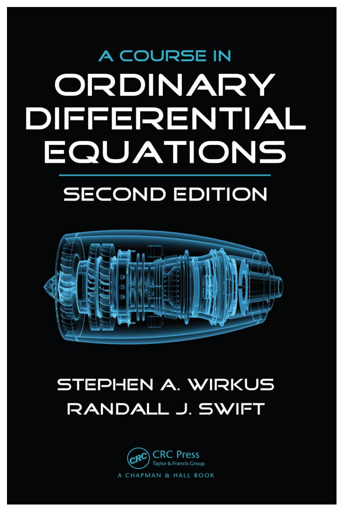
A Solution Manual For

A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition



Nasser M. Abbasi

March 4, 2026   Compiled on March 4, 2026 at 6:26pm

Contents

Chapter 1

Lookup tables for all problems in current book

Local contents

1.1 Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
9487	1. Using series method	$y' = y^2 - x$ $y(0) = 1$ Series expansion around $x = 0$.	✓	✓	✓	✓
9488	1. direct method	$y' = y^2 - x$ $y(0) = 1$	✓	✓	✓	✗
9489	2. Using series method	$y' - 2y = x^2$ $y(1) = 1$ Series expansion around $x = 1$.	✓	✓	✓	✓
9490	2. direct method	$y' - 2y = x^2$ $y(1) = 1$	✓	✓	✓	✓
9491	3. series method	$y' = y + x\,e^y$ $y(0) = 0$ Series expansion around $x = 0$.	✓	✓	✗	✓
9492	3. direct method	$y' = y + x\,e^y$ $y(0) = 0$	✗	✗	✗	✗

Chapter 2

Book Solved Problems

Local contents

2.1 Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603

Local contents

2.1.1 Problem 1. Using series method

Local contents

Internal problem ID [9487]
Book : A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section : Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number : 1. Using series method
Date solved : `Sunday, March 01, 2026 at 04:35:34 AM`
CAS classification : `[[_Riccati, _special]]`

$$
\begin{aligned}
y' &= -x + y^2 \\
y(0) &= 1
\end{aligned}
$$

Series expansion around $x = 0$.

Entering first order ode series solver taylor

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\Big|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\Big|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\Big|_{x_0, y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n=1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f \end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}F_n|_{x_0,y_0} \tag{6}$$

Hence

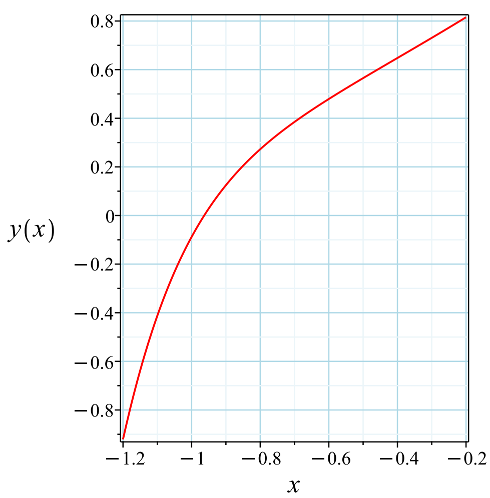
$$\begin{aligned} F_0 &= -x + y^2 \\ F_1 &= \frac{dF_0}{dx} \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}F_0 \\ &= 2y^3 - 2xy - 1 \\ F_2 &= \frac{dF_1}{dx} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}F_1 \\ &= 6y^4 - 8xy^2 + 2x^2 - 2y \\ F_3 &= \frac{dF_2}{dx} \\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}F_2 \\ &= 24y^5 - 40y^3x + 16yx^2 - 10y^2 + 6x \\ F_4 &= \frac{dF_3}{dx} \\ &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}F_3 \\ &= 120y^6 - 240xy^4 + 136x^2y^2 - 60y^3 - 16x^3 + 52xy + 6 \\ F_5 &= \frac{dF_4}{dx} \\ &= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}F_4 \\ &= 720y^7 - 1680xy^5 - 420y^4 + 1232y^3x^2 + 504xy^2 + \left(-272x^3 + 52\right)y - 100x^2 \\ F_6 &= \frac{dF_5}{dx} \\ &= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}F_5 \\ &= 5040y^8 - 13440y^6x - 3360y^5 + 12096y^4x^2 + 5152y^3x + \left(-3968x^3 + 556\right)y^2 - 1824yx^2 + 272x^4 \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 1$ gives

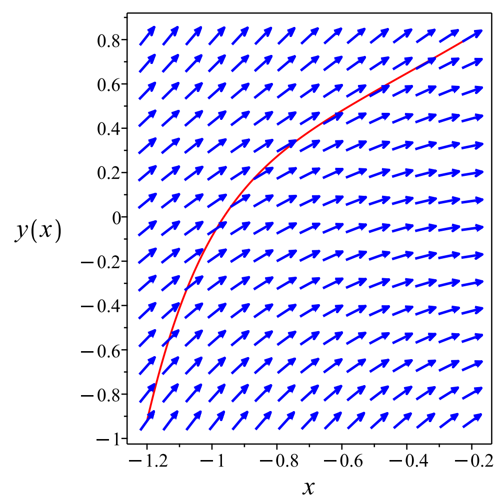
$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_2 &= 4 \\
F_3 &= 14 \\
F_4 &= 66 \\
F_5 &= 352 \\
F_6 &= 2236
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

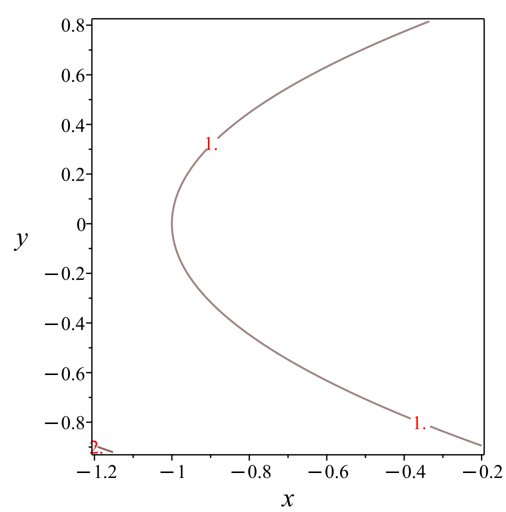
$$
y = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \frac{11x^5}{20} + \frac{22x^6}{45} + \frac{559x^7}{1260} + O(x^8)
$$



(a) Solution plot for $y' = -x + y^2$



(b) Direction fields for $y' = -x + y^2$



(c) Isoclines for $y' = -x + y^2$

2.1.1.1 ✓ Maple. `Time used: 0.003 (sec). Leaf size: 24`

```
Order:=8;
ode:=diff(y(x),x) = y(x)^2-x;
ic:=[y(0) = 1];
dsolve([ode,op(ic)],y(x),type='series',x=0);
```

$$
y = 1 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{7}{12}x^4 + \frac{11}{20}x^5 + \frac{22}{45}x^6 + \frac{559}{1260}x^7 + \mathrm{O}\left(x^8\right)
$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful
```

2.1.1.2 ✓ Mathematica. Time used: 8.172 (sec). Leaf size: 1113

```
ode=D[y[x],x]==y[x]^2-x;
ic={y[0]==1};
AsymptoticDSolveValue[{ode,ic},y[x],{x,0,7}]
```

Too large to display

2.1.1.3 ✓ Sympy. Time used: 0.273 (sec). Leaf size: 48

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(x - y(x)**2 + Derivative(y(x), x),0)
ics = {y(0): 1}
dsolve(ode,func=y(x),ics=ics,hint="1st_power_series",x0=0,n=8)
```

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \frac{11x^5}{20} + \frac{22x^6}{45} + \frac{559x^7}{1260} + O(x^8)$$

```
Python version: 3.12.3 (main, Aug 14 2025, 17:47:21) [GCC 13.3.0]
Sympy version 1.14.0
```

```
classify_ode(ode,func=y(x))

('1st_rational_riccati', '1st_power_series', 'lie_group')
```

2.1.2 Problem 1. direct method

Local contents

Internal problem ID [9488]
Book : A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section : Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number : 1. direct method
Date solved : `Sunday, March 01, 2026 at 04:35:38 AM`
CAS classification : `[[_Riccati, _special]]`

2.1.2.1 Existence and uniqueness analysis

$$
\begin{aligned}
y' &= -x + y^2 \\
y(0) &= 1
\end{aligned}
$$

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x, y) \\
&= y^2 - x
\end{aligned}
$$

The x domain of $f(x, y)$ when $y = 1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - x\right) \\
&= 2y
\end{aligned}
$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

0.477 (sec)

2.1.2.2 Solved using first_order_ode_riccati

Entering first order ode riccati solver

$$y' = -x + y^2$$
$$y(0) = 1$$

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -x + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -x + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -x$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -x \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - x u(x) = 0$$

Entering second order Airy solver

This is Airy ODE. It has the general form

$$a \frac{d^2 u}{dx^2} + b \frac{du}{dx} + c x u = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= 0 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$u = c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Taking derivative gives

$$u'(x) = -c_1 (-1)^{1/3} \operatorname{AiryAi}\left(1, -x(-1)^{1/3}\right) - c_2 (-1)^{1/3} \operatorname{AiryBi}\left(1, -x(-1)^{1/3}\right) \tag{4}$$

Substituting equations (3,4) into (1) results in

$$y = \frac{-u'}{f_2 u}$$

$$y = \frac{-u'}{u}$$

$$y = -\frac{-c_1(-1)^{1/3}\,\mathrm{AiryAi}\left(1,-x(-1)^{1/3}\right) - c_2(-1)^{1/3}\,\mathrm{AiryBi}\left(1,-x(-1)^{1/3}\right)}{c_1\,\mathrm{AiryAi}\left(-x\,(-1)^{1/3}\right) + c_2\,\mathrm{AiryBi}\left(-x\,(-1)^{1/3}\right)}$$

Doing change of constants, the above solution becomes

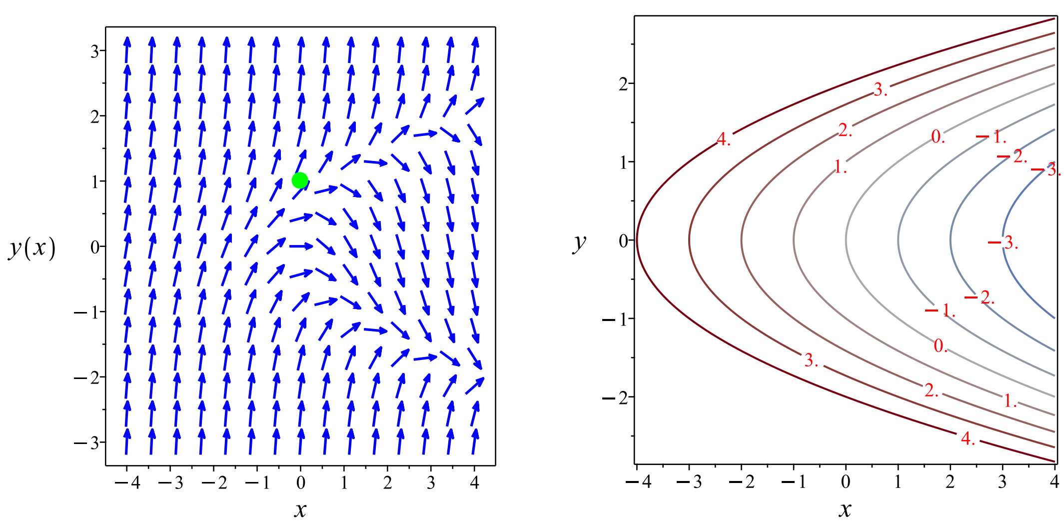
$$y = -\frac{-(-1)^{1/3}\,\mathrm{AiryAi}\left(1,-x(-1)^{1/3}\right) - c_3(-1)^{1/3}\,\mathrm{AiryBi}\left(1,-x(-1)^{1/3}\right)}{\mathrm{AiryAi}\left(-x\,(-1)^{1/3}\right) + c_3\,\mathrm{AiryBi}\left(-x\,(-1)^{1/3}\right)}$$

Simplifying the above gives

$$y = \frac{(1+i\sqrt{3})\left(\mathrm{AiryBi}\left(1,-\frac{(1+i\sqrt{3})x}{2}\right)c_3 + \mathrm{AiryAi}\left(1,-\frac{(1+i\sqrt{3})x}{2}\right)\right)}{2c_3\,\mathrm{AiryBi}\left(-\frac{(1+i\sqrt{3})x}{2}\right) + 2\,\mathrm{AiryAi}\left(-\frac{(1+i\sqrt{3})x}{2}\right)}$$

Solving for constant of integration from initial conditions gives

$$y = \frac{9(1+i\sqrt{3})\left(\left(\left(i3^{1/6}+\frac{3^{2/3}}{3}\right)\Gamma\left(\frac{2}{3}\right)^2-\frac{4\,3^{5/6}\pi}{9}\right)\mathrm{AiryAi}\left(1,-\frac{(1+i\sqrt{3})x}{2}\right)+\frac{\mathrm{AiryBi}\left(1,-\frac{(1+i\sqrt{3})x}{2}\right)\left((i3^{2/3}+3\cdots\right.}{3}\cdots\right.}{\left(\left(6\,3^{2/3}+18i3^{1/6}\right)\Gamma\left(\frac{2}{3}\right)^2-8\,3^{5/6}\pi\right)\mathrm{AiryAi}\left(-\frac{(1+i\sqrt{3})x}{2}\right)+6\left(\left(i3^{2/3}+3^{1/6}\right)\Gamma\left(\frac{2}{3}\right)^2+\frac{4\,3^{1/3}\pi}{3}\right)\mathrm{Airy}\cdots}$$



(a) Direction field $y' = -x + y^2$

(b) Isoclines for $y' = -x + y^2$

Summary of solutions found

$$y = \frac{9(1+i\sqrt{3})\left(\left(\left(i3^{1/6}+\frac{3^{2/3}}{3}\right)\Gamma\left(\frac{2}{3}\right)^2-\frac{4\,3^{5/6}\pi}{9}\right)\mathrm{AiryAi}\left(1,-\frac{(1+i\sqrt{3})x}{2}\right)+\frac{\mathrm{AiryBi}\left(1,-\frac{(1+i\sqrt{3})x}{2}\right)\left((i3^{2/3}+3\cdots\right.}{3}\cdots\right.}{\left(\left(6\,3^{2/3}+18i3^{1/6}\right)\Gamma\left(\frac{2}{3}\right)^2-8\,3^{5/6}\pi\right)\mathrm{AiryAi}\left(-\frac{(1+i\sqrt{3})x}{2}\right)+6\left(\left(i3^{2/3}+3^{1/6}\right)\Gamma\left(\frac{2}{3}\right)^2+\frac{4\,3^{1/3}\pi}{3}\right)\mathrm{Airy}\cdots}$$

2.1.2.3 ✓ Maple. Time used: 0.115 (sec). Leaf size: 89

```
ode:=diff(y(x),x) = y(x)^2-x;
ic:=[y(0) = 1];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = \frac{-2\, 3^{5/6}\pi \operatorname{AiryAi}(1,x) - 3\Gamma\left(\frac{2}{3}\right)^2 3^{2/3} \operatorname{AiryAi}(1,x) - 3\, 3^{1/6}\Gamma\left(\frac{2}{3}\right)^2 \operatorname{AiryBi}(1,x) + 2\pi 3^{1/3} \operatorname{AiryBi}(1,x}{2\, 3^{5/6}\pi \operatorname{AiryAi}(x) + 3\Gamma\left(\frac{2}{3}\right)^2 3^{2/3} \operatorname{AiryAi}(x) + 3\, 3^{1/6}\Gamma\left(\frac{2}{3}\right)^2 \operatorname{AiryBi}(x) - 2\pi 3^{1/3} \operatorname{AiryBi}(x)}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful
```

2.1.2.4 ✓ Mathematica. Time used: 0.471 (sec). Leaf size: 164

```
ode=D[y[x],x]==y[x]^2-x;
ic={y[0]==1};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{-3}\,\text{Gamma}\left(\frac{2}{3}\right)\left(ix^{3/2}\,\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}ix^{3/2}\right) - ix^{3/2}\,\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}ix^{3/2}\right) + \text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}ix^{3/}\right.\right.}{2x\left(\text{Gamma}\left(\frac{1}{3}\right)\text{BesselJ}\left(\frac{1}{3},\frac{2}{3}ix^{3/2}\right) - \sqrt[3]{-3}\,\text{Gamma}\left(\frac{2}{3}\right)\text{Bessel}\right.}$$

2.1.2.5 ✗ Sympy

```python
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(x - y(x)**2 + Derivative(y(x), x),0)
ics = {y(0): 1}
dsolve(ode,func=y(x),ics=ics)
```

```
TypeError : bad operand type for unary -: list
```

```
Python version: 3.12.3 (main, Aug 14 2025, 17:47:21) [GCC 13.3.0]
Sympy version 1.14.0
```

```
classify_ode(ode,func=y(x))

('1st_rational_riccati', '1st_power_series', 'lie_group')
```

2.1.3 Problem 2. Using series method

Local contents

Internal problem ID [9489]
Book : A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section : Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number : 2. Using series method
Date solved : `Sunday, March 01, 2026 at 04:35:50 AM`
CAS classification : `[[_linear, 'class A']]`

$$y' - 2y = x^2$$
$$y(1) = 1$$

Series expansion around $x = 1$.

Entering first order ode series solver

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$\alpha = x - 1$$

The ode is converted to be in terms of the new independent variable α. This results in

$$\frac{d}{d\alpha}y(\alpha) - 2y(\alpha) = (\alpha + 1)^2$$

Entering first order ode series solver taylor

With its expansion point and initial conditions now at $\alpha = 0$. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f &(1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f &(2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f &(3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y) &(4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 &(5)
\end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n\big|_{x_0,y_0} \tag{6}$$

Hence

$$
\begin{aligned}
F_0 &= \alpha^2 + 2\alpha + 2y(\alpha) + 1 \\
F_1 &= \frac{dF_0}{d\alpha} \\
&= \frac{\partial F_0}{\partial \alpha} + \frac{\partial F_0}{\partial y} F_0 \\
&= 6\alpha + 4 + 2\alpha^2 + 4y(\alpha) \\
F_2 &= \frac{dF_1}{d\alpha} \\
&= \frac{\partial F_1}{\partial \alpha} + \frac{\partial F_1}{\partial y} F_1 \\
&= 10 + 12\alpha + 4\alpha^2 + 8y(\alpha) \\
F_3 &= \frac{dF_2}{d\alpha} \\
&= \frac{\partial F_2}{\partial \alpha} + \frac{\partial F_2}{\partial y} F_2 \\
&= 20 + 24\alpha + 8\alpha^2 + 16y(\alpha) \\
F_4 &= \frac{dF_3}{d\alpha} \\
&= \frac{\partial F_3}{\partial \alpha} + \frac{\partial F_3}{\partial y} F_3 \\
&= 40 + 48\alpha + 16\alpha^2 + 32y(\alpha) \\
F_5 &= \frac{dF_4}{d\alpha} \\
&= \frac{\partial F_4}{\partial \alpha} + \frac{\partial F_4}{\partial y} F_4 \\
&= 80 + 96\alpha + 32\alpha^2 + 64y(\alpha) \\
F_6 &= \frac{dF_5}{d\alpha} \\
&= \frac{\partial F_5}{\partial \alpha} + \frac{\partial F_5}{\partial y} F_5 \\
&= 160 + 192\alpha + 64\alpha^2 + 128y(\alpha)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $\alpha(0) = 0$ and $y(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 3 \\
F_1 &= 8 \\
F_2 &= 18 \\
F_3 &= 36 \\
F_4 &= 72 \\
F_5 &= 144 \\
F_6 &= 288
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y(\alpha) = 3\alpha^3 + 4\alpha^2 + 3\alpha + 1 + \frac{3\alpha^4}{2} + \frac{3\alpha^5}{5} + \frac{\alpha^6}{5} + \frac{2\alpha^7}{35} + O(\alpha^8)
$$

Since $\alpha = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
\frac{d}{d\alpha} y(\alpha) + q(\alpha) y(\alpha) &= p(\alpha) \\
\frac{d}{d\alpha} y(\alpha) - 2y(\alpha) &= (\alpha + 1)^2
\end{aligned}
$$

Where

$$
\begin{aligned}
q(\alpha) &= -2 \\
p(\alpha) &= (\alpha+1)^2
\end{aligned}
$$

Next, the type of the expansion point $\alpha = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $\alpha = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $\alpha = 0$ is called an ordinary point $q(\alpha)$ has a Taylor series expansion around the point $\alpha = 0$. $\alpha = 0$ is called a regular singular point if $q(\alpha)$ is not not analytic at $\alpha = 0$ but $\alpha q(\alpha)$ has Taylor series expansion. And finally, $\alpha = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $\alpha = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

Entering first order ode series solver ordinary point solver

$$
y(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^n
$$

Then

$$
\frac{d}{d\alpha} y(\alpha) = \sum_{n=1}^{\infty} n a_n \alpha^{n-1}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=1}^{\infty} n a_n \alpha^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n \alpha^n\right) = (\alpha+1)^2 \tag{1}
$$

Expanding $(\alpha+1)^2$ as Taylor series around $\alpha = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
(\alpha+1)^2 &= \alpha^2 + 2\alpha + 1 + \dots \\
&= \alpha^2 + 2\alpha + 1
\end{aligned}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n \alpha^{n-1}\right) + \sum_{n=0}^{\infty} (-2 a_n \alpha^n) = \alpha^2 + 2\alpha + 1 \tag{2}
$$

The next step is to make all powers of α be n in each summation term. Going over each summation term above with power of α in it which is not already α^n and adjusting the power and the corresponding index gives

$$
\sum_{n=1}^{\infty} n a_n \alpha^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} \alpha^n
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of α are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} \alpha^n\right) + \sum_{n=0}^{\infty} (-2 a_n \alpha^n) = \alpha^2 + 2\alpha + 1 \tag{3}
$$

For $0 \leq n$, the recurrence equation is

$$
((n+1)\, a_{n+1} - 2a_n)\, \alpha^n = \alpha^2 + 2\alpha + 1 \tag{4}
$$

For $n = 0$ the recurrence equation gives

$$(a_1 - 2a_0)\,1 = 1$$
$$a_1 - 2a_0 = 1$$

Which after substituting the earlier terms found becomes

$$a_1 = 1 + 2a_0$$

For $n = 1$ the recurrence equation gives

$$(2a_2 - 2a_1)\,\alpha = 2\alpha$$
$$2a_2 - 2a_1 = 2$$

Which after substituting the earlier terms found becomes

$$a_2 = 2 + 2a_0$$

For $n = 2$ the recurrence equation gives

$$(3a_3 - 2a_2)\,\alpha^2 = \alpha^2$$
$$3a_3 - 2a_2 = 1$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{5}{3} + \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$(4a_4 - 2a_3)\,\alpha^3 = 0$$
$$4a_4 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5}{6} + \frac{2a_0}{3}$$

For $n = 4$ the recurrence equation gives

$$(5a_5 - 2a_4)\,\alpha^4 = 0$$
$$5a_5 - 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{3} + \frac{4a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$(6a_6 - 2a_5)\,\alpha^5 = 0$$
$$6a_6 - 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{9} + \frac{4a_0}{45}$$

For $n = 6$ the recurrence equation gives

$$\begin{aligned}(7a_7 - 2a_6)\,\alpha^6 &= 0\\ 7a_7 - 2a_6 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{2}{63} + \frac{8a_0}{315}$$

For $n = 7$ the recurrence equation gives

$$\begin{aligned}(8a_8 - 2a_7)\,\alpha^7 &= 0\\ 8a_8 - 2a_7 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{1}{126} + \frac{2a_0}{315}$$

And so on. Therefore the solution is

$$\begin{aligned}y(\alpha) &= \sum_{n=0}^{\infty} a_n \alpha^n\\ &= a_3\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned}y(\alpha) = a_0 + (1+2a_0)\,\alpha + (2+2a_0)\,\alpha^2 + \left(\frac{5}{3} + \frac{4a_0}{3}\right)\alpha^3 + \left(\frac{5}{6} + \frac{2a_0}{3}\right)\alpha^4\\ + \left(\frac{1}{3} + \frac{4a_0}{15}\right)\alpha^5 + \left(\frac{1}{9} + \frac{4a_0}{45}\right)\alpha^6 + \left(\frac{2}{63} + \frac{8a_0}{315}\right)\alpha^7 + \dots\end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned}y(\alpha) = \left(1 + 2\alpha + 2\alpha^2 + \frac{4}{3}\alpha^3 + \frac{2}{3}\alpha^4 + \frac{4}{15}\alpha^5 + \frac{4}{45}\alpha^6 + \frac{8}{315}\alpha^7\right) a_0\\ + \alpha + 2\alpha^2 + \frac{5\alpha^3}{3} + \frac{5\alpha^4}{6} + \frac{\alpha^5}{3} + \frac{\alpha^6}{9} + \frac{2\alpha^7}{63} + O(\alpha^8)\end{aligned} \tag{3}$$

At $\alpha = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

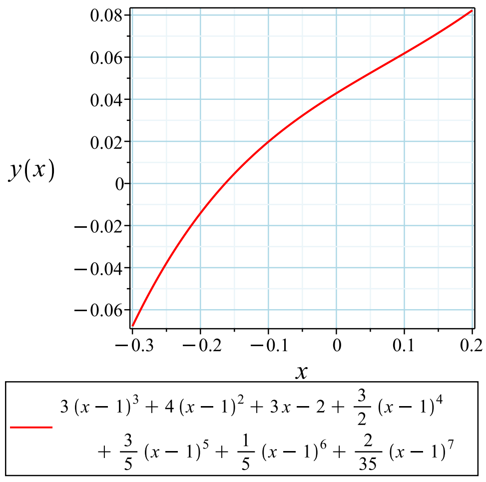
$$y(\alpha) = 3\alpha^3 + 4\alpha^2 + 3\alpha + 1 + \frac{3\alpha^4}{2} + \frac{3\alpha^5}{5} + \frac{\alpha^6}{5} + \frac{2\alpha^7}{35} + O(\alpha^8)$$

Hence the solution is

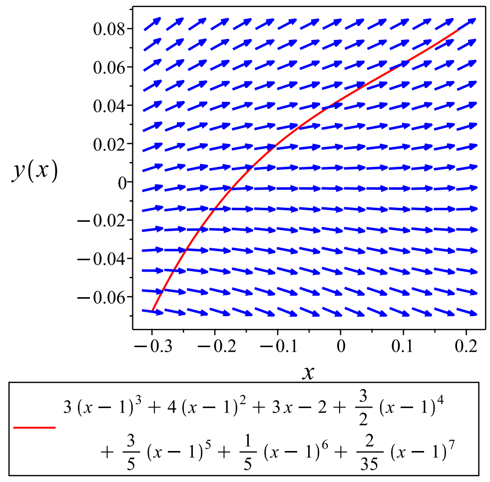
$$\begin{aligned}y = 3(x-1)^3 + 4(x-1)^2 + 3x - 2 + \frac{3(x-1)^4}{2}\\ + \frac{3(x-1)^5}{5} + \frac{(x-1)^6}{5} + \frac{2(x-1)^7}{35} + O\left((x-1)^8\right)\end{aligned}$$

Hence the solution is

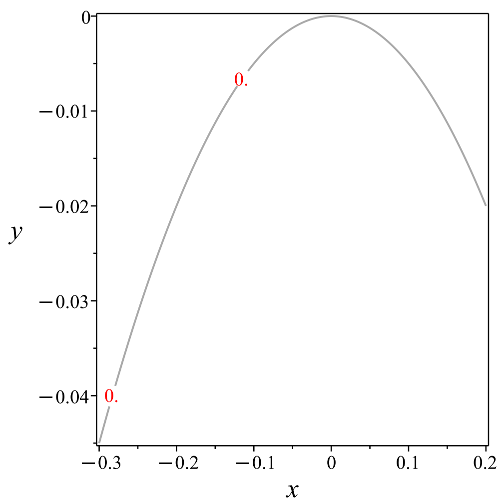
$$y = 3(x-1)^3 + 4(x-1)^2 + 3x - 2 + \frac{3(x-1)^4}{2} + \frac{3(x-1)^5}{5} + \frac{(x-1)^6}{5} + \frac{2(x-1)^7}{35} + O\left((x-1)^8\right)$$



(a) Solution plot for $y' - 2y = x^2$



(b) Direction fields for $y' - 2y = x^2$



(c) Isoclines for $y' - 2y = x^2$

2.1.3.1 ✓ Maple. Time used: 0.001 (sec). Leaf size: 24

```
Order:=8;
ode:=diff(y(x),x)-2*y(x) = x^2;
ic:=[y(1) = 1];
dsolve([ode,op(ic)],y(x),type='series',x=1);
```

$$y = 1 + 3(x-1) + 4(x-1)^2 + 3(x-1)^3 + \frac{3}{2}(x-1)^4 + \frac{3}{5}(x-1)^5 + \frac{1}{5}(x-1)^6 + \frac{2}{35}(x-1)^7 + \mathrm{O}\left((x-1)^8\right)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
```

Maple step by step

Let's solve
$\left[\frac{d}{dx}y(x) - 2y(x) = x^2, y(1) = 1\right]$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = 2y(x) + x^2$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
$\frac{d}{dx}y(x) - 2y(x) = x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$\mu(x)\left(\frac{d}{dx}y(x) - 2y(x)\right) = \mu(x)\, x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
$\mu(x)\left(\frac{d}{dx}y(x) - 2y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
$\frac{d}{dx}\mu(x) = -2\mu(x)$
- Solve to find the integrating factor
$\mu(x) = e^{-2x}$
- Integrate both sides with respect to x
$\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \mu(x)\, x^2 dx + C1$
- Evaluate the integral on the lhs
$y(x)\,\mu(x) = \int \mu(x)\, x^2 dx + C1$
- Solve for $y(x)$
$y(x) = \frac{\int \mu(x) x^2 dx + C1}{\mu(x)}$
- Substitute $\mu(x) = e^{-2x}$
$y(x) = \frac{\int e^{-2x} x^2 dx + C1}{e^{-2x}}$
- Evaluate the integrals on the rhs
$y(x) = \frac{-\frac{(2x^2+2x+1)e^{-2x}}{4} + C1}{e^{-2x}}$
- Simplify
$y(x) = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} + C1\, e^{2x}$
- Use initial condition $y(1) = 1$
$1 = -\frac{5}{4} + C1\, e^2$
- Solve for $\_C1$
$C1 = \frac{9}{4\, e^2}$
- Substitute $\_C1 = \frac{9}{4e^2}$ into general solution and simplify
$y(x) = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} + \frac{9\, e^{-2+2x}}{4}$
- Solution to the IVP
$y(x) = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} + \frac{9\, e^{-2+2x}}{4}$

2.1.3.2 ✓ Mathematica. Time used: 0.008 (sec). Leaf size: 60

```
ode=D[y[x],x]-2*y[x]==x^2;
ic={y[1]==1};
AsymptoticDSolveValue[{ode,ic},y[x],{x,1,7}]
```

$$y(x) \to \frac{2}{35}(x-1)^7 + \frac{1}{5}(x-1)^6 + \frac{3}{5}(x-1)^5 + \frac{3}{2}(x-1)^4 + 3(x-1)^3 + 4(x-1)^2 + 3(x-1) + 1$$

2.1.3.3 ✓ Sympy. Time used: 0.269 (sec). Leaf size: 56

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x**2 - 2*y(x) + Derivative(y(x), x),0)
ics = {y(1): 1}
dsolve(ode,func=y(x),ics=ics,hint="1st_power_series",x0=1,n=8)
```

$$y(x) = -2 + 4(x-1)^2 + 3(x-1)^3 + \frac{3(x-1)^4}{2} + \frac{3(x-1)^5}{5} + \frac{(x-1)^6}{5} + \frac{2(x-1)^7}{35} + 3x + O(x^8)$$

```
Python version: 3.12.3 (main, Aug 14 2025, 17:47:21) [GCC 13.3.0]
Sympy version 1.14.0
```

```
classify_ode(ode,func=y(x))

('1st_exact', '1st_linear', 'Bernoulli', 'almost_linear', '1st_power_series
   ', 'lie_group', 'nth_linear_constant_coeff_undetermined_coefficients',
   'nth_linear_constant_coeff_variation_of_parameters', '1
   st_exact_Integral', '1st_linear_Integral', 'Bernoulli_Integral', '
   almost_linear_Integral', '
   nth_linear_constant_coeff_variation_of_parameters_Integral')
```

2.1.4 Problem 2. direct method

Local contents

Internal problem ID [9490]
Book : A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section : Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number : 2. direct method
Date solved : `Sunday, March 01, 2026 at 04:35:54 AM`
CAS classification : `[[_linear, ‘class A‘]]`

2.1.4.1 Existence and uniqueness analysis

$$
\begin{aligned}
y' - 2y &= x^2 \\
y(1) &= 1
\end{aligned}
$$

This is a linear ODE. In canonical form it is written as

$$
y' + q(x)y = p(x)
$$

Where here

$$
\begin{aligned}
q(x) &= -2 \\
p(x) &= x^2
\end{aligned}
$$

Hence the ode is

$$
y' - 2y = x^2
$$

The domain of $q(x) = -2$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 1$ is inside this domain. The domain of $p(x) = x^2$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

0.289 (sec) Entering first order ode linear solver

2.1.4.2 Solved using first_order_ode_linear

$$y' - 2y = x^2$$
$$y(1) = 1$$

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -2$$
$$p(x) = x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= e^{\int (-2)dx} \\ &= e^{-2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(x^2\right) \\ \frac{d}{dx}\left(e^{-2x}y\right) &= \left(e^{-2x}\right)\left(x^2\right) \\ d\left(e^{-2x}y\right) &= \left(e^{-2x}x^2\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-2x}y &= \int e^{-2x}x^2\, dx \\ &= -\frac{(2x^2+2x+1)\,e^{-2x}}{4} + c_1\end{aligned}$$

Dividing throughout by the integrating factor e^{-2x} gives the final solution

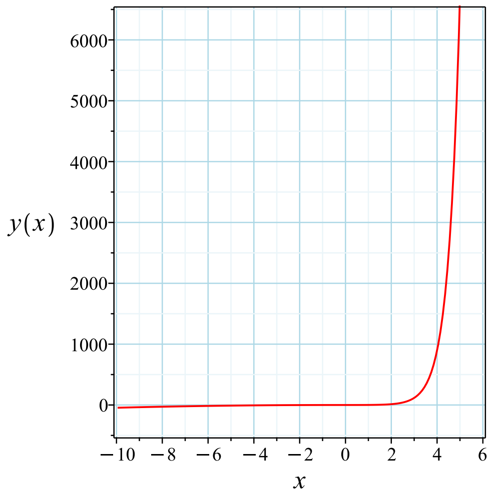
$$y = e^{2x}\left(-\frac{(2x^2+2x+1)\,e^{-2x}}{4} + c_1\right)$$

Simplifying the above gives

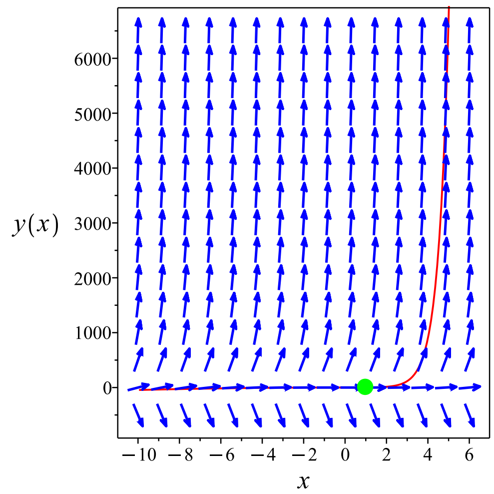
$$y = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} + c_1\,e^{2x}$$

Solving for initial conditions the solution becomes

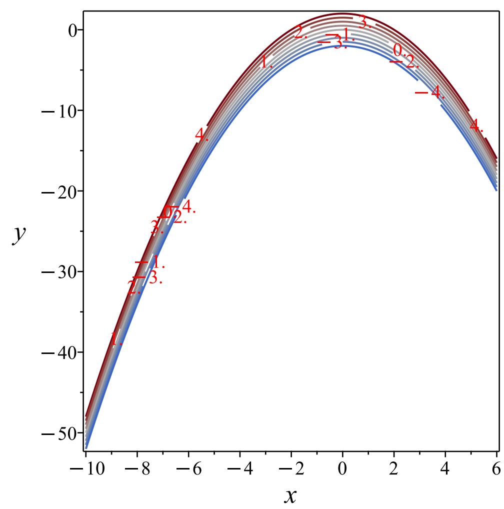
$$y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$



(a) Solution plot for $y' - 2y = x^2$



(b) Direction fields for $y' - 2y = x^2$



(c) Isoclines for $y' - 2y = x^2$

Summary of solutions found

$$y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

0.462 (sec) Entering first order ode exact solver

2.1.4.3 Solved using first_order_ode_exact

$$y' - 2y = x^2$$
$$y(1) = 1$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= \left(x^2 + 2y\right)\mathrm{d}x \\ \left(-x^2 - 2y\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -x^2 - 2y \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - 2y\right) \\ &= -2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-2) - (0)) \\ &= -2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -2\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2x} \\ &= \mathrm{e}^{-2x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= e^{-2x}(-x^2 - 2y) \\ &= (-x^2 - 2y)\,e^{-2x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= e^{-2x}(1) \\ &= e^{-2x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\ \left((-x^2 - 2y)\,e^{-2x}\right) + \left(e^{-2x}\right)\frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y}\,dy &= \int \overline{N}\,dy \\ \int \frac{\partial \phi}{\partial y}\,dy &= \int e^{-2x}\,dy\end{aligned}$$

$$\phi = e^{-2x}y + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -2\,e^{-2x}y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = (-x^2 - 2y)\,e^{-2x}$. Therefore equation (4) becomes

$$(-x^2 - 2y)\,e^{-2x} = -2\,e^{-2x}y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -e^{-2x}x^2$$

Integrating the above w.r.t x gives

$$\begin{aligned}\int f'(x)\,dx &= \int \left(-e^{-2x}x^2\right)dx \\ f(x) &= \frac{(2x^2 + 2x + 1)\,e^{-2x}}{4} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-2x} y + \frac{(2x^2 + 2x + 1)\,\mathrm{e}^{-2x}}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-2x} y + \frac{(2x^2 + 2x + 1)\,\mathrm{e}^{-2x}}{4}$$

Simplifying the above gives

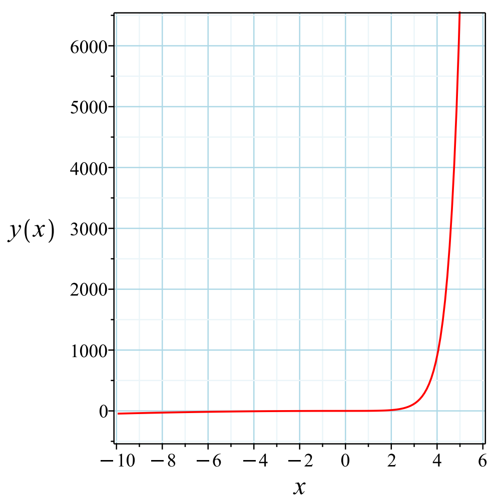
$$\frac{\mathrm{e}^{-2x}(2x^2 + 4y + 2x + 1)}{4} = c_1$$

Solving for initial conditions the solution becomes

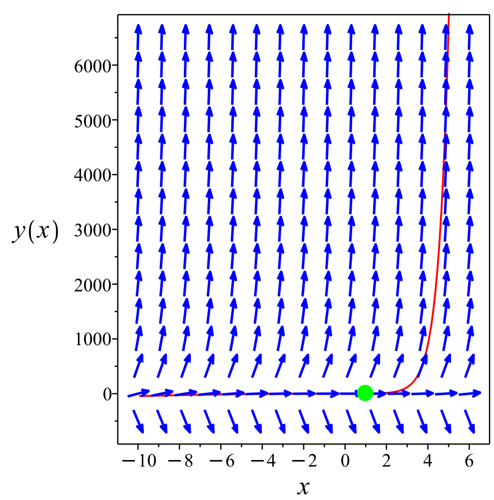
$$\frac{\mathrm{e}^{-2x}(2x^2 + 4y + 2x + 1)}{4} = \frac{9\,\mathrm{e}^{-2}}{4}$$

Solving for y gives

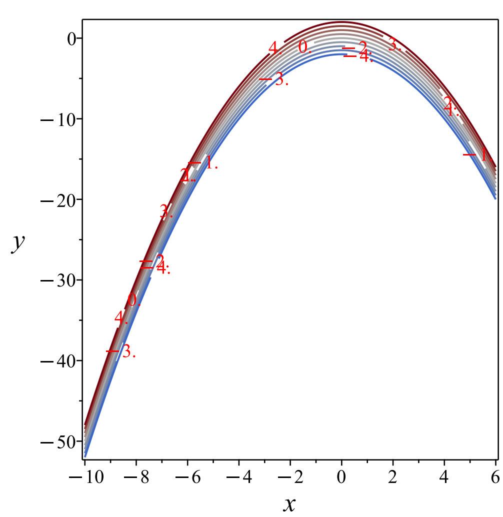
$$y = \frac{(-2\,\mathrm{e}^{-2x} x^2 - 2\,\mathrm{e}^{-2x} x + 9\,\mathrm{e}^{-2} - \mathrm{e}^{-2x})\,\mathrm{e}^{2x}}{4}$$



(a) Solution plot for $y' - 2y = x^2$



(b) Direction fields for $y' - 2y = x^2$



(c) Isoclines for $y' - 2y = x^2$

Summary of solutions found

$$y = \frac{(-2\,\mathrm{e}^{-2x} x^2 - 2\,\mathrm{e}^{-2x} x + 9\,\mathrm{e}^{-2} - \mathrm{e}^{-2x})\,\mathrm{e}^{2x}}{4}$$

1.384 (sec) Entering first order ode LIE solver

2.1.4.4 Solved using first_order_ode_LIE

$$y' - 2y = x^2$$
$$y(1) = 1$$

Writing the ode as

$$y' = x^2 + 2y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (x^2 + 2y)(b_3 - a_2) - (x^2 + 2y)^2 a_3 - 2x(xa_2 + ya_3 + a_1) - 2xb_2 - 2yb_3 - 2b_1 = 0 \tag{5E}$$

Putting the above in normal form gives

$$-x^4 a_3 - 4x^2 y a_3 - 3x^2 a_2 + x^2 b_3 - 2xya_3 - 4y^2 a_3 - 2xa_1 - 2xb_2 - 2ya_2 - 2b_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-x^4 a_3 - 4x^2 y a_3 - 3x^2 a_2 + x^2 b_3 - 2xya_3 - 4y^2 a_3 - 2xa_1 - 2xb_2 - 2ya_2 - 2b_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3 v_1^4 - 4a_3 v_1^2 v_2 - 3a_2 v_1^2 - 2a_3 v_1 v_2 - 4a_3 v_2^2 + b_3 v_1^2 - 2a_1 v_1 - 2a_2 v_2 - 2b_2 v_1 - 2b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-a_3v_1^4 - 4a_3v_1^2v_2 + (-3a_2 + b_3)\, v_1^2 - 2a_3v_1v_2 + (-2a_1 - 2b_2)\, v_1 - 4a_3v_2^2 - 2a_2v_2 - 2b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_2 &= 0 \\ -4a_3 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ -2a_1 - 2b_2 &= 0 \\ -3a_2 + b_3 &= 0 \\ -2b_1 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -2b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 2b_1 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -2 \\ \eta &= 2x + 1 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{2x + 1}{-2} \\ &= -x - \frac{1}{2} \end{aligned}$$

This is easily solved to give

$$y = -\frac{1}{2}x^2 - \frac{1}{2}x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{1}{2}x^2 + \frac{1}{2}x + y$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{-2} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= -\frac{x}{2} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x^2 + 2y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= x + \frac{1}{2} \\ R_y &= 1 \\ S_x &= -\frac{1}{2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x^2 + 2x + 4y + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{4R+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

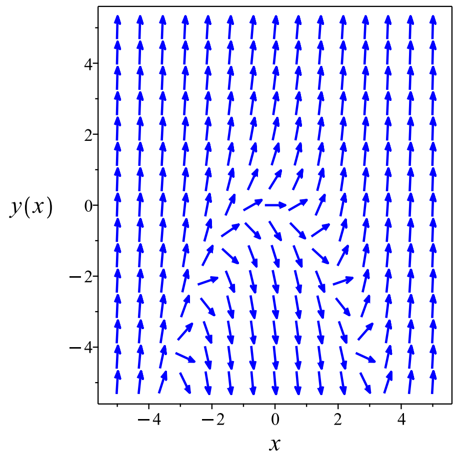
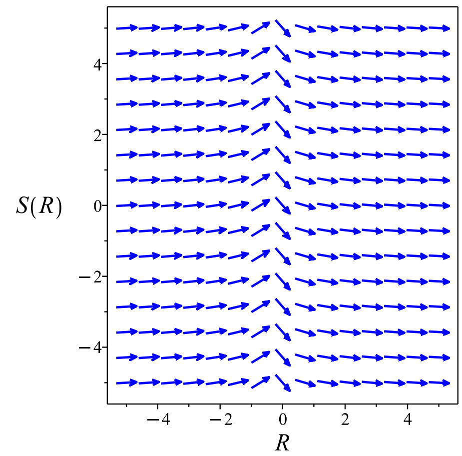
$$\begin{aligned} \int dS &= \int -\frac{1}{4R+1}\, dR \\ S(R) &= -\frac{\ln(4R+1)}{4} + c_2 \end{aligned}$$

$$S(R) = -\frac{\ln(4R+1)}{4} + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\frac{x}{2} = -\frac{\ln(2x^2 + 4y + 2x + 1)}{4} + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

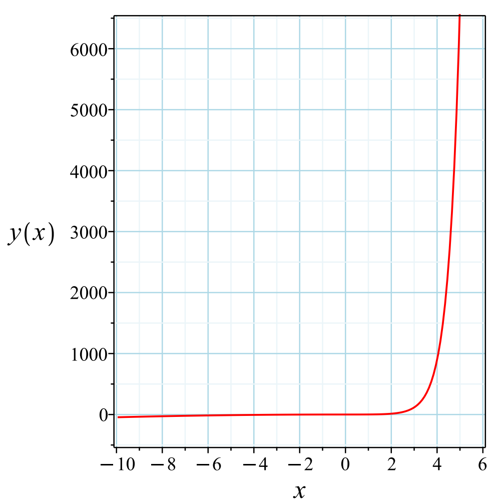
Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 + 2y$ 	$R = \frac{1}{2}x^2 + \frac{1}{2}x + y$ $S = -\frac{x}{2}$	$\frac{dS}{dR} = -\frac{1}{4R+1}$ 

Solving for initial conditions the solution becomes

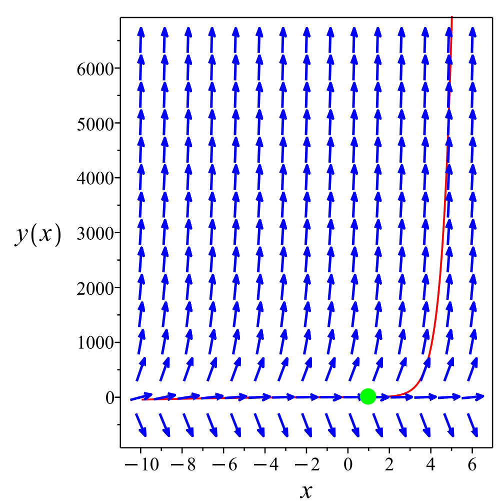
$$-\frac{x}{2} = -\frac{\ln(2x^2 + 4y + 2x + 1)}{4} - \frac{1}{2} + \frac{\ln(3)}{2}$$

Solving for y gives

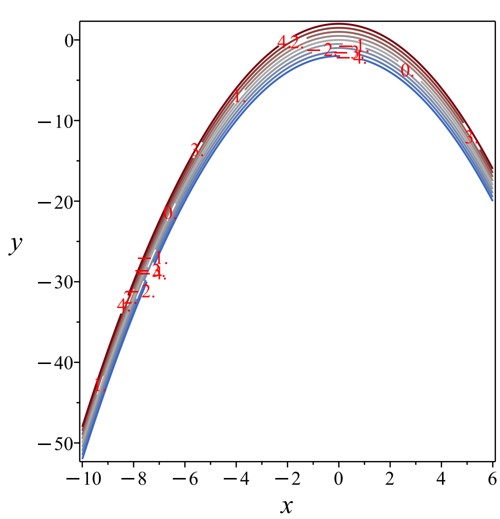
$$y = \frac{9\,e^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$



(a) Solution plot for $y' - 2y = x^2$



(b) Direction fields for $y' - 2y = x^2$



(c) Isoclines for $y' - 2y = x^2$

Summary of solutions found

$$y = \frac{9\,\mathrm{e}^{2x-2}}{4} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

2.1.4.5 ✓ Maple. Time used: 0.017 (sec). Leaf size: 22

```
ode:=diff(y(x),x)-2*y(x) = x^2;
ic:=[y(1) = 1];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} + \frac{9\,\mathrm{e}^{-2+2x}}{4}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
```

Maple step by step

Let's solve
$\left[\frac{d}{dx}y(x) - 2y(x) = x^2, y(1) = 1\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = 2y(x) + x^2$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) - 2y(x) = x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) - 2y(x)\right) = \mu(x)\,x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) - 2y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = -2\mu(x)$
- Solve to find the integrating factor
  $\mu(x) = \mathrm{e}^{-2x}$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \mu(x)\,x^2 dx + C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x) = \int \mu(x)\,x^2 dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int \mu(x)x^2 dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-2x}$
  $y(x) = \frac{\int \mathrm{e}^{-2x}x^2 dx + C1}{\mathrm{e}^{-2x}}$
- Evaluate the integrals on the rhs
  $y(x) = \frac{-\frac{(2x^2+2x+1)\mathrm{e}^{-2x}}{4} + C1}{\mathrm{e}^{-2x}}$
- Simplify
  $y(x) = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} + C1\,\mathrm{e}^{2x}$
- Use initial condition $y(1) = 1$
  $1 = -\frac{5}{4} + C1\,\mathrm{e}^{2}$

- Solve for $\_C1$
  $C1 = \frac{9}{4\,e^{2}}$
- Substitute $\_C1 = \frac{9}{4e^{2}}$ into general solution and simplify
  $y(x) = -\frac{x^{2}}{2} - \frac{x}{2} - \frac{1}{4} + \frac{9\,e^{-2+2x}}{4}$
- Solution to the IVP
  $y(x) = -\frac{x^{2}}{2} - \frac{x}{2} - \frac{1}{4} + \frac{9\,e^{-2+2x}}{4}$

2.1.4.6 ✓ Mathematica. Time used: 0.026 (sec). Leaf size: 37

```
ode=D[y[x],x]-2*y[x]==x^2;
ic={y[1]==1};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{2x-2}\left(e^{2}\int_{1}^{x} e^{-2K[1]}K[1]^{2}dK[1] + 1\right)$$

2.1.4.7 ✓ Sympy. Time used: 0.179 (sec). Leaf size: 26

```python
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x**2 - 2*y(x) + Derivative(y(x), x),0)
ics = {y(1): 1}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = -\frac{x^{2}}{2} - \frac{x}{2} + \frac{9e^{2x}}{4e^{2}} - \frac{1}{4}$$

```
Python version: 3.12.3 (main, Aug 14 2025, 17:47:21) [GCC 13.3.0]
Sympy version 1.14.0
```

```
classify_ode(ode,func=y(x))

('1st_exact', '1st_linear', 'Bernoulli', 'almost_linear', '1st_power_series
    ', 'lie_group', 'nth_linear_constant_coeff_undetermined_coefficients',
    'nth_linear_constant_coeff_variation_of_parameters', '1
    st_exact_Integral', '1st_linear_Integral', 'Bernoulli_Integral', '
    almost_linear_Integral', '
    nth_linear_constant_coeff_variation_of_parameters_Integral')
```

2.1.5 Problem 3. series method

Local contents

Internal problem ID [9491]
Book : A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section : Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number : 3. series method
Date solved : `Sunday, March 01, 2026 at 04:36:02 AM`
CAS classification : `[‘y=_G(x,y’)‘]`

$$y' = y + x\,\mathrm{e}^{y}$$
$$y(0) = 0$$

Series expansion around $x = 0$.

Entering first order ode series solver taylor

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \end{aligned} \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$F_0 = f(x, y) \tag{4}$$

$$\begin{aligned} F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \end{aligned} \tag{5}$$

For example, for $n=1$ we see that

$$\begin{aligned}F_1 &= \frac{d}{dx}(F_0)\\ &= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned}F_2 &= \frac{d}{dx}(F_1)\\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0\\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f\\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}F_n|_{x_0,y_0} \tag{6}$$

Hence

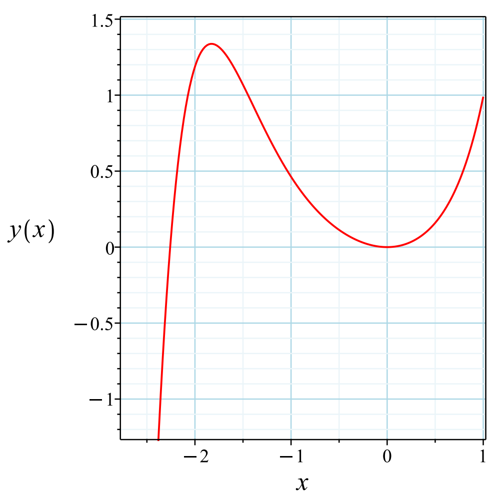
$$\begin{aligned}F_0 &= y + x\,\mathrm{e}^{y}\\ F_1 &= \frac{dF_0}{dx}\\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}F_0\\ &= x^2\mathrm{e}^{2y} + (xy + x + 1)\,\mathrm{e}^{y} + y\\ F_2 &= \frac{dF_1}{dx}\\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}F_1\\ &= 2\left(\mathrm{e}^{y}x^3 + yx^2 + x\right)\mathrm{e}^{2y} + \left(yx^2 + 2x^2 + x\right)\mathrm{e}^{2y} + \left(1 + xy^2 + 2y(x+1) + x\right)\mathrm{e}^{y} + y\\ F_3 &= \frac{dF_2}{dx}\\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}F_2\\ &= \mathrm{e}^{y}(1+y)^2 + \left(6\,\mathrm{e}^{2y}x^4 + \left(12x^3y + 7x^3 + 12x^2\right)\mathrm{e}^{y} + 6x^2y^2 + \left(7x^2 + 12x\right)y + 4x + 3\right)\mathrm{e}^{2y} + (1 +\\ F_4 &= \frac{dF_3}{dx}\\ &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}F_3\\ &= \left(4 + 3\left(12yx^4 + 11x^4 + 12x^3\right)\mathrm{e}^{2y} + \left(50y^2x^3 + \left(69x^3 + 100x^2\right)y + 17x^3 + 49x^2 + 30x\right)\mathrm{e}^{y} + 14y\\ F_5 &= \frac{dF_4}{dx}\\ &= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}F_4\\ &= \left(\left(150y^2x^4 + \left(307x^4 + 300x^3\right)y + 120x^4 + 247x^3 + 90x^2\right)\mathrm{e}^{2y} + \left(180x^3y^3 + \left(438x^3 + 540x^2\right)y^2\right.\right.\\ F_6 &= \frac{dF_5}{dx}\\ &= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y}F_5\\ &= \left(18 + \left(119 + 62y^4x^2 + 8\left(51x^2 + 31x\right)y^3 + 2\left(387x^2 + 511x + 85\right)y^2 + 2\left(236x^2 + 505x + 180\right)\right.\right.\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 0$ gives

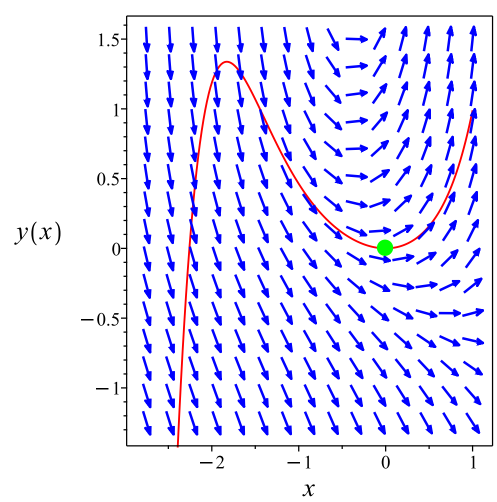
$$F_0 = 0$$
$$F_1 = 1$$
$$F_2 = 1$$
$$F_3 = 4$$
$$F_4 = 8$$
$$F_5 = 43$$
$$F_6 = 151$$

Substituting all the above in (6) and simplifying gives the solution as

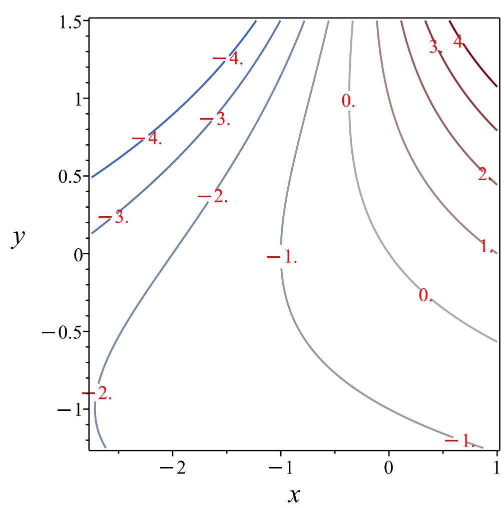
$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{15} + \frac{43x^6}{720} + \frac{151x^7}{5040} + O(x^8)$$



(a) Solution plot for $y' = y + x\,e^y$



(b) Direction fields for $y' = y + x\,e^y$



(c) Isoclines for $y' = y + x\,e^y$

2.1.5.1 ✓ Maple. `Time used: 0.002 (sec). Leaf size: 20`

```
Order:=8;
ode:=diff(y(x),x) = y(x)+x*exp(y(x));
ic:=[y(0) = 0];
dsolve([ode,op(ic)],y(x),type='series',x=0);
```

$$y = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{15}x^5 + \frac{43}{720}x^6 + \frac{151}{5040}x^7 + O\left(x^8\right)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
   -> Computing symmetries using: way = 3
   -> Computing symmetries using: way = 4
   -> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type
```

2.1.5.2 ✗ Mathematica

```
ode=D[y[x],x]==y[x]+x*Exp[y[x]];
ic={y[0]==0};
AsymptoticDSolveValue[{ode,ic},y[x],{x,0,7}]
```

Not solved

2.1.5.3 ✓ Sympy. Time used: 0.423 (sec). Leaf size: 39

```python
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x*exp(y(x)) - y(x) + Derivative(y(x), x),0)
ics = {y(0): 0}
dsolve(ode,func=y(x),ics=ics,hint="1st_power_series",x0=0,n=8)
```

$$y(x) = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{15} + \frac{43x^6}{720} + \frac{151x^7}{5040} + O(x^8)$$

```
Python version: 3.12.3 (main, Aug 14 2025, 17:47:21) [GCC 13.3.0]
Sympy version 1.14.0
```

```
classify_ode(ode,func=y(x))

('1st_power_series', 'lie_group')
```

2.1.6 Problem 3. direct method

Local contents

Internal problem ID [9492]
Book : A course in Ordinary Differential Equations. by Stephen A. Wirkus, Randall J. Swift. CRC Press NY. 2015. 2nd Edition
Section : Chapter 8. Series Methods. section 8.2. The Power Series Method. Problems Page 603
Problem number : 3. direct method
Date solved : `Sunday, March 01, 2026 at 04:36:07 AM`
CAS classification : `[‘y=_G(x,y’)‘]`

$$
\begin{aligned}
y' &= y + x\,\mathrm{e}^{y} \\
y(0) &= 0
\end{aligned}
$$

2.1.6.1 Existence and uniqueness analysis

$$
\begin{aligned}
y' &= y + x\,\mathrm{e}^{y} \\
y(0) &= 0
\end{aligned}
$$

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= y + \mathrm{e}^{y} x
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=0$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y + \mathrm{e}^{y} x) \\
&= 1 + \mathrm{e}^{y} x
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=0$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

Unknown ode type.

2.1.6.2 ✗ Maple

```
ode:=diff(y(x),x) = y(x)+x*exp(y(x));
ic:=[y(0) = 0];
dsolve([ode,op(ic)],y(x), singsol=all);
```

No solution found

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
   -> Computing symmetries using: way = 3
   -> Computing symmetries using: way = 4
   -> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type
```

2.1.6.3 ✗ Mathematica

```
ode=D[y[x],x]==y[x]+x*Exp[y[x]];
ic={y[0]==0};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.1.6.4 ✗ Sympy

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x*exp(y(x)) - y(x) + Derivative(y(x), x),0)
ics = {y(0): 0}
dsolve(ode,func=y(x),ics=ics)
```

```
TypeError : argument of type NegativeOne is not iterable
```

```
Python version: 3.12.3 (main, Aug 14 2025, 17:47:21) [GCC 13.3.0]
Sympy version 1.14.0
```

```
classify_ode(ode,func=y(x))

('1st_power_series', 'lie_group')
```