2(x) &= C y_1(x) \ln\left(x\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)\left(n+r-1\right) - 2a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-1}}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{r^2 - r - 2}$$

Which for the root $r = 3$ becomes

$$a_1 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r^4 - 5r^2 + 4}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{1}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r^4 - 5r^2 + 4)\, r\, (r + 3)}$$

Which for the root $r = 3$ becomes

$$a_3 = -\frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$-\frac{1}{720}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^4 - 5r^2 + 4)\, r\, (r + 3)\, (r^2 + 5r + 4)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{20160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$-\frac{1}{720}$
a_4	$\frac{1}{(r^4-5r^2+4)r(r+3)(r^2+5r+4)}$	$\frac{1}{20160}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(r + 1)^2\, (r - 2)\, (r + 2)^2\, (-1 + r)\, r\, (r + 3)\, (r + 4)\, (r + 5)}$$

Which for the root $r = 3$ becomes

$$a_5 = -\frac{1}{806400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$-\frac{1}{720}$
a_4	$\frac{1}{(r^4-5r^2+4)r(r+3)(r^2+5r+4)}$	$\frac{1}{20160}$
a_5	$-\frac{1}{(r+1)^2(r-2)(r+2)^2(-1+r)r(r+3)(r+4)(r+5)}$	$-\frac{1}{806400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^3\left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= -\frac{1}{(r^4 - 5r^2 + 4)\, r\, (r + 3)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{1}{(r^4 - 5r^2 + 4)\, r\, (r + 3)} &= \lim_{r \to 0} -\frac{1}{(r^4 - 5r^2 + 4)\, r\, (r + 3)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $xy'' - 2y' + y = 0$ gives

$$\begin{aligned}
&\left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x - 2C y_1'(x) \ln(x) \\
&- \frac{2C y_1(x)}{x} - 2\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left(y_1''(x)\, x + y_1(x) - 2y_1'(x)\right) \ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x - \frac{2y_1(x)}{x}\right) C \\
&+ \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x \\
&- 2\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\, x + y_1(x) - 2y_1'(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x - \frac{2y_1(x)}{x}\right) C \\ &+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x \\ &- 2\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x - 3\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} \\ &+ \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) x^2 - 2\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} \\ &= 0\end{aligned} \tag{9}$$

Since $r_1 = 3$ and $r_2 = 0$ then the above becomes

$$\begin{aligned}&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{2+n} a_n(n+3)\right) x - 3\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right)\right) C}{x} \\ &+ \frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2 - 2\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x + \left(\sum_{n=0}^{\infty} b_n x^n\right) x}{x} = 0\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2C\, x^{2+n} a_n(n+3)\right) + \sum_{n=0}^{\infty}\left(-3C\, x^{2+n} a_n\right) \\ &+ \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n(n-1)\right) + \sum_{n=0}^{\infty}\left(-2x^{n-1} b_n n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{2+n} a_n (n+3) &= \sum_{n=3}^{\infty} 2C a_{n-3} n\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-3C\, x^{2+n} a_n\right) &= \sum_{n=3}^{\infty} \left(-3C a_{n-3} x^{n-1}\right) \\
\sum_{n=0}^{\infty} b_n x^n &= \sum_{n=1}^{\infty} b_{n-1} x^{n-1}
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=3}^{\infty} 2C a_{n-3} n\, x^{n-1}\right) + \sum_{n=3}^{\infty} \left(-3C a_{n-3} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n-1} b_n n\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0 \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-2b_1 + b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1 + 1 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{1}{2}$$

For $n = 2$, Eq (2B) gives

$$-2b_2 + b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_2 + \frac{1}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{4}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$3C + \frac{1}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{12}$$

For $n = 4$, Eq (2B) gives

$$5Ca_1 + b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$4b_4 + \frac{5}{48} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{5}{192}$$

For $n = 5$, Eq (2B) gives

$$7Ca_2 + b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$10b_5 - \frac{13}{320} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{13}{3200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{12}$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = &-\frac{1}{12}\left(x^3\left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6)\right)\right)\ln(x) \\ &+ 1 + \frac{x}{2} + \frac{x^2}{4} - \frac{5x^4}{192} + \frac{13x^5}{3200} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6)\right) \\
&\quad + c_2 \left(-\frac{1}{12}\left(x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6)\right)\right) \ln(x) + 1 \right. \\
&\qquad \left. + \frac{x}{2} + \frac{x^2}{4} - \frac{5x^4}{192} + \frac{13x^5}{3200} + O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(x^{-1+m} m(-1+m) - 2m\, x^{-1+m}\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\boxed{
\begin{aligned}
a_1 &= -\frac{a_0}{r^2 - r - 2} \\
a_2 &= \frac{a_0}{r^4 - 5r^2 + 4} \\
a_3 &= -\frac{a_0}{(r^4 - 5r^2 + 4) r (r+3)} \\
a_4 &= \frac{a_0}{(r^4 - 5r^2 + 4) r (r+3) (r^2 + 5r + 4)} \\
a_5 &= -\frac{a_0}{(r+1)^2 (r-2) (r+2)^2 (-1+r) r (r+3) (r+4) (r+5)}
\end{aligned}
}
$$

Expanding the rhs of the ode $\cos(x)$ in series gives

$$
\cos(x) = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1
$$

Since the $F = \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = \frac{x^4}{24}$ by solving the balance equation

$$
\left(x^{-1+m} m(-1+m) - 2m\, x^{-1+m}\right) c_0 = \frac{x^4}{24}
$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{240} \\ m &= 5 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+5} \end{aligned}$$

Where in the above $c_0 = \frac{1}{240}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 5$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{240}$ and $r = m$ or $r = 5$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{240}$
$c_1 = -\frac{1}{4320}$
$c_2 = \frac{1}{120960}$
$c_3 = -\frac{1}{4838400}$
$c_4 = \frac{1}{261273600}$
$c_5 = -\frac{1}{18289152000}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^5 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$y_p = x^5 \left(\frac{1}{240} - \frac{1}{4320} x + \frac{1}{120960} x^2 - \frac{1}{4838400} x^3 + \frac{1}{261273600} x^4 - \frac{1}{18289152000} x^5 \right)$$

$$= \frac{1}{240}x^5 - \frac{1}{4320}x^6 + \frac{1}{120960}x^7 - \frac{1}{4838400}x^8 + \frac{1}{261273600}x^9 - \frac{1}{18289152000}x^{10}$$

Unable to solve the balance equation $(x^{-1+m}m(-1+m) - 2m\, x^{-1+m})\, c_0$ for c_0 and x. No particular solution exists.

Failed to convert RHS $\cos(x)$ to series in order to find particular solution. Unable to solve. Terminating Unable to find the particular solution or no solution exists.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x*diff(y(x),x$2)-2*diff(y(x),x)+y(x)=cos(x),y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 312

```
AsymptoticDSolveValue[x*y''[x]-2*y'[x]+y[x]==Cos[x],y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\left(x^4\left(\frac{\log(x)}{48}-\frac{5}{192}\right)-\frac{1}{12}x^3\log(x)+\frac{x^2}{4}+\frac{x}{2}+1\right) \\
& + c_2\left(-\frac{x^5}{806400}+\frac{x^4}{20160}-\frac{x^3}{720}+\frac{x^2}{40}-\frac{x}{4}+1\right)x^3+\left(-\frac{x^5}{806400}+\frac{x^4}{20160}-\frac{x^3}{720}\right. \\
& \left.+\frac{x^2}{40}-\frac{x}{4}+1\right)x^3\left(\frac{x^6\left(-20160\log^2(x)+141222\log(x)-201569\right)}{3135283200}\right. \\
& +\frac{x^5(22277-114360\log(x))}{435456000}+\frac{x^4(69541-29064\log(x))}{34836480} \\
& +\frac{x^3(1860\log(x)+193)}{388800}-\frac{1}{6x^2}+\frac{x^2(4\log(x)-23)}{1152}-\frac{1}{6x}+\frac{1}{36}x(-\log(x)-2) \\
& \left.-\frac{\log(x)}{12}\right)+\left(\frac{x^6(5791-672\log(x))}{8709120}-\frac{589x^5}{302400}-\frac{89x^4}{8640}+\frac{19x^3}{360}+\frac{x^2}{24}\right. \\
& \left.-\frac{x}{3}\right)\left(x^4\left(\frac{\log(x)}{48}-\frac{5}{192}\right)-\frac{1}{12}x^3\log(x)+\frac{x^2}{4}+\frac{x}{2}+1\right)
\end{aligned}
$$

1.23 problem 3.50

Internal problem ID [5502]
Internal file name [`OUTPUT/4750_Sunday_June_05_2022_03_04_56_PM_66020216/index.tex`]

Book: Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag. Springer October 29, 1999
Section: Chapter 3. APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. page 136
Problem number: 3.50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

Unable to solve or complete the solution.

$$\boxed{y' - \frac{y}{x} = \cos(x)}$$

With the expansion point for the power series method at $x = 0$.

1.23.1 Solving as series ode

Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - \frac{y}{x} &= \cos(x) \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -\frac{1}{x} \\ p(x) &= \cos(x) \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not.

Since $x = 0$ is not an ordinary point, we now check to see if it is a regular singular point. $xq(x) = -1$ has a Taylor series around $x = 0$. Since $x = 0$ is regular singular point, then Frobenius power series is used. Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y' - \frac{y}{x} = 0$,and y_p is a particular solution to the inhomogeneous ode. First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) - \frac{\sum_{n=0}^{\infty} a_n x^{n+r}}{x} = 0 \tag{1}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) - \frac{\sum_{n=0}^{\infty} a_n x^{n+r}}{x} = 0 \tag{1}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + -1 \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \frac{1}{x} \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n\, =0}^{\infty} \left(-x^{n+r-1} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n\, =0}^{\infty} \left(-x^{n+r-1} a_n\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2) this gives

$$(n+r)\, a_n x^{n+r-1} - x^{n+r-1} a_n = 0$$

When $n = 0$ the above becomes

$$r a_0 x^{-1+r} - x^{-1+r} a_0 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(x^{-1+m} m - x^{-1+m}\right) c_0 = \cos(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)\, x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$-1+r = 0$$

Solving for r gives the root of the indicial equation as

$$r = 1$$

We start by finding y_h. From the above we see that there is no recurrence relation since there is only one summation term. Therefore all a_n terms are zero except for a_0. Hence

$$y_h = a_0 x^r$$

Therefore the homogeneous solution is

$$y_h(x) = a_0\big(x + O(x^6)\big)$$

Unable to solve the balance equation $(x^{-1+m}m - x^{-1+m})\, c_0 = \cos(x)$ for c_0 and x. No particular solution exists.

Unable to find the particular solution. No solution exist.

Verification of solutions N/A

1.23.2 Maple step by step solution

Let's solve

$y' - \frac{y}{x} = \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + \cos(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x) \cos(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x) \cos(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) \cos(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{\cos(x)}{x} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\mathrm{Ci}(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(diff(y(x),x)-y(x)/x=cos(x),y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 34

```
AsymptoticDSolveValue[y'[x]-y[x]/x==Cos[x],y[x],{x,0,5}]
```

$$y(x) \to x\left(-\frac{x^6}{4320} + \frac{x^4}{96} - \frac{x^2}{4} + \log(x)\right) + c_1 x$$